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K-FINITE SOLUTIONS TO CONFORMALLY INVARIANT SYSTEMS OF DIFFERENTIAL EQUATIONS

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Abstract. Let G be a connected semisimple linear real Lie group, and Q (resp. K) a real parabolic subgroup (resp. maximal compact subgroup) of G. The space of K-finite solutions to a conformally invariant system of differential equations on a line bundle over the real flag manifold G/Q is studied. The general theory is then applied to certain second order systems on the flag manifold that corresponds to the Heisenberg parabolic subgroup in a split simple Lie group.

1. Introduction. Suppose that $\mathcal{E} \to M$ is a vector bundle over a manifold M and $\mathfrak{g}$ is a Lie algebra of first-order differential operators that act on sections of $\mathcal{E}$. A linearly independent list $D_1, \dots, D_m$ of linear differential operators on sections of $\mathcal{E}$ is said to be *conformally invariant* with respect to $\mathfrak{g}$ if there are maps $C_{ji} : \mathfrak{g} \to C^\infty(M)$ such that

$$[X, D_i] = \sum_{j=1}^{m} C(X)_{ji} D_j$$

for all $1 \le i \le m$ and $X \in \mathfrak{g}$. Such systems were investigated in [2], where the reader may find further discussion of their significance and some references to earlier work on them.

Given a conformally invariant system of differential operators, one may investigate the subspace of its solutions inside various spaces of sections of the bundle $\mathcal{E}$. Normally, the subspace of solutions will depend very strongly on what space of sections has been chosen for study. When the Lie algebra $\mathfrak{g}$ derives from a semisimple real Lie group that acts on $\mathcal{E} \to M$ then one natural space of sections to choose is the space of K-finite sections, where $K \subset G$ is a maximal compact subgroup. This space is contained in many other natural spaces of sections, so that determination of the K-finite solutions is a useful starting point for studying solutions in more general spaces, and finiteness under K imposes strong restrictions that help to render the problem tractable.

In the present work, we specialize to the situation where $M = G/\bar{Q}$ is a real flag manifold and the vector bundle $\mathcal{E}$ is a homogeneous line bundle over M. In this setting, the problem of determining all K-finite solutions of a conformally invariant system can be reduced to a purely algebraic one. We describe this reduction of the problem in Section 2. The framework that is laid out here will be used in subsequent work to investigate various specific conformally invariant systems. As a first illustration of its use, it is applied to some of the

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systems that were constructed in [1]. In that work, a number of conformally invariant systems were constructed in the case where $\bar{Q}$ is a parabolic subgroup of Heisenberg type. The reader should note one slightly subtle point about the relationship between that construction and the problem considered here. The method for constructing conformally invariant systems that was used in [1] makes no reference to the real group G, but rather takes place in the Weyl algebra of the radical of a complex parabolic subalgebra of $\mathfrak{g}$. Once the system is in hand, it gives rise to a conformally invariant system on the flag manifold $G/\bar{Q}$ for every real form of the complex group associated to $\mathfrak{g}$ in which there is a real parabolic subgroup of the appropriate type. Although the construction is insensitive to which real form is taken, the space of K-finite solutions depends strongly on the real form, and so each system has many K-finite solution spaces associated with it. The project of studying spaces of K-finite solutions to interesting conformally invariant systems is continued in [4] and [5].

In Section 3 we take G to be the split real form in each type and investigate the systems that were labeled Ω_2 in [1]. In Section 4 we look in more detail at the K-finite solutions to the Ω_2 system for the split linear group of type C. Although this system is anomalous in several distinct ways and the determination of its K-finite solutions is dramatically easier than in all other cases, it is hoped that the exercise will still be instructive.

2. General considerations. The purpose of this section is to show how to reduce the problem of finding all K-finite solutions to a conformally invariant system of differential equations on a line bundle over a real flag manifold to a purely algebraic problem.

Let G be a connected linear Lie group with Lie algebra $\mathfrak{g}_0$. We assume that the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is semisimple. (The convention according to which the name of a real object includes a zero subscript that is removed to name the complexification of the object will be systematically employed.) Let $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ be a Cartan decomposition of $\mathfrak{g}_0$, K the maximal compact subgroup of G whose Lie algebra is $\mathfrak{k}_0$, and $\mathfrak{a}_0 \subset \mathfrak{p}_0$ a maximal abelian subspace.

We write $R(\mathfrak{g}_0, \mathfrak{a}_0)$ for the set of roots of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$, choose a positive system $R_+(\mathfrak{g}_0, \mathfrak{a}_0)$, and let $R_s(\mathfrak{g}_0, \mathfrak{a}_0)$ be the resulting set of simple roots. (Similar notation will be used for other roots systems.) To each subset of $R_s(\mathfrak{g}_0, \mathfrak{a}_0)$ is associated a real parabolic subalgebra of $\mathfrak{g}_0$, together with a Levi decomposition of this subalgebra. We shall call these subalgebras and Levi decompositions standard.

Let $\mathfrak{q}_0$ be a standard real parabolic subalgebra of $\mathfrak{g}_0$, $\mathfrak{q}_0 = \mathfrak{l}_0 \oplus \mathfrak{n}_0$ be its standard Levi decomposition, and use an overline to denote opposition, so that, for example, $\bar{\mathfrak{q}}_0$ denotes the real parabolic subalgebra opposite to $\mathfrak{q}_0$. Let $\bar{Q} = N_G(\bar{\mathfrak{q}}_0)$ be the real parabolic subgroup of G associated to $\bar{\mathfrak{q}}_0$ and $\bar{Q} = L\bar{N}$ be the Levi decomposition of $\bar{Q}$ that is consistent with the standard Levi decomposition of $\bar{\mathfrak{q}}_0$.

Let $\chi : L \to \boldsymbol{R}^\times$ be an analytic homomorphism. We may extend χ to be trivial on $\bar{N}$ and so obtain an analytic homomorphism $\chi : \bar{Q} \to \boldsymbol{R}^\times$. Associated to this homomorphism is a homogeneous line bundle $\mathcal{L} \to G/\bar{Q}$. The total space of this bundle may be constructed as the quotient of $G \times \boldsymbol{C}$ by the equivalence relation $(g\bar{q}, z) \sim (g, \chi(\bar{q})^{-1}z)$ for $g \in G$, $\bar{q} \in \bar{Q}$, and $z \in \boldsymbol{C}$. The space $\Gamma(U, \mathcal{L})$ of smooth sections of $\mathcal{L}$ over an open set $U \subset G/\bar{Q}$ may be identified with the space of smooth functions $\varphi : W \to \boldsymbol{C}$ that satisfy $\varphi(g\bar{q}) = \chi(\bar{q})\varphi(g)$ for $g \in W$ and $\bar{q} \in \bar{Q}$, where W is the preimage of U under the canonical projection $G \to G/\bar{Q}$. The space $\Gamma(\mathcal{L})$ of smooth global sections of $\mathcal{L}$ under the left-translation action of G gives a model of the smooth induced representation $\mathrm{Ind}(G, \bar{Q}, \chi^{-1})$. We shall denote the resulting representation of G by π and the derived representation of $\mathfrak{g}$ by Π. The latter action realizes $\mathfrak{g}$ as an algebra of first order differential operators on $G/\bar{Q}$ and hence extends to $\Gamma(U, \mathcal{L})$ for any open set $U \subset G/\bar{Q}$. (This and similar extensions will henceforth be made silently.)

It follows from the theory of the Iwasawa decomposition that the map $h(K \cap L) \mapsto h\bar{Q}$ is a diffeomorphism from $K/(K \cap L)$ to $G/\bar{Q}$. The objects that were constructed above on $G/\bar{Q}$ may be transported to $K/(K \cap L)$ via this diffeomorphism. To avoid overburdening the notation, we shall typically use the same symbol for an object on $G/\bar{Q}$ and its pullback to $K/(K \cap L)$, allowing context to remove the ambiguity. In particular, we obtain a model of the smooth induced representation $\mathrm{Ind}(G, \bar{Q}, \chi^{-1})$ in the space of smooth functions φ on K that satisfy the transformation law $\varphi(kl) = \chi(l)\varphi(k)$ for $k \in K$ and $l \in K \cap L$. Note that χ is trivial on $(K \cap L)^{\circ}$, the connected component of the identity in $K \cap L$, and that $\chi(K \cap L) \subset \{\pm 1\}$.

We have $\mathfrak{g} = \mathfrak{k} + \bar{\mathfrak{q}}$ and $\mathfrak{g} = (\mathfrak{k} \cap \mathfrak{l})^{\perp} \oplus \bar{\mathfrak{q}}$, where the orthogonal complement in the first summand is with respect to the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{k}$. Thus we may express any element of $\mathfrak{g}$ as the sum of an element of $\mathfrak{k}$ and an element of $\bar{\mathfrak{q}}$, and we may insist that the element of $\mathfrak{k}$ lie in $(\mathfrak{k} \cap \mathfrak{l})^{\perp}$ if uniqueness of the expression is desired. In using this decomposition, we shall often start with the more restrictive unique version, and then observe that the result would be unaffected if the less restrictive version were used instead. For calculation, the less restrictive version may be more convenient.

We write $d\chi : \bar{\mathfrak{q}} \to \boldsymbol{C}$ for the differential of χ and denote by $\boldsymbol{C}_{d\chi}$ the one-dimensional $\bar{\mathfrak{q}}$-module on which $\bar{\mathfrak{q}}$ acts via $d\chi$. Since χ is trivial on $(K \cap L)^{\circ}$, $d\chi$ vanishes on $\mathfrak{k} \cap \mathfrak{l}$ and the restriction of $\boldsymbol{C}_{d\chi}$ to $\mathfrak{k} \cap \mathfrak{l}$ is the trivial $(\mathfrak{k} \cap \mathfrak{l})$-module.

LEMMA 2.1. *The $\boldsymbol{C}$-linear map*

$$\iota : \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k}\cap\mathfrak{l})} \boldsymbol{C} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$$

given on simple tensors by $\iota(u \otimes 1) = u \otimes 1$ is an isomorphism of $\mathcal{U}(\mathfrak{k})$-modules.

PROOF. We have observed that the restriction of $\boldsymbol{C}_{d\chi}$ to $\mathfrak{k} \cap \mathfrak{l}$ is trivial and it follows that ι is well-defined. Let us fix an ordered basis for $(\mathfrak{k} \cap \mathfrak{l})^{\perp}$. This basis may be extended to a basis for $\mathfrak{k}$ by adjoining a basis for $\mathfrak{k} \cap \mathfrak{l}$ or to a basis for $\mathfrak{g}$ by adjoining a basis for $\bar{\mathfrak{q}}$. By the PBW Theorem applied to $\mathcal{U}(\mathfrak{k})$ and to $\mathcal{U}(\mathfrak{g})$, it follows that ι is an isomorphism of vector spaces. Finally, ι is a $\mathfrak{k}$-module homomorphism by direct calculation. □

The functional $d\chi$ is invariant under the adjoint action of L on $\bar{\mathfrak{q}}$ and it follows that L acts adjointly on the module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$. Similarly, $K \cap L$ acts adjointly on the module $\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k}\cap\mathfrak{l})} \boldsymbol{C}$. The map ι defined in Lemma 2.1 is equivariant for $K \cap L$.

