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k-Path-Connectivity of Completely Balanced Tripartite Graphs

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Abstract: For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of a size at least 2, a path in G is said to be an S-path if it connects all vertices of S. Two S-paths P_1 and P_2 are said to be *internally disjoint* if $E(P_1) \cap E(P_2) = \emptyset$ and $V(P_1) \cap V(P_2) = S$; that is, they share no vertices and edges apart from S. Let $\pi_G(S)$ denote the maximum number of internally disjoint S-paths in G. The k-path-connectivity $\pi_k(G)$ of G is then defined as the minimum $\pi_G(S)$, where S ranges over all k-subsets of $V(G)$. In this paper, we study the k-path-connectivity of the complete balanced tripartite graph $K_{n,n,n}$ and obtain $\pi_k(K_{n,n,n}) = \left\lfloor \frac{2n}{k-1} \right\rfloor$ for $3 \leq k \leq n$.

Keywords: path-connectivity; internally disjoint paths; complete balanced tripartite graphs

MSC: 05C38; 05C40

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1. Introduction

An interconnection network is usually modeled by a connected graph $G = (V, E)$, where vertices represent processors and edges represent communication links between processors. Connectivity is an important parameter to evaluate the reliability and fault tolerance of a network. For a graph G, the *connectivity* $\kappa(G)$ is defined as the minimum cardinality of a subset V' of vertices of G such that $G - V'$ is disconnected or trivial. An equivalent definition of connectivity was given in [1]. For each 2-subset $S = \{u, v\}$ of vertices of G, let $\kappa(S)$ denote the maximum number of internally disjoint (u, v)-paths in G. Then, $\kappa(G) = \min\{\kappa(S) | S \subseteq V \text{ and } |S| = 2\}$.

There exist many generalizations of the classical connectivity, such as conditional connectivity [2], component connectivity [3], tree-connectivity [4,5] and rainbow connectivity [6]. In particular, Hager [7] introduced the concept of path-connectivity, which concerns paths connecting any k vertices in G and not only any two. Given a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of a size at least 2, a path in G is said to be an S-path if it connects all vertices of S. Two S paths P_1 and P_2 are said to be *internally disjoint* if $E(P_1) \cap E(P_2) = \emptyset$ and $V(P_1) \cap V(P_2) = S$; that is, they share no vertices and edges apart from S. Let $\pi_G(S)$ denote the maximum number of internally disjoint S-paths in G. The k-path connectivity of G, denoted by $\pi_k(G)$, is then defined as $\pi_k(G) = \min\{\pi_G(S) | S \subseteq V(G) \textit{ and } |S| = k\}$, where $2 \leq k \leq n$. Clearly, $\pi_2(G)$ is exactly the classical connectivity $\kappa(G)$, and $\pi_n(G)$ is exactly the maximum number of edge-disjoint Hamiltonian paths in G.

In [7], Hager studied the sufficient conditions for $\pi_k(G)$ to be at least ℓ in terms of $\kappa(G)$. Hager conjectured that if G is a graph with $\kappa(G) \geq \ell(k-1)$ for $k \geq 2$ and $\ell \geq 1$, then $\pi_k(G) \geq \ell$; moreover, the bound is sharp. He confirmed the conjecture for $2 \leq k \leq 4$. Recently, Li et al. [8] showed that this conjecture also is true for $k = 5$. Moreover, they studied the complexity of the path-connectivity. With their conclusions, it is difficult to obtain $\pi_k(G)$ for general G and $k \geq 5$. In [9,10], the path connectivity of lexicographic product graphs was investigated. For special classes of graphs, the exact values of $\pi_k(G)$ were obtained for complete graphs [7] and complete bipartite graphs [7,11].

A complete multipartite graph is *balanced* if the partite sets all have the same cardinality. In this paper, we study the k-path-connectivity of the complete balanced tripartite graph $K_{n,n,n}$ and obtain $\pi_k(K_{n,n,n}) = \left\lfloor \frac{2n}{k-1} \right\rfloor$, for $3 \leq k \leq n$. Moreover, our result implies that Hager's conjecture is true for $K_{n,n,n}$ and $3 \leq k \leq n$.

2. Main Result

We first introduce some notations and terminology that will be used throughout the paper.

The subgraph of G induced by a vertex set $U \subseteq V(G)$ is denoted by $G[U]$. A subset S of V is called an independent set of G if no two vertices of S are adjacent in G. For any two vertices $x, y \in V(G)$, an xy-path is a path starting at x and ending at y. For convenience, let $[x_1, x_n] = \{x_1, \ldots, x_n\}$. We refer the reader to [12] for the notations and terminology not defined in this paper.

Now we provide our main result.

Theorem 1. *Given any positive integer $n \geq 2$, let $K_{n,n,n}$ denote a complete balanced tripartite graph in which each partite set contains exactly n vertices. Then, we have the following.*

$$\pi_k(K_{n,n,n}) = \left\lfloor \frac{2n}{k-1} \right\rfloor, \text{ for } 3 \leq k \leq n.$$

Proof. Suppose that X, Y, and Z are the three parts of $K_{n,n,n}$, where $X = [x_1, x_n]$, $Y = [y_1, y_n]$, and $Z = [z_1, z_n]$. Let $G = K_{n,n,n}$ and S be any subset of $V(G)$ of cardinality k. By the symmetry of $K_{n,n,n}$, we can assume that $S \cap X = A = [x_1, x_a]$, $S \cap Y = B = [y_1, y_b]$, $S \cap X = C = [z_1, z_c]$. Obviously, $a + b + c = k$.

Remember that, when we construct internally disjoint S-paths, each vertex in $V(G) \setminus S$ can appear on one S-path at most. We distinguish three cases as follows.

Case 1: $a = b = 0$ and $c = k$.

In this case, $S \subseteq Z$. Therefore, each vertex in S is adjacent to all the vertices in $X \cup Y$, which means that we can use any $k-1$ vertices of $X \cup Y$ to connect all vertices in S into an S-path. On the other hand, since S is an independent set, each S-path needs at least $k-1$ vertices of $X \cup Y$. Thus, $\pi(S) = \left\lfloor \frac{|X \cup Y|}{k-1} \right\rfloor = \left\lfloor \frac{2n}{k-1} \right\rfloor$.

Case 2: $1 \leq a \leq b \leq c$.

Note that $3 \leq k = a + b + c \leq n$. We will show $\pi(S) \geq \left\lfloor \frac{2n}{k-1} \right\rfloor$ in this case by constructing $\left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor + 2$ internally disjoint S-paths and prove that $\left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor + 2 \geq \left\lfloor \frac{2n}{k-1} \right\rfloor$. We divide the construction process into four steps. In Steps 1 and 2, we will construct two S-paths mainly by using some edges in $G[S]$ and some vertices in $Z \setminus C$. In Steps 3 and 4, we will use $n-c-1$ vertices from $X \setminus A$ and $n-c-1$ vertices from $Y \setminus B$ to construct $\left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor$ internally disjoint S-paths. On these S-paths, any two vertices of S are connected by the vertices from $X \setminus A$ and $Y \setminus B$.

Step 1: Construct the first S-path P_1.

Firstly, by using vertices $z_{c+1}, \ldots, z_{c+b-1}$ in $Z \setminus C$, we can connect all vertices $y_1, \ldots, y_b$ of B into a path, denoted by P_{11}, i.e., $P_{11} = y_1 z_{c+1} y_2 z_{c+2} \ldots y_{b-1} z_{c+b-1} y_b$.

Since $c \geq a$, $A \subseteq \{x_1, \ldots, x_c\}$. Note that $G[\{x_1, \ldots, x_c\} \cup C] = K_{c,c}$. Thus, there must exist a path, denoted by P_{12}, connecting all the vertices of $A \cup C$ in $G[\{x_1, \ldots, x_c\} \cup C]$. More specifically, let $P_{12} = z_1 x_1 z_2 \ldots x_{c-1} z_c x_c$.

Finally, using the vertex x_{c+1} to connect y_b and z_1, we obtain the first S-path P_1, i.e., $P_1 = P_{11} \cup \{y_b x_{c+1} z_1\} \cup P_{12}$.

Step 2: Construct the second S-path P_2.

Firstly, by using the vertices $z_{c+b}, \ldots, z_{c+b+a-2}$ in $Z \setminus C$, we can connect all the vertices $x_1, \ldots, x_a$ of A into a path, denoted by P_{21}, i.e., $P_{21} = x_1 z_{b+c} \ldots x_{a-1} z_{a+b+c-2} x_a$.

Since $c \geq b$, $B \subseteq \{y_1, \ldots, y_c\}$. Similarly, in $G[\{y_1, \ldots, y_c\} \cup C]$ there must exist a path, denoted by P_{22}, connecting all the vertices of $B \cup C$. More specifically, let $P_{22} = z_1 y_1 z_2 \ldots y_{c-1} z_c y_c$.

Finally, using the vertex y_{c+1} to connect x_a and z_1, we obtain the second S-path P_2, i.e., $P_2 = P_{21} \cup \{x_a y_{c+1} z_1\} \cup P_{22}$.