Let σ be an irreducible representation of K. Write E_{σ} for the space on which σ is realized and fix a non-zero K-invariant Hermitian form $\langle\, \cdot\,, \,\cdot\, \rangle_{\sigma}$ on E_{σ}. We take this form to be linear in its first argument and conjugate linear in its second argument. For $\xi_1, \xi_2 \in E_\sigma$, we define a function $\psi_\sigma(\xi_1, \xi_2)$ on K by

$$\psi_\sigma(\xi_1, \xi_2)(k) = \langle \xi_1, \sigma(k)\xi_2 \rangle_\sigma .$$

This function may be regarded as an element of $\Gamma(\mathcal{L})$ if and only if $\xi_2 \in E_\sigma^{(K\cap L,\chi)}$, the subspace of E_σ consisting of those vectors that transform by χ under $K \cap L$. Every K-finite vector in $\Gamma(\mathcal{L})$ is a finite linear combination of various of the vectors $\psi_\sigma(\xi_1, \xi_2)$ with $\xi_2 \in E_\sigma^{(K\cap L,\chi)}$.

We now wish to apply the framework and results of [2] to the present situation. We obtain the strongest results by restricting attention to the dense open set $U \subset K/(K \cap L)$ that corresponds to $N\bar{Q}/\bar{Q} \subset G/\bar{Q}$, since this places us in the situation studied in Sections 5, 6 and 7 of [2]. The restriction is harmless for the purpose at hand. Note that we may identify U with N as $\mathfrak{g}$-manifolds, and we shall do so when convenient.

As in [2], we let $\boldsymbol{D}(\mathcal{L}|_U)$ denote the algebra of linear differential operators on $\mathcal{L}|_U$. The conformally invariant systems that we shall consider will be composed of operators that lie in the subalgebra $\boldsymbol{D}(\mathcal{L}|_U)^{\mathfrak{n}}$ that consists of all elements of $\boldsymbol{D}(\mathcal{L}|_U)$ that commute with $\Pi(X)$ for all $X \in \mathfrak{n}$. As explained in Section 5 of [2], there is an isomorphism $\Lambda \mapsto D_\Lambda$ from $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi} \otimes_{\boldsymbol{C}} \boldsymbol{C}_{-d\chi}$ to $\boldsymbol{D}(\mathcal{L}|_U)^{\mathfrak{n}}$. Let $u \mapsto u^\dagger$ be the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ that satisfies $Y^\dagger = -Y$ for all $Y \in \mathfrak{g}$. By following the construction of D_Λ that is explained in [2], we find that if $\Lambda = u \otimes 1 \otimes 1$ then

$$(D_\Lambda \bullet \varphi)(n) = \big(\Pi(u^\dagger) \bullet (\ell_{n^{-1}}\varphi)\big)(e) .$$

Here $\ell_{n^{-1}}$ denotes left translation by n^{-1} and $\bullet$ designates application of a differential operator to an element of its domain. Let $u \mapsto \bar{u}$ be the conjugate linear map from $\mathcal{U}(\mathfrak{g})$ to $\mathcal{U}(\mathfrak{g})$ that is induced by the complex conjugation of $\mathfrak{g}$ with respect to the real structure $\mathfrak{g}_0$.

LEMMA 2.2. *Let $u \in \mathcal{U}(\mathfrak{k})$ and write $\Lambda = \iota(u\otimes 1)\otimes 1$, where ι is the map introduced in Lemma 2.1. Let (σ, E_σ) be an irreducible representation of K, $\xi_1 \in E_\sigma$, and $\xi_2 \in E_\sigma^{(K\cap L,\chi)}$. Then we have*

$$\big(D_\Lambda \bullet \psi_\sigma(\xi_1, \xi_2)\big)(e) = \langle \xi_1, d\sigma(\bar{u})\xi_2 \rangle_\sigma .$$

PROOF. Let $Z \in \mathfrak{k}$. Then, directly from the definitions, we find that

$$\Pi(Z) \bullet \psi_\sigma(\xi_1, \xi_2) = \psi_\sigma(d\sigma(Z)\xi_1, \xi_2)$$

and it follows that

$$\Pi(u) \bullet \psi_\sigma(\xi_1, \xi_2) = \psi_\sigma(d\sigma(u)\xi_1, \xi_2)$$

for all $u \in \mathcal{U}(\mathfrak{k})$. The above expression for D_Λ now gives

$$\big(D_\Lambda \bullet \psi_\sigma(\xi_1, \xi_2)\big)(e) = \langle d\sigma(u^\dagger)\xi_1, \xi_2 \rangle_\sigma .$$

It remains to observe that

$$\langle d\sigma(u^\dagger)\xi_1, \xi_2 \rangle_\sigma = \langle \xi_1, d\sigma(\bar{u})\xi_2 \rangle_\sigma ,$$

because the Hermitian form is K-invariant. □

We continue in the framework established in [2]. Let $D_1, \ldots, D_m \in \boldsymbol{D}(\mathcal{L}|_U)$ be a straight, L-stable, homogeneous, conformally invariant system. By definition, there is a homomorphism $c : L \to \mathrm{GL}(m, \boldsymbol{C})$ such that

$$\pi(l) \circ D_i \circ \pi(l^{-1}) = \sum_{j=1}^{m} c(l)_{ji} D_j \tag{1}$$

for $1 \leq i \leq m$ and $l \in L$. We choose $\Lambda_i \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi} \otimes_{\boldsymbol{C}} \boldsymbol{C}_{-d\chi}$ such that $D_{\Lambda_i} = D_i$ for $1 \leq i \leq m$ and let $\Lambda_i = v_i \otimes 1$ with $v_i \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$. Note that the v_i are uniquely determined by this description. We have $d\chi(\mathrm{Ad}(l)X) = d\chi(X)$ for $X \in \bar{\mathfrak{q}}$, and consequently L acts on the module $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$ via $\mathrm{Ad} \otimes \chi$. In terms of this action, (1) becomes

$$l v_i = \chi(l) \sum_{j=1}^{m} c_{ji}(l) v_j \,. \tag{2}$$

If $g \in NL\bar{N}$ then there is a unique factorization

$$g = \zeta(g) a(g) \bar{\zeta}(g)$$

with $\zeta(g) \in N$, $a(g) \in L$, and $\bar{\zeta}(g) \in \bar{N}$. For any $g \in G$, the set $U_g = N\bar{Q}/\bar{Q} \cap gN\bar{Q}/\bar{Q}$ is open and dense in $G/\bar{Q}$. If $n \in N$ and $n\bar{Q} \in U_g$ then $g^{-1}n \in NL\bar{N}$ and so $g^{-1}n$ has a factorization of the above type.

PROPOSITION 2.3. *Let $g \in G$. Then we have*

$$\pi(g) \circ D_i \circ \pi(g^{-1}) = \sum_{j=1}^{m} c_{ji}(a(g^{-1}n)^{-1}) D_j$$

on the set U_g. In particular, the space of all $\varphi \in \Gamma(\mathcal{L})$ such that $D_i \bullet \varphi = 0$ for all $1 \leq i \leq m$ is invariant under G.

PROOF. We make use of the notation introduced above. Let $F \subset \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$ be the span of the elements $v_1, \ldots, v_m$. Then, by (2), F is stable under the action of L on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$ and we denote by η the representation of L that is afforded by this space. The infinitesimal representation $d\eta$ may be extended by zero on $\bar{\mathfrak{n}}$ to yield a representation of $\bar{\mathfrak{q}}$. It follows from Theorem 19 in [2] that $\bar{\mathfrak{n}}F = \{0\}$ and consequently there is a map

$$T \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} F_{d\eta}, \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi})$$

such that $T(y \otimes v) = yv$ for $y \in \mathcal{U}(\mathfrak{g})$ and $v \in F$. By Lemma 2.4 in [3], the map T corresponds to a differential intertwining operator $\check{T}$ from the smooth induced representation $\mathrm{Ind}(G, \bar{Q}, \chi^{-1})$ to the smooth induced representation $\mathrm{Ind}(G, \bar{Q}, \eta^*)$, where η^* denotes the contragradient of η. If we make the usual identification of vectors in these smooth induced representations with functions on N then we have

$$\check{T}(\varphi) = \sum_{j=1}^{m} (D_j \bullet \varphi) v_j^* ,$$

where $v_1^*, \ldots, v_m^*$ is the basis of F^* dual to $v_1, \ldots, v_m$.

Let π_η denote the induced representation from η^*. Since $\check{T}$ is an intertwining operator, we have $\pi_\eta(g^{-1}) \circ \check{T} = \check{T} \circ \pi(g^{-1})$. Let $\varphi \in \Gamma(\mathcal{L})$ and $n \in N$ with $n\bar{Q} \in U_{g^{-1}}$. Then

$$\begin{aligned}\big(\pi_\eta(g^{-1})\check{T}(\varphi)\big)(n) &= \pi_\eta(g^{-1})\Big(\sum_{j=1}^{m}(D_j \bullet \varphi) v_j^*\Big)(n) \\ &= \sum_{j=1}^{m}(D_j \bullet \varphi)(\zeta(gn))\eta^*(a(gn)^{-1}) v_j^* \\ &= \chi(a(gn)) \sum_{j,p=1}^{m} (D_j \bullet \varphi)(\zeta(gn)) c_{jp}(a(gn)) v_p^*\end{aligned}$$

and the equality of this and $(\check{T}(\pi(g^{-1})\varphi))(n)$ implies that

$$\big(D_i \bullet (\pi(g^{-1})\varphi)\big)(n) = \chi(a(gn)) \sum_{j=1}^{m} c_{ji}(a(gn))(D_j \bullet \varphi)(\zeta(gn)) . \tag{3}$$

It follows directly from the definitions that if $n\bar{Q} \in U_g$ then we have $\zeta(gn) \in U_{g^{-1}}$, $\zeta(g\zeta(g^{-1}n)) = n$, and $a(g\zeta(g^{-1}n)) = a(g^{-1}n)^{-1}$. Thus if $n\bar{Q} \in U_g$ then

$$\begin{aligned}&\big(\pi(g)(D_i \bullet (\pi(g^{-1})\varphi))\big)(n) \\ &\quad = \chi(a(g^{-1}n))\big(D_i \bullet (\pi(g^{-1})\varphi)\big)(\zeta(g^{-1}n)) \\ &\quad = \chi(a(g^{-1}n))\chi(a(g\zeta(g^{-1}n))) \sum_{j=1}^{m} c_{ji}(a(g\zeta(g^{-1}n)))(D_j \bullet \varphi)(\zeta(g\zeta(g^{-1}n))) \\ &\quad = \chi(a(g^{-1}n))\chi(a(g^{-1}n)^{-1}) \sum_{j=1}^{m} c_{ji}(a(g^{-1}n)^{-1})(D_j \bullet \varphi)(n) \\ &\quad = \sum_{j=1}^{m} c_{ji}(a(g^{-1}n)^{-1})(D_j \bullet \varphi)(n) ,\end{aligned}$$

where we have used (3) from the second line to the third. This evaluation establishes the required equality for $\varphi \in \Gamma(\mathcal{L})$, but it automatically extends to an identity of differential operators on U_g because the set of restrictions of elements of $\Gamma(\mathcal{L})$ to U_g is dense in the space of smooth functions on U_g with the smooth topology. Moreover, since U_g is dense in $N\bar{Q}/\bar{Q}$, it follows from the identity that was just derived that if $D_i \bullet \varphi = 0$ for $1 \leq i \leq m$ then $D_i \bullet (\pi(g^{-1})\varphi) = 0$ for all $g \in G$ and $1 \leq i \leq m$. This observation establishes the second claim and completes the proof. $\square$

It follows from Proposition 2.3 and general properties of induced representations that

$$\Gamma(\mathcal{L})^D = \{\varphi \in \Gamma(\mathcal{L})\,;\; D_i \bullet \varphi = 0 \;\text{ for }\; 1 \le i \le m\}$$

is a closed, G-invariant subspace of $\Gamma(\mathcal{L})$, and affords a smooth, admissible representation of G.

COROLLARY 2.4. *With $D_1, \dots, D_m$ as above, suppose in addition that $D_i \bullet 1 = 0$ for $1 \le i \le m$. Then, for all $g \in G$, the function $n \mapsto \chi(a(g^{-1}n))$ is a solution to the system $D_1, \dots, D_m$ on U_g.*

PROOF. This follows from Proposition 2.3 on observing that the specified function is $\pi(g)1$. □

The special solutions identified in Corollary 2.4 are not usually restrictions of elements of $\Gamma(\mathcal{L})$. They are also not usually locally integrable, but they can be regarded as defining distributions on $N\bar{Q}/\bar{Q}$ by a suitable analytic continuation procedure, and it is an interesting problem to compute the result of applying the operators D_i to these distributions. We shall consider this problem elsewhere.

With $D_1, \dots, D_m$ as above, let us choose $u_1, \dots, u_m \in \mathcal{U}(\mathfrak{k})$ such that

$$D_i = D_{\iota(u_i \otimes 1) \otimes 1}$$

for $1 \le i \le m$.

LEMMA 2.5. *Let (σ, E_σ) be an irreducible representation of K, $\xi_1 \in E_\sigma$, and $\xi_2 \in E_\sigma^{(K \cap L, \chi)}$. If $k \in K \cap N\bar{Q}$ then*

$$\big(D_i \bullet \psi_\sigma(\xi_1, \xi_2)\big)(k) = \sum_{j=1}^{m} c_{ji}(a(k)^{-1})\psi_\sigma(\xi_1, d\sigma(\bar{u}_j)\xi_2)(k)\,.$$

PROOF. We have

$$\begin{aligned}
\big(D_i \bullet \psi_\sigma(\xi_1, \xi_2)\big)(k) &= \big(\pi(k^{-1})(D_i \bullet \psi_\sigma(\xi_1, \xi_2))\big)(e) \\
&= \big((\pi(k^{-1}) \circ D_i \circ \pi(k))(\psi_\sigma(\sigma(k^{-1})\xi_1, \xi_2))\big)(e) \\
&= \sum_{j=1}^{m} c_{ji}(a(k)^{-1})\big(D_j \bullet \psi_\sigma(\sigma(k^{-1})\xi_1, \xi_2)\big)(e) \\
&= \sum_{j=1}^{m} c_{ji}(a(k)^{-1})\langle \sigma(k^{-1})\xi_1, d\sigma(\bar{u}_j)\xi_2\rangle_\sigma \\
&= \sum_{j=1}^{m} c_{ji}(a(k)^{-1})\langle \xi_1, \sigma(k)d\sigma(\bar{u}_j)\xi_2\rangle_\sigma \\
&= \sum_{j=1}^{m} c_{ji}(a(k)^{-1})\psi_\sigma\big(\xi_1, d\sigma(\bar{u}_j)\xi_2\big)(k)\,,
\end{aligned}$$

where we have used Proposition 2.3 from the third line to the fourth, and Lemma 2.2 from the fourth line to the fifth. □

Let $\hat{K}$ denote the set of isomorphism classes of irreducible representations of K and for $\sigma \in \hat{K}$ set

$$\boldsymbol{M}_\chi(\sigma) = \{\xi \in E_\sigma^{(K\cap L,\chi)}\,;\ d\sigma(\bar{u}_i)\xi = 0 \ \text{ for } \ 1 \le i \le m\}\,.$$

For any admissible Frechet representation r of G, let $\mathrm{HC}(r)$ denote the underlying Harish-Chandra module.

THEOREM 2.6. *As representations of K, we have*

$$\mathrm{HC}(\Gamma(\mathcal{L})^D) \cong \bigoplus_{\sigma\in\hat{K}} \sigma \otimes \overline{\boldsymbol{M}_\chi(\sigma)}\,.$$

On the summand corresponding to σ, the isomorphism from right to left satisfies $\xi_1 \otimes \xi_2 \mapsto \psi_\sigma(\xi_1, \xi_2)$.

PROOF. As usual,

$$\mathrm{HC}(\Gamma(\mathcal{L})) \cong \bigoplus_{\sigma\in\hat{K}} \sigma \otimes \bar{E}_\sigma^{(K\cap L,\chi)}$$

and this decomposition corresponds to an isomorphism

$$\bar{E}_\sigma^{(K\cap L,\chi)} \cong \mathrm{Hom}_K(\sigma, \mathrm{HC}(\Gamma(\mathcal{L})))$$

of complex vector spaces. In order to establish the claimed isomorphism, we merely have to verify that the subspace $\overline{\boldsymbol{M}_\chi(\sigma)}$ of $\bar{E}_\sigma^{(K\cap L,\chi)}$ corresponds to the subspace

$$\mathrm{Hom}_K(\sigma, \mathrm{HC}(\Gamma(\mathcal{L})^D)) \subset \mathrm{Hom}_K(\sigma, \mathrm{HC}(\Gamma(\mathcal{L})))\,.$$

This correspondence is immediate from Lemma 2.5, since $K \cap N\bar{Q}$ is dense in K. □

For $\sigma \in \hat{K}$, it is convenient to consider

$$\boldsymbol{M}(\sigma) = \{\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}\,;\ d\sigma(\bar{u}_i)\xi = 0 \ \text{ for } 1 \le i \le m\}.$$

This space is stable under the action of $K \cap L$, and $(K \cap L)^\circ$ acts trivially on it. It is known that the component group of $K \cap L$ is an elementary abelian 2-group. Consequently, $\boldsymbol{M}(\sigma)$ decomposes into a direct sum of eigenspaces for $K \cap L$, with each eigenspace corresponding to a sign character of $K \cap L$. The inclusion $K \cap L \to L$ induces an isomorphism of components groups, and so these sign characters may also be regarded as sign characters of L. Thus if χ is a real-valued analytic character of L then we have

$$\boldsymbol{M}(\sigma) = \bigoplus_{\varepsilon} \boldsymbol{M}_{\varepsilon\chi}(\sigma)\,,$$

where the sum is over all sign characters of L. Note that the action of $\mathfrak{g}$ on $\Gamma(\mathcal{L})$ depends only the restriction of χ to L°, as does the conformal invariance of a system of operators $D_1, \dots, D_m$ on $\mathcal{L}|_U$ with $U = N\bar{Q}/\bar{Q}$. Thus an element of $\boldsymbol{M}_{\varepsilon\chi}(\sigma)$ corresponds to an embedding of σ into $\Gamma(\mathcal{L})^D$, where $\mathcal{L}$ corresponds to $\varepsilon\chi$ rather than χ. For this reason, we shall call a $(K \cap L)$-eigenvector in $\boldsymbol{M}(\sigma)$ an *embedding vector*.

We now consider the action of $\mathfrak{p}$ on $\mathrm{HC}(\Gamma(\mathcal{L})^D)$. To formulate the result, it is convenient to extend the symbols ψ_σ, $\boldsymbol{M}_\chi(\sigma)$, and $\boldsymbol{M}(\sigma)$ to allow σ to be a direct sum of finitely-many irreducible representations, which may be done in the obvious way. We also confuse the space $\mathfrak{p}$ and the representation $(\mathrm{Ad}, \mathfrak{p})$ of K when convenient. The form $\mathbf{B}$ is positive definite and K-invariant on $\mathfrak{p}_0$, and there is a K-invariant Hermitian form $\langle \cdot\,, \cdot \rangle_{\mathfrak{p}}$ on $\mathfrak{p} = \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{p}_0$ that satisfies

$$\langle z_1 \otimes Y_1, z_2 \otimes Y_2 \rangle_{\mathfrak{p}} = z_1 \bar{z}_2 \mathbf{B}(Y_1, Y_2)$$

for $z_1, z_2 \in \boldsymbol{C}$ and $Y_1, Y_2 \in \mathfrak{p}_0$. Let $\{W_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{p}_0$ with respect to $\mathbf{B}$. For each $1 \leq i \leq n$, write $W_i = Z_i + U_i$ with $Z_i \in \mathfrak{k}_0$ and $U_i \in \bar{\mathfrak{q}}_0$. Given $\sigma \in \hat{K}$, define a map $R(\sigma) : E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \to \mathfrak{p} \otimes E_\sigma$ by

$$R(\sigma)\xi = \sum_{i=1}^{n} W_i \otimes (d\sigma(Z_i) + d\chi(U_i))\xi \tag{4}$$

for $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$. Note that, from the bilinearity of the tensor product, $R(\sigma)$ is independent of the choice of $\{W_i\}_{i=1}^n$. We take the standard choice of K-invariant Hermitian form on $\mathfrak{p} \otimes E_\sigma$ to be the one that satisfies

$$\langle Y_1 \otimes \xi_1, Y_2 \otimes \xi_2 \rangle_{\mathfrak{p}\otimes\sigma} = \langle Y_1, Y_2 \rangle_{\mathfrak{p}} \langle \xi_1, \xi_2 \rangle_\sigma$$

for $Y_1, Y_2 \in \mathfrak{p}$ and $\xi_1, \xi_2 \in E_\sigma$.

LEMMA 2.7. *The map $R(\sigma) : E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \to \mathfrak{p} \otimes E_\sigma$ is $(K \cap L)$-intertwining.*

PROOF. For $k \in K \cap L$, a brief calculation yields

$$R(\sigma)\sigma(k)\xi = (\mathrm{Ad} \otimes \sigma)(k) \sum_{i=1}^{n} \mathrm{Ad}(k^{-1})W_i \otimes (d\sigma(\mathrm{Ad}(k^{-1})Z_i) + d\chi(U_i))\xi\,.$$

Now $\{\mathrm{Ad}(k^{-1})W_i\}_{i=1}^n$ is an orthonormal basis for $\mathfrak{p}_0$, $\mathrm{Ad}(k^{-1})W_i = \mathrm{Ad}(k^{-1})Z_i + \mathrm{Ad}(k^{-1})U_i$ is a decomposition of $\mathrm{Ad}(k^{-1})W_i$ into the sum of an element of $\mathfrak{k}_0$ and an element of $\bar{\mathfrak{q}}_0$, and $d\chi(U_i) = d\chi(\mathrm{Ad}(k^{-1})U_i)$ for $1 \leq i \leq n$. These observations and the independence of $R(\sigma)$ of the choice of the orthonormal basis used to compute it imply that $R(\sigma)\sigma(k) = (\mathrm{Ad} \otimes \sigma)(k)R(\sigma)$, as claimed. □

Note that Lemma 2.7 implies that

$$R(\sigma)(E_\sigma^{(K\cap L,\chi)}) \subset (\mathfrak{p} \otimes E_\sigma)^{(K\cap L,\chi)}\,. \tag{5}$$

Although $R(\sigma)$ has been defined on $E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$, only its restriction to $E_\sigma^{(K\cap L,\chi)}$ is of interest for the present purposes. We shall shortly see that a containment similar to (5) also holds for the subspaces of embedding vectors.

Suppose that $\xi_1 \in E_\sigma$ and $\xi_2 \in E_\sigma^{(K\cap L,\chi)}$, so that $\psi_\sigma(\xi_1, \xi_2) \in \Gamma(\mathcal{L})$. Let $Y \in \mathfrak{p}_0$ and $k \in K$. Then we may find $\varepsilon > 0$ and smooth maps $\kappa : (-\varepsilon, \varepsilon) \to K$ and $\bar{q} : (-\varepsilon, \varepsilon) \to \bar{Q}$

such that $\kappa(0) = \bar{q}(0) = e$ and

$$\exp(-tY)k = k\kappa(t)\bar{q}(t) \tag{6}$$

for $|t| < \varepsilon$. Directly from the definition of the action, we have

$$\begin{aligned}\big(\Pi(Y)\bullet\psi_\sigma(\xi_1, \xi_2)\big)(k) &= d\chi(\bar{q}'(0))\psi_\sigma(\xi_1, \xi_2)(k) + \frac{d}{dt}\bigg|_{t=0} \psi_\sigma(\xi_1, \xi_2)(k\kappa(t)) \\ &= d\chi(\bar{q}'(0))\psi_\sigma(\xi_1, \xi_2)(k) + \langle \xi_1, \sigma(k) d\sigma(\kappa'(0))\xi_2\rangle_\sigma .\end{aligned}$$

By rearranging and differentiating (6), we obtain

$$-\mathrm{Ad}(k^{-1})Y = \kappa'(0) + \bar{q}'(0) \in \mathfrak{k}_0 + \bar{\mathfrak{q}}_0 .$$

As we have remarked above, such a decomposition of $-\mathrm{Ad}(k^{-1})Y$ is not unique, but the non-uniqueness is irrelevant for the determination of $\Pi(Y)\bullet\psi_\sigma(\xi_1, \xi_2)$. By taking advantage of the orthonormal basis $\{W_i\}_{i=1}^n$ that we have chosen above, we may write

$$\begin{aligned}\mathrm{Ad}(k^{-1})Y &= \sum_{i=1}^n \mathbf{B}(\mathrm{Ad}(k^{-1})Y, W_i)W_i \\ &= \sum_{i=1}^n \mathbf{B}(\mathrm{Ad}(k^{-1})Y, W_i)Z_i + \sum_{i=1}^n \mathbf{B}(\mathrm{Ad}(k^{-1})Y, W_i)U_i \\ &= \sum_{i=1}^n \mathbf{B}(Y, \mathrm{Ad}(k)W_i)Z_i + \sum_{i=1}^n \mathbf{B}(Y, \mathrm{Ad}(k)W_i)U_i ,\end{aligned}$$

and taking the resulting possibilities for $\kappa'(0)$ and $\bar{q}'(0)$, we obtain

$$\begin{aligned}&-\big(\Pi(Y)\bullet\psi_\sigma(\xi_1, \xi_2)\big)(k) \\ &\quad= \sum_{i=1}^n \mathbf{B}(Y, \mathrm{Ad}(k)W_i)(d\chi(U_i)\langle \xi_1, \sigma(k)\xi_2\rangle_\sigma + \langle \xi_1, \sigma(k)d\sigma(Z_i)\xi_2\rangle_\sigma) \\ &\quad= \sum_{i=1}^n \langle Y, \mathrm{Ad}(k)W_i\rangle_{\mathfrak{p}} \langle \xi_1, \sigma(k)(d\sigma(Z_i) + d\chi(U_i))\xi_2\rangle_\sigma \\ &\quad= \langle Y \otimes \xi_1, (\mathrm{Ad} \otimes \sigma)(k)R(\sigma)\xi_2\rangle_{\mathfrak{p}\otimes\sigma} \\ &\quad= \psi_{\mathfrak{p}\otimes\sigma}(Y \otimes \xi_1, R(\sigma)\xi_2)(k) .\end{aligned}$$

From this calculation, we conclude that

$$\Pi(Y)\bullet\psi_\sigma(\xi_1, \xi_2) = -\psi_{\mathfrak{p}\otimes\sigma}(Y \otimes \xi_1, R(\sigma)\xi_2)$$

for $Y \in \mathfrak{p}_0$. Since both sides are complex-linear in Y, this evaluation extends to all $Y \in \mathfrak{p}$. If we take $\xi_2 \in \boldsymbol{M}_\chi(\sigma)$, so that $\psi_\sigma(\xi_1, \xi_2) \in \Gamma(\mathcal{L})^D$ then, by the conformal invariance of the system, $\Pi(Y)\bullet\psi_\sigma(\xi_1, \xi_2) \in \Gamma(\mathcal{L})^D$ also, and we conclude that $R(\sigma)\xi_2 \in \boldsymbol{M}_\chi(\mathfrak{p} \otimes \sigma)$. That is, we have

$$R(\sigma)\boldsymbol{M}_\chi(\sigma) \subset \boldsymbol{M}_\chi(\mathfrak{p} \otimes \sigma) .$$

The action of $R(\sigma)$ on embedding vectors is a skeletal version of the action of $\mathfrak{p}$ on $\Gamma(\mathcal{L})^D$ and, in favorable cases, may be used to study reducibility questions for this module.

3. Conformally invariant systems attached to the Heisenberg parabolic subalgebra in the split real form. In order to apply the results of Section 2 we require specific examples of conformally invariant systems on real flag manifolds. In [1] numerous examples of such systems were constructed for the case where the parabolic subalgebra belongs to the unique conjugacy class of Heisenberg parabolic subalgebras in $\mathfrak{g}$. If $\mathfrak{g}_0$ is a particular real form of $\mathfrak{g}$ for which some such parabolic subalgebra is real then these systems give rise to conformally invariant systems on $G/\bar{Q}$. Although these systems are constructed without reference to a real form, we cannot discuss their K-finite solutions until a real form has been fixed. The purpose of this section is to study these conformally invariant systems on the split real form of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank greater than one. Choose a Cartan subalgebra $\mathfrak{h}$, let $R = R(\mathfrak{g}, \mathfrak{h})$ be the resulting root system, fix a positive system $R_+ \subset R$, and let $R_s \subset R_+$ be the set of simple roots. Let $\mathbf{B}$ be a positive multiple of the Killing form of $\mathfrak{g}$ and let $(\,\cdot\,,\,\cdot\,)$ denote the inner product on $\mathfrak{h}^*$ induced by $\mathbf{B}$. The normalization of $\mathbf{B}$ is fixed by requiring that the length of the long roots in R is $\sqrt{2}$, with all roots being considered long in the simply-laced case. We choose a Chevalley system in $\mathfrak{g}$ that is normalized to satisfy conditions (C1) through (C9) in Section 2 of [1]. These normalization conditions will be recalled as they are required. For each $\alpha \in R$, we have a root vector X_α and an element $H_\alpha \in \mathfrak{h}$ such that $X_\alpha, H_\alpha, X_{-\alpha}$ is an $\mathfrak{sl}(2)$-triple. The chosen normalization of $\mathbf{B}$ implies that $\mathbf{B}(X_\alpha, X_{-\alpha}) = 2/\|\alpha\|^2$ for all $\alpha \in R$.

Let $\mathfrak{g}_0$ be the $\boldsymbol{R}$-span of the X_α and the H_α. Then $\mathfrak{g}_0$ is a split real form of $\mathfrak{g}$ with Cartan involution θ determined by $\theta(X_\alpha) = -X_{-\alpha}$ and $\theta(H) = -H$ for $H \in \mathfrak{h}_0$. For $\alpha \in R$, let $Z_\alpha = X_\alpha - X_{-\alpha}$. Then $Z_\alpha \in \mathfrak{k}_0$ for all $\alpha \in R$, the set $\{Z_\alpha\,;\ \alpha \in R_+\}$ is an $\boldsymbol{R}$-basis for $\mathfrak{k}_0$, and we have

$$[Z_\alpha, Z_\beta] = N_{\alpha,\beta} Z_{\alpha+\beta} - N_{\alpha,-\beta} Z_{\alpha-\beta}\,,$$

where $N_{\alpha,\beta}$ is the structure constant defined by $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha + \beta \in R$ and $N_{\alpha,\beta} = 0$ otherwise.

Let $\gamma \in R_+$ be the highest root and $\mathfrak{q}$ the standard parabolic subalgebra corresponding to the set $\{\alpha \in R_s\,;\ (\alpha, \gamma) = 0\}$. Then $\mathfrak{q}_0$ is a real Heisenberg parabolic subalgebra (that is, $\mathfrak{n}_0$ is a nilpotent algebra of length two with one-dimensional center). Let $\mathfrak{c}_0 \subset \mathfrak{h}_0$ be the center of $\mathfrak{l}_0$. Then $\mathfrak{l}_0 = \mathfrak{c}_0 \oplus \mathfrak{m}_0$ with $\mathfrak{m}_0$ a semisimple ideal of $\mathfrak{l}_0$. Suppose that G is a connected linear Lie group with Lie algebra $\mathfrak{g}_0$ and let C and M° be the connected subgroups of G corresponding to $\mathfrak{c}_0$ and $\mathfrak{m}_0$, respectively. The Langlands decomposition of the Heisenberg parabolic subgroup $Q = N_G(\mathfrak{q}_0)$ is then $Q = MCN$, where $M = Z_K(\mathfrak{c}_0)M^\circ$. The standard Levi component of Q is $L = MC$ and there is a real-valued analytic character ν of L defined by $\mathrm{Ad}(l)X_\gamma = \nu(l)X_\gamma$. The restriction of the root γ to $\mathfrak{c}$ extends to a Lie algebra homomorphism $\gamma : \mathfrak{l} \to \boldsymbol{C}$ by making it zero on $\mathfrak{m}$ and $d\nu = \gamma$. If $\chi : L \to \boldsymbol{R}^\times$ is an analytic character such that $d\chi$ is a multiple of γ then

$$\chi = \varepsilon|\nu|^s$$

for some $s \in \boldsymbol{R}$ and some sign character $\varepsilon : L \to \{\pm 1\}$. We shall always take $\chi = \chi(s, \varepsilon)$ in this form when applying the results of Section 2 to the present situation. Note that we have $d\chi(s, \varepsilon) = s\gamma$.

The conformally invariant systems that we wish to study here always consist of operators $D_{y\otimes 1}$ for various $y \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$. The space $\boldsymbol{D}(\mathcal{L})^{\mathfrak{n}}$ is closed under composition, and it is convenient to have a description of the element $y \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{d\chi}$ such that $D_{y\otimes 1} = D_{y_1\otimes 1} \circ D_{y_2\otimes 1}$. The decomposition $\mathfrak{g} = \mathfrak{n} \oplus \bar{\mathfrak{q}}$ and the PBW Theorem imply that we may choose $y_1 = x_1 \otimes 1$ and $y_2 = x_2 \otimes 1$ with $x_1, x_2 \in \mathcal{U}(\mathfrak{n})$. If we do so then it is a consequence of Proposition 16 in [2] that $y = x_1x_2 \otimes 1$. Of course, this relation does not generally persist if we choose x_1 and x_2 without the restriction that they lie in $\mathcal{U}(\mathfrak{n})$.

In order to prepare for the definition of the conformally invariant systems that we shall study we must first recall some further notation, mostly drawn from [1]. There is a grading $\mathfrak{g} = \mathfrak{g}(-2) \oplus \mathfrak{g}(-1) \oplus \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$, where $\mathfrak{g}(j)$ is the j-eigenspace of $\mathrm{ad}(H_\gamma)$. We have $\mathfrak{l} = \mathfrak{g}(0)$, and we follow [1] in writing $V^+ = \mathfrak{g}(1)$ and $V^- = \mathfrak{g}(-1)$. Note that $\mathfrak{g}_\gamma = \mathfrak{g}(2)$ and $\mathfrak{g}_{-\gamma} = \mathfrak{g}(-2)$. For any $\mathfrak{h}$-stable subspace E of $\mathfrak{g}$, we write $R(E)$ for the set of roots α such that $\mathfrak{g}_\alpha \subset E$. If $\alpha \in R(V^+)$ then $\gamma - \alpha \in R(V^+)$ and we write $\alpha' = \gamma - \alpha$. The space V^+ is an $\mathfrak{l}$-module under the adjoint action and we define matrix coefficients for this module by

$$[Y, X_\alpha] = \sum_{\mu \in R(V^+)} M_{\alpha\mu}(Y) X_\mu \tag{7}$$

for $Y \in \mathfrak{l}$ and $\alpha \in R(V^+)$.

LEMMA 3.1. *Let $Y \in \mathfrak{l}$ and $\alpha, \beta \in R(V^+)$. Then we have*

$$M_{\alpha\beta}(Y) = \begin{cases} 0 & \text{if } \alpha \neq \beta \text{ and } \alpha - \beta \notin R\,, \\ \frac{1}{2}\|\beta\|^2 N_{\alpha,-\beta}\mathbf{B}(Y, X_{\alpha-\beta}) & \text{if } \alpha \neq \beta \text{ and } \alpha - \beta \in R\,, \\ \frac{1}{2}\|\alpha\|^2 \mathbf{B}(Y, H_\alpha) & \text{if } \alpha = \beta\,. \end{cases}$$

PROOF. By applying $\mathbf{B}(-, X_{-\beta})$ to both sides of (7) we obtain

$$M_{\alpha\beta}(Y) = \frac{1}{2}\|\beta\|^2 \mathbf{B}([Y, X_\alpha], X_{-\beta})\,.$$

The invariance of $\mathbf{B}$ allows this equation to be written as

$$M_{\alpha\beta}(Y) = \frac{1}{2}\|\beta\|^2 \mathbf{B}(Y, [X_\alpha, X_{-\beta}])$$

and the claims follow on considering the various possibilities for $[X_\alpha, X_{-\beta}]$. □

The systems that we wish to consider are those labeled Ω_2 in [1]. The operators in these systems are associated to elements of the ideal

$$\mathfrak{l}^{\gamma} = \{Y \in \mathfrak{l};\ \gamma(Y) = 0\}$$

of $\mathfrak{l}$. If $Y \in \mathfrak{l}^{\gamma}$ then we define

$$\omega_2(Y) = \frac{1}{2} \sum_{\alpha,\beta \in R(V^+)} N_{\beta,\beta'}^{-1} M_{\beta'\alpha}(Y) X_\beta X_\alpha \otimes 1 \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{s\gamma}$$

and

$$\Omega_2(Y) = D_{\omega_2(Y)\otimes 1}\,.$$

If $\mathfrak{r} \subset \mathfrak{l}^{\gamma}$ is an irreducible ideal then it follows from Theorems 5.2 and 5.3 in [1] that there is a unique $s \in \boldsymbol{R}$ such that the system $\Omega_2(Y)$ as Y runs over a basis of $\mathfrak{r}$ is conformally invariant. The value of s corresponding to each irreducible ideal in $\mathfrak{l}^{\gamma}$ may be found in the table in Subsection 8.10 in [1]. Note, however, that the columns for the systems Ω_2^{big} and Ω_2^{small} in types B and D were inadvertently transposed in that table. Inspection of the table shows that the only Ω_2 system for which the corresponding value of s is zero is Ω_2^{big} for algebras of type A. When we discuss a particular conformally invariant system below, it will always be understood that s takes the value associated with this system.

PROPOSITION 3.2. *Let $Y \in \mathfrak{l}^{\gamma}$ and define*

$$\Upsilon_2(Y) = \frac{1}{2} \sum_{\alpha,\beta \in R(V^+)} N_{\beta,\beta'}^{-1} M_{\beta'\alpha}(Y) Z_\beta Z_\alpha - s \sum_{\beta \in R(V^+)} N_{\beta,\beta'}^{-1} \|\beta\|^{-2} M_{\beta'\beta}(Y) \in \mathcal{U}(\mathfrak{k})\,.$$

Then $\iota(\Upsilon_2(Y) \otimes 1) = \omega_2(Y)$, where

$$\iota : \mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k}\cap\mathfrak{l})} \boldsymbol{C} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{s\gamma}$$

is the map from Lemma 2.1.

PROOF. In $\mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\bar{\mathfrak{q}})} \boldsymbol{C}_{s\gamma}$ we have

$$\begin{aligned} X_\beta X_\alpha \otimes 1 &= (Z_\beta + X_{-\beta})(Z_\alpha + X_{-\alpha}) \otimes 1 \\ &= Z_\beta Z_\alpha \otimes 1 + X_{-\beta} Z_\alpha \otimes 1 \\ &= Z_\beta Z_\alpha \otimes 1 - [Z_\alpha, X_{-\beta}] \otimes 1 \\ &= Z_\beta Z_\alpha \otimes 1 - [X_\alpha, X_{-\beta}] \otimes 1\,. \end{aligned}$$

Now $\gamma([X_\alpha, X_{-\beta}]) = 0$ unless $\alpha = \beta$ and so

$$\begin{aligned} X_\beta X_\alpha \otimes 1 &= Z_\beta Z_\alpha \otimes 1 - \delta_{\alpha\beta} H_\beta \otimes 1 \\ &= Z_\beta Z_\alpha - s\delta_{\alpha\beta}\gamma(H_\beta) 1 \otimes 1 \\ &= Z_\beta Z_\alpha - \frac{2s}{\|\beta\|^2} \delta_{\alpha\beta} 1 \otimes 1\,, \end{aligned}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. In this calculation, we have used the facts that $(\beta, \gamma) = 1$ for all $\beta \in R(V^+)$, and $\mu(H_\nu) = 2(\mu, \nu)\|\nu\|^{-2}$ for any $\mu, \nu \in R$. (The latter is Condition

(C5) from Section 2 in [1].) By substituting this evaluation of $X_\beta X_\alpha \otimes 1$ into the definition of $\omega_2(Y)$, we obtain the asserted equality. $\square$

Note that $\overline{\Upsilon_2(Y)} = \Upsilon_2(\bar{Y})$ for all $Y \in \mathfrak{l}^\gamma$. Each of the Ω_2 systems that we consider has a basis coming from elements of $\mathfrak{l}_0^\gamma$ and is hence closed under conjugation. We make repeated use of this observation in what follows.

LEMMA 3.3. *Let $\beta \in R(V^+)$. Then $\beta' - \beta$ is a root if and only if β is short. If β is short then $\beta' - \beta$ is long unless R has type* G_2, *in which case $\beta' - \beta$ is short.*

PROOF. We first consider the root system G_2. Let $R_s = \{\lambda, \sigma\}$, with λ long and σ short. Then $R(V^+) = \{\lambda, \lambda + \sigma, \lambda + 2\sigma, \lambda + 3\sigma\}$ and $\gamma = 2\lambda + 3\sigma$. By using these facts, it is easy to verify the claims in this case.

We now assume that R does not have type G_2, so that the long roots have length $\sqrt{2}$ and the short roots, if they are present, have length 1. Note that γ is always long. If $\beta \in R(V^+)$ is a long root then

$$\|\beta' - \beta\|^2 = (\gamma - 2\beta, \gamma - 2\beta) = \|\gamma\|^2 + 4\|\beta\|^2 - 4(\gamma, \beta) = 6\,,$$

since $(\gamma, \beta) = 1$. This implies that $\beta' - \beta$ is not a root. Now suppose that β is short. Then

$$s_\beta(\gamma) = \gamma - \frac{2(\gamma, \beta)}{\|\beta\|^2}\beta = \gamma - 2\beta = \beta' - \beta$$

is a root. Moreover, it has the same length as γ and hence is long. $\square$

In what follows, let $\mathrm{CT} : \mathcal{U}(\mathfrak{k}) \to \boldsymbol{C}$ be the algebra homomorphism that is induced by the zero Lie algebra homomorphism $\mathfrak{k} \to \boldsymbol{C}$. It follows from Proposition 3.2 that

$$\mathrm{CT}(\Upsilon_2(Y)) = -s \sum_{\beta \in R(V^+)} N_{\beta,\beta'}^{-1} \|\beta\|^{-2} M_{\beta'\beta}(Y)$$

for all $Y \in \mathfrak{l}^\gamma$.

COROLLARY 3.4. *We have* $\mathrm{CT}(\Upsilon_2(Y)) = 0$ *for all $Y \in \mathfrak{l}^\gamma$ if and only if R is simply laced.*

PROOF. If R is simply laced then it follows from Lemmas 3.1 and 3.3 that $M_{\beta'\beta}(Y) = 0$ for all $\beta \in R(V^+)$. (Note that R is reduced, so that $\beta' \neq \beta$ for all $\beta \in R(V^+)$.) This gives one implication. For the other, we require the fact (which may be verified case-by-case) that if R is not simply laced then $R(V^+)$ contains short roots. Suppose that $\alpha \in R(V^+)$ is such a root and take $Y = X_{-(\alpha' - \alpha)}$. By noting that $\beta' - \beta = \alpha' - \alpha$ implies that $\beta = \alpha$, and once again using Lemmas 3.1 and 3.3, we find that

$$\sum_{\beta \in R(V^+)} N_{\beta,\beta'}^{-1} \|\beta\|^{-2} M_{\beta'\beta}(X_{-(\alpha'-\alpha)}) = \frac{N_{\alpha,\alpha'}^{-1} N_{\alpha',-\alpha}}{\|\alpha' - \alpha\|^2}\,,$$

which is non-zero. It has been observed above that $s \neq 0$ for all Ω_2 systems on all non-simply-laced algebras, and the required conclusion follows. $\square$

We can be slightly more precise about the conclusion of Corollary 3.4. The list of irreducible non-simply-laced root systems is B_r ($r \geq 3$), C_r ($r \geq 2$), F_4, and G_2. The algebra $\mathfrak{l}^\gamma$ has two simple ideals when R has type B_r, and is simple otherwise. With respect to the standard model of the B_r root system inside $\boldsymbol{R}^r$ and the standard choice of positive system, we have

$$R(V^+) = \{e_1, e_2\} \cup \{e_1 \pm e_j\,;\, 3 \leq j \leq r\} \cup \{e_2 \pm e_j\,;\, 3 \leq j \leq r\}\,.$$

The only short roots in $R(V^+)$ are e_1 and e_2 and it follows that, of the two simple ideals in $\mathfrak{l}^\gamma$, the functional $Y \mapsto \mathrm{CT}(\varUpsilon_2(Y))$ is non-zero only on the one generated by the root vectors $X_{\pm(e_1-e_2)}$. This ideal corresponds to the system that is called $\Omega_2^{\mathrm{small}}$ in Section 8 of [1].

Let σ_0 denote the trivial representation of K. Suppose that $D_1, \dots, D_m$ is a conformally invariant system and that $u_i \in \mathcal{U}(\mathfrak{k})$ corresponds to the operator D_i, as above. Then $1 \in \boldsymbol{M}(\sigma_0)$ if and only if $\mathrm{CT}(\bar{u}_i) = 0$ for all $1 \leq i \leq m$. Thus one consequence of the above discussion is that we have determined which of the Ω_2 systems admit the constant function 1 on K as a solution. In case this function does lie in the solution space, it will generate a subrepresentation with a spherical representation as a quotient. Thus we have established the following result.

THEOREM 3.5. *Let D denote one of the Ω_2 systems considered above and s the corresponding special value. Then the solution space $\Gamma(\mathcal{L})^D$ contains an irreducible spherical subquotient for $\chi = \chi(s, \varepsilon_0)$, with ε_0 the trivial character, if and only if either the root system R is simply laced or R has type B_r with $r \geq 3$ and $D = \Omega_2^{\mathrm{big}}$.*

The only finite-dimensional irreducible spherical representation is the trivial representation. The constant function 1 on $K/(K \cap L)$ is fixed under the action of G on $\Gamma(\mathcal{L})$ if and only if $s = 0$. The only Ω_2 system for which $s = 0$ is Ω_2^{big} in type A. Thus the spherical subquotient considered in Theorem 3.5 is a one-dimensional subrepresentation in this case. Otherwise, it is not finite-dimensional, although this requires an additional argument.

LEMMA 3.6. *Let $Z \in \mathfrak{k} \cap \mathfrak{l}$ and $Y \in \mathfrak{l}^\gamma$. Then*

$$Z\varUpsilon_2(Y) \in \varUpsilon_2([Z, Y]) + \mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})\,,$$

where $\mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})$ denotes the left ideal in $\mathcal{U}(\mathfrak{k})$ generated by the elements of $\mathfrak{k} \cap \mathfrak{l}$.

PROOF. In the discussion preceding the statement of Theorem 5.2 in [1], it is observed that $\Omega_2(\mathrm{Ad}(l)Y) = \nu(l)^{-1} l \cdot \Omega_2(Y)$ for $l \in L$ and $Y \in \mathfrak{l}^\gamma$, where ν is the character defined by $\mathrm{Ad}(l)X_\gamma = \nu(l)X_\gamma$. For $l \in (K \cap L)^\circ$, this reduces to $\Omega_2(\mathrm{Ad}(l)Y) = l \cdot \Omega_2(Y)$, and it follows that $\omega_2(\mathrm{Ad}(l)Y) = (\mathrm{Ad}(l) \otimes I)(\omega_2(Y))$ under the same assumption on l. By taking $l = \exp(tZ)$ in this relation, differentiating with respect to t, and setting $t = 0$, we obtain $\omega_2([Z, Y]) = Z\omega_2(Y)$. The $(K \cap L)$-equivariance of the map ι now implies that $\varUpsilon_2([Z, Y]) \otimes 1 = Z(\varUpsilon_2(Y) \otimes 1)$ in the module $\mathcal{U}(\mathfrak{k}) \otimes_{\mathcal{U}(\mathfrak{k}\cap\mathfrak{l})} \boldsymbol{C}$. This module is isomorphic to the quotient module $\mathcal{U}(\mathfrak{k})/\mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})$ via the map $y \otimes 1 \mapsto y + \mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})$ and the claim follows from this. $\square$

It is consequence of Lemma 3.6 that in order to show that a vector annihilated by $\mathfrak{k} \cap \mathfrak{l}$ is an embedding vector it is sufficient to verify that it is annihilated by $\Upsilon_2(Y)$ for each Y in a $(\mathfrak{k} \cap \mathfrak{l})$-generating set for the ideal that defines the system. In some cases, this dramatically reduces the number of equations that must be checked.

Because $\mathfrak{k} \cap \mathfrak{l}$ is reductive, $\mathfrak{l}^\gamma$ decomposes canonically into a direct sum of isotypic subspaces and, in particular, there is a canonical projection $\mathfrak{l}^\gamma \to (\mathfrak{l}^\gamma)^{\mathfrak{k}\cap\mathfrak{l}}$ onto the trivial isotypic subspace.

LEMMA 3.7. *The functional $Y \mapsto \mathrm{CT}(\Upsilon_2(Y))$ on $\mathfrak{l}^\gamma$ factors through the canonical projection map $\mathfrak{l}^\gamma \to (\mathfrak{l}^\gamma)^{\mathfrak{k}\cap\mathfrak{l}}$.*

PROOF. Let $Z \in \mathfrak{k} \cap \mathfrak{l}$ and $Y \in \mathfrak{l}^\gamma$. By Lemma 3.6, $Z\Upsilon_2(Y) \in \Upsilon_2([Z, Y]) + \mathcal{U}(\mathfrak{k})(\mathfrak{k} \cap \mathfrak{l})$, and if we apply the algebra homomorphism CT to this membership relation, we conclude that $\mathrm{CT}(\Upsilon_2([Z, Y])) = 0$. This identity is equivalent to the required factorization claim. □

In addition to the trivial representation, a natural place to look for embedding vectors is the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{k}$. The following result identifies the space of candidates for such embedding vectors. It will also have further applications later in the development of the theory. In the excluded special case, $\mathfrak{k} \cap \mathfrak{l} = \{0\}$ and so $\mathfrak{g}^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{g}$.

PROPOSITION 3.8. *Suppose that $\mathfrak{g}$ is not of type A_2. Then we have*

$$\mathfrak{k} \cap \mathfrak{g}^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{z}(\mathfrak{k} \cap \mathfrak{l}) \oplus \boldsymbol{C} Z_\gamma$$

and

$$\mathfrak{p} \cap \mathfrak{g}^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{z}(\mathfrak{l}) \oplus \boldsymbol{C} W_\gamma$$

with $W_\gamma = X_\gamma + X_{-\gamma}$.

PROOF. The decomposition $\mathfrak{g} = \mathfrak{g}_{-\gamma} \oplus V^- \oplus \mathfrak{l} \oplus V^+ \oplus \mathfrak{g}_\gamma$ is $(\mathfrak{k} \cap \mathfrak{l})$-invariant, and it follows that $\mathfrak{g}^{\mathfrak{k}\cap\mathfrak{l}}$ has a corresponding direct sum decomposition. The first step is to show that $(V^\pm)^{\mathfrak{k}\cap\mathfrak{l}} = \{0\}$. Since V^+ and V^- are dual $(\mathfrak{k} \cap \mathfrak{l})$-modules via $\mathbf{B}$, it suffices to show that $(V^+)^{\mathfrak{k}\cap\mathfrak{l}} = \{0\}$. In type $\mathrm{A_r}$ with $r \geq 3$, $\mathfrak{l}$ is isomorphic to $\mathfrak{gl}(1)^{\oplus 2} \oplus \mathfrak{sl}(r-1)$, the restriction of V^+ to the $\mathfrak{sl}(r-1)$ summand is isomorphic to the sum of the standard representation and its dual, and $\mathfrak{k} \cap \mathfrak{l}$ is isomorphic to $\mathfrak{so}(r-1)$ embedded in the standard way in $\mathfrak{sl}(r-1)$. The claim follows in this case. Now suppose that $\mathfrak{g}$ does not have type A. Then the Heisenberg parabolic subalgebra is maximal and so V^+ is an irreducible representation of $\mathfrak{l}$. The identity $X_{-\lambda} = X_\lambda - Z_\lambda$ for λ a positive root of $\mathfrak{l}$, together with a standard induction argument, implies that an $\mathfrak{l}$-highest weight vector in V^+ is a $(\mathfrak{k} \cap \mathfrak{l})$-cyclic vector for V^+. Hence the multiplicity of the trivial representation of $\mathfrak{k} \cap \mathfrak{l}$ in V^+ is at most one. Now the map $(X, Y) \mapsto X_\gamma^*([X, Y])$ is a non-degenerate $(\mathfrak{k} \cap \mathfrak{l})$-invariant alternating form on V^+. For general reasons, $(V^+)^{\mathfrak{k}\cap\mathfrak{l}}$ pairs trivially with all other $(\mathfrak{k} \cap \mathfrak{l})$-isotypes in V^+ under this form. But a one-dimensional space supports no non-zero alternating form and we conclude that $(V^+)^{\mathfrak{k}\cap\mathfrak{l}} = \{0\}$, as claimed.

Observe that $\mathfrak{k} \cap \mathfrak{l}$ acts trivially on

$$\mathfrak{g}_{-\gamma} \oplus \mathfrak{g}_\gamma = \boldsymbol{C} Z_\gamma \oplus \boldsymbol{C} W_\gamma\ ,$$

with $Z_\gamma \in \mathfrak{k}$ and $W_\gamma \in \mathfrak{p}$. To complete the determination of $\mathfrak{g}^{\mathfrak{k}\cap\mathfrak{l}}$, it remains to determine $\mathfrak{l}^{\mathfrak{k}\cap\mathfrak{l}}$. We may write $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{s}$, where $\mathfrak{s}$ is a semisimple ideal in $\mathfrak{l}$. Both $\mathfrak{z}(\mathfrak{l})$ and $\mathfrak{s}$ are stable under θ and $\mathfrak{z}(\mathfrak{l}) \subset \mathfrak{h} \subset \mathfrak{p}$. We conclude that $\mathfrak{k}\cap\mathfrak{l} \subset \mathfrak{s}$ and that $\mathfrak{s} = (\mathfrak{k}\cap\mathfrak{l}) \oplus (\mathfrak{s}\cap\mathfrak{p})$ is the Cartan decomposition of $\mathfrak{s}$. Because $\mathfrak{s}$ is semisimple, the trivial representation of $\mathfrak{k}\cap\mathfrak{l}$ does not appear in $\mathfrak{s}\cap\mathfrak{p}$, and so $\mathfrak{p}\cap\mathfrak{l}^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{z}(\mathfrak{l})$. By definition, $(\mathfrak{k}\cap\mathfrak{l})^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{z}(\mathfrak{k}\cap\mathfrak{l})$, and it follows that $\mathfrak{k}\cap\mathfrak{l}^{\mathfrak{k}\cap\mathfrak{l}} = \mathfrak{z}(\mathfrak{k}\cap\mathfrak{l})$. □

In keeping with the notation introduced in the statement of Proposition 3.8, we set $W_\alpha = X_\alpha + X_{-\alpha}$ for any $\alpha \in R$. Note that $\mathfrak{p}_0$ is the direct sum of $\mathfrak{h}_0$ and the $\boldsymbol{R}$-span of the set $\{W_\alpha\ ;\ \alpha \in R_+\}$.

LEMMA 3.9. *If $Y \in \mathfrak{k}\cap\mathfrak{l}$ and $\alpha \in R(V^+)$ then*

$$[Y, Z_\alpha] = \sum_{\mu\in R(V^+)} M_{\alpha\mu}(Y) Z_\mu\,.$$

If $Y \in \mathfrak{p}\cap\mathfrak{l}$ and $\alpha \in R(V^+)$ then

$$[Y, W_\alpha] = \sum_{\mu\in R(V^+)} M_{\alpha\mu}(Y) Z_\mu\,.$$

PROOF. Suppose that $Y \in \mathfrak{l}$ is either in $\mathfrak{k}$ or in $\mathfrak{p}$. By applying the Cartan involution to the identity (7), we obtain

$$[Y, X_{-\alpha}] = \pm \sum_{\mu\in R(V^+)} M_{\alpha\mu}(Y) X_{-\mu}\,,$$

with the upper sign if $Y \in \mathfrak{k}$ and the lower if $Y \in \mathfrak{p}$. Subtracting this from or adding it to (7), according to the sign, yields the required identities. □

Lemma 3.9 allows us to write $\Upsilon_2(Y)$ for $Y \in \mathfrak{k}\cap\mathfrak{l}^\gamma$ in the more tractable form

$$\Upsilon_2(Y) = \frac{1}{2}\sum_{\beta\in R(V^+)} N^{-1}_{\beta,\beta'} Z_\beta [Y, Z_{\beta'}] - s \sum_{\beta\in R(V^+)} N^{-1}_{\beta,\beta'} \|\beta\|^{-2} M_{\beta'\beta}(Y)\,. \tag{8}$$

It leads to a more substantial reduction for that part of an Ω_2 system that comes from $\mathfrak{p}\cap\mathfrak{l}^\gamma$.

PROPOSITION 3.10. *Let $\mathfrak{r} \subset \mathfrak{l}^\gamma$ be an irreducible ideal. Suppose that $\sigma \in \hat{K}$ and that $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$. Then $d\sigma(\Upsilon_2(Y))\xi = 0$ for all $Y \in \mathfrak{p}\cap\mathfrak{r}$ if and only if ξ is annihilated by the operator*

$$\Upsilon_2(Y) = \frac{1}{2}\sum_{\beta\in R(V^+)} N^{-1}_{\beta,\beta'} \beta'(Y) Z_\beta Z_{\beta'}$$

for all $Y \in \mathfrak{h}\cap\mathfrak{r}$.

PROOF. We begin by deriving the expression for $\Upsilon_2(Y)$ when $Y \in \mathfrak{h}^\gamma$. In cases other than type A, the Heisenberg parabolic is maximal and so $\mathfrak{z}(\mathfrak{l}) = \boldsymbol{C}H_\gamma$. It follows from this and Proposition 3.8 that $\mathfrak{p}\cap(\mathfrak{l}^\gamma)^{\mathfrak{k}\cap\mathfrak{l}} = \{0\}$. Now $\mathfrak{p}\cap\mathfrak{l}^\gamma$ is a $(\mathfrak{k}\cap\mathfrak{l})$-submodule of $\mathfrak{l}^\gamma$ and so the projection of $\mathfrak{p}\cap\mathfrak{l}^\gamma$ to $(\mathfrak{l}^\gamma)^{\mathfrak{k}\cap\mathfrak{l}}$ is zero. By Lemma 3.7, we conclude that $\mathrm{CT}(\Upsilon_2(Y)) = 0$ for all $Y \in \mathfrak{p}\cap\mathfrak{l}^\gamma$. The same conclusion holds in type A, since it is simply laced and so Corollary 3.4

applies. If $Y \in \mathfrak{h}$ then $[Y, W_{\beta'}] = \beta'(Y)Z_{\beta'}$. By combining these observations with Lemma 3.9 and Proposition 3.2 we obtain the required expression for $\Upsilon_2(Y)$ when $Y \in \mathfrak{h}^{\gamma}$.

By Lemma 3.6, the proof will be complete once we know that $\mathfrak{h} \cap \mathfrak{r}$ generates $\mathfrak{p} \cap \mathfrak{r}$ as an $\mathcal{U}(\mathfrak{k} \cap \mathfrak{l})$-module. We may find a basis for $\mathfrak{r}$ consisting of certain elements of $\mathfrak{h}$ and certain root vectors X_λ with $\lambda \in R(\mathfrak{l})$. If $X_\lambda \in \mathfrak{r}$ then $H_\lambda = [X_\lambda, X_{-\lambda}] \in \mathfrak{h} \cap \mathfrak{r}$. The identity $[Z_\lambda, H_\lambda] = -2W_\lambda$ then shows that W_λ belongs to the $\mathcal{U}(\mathfrak{k} \cap \mathfrak{l})$-submodule of $\mathfrak{p} \cap \mathfrak{r}$ generated by $\mathfrak{h} \cap \mathfrak{r}$, and the claim follows. □

For later use, we derive a more explicit expression for the operator $R(\sigma)$ for the split real form.

LEMMA 3.11. *Let $\sigma \in \hat{K}$ and $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$. Then*

$$R(\sigma)\xi = sH_\gamma \otimes \xi + \frac{1}{4} \sum_{\alpha \in R(\mathfrak{n})} \|\alpha\|^2 W_\alpha \otimes d\sigma(Z_\alpha)\xi \,.$$

PROOF. We must produce an orthonormal basis for $\mathfrak{p}_0$. We begin with the vector H_γ. This has square-length 2 and its perp in $\mathfrak{h}_0$ is precisely $\mathfrak{h}_0^\gamma$. We may choose $H_1, \dots, H_{r-1} \in \mathfrak{h}_0^\gamma$ an orthonormal basis for $\mathfrak{h}_0^\gamma$. Note that these vectors contribute zero to the sum (4), since they lie in $\bar{\mathfrak{q}}_0$ and $d\chi(H_i) = s\gamma(H_i) = 0$ for $1 \le i \le r-1$. Let $\alpha \in R_+$. The length of the vector W_α is $2/\|\alpha\|$ and so

$$\left\{\frac{1}{\sqrt{2}} H_\gamma\right\} \cup \{H_i \,;\ 1 \le i \le r-1\} \cup \left\{\frac{\|\alpha\|}{2} W_\alpha \,;\ \alpha \in R_+\right\}$$

is the required orthonormal basis. We may take

$$\frac{\|\alpha\|}{2} W_\alpha = \frac{\|\alpha\|}{2} Z_\alpha + \|\alpha\| X_{-\alpha}$$

as the $\mathfrak{k}_0 + \bar{\mathfrak{q}}_0$ decomposition of $(\|\alpha\|/2)W_\alpha$. The required evaluation now follows from (4), and the facts that $d\chi(H_\gamma) = 2s$, $d\chi(X_{-\alpha}) = 0$ for all $\alpha \in R_+$, and $d\sigma(Z_\alpha)\xi = 0$ for all $\alpha \in R_+(\mathfrak{l})$. □

4. Embedding vectors for the Ω_2 system in type C. In this section we determine the complete set of embedding vectors for the unique Ω_2 system on the simple algebra of type C_r ($r \ge 2$). As we have already mentioned, this system is quite anomalous among the Ω_2 systems.

Let $G = \mathrm{Sp}(2r, \boldsymbol{R})$ be the real symplectic group of rank r. In order to have a consistent set of structure constants for $\mathfrak{g} = \mathfrak{sp}(2r, \boldsymbol{C})$ available, it is convenient to choose a specific Chevalley system in $\mathfrak{g}$. Let us choose an index set $I = \{1, \dots, r\} \cup \{\bar{1}, \dots, \bar{r}\}$ and identify $\mathfrak{g}$ with the isotropy algebra of the form

$$\langle x, y\rangle = \sum_{a=1}^{r} (x_a y_{\bar{a}} - x_{\bar{a}} y_a)$$

on $\boldsymbol{C}^I$. With the standard notation for elementary linear maps on $\boldsymbol{C}^I$, we take $\mathfrak{h}$ to be spanned by the set $\{E_{aa} - E_{\bar{a}\bar{a}} \,;\ 1 \le a \le r\}$ and $\{e_a \,;\ 1 \le a \le r\}$ to be the dual set. The roots are

$R = \{\pm 2e_a\,;\ 1 \le a \le r\} \cup \{e_a \pm e_b\,;\ 1 \le a \ne b \le r\}$ and we take root vectors as follows:

$$\begin{aligned} X_{2e_a} &= E_{a\bar{a}}\,, \\ X_{-2e_a} &= E_{\bar{a}a}\,, \\ X_{e_a-e_b} &= E_{ab} - E_{\bar{b}\bar{a}}\,, \\ X_{e_a+e_b} &= E_{a\bar{b}} + E_{b\bar{a}}\,, \\ X_{-(e_a+e_b)} &= E_{\bar{a}b} + E_{\bar{b}a}\,. \end{aligned}$$

The standard inner product on $\mathfrak{h}^*$ satisfies $(e_a, e_b) = \delta_{ab}/2$ and the corresponding form $\mathbf{B}$ is $\mathbf{B}(X, Y) = \mathrm{Tr}(XY)$. The highest root is $\gamma = 2e_1$ and the set $R(V^+)$ is $\{e_1 \pm e_a\,;\ 2 \le a \le r\}$. If we let $S = \{e_1 - e_a\,;\ 2 \le a \le r\}$ then $R(V^+) = S \cup S'$ and $N_{\beta,\beta'} = 2$ for all $\beta \in S$. The set $R(\mathfrak{l})$ is $\{2e_a\,;\ 2 \le a \le r\} \cup \{e_a \pm e_b\,;\ 2 \le a \ne b \le r\}$.

The algebra $\mathfrak{k}_0$ is isomorphic to $\mathfrak{u}(r)$ with center spanned by the element

$$U_0 = \sum_{a=1}^{r} Z_{2e_a}\,.$$

The algebra $\mathfrak{k}_0 \cap \mathfrak{l}_0$ is isomorphic to $\mathfrak{u}(r-1)$ with center spanned by the element

$$V_0 = \sum_{a=2}^{r} Z_{2e_a}\,.$$

The $\mathcal{U}(\mathfrak{k} \cap \mathfrak{l})$-module $\mathfrak{p} \cap \mathfrak{l}^\gamma$ is cyclic and generated by the element H_{2e_2}. The $\mathcal{U}(\mathfrak{k} \cap \mathfrak{l})$-module $\mathfrak{k} \cap \mathfrak{l}$ is generated by the element V_0 if $r = 2$ and by the elements V_0 and $Z_{e_2-e_3}$ if $r \ge 3$. It follows that if $\sigma \in \hat{K}$ and $\xi \in E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$ then ξ is an embedding vector for the Ω_2 system if and only if it is annihilated by $\Upsilon_2(H_{2e_2})$, $\Upsilon_2(V_0)$, and $\Upsilon_2(Z_{e_2-e_3})$, with the third element present only when $r \ge 3$. From the table in Section 8 of [1], we have $s = -1/2$ for this system. By using Lemma 3.1, Equation (8), and the expression for $\Upsilon_2(Y)$ for $Y \in \mathfrak{h}^\gamma$ given in Proposition 3.10, we find that

$$\begin{aligned} \Upsilon_2(H_{2e_2}) &= \frac{1}{4}(Z_{e_1-e_2}Z_{e_1+e_2} + Z_{e_1+e_2}Z_{e_1-e_2})\,, \\ \Upsilon_2(V_0) &= \frac{1}{4}\sum_{a=2}^{r} Z_{e_1-e_a}^2 + \frac{1}{4}\sum_{a=2}^{r} Z_{e_1+e_a}^2 + \frac{r-1}{2}\,, \\ \Upsilon_2(Z_{e_2-e_3}) &= \frac{1}{4}(Z_{e_1+e_2}Z_{e_1-e_3} + Z_{e_1-e_3}Z_{e_1+e_2} - Z_{e_1-e_2}Z_{e_1+e_3} - Z_{e_1+e_3}Z_{e_1-e_2})\,. \end{aligned}$$

The elements $Z_{2e_a} \in \mathfrak{k}_0$ span a Cartan subalgebra in $\mathfrak{k}_0$. We let $\varepsilon_1, \ldots, \varepsilon_r$ denote the elements dual to $Z_{2e_1}, \ldots, Z_{2e_r}$, and order them in the usual way. It is well known that $\sigma \in \hat{K}$ satisfies $E_\sigma^{\mathfrak{k}\cap\mathfrak{l}} \ne \{0\}$ if and only if the highest weight of σ is either $im\varepsilon_1$ or $-im\varepsilon_r$ for some $m \in \boldsymbol{N}$, and that if σ has such a highest weight then $E_\sigma^{\mathfrak{k}\cap\mathfrak{l}}$ is one-dimensional. We denote by $\sigma(m) \in \hat{K}$ the irreducible K-module with highest weight $im\varepsilon_1$ if $m \ge 0$ and $im\varepsilon_r$ if $m < 0$. The space $\boldsymbol{C}^I$ introduced above is naturally a G-module and its restriction to K is isomorphic to $\sigma(1) \oplus \sigma(-1)$. Let $\{\xi_a, \xi_{\bar{a}}\,;\ 1 \le a \le r\}$ be the canonical basis of $\boldsymbol{C}^I$. Then $\sigma(1)$ is realized on the span of the set $\{\xi_a + i\xi_{\bar{a}}\,;\ 1 \le a \le r\}$ and $\sigma(-1)$ is realized on the span of the set

$\{\xi_a - i\xi_{\bar{a}}\,;\ 1 \le a \le r\}$. The vectors $\xi_1 \pm i\xi_{\bar{1}}$ span the spaces $E^{\mathfrak{k}\cap\mathfrak{l}}_{\sigma(\pm 1)}$. We already know that $\sigma(0)$ contains no embedding vectors. If $m > 0$ then the submodule of the symmetric product $\vee^m\sigma(\pm 1)$ generated by the vector $(\xi_1 \pm i\xi_{\bar{1}})^m$ is isomorphic to $\sigma(\pm m)$, and this generating vector also spans $E^{\mathfrak{k}\cap\mathfrak{l}}_{\sigma(\pm m)}$. Direct calculation gives

$$Z^2_{e_1-e_a}\cdot(\xi_1 \pm i\xi_{\bar{1}})^m = -m(\xi_1 \pm i\xi_{\bar{1}})^m + m(m-1)(\xi_1 \pm i\xi_{\bar{1}})^{m-2}(\xi_a \pm i\xi_{\bar{a}})^2$$

and

$$Z^2_{e_1+e_a}\cdot(\xi_1 \pm i\xi_{\bar{1}})^m = -m(\xi_1 \pm i\xi_{\bar{1}})^m - m(m-1)(\xi_1 \pm i\xi_{\bar{1}})^{m-2}(\xi_a \pm i\xi_{\bar{a}})^2$$

and so

$$\Upsilon_2(V_0)\cdot(\xi_1 \pm i\xi_{\bar{1}})^m = (1-m)\frac{r-1}{2}(\xi_1 \pm i\xi_{\bar{1}})^m\,.$$

It follows from this that $\sigma(\pm m)$ does not contain an embedding vector unless $m = 1$. The vectors $\xi_1 \pm i\xi_{\bar{1}}$ are both annihilated by $\Upsilon_2(V_0)$, and it is easy to check that they are also annihilated by $\Upsilon_2(H_{2e_2})$, and by $\Upsilon_2(Z_{e_2-e_3})$ when this element is present in the system. We have established the following result. Note that, in other examples, embedding vectors are not usually weight vectors; this is one anomalous feature of this example.

THEOREM 4.1. *The space of K-finite solutions to the Ω_2 system in type* C *is isomorphic as a K-module to $\sigma(1) \oplus \sigma(-1)$. The space of embedding vectors in $\sigma(1)$ coincides with the highest weight space and the space of embedding vectors in $\sigma(-1)$ coincides with the lowest weight space.*

A calculation using Lemma 3.11 reveals that

$$\begin{aligned} &R(\sigma(\pm 1))(\xi_1 \pm i\xi_{\bar{1}}) \\ &\quad = -\frac{1}{2}(H_\gamma \mp iW_\gamma)\otimes(\xi_1 \pm i\xi_{\bar{1}}) - \frac{1}{4}\sum_{a=2}^{r}(W_{e_1-e_a} \mp iW_{e_1+e_a})\otimes(\xi_a \pm i\xi_{\bar{a}})\,. \end{aligned}$$

From this evaluation, one may verify that $R(\sigma(1))(\xi_1 + i\xi_{\bar{1}})$ is a lowest weight vector with weight $-i\varepsilon_1$ and that $R(\sigma(-1))(\xi_1 - i\xi_{\bar{1}})$ is a highest weight vector with weight $i\varepsilon_1$. With a slight extension of our earlier notation, we have the decompositions $\mathfrak{p} \cong \sigma(2i\varepsilon_1)\oplus\sigma(-2i\varepsilon_r)$,

$$\mathfrak{p}\otimes\sigma(1) \cong \sigma(3i\varepsilon_1)\oplus\sigma(2i\varepsilon_1 + i\varepsilon_2)\oplus\sigma(i\varepsilon_1 - 2i\varepsilon_r)\oplus\sigma(-1)\,,$$

and

$$\mathfrak{p}\otimes\sigma(-1) \cong \sigma(-3i\varepsilon_r)\oplus\sigma(-i\varepsilon_{r-1} - 2i\varepsilon_r)\oplus\sigma(2i\varepsilon_1 - i\varepsilon_r)\oplus\sigma(1)\,.$$

In the latter two decompositions, only the last summand contains non-zero embedding vectors. Thus, from general properties of the map $R(\sigma)$ that were noted in Section 2, $R(\sigma(\pm 1))(\xi_1 \pm i\xi_{\bar{1}})$ must lie in the subspace of $\mathfrak{p}\otimes E_{\sigma(\pm 1)}$ that corresponds to these summands. This is confirmed by the above evaluation of $R(\sigma(\pm 1))(\xi_1 \pm i\xi_{\bar{1}})$, which also reveals that $R(\sigma(\pm 1))(\xi_1 \pm i\xi_{\bar{1}})$ is non-zero. These observations confirm directly the simple fact that the representation of G on the space $\Gamma(\mathcal{L})^D$ is irreducible.

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