Remark. *After the first two steps, we have found two S-paths, which are obviously internally disjoint. Moreover, there are* $n-c-1$ *unused vertices in* $X\backslash A$ *(namely,* $x_{c+2}, \ldots, x_n$*),* $n-c-1$ *unused vertices in* $Y\backslash B$ *(namely,* $y_{c+2}, \ldots, y_n$*) and* $n-k+2$ *unused vertices in* $Z\backslash C$*. Set* $A' = [x_{c+2}, x_n]$ *and* $B' = [y_{c+2}, y_n]$*.*

Step 3: Construct the next $2l$ S-paths, where $l = \left\lfloor \frac{n-c-1}{k-1} \right\rfloor$.

Note that, if $l = 0$, proceed directly to Step 4. Thus, we assume that $l \geq 1$. We now provide a method to construct S-paths in pairs. The outline of the method is as follows.

Firstly, we take $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices from A' and $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices from B'. Then, using the $k-1$ vertices in total, connect all the vertices of S into an S-path. Next, we take $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices from A' and $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices from B'. Using the $k-1$ vertices in total, construct another S-path. Thus, by $\left\lceil \frac{k-1}{2} \right\rceil + \left\lfloor \frac{k-1}{2} \right\rfloor = k-1$ vertices in A' and $\left\lfloor \frac{k-1}{2} \right\rfloor + \left\lceil \frac{k-1}{2} \right\rceil = k-1$ vertices in B', we can obtain a pair of S-paths. By repeating this process, we can construct $l = \left\lfloor \frac{n-c-1}{k-1} \right\rfloor$ pairs of S-paths in this step.

Now, we construct the S-paths P_3 and P_4 to illustrate the specific method. Note that, since $a \geq 1$, $\left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{a+b+c-1}{2} \right\rfloor \geq \left\lfloor \frac{b+c}{2} \right\rfloor \geq b \geq a$.

The construction of P_3.

Firstly, by using $b-1$ vertices $x_{c+2}, \ldots, x_{c+b}$ in A', connect all vertices $y_1, \ldots, y_b$ of B into a path, denoted by P_{31}, i.e., $P_{31} = y_1 x_{c+2} y_2 x_{c+3} \ldots x_{c+b} y_b$.

Similarly, by using $a-1$ vertices $y_{c+2}, \ldots, y_{c+a}$ in B', connect all vertices $x_1, \ldots, x_a$ of A into a path, denoted by P_{32}, i.e., $P_{32} = x_1 y_{c+2} x_2 y_{c+3} \ldots y_{c+a} x_a$.

Then, join the vertices y_b and z_1 by vertex x_{c+b+1}. Moreover, join vertices x_1 and z_c by vertex y_{c+a+1}.

Next, we take $\left\lceil \frac{k-1}{2} \right\rceil - b$ unused vertices $\left[x_{c+b+2}, x_{c+1+\lceil \frac{k-1}{2} \rceil}\right]$ from A' and take $\left\lfloor \frac{k-1}{2} \right\rfloor - a$ unused vertices $\left[y_{c+a+2}, y_{c+1+\lfloor \frac{k-1}{2} \rfloor}\right]$ from B'. Since each vertex in $A' \bigcup B'$ is adjacent to all the vertices in C, using the $k-1-a-b = c-1$ vertices in total, we can connect all the vertices of C into a $z_1 z_c$-path P_{33}.

Now, we obtain the third S-path $P_3 = P_{31} \cup \{y_b x_{c+b+1} z_1\} \cup P_{33} \cup \{z_c y_{c+a+1} x_1\} \cup P_{32}$.

The construction of P_4 is similar. The only difference is that the subpath P_{43} is constructed by $\left\lfloor \frac{k-1}{2} \right\rfloor - b$ unused vertices in A' and $\lceil \frac{k-1}{2} \rceil - a$ unused vertices in B'. It follows that the fourth S-path P_4 uses $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices in A' and $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices in B', respectively.

Step 4: Construct the last path if necessary.

Let $d = n-c-1-l(k-1)$. Thus, there are d unused vertices in A' and B', respectively. Since $l = \left\lfloor \frac{n-c-1}{k-1} \right\rfloor$, $0 \leq d < k-1$. Now, according to the value of d, we distinguish two cases.

If $0 \leq d < \left\lceil \frac{k-1}{2} \right\rceil$, then $2l = 2\left\lfloor \frac{n-c-1}{k-1} \right\rfloor = 2\left\lfloor \frac{l(k-1)+d}{k-1} \right\rfloor = \left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor$. In this case, we stop constructing any new S-path.

If $\left\lceil \frac{k-1}{2} \right\rceil \leq d < k-1$, then $2l = 2\left\lfloor \frac{n-c-1}{k-1} \right\rfloor = \left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor - 1$. Since $d \geq \left\lceil \frac{k-1}{2} \right\rceil$, we can take $\left\lceil \frac{k-1}{2} \right\rceil$ and $\left\lfloor \frac{k-1}{2} \right\rfloor$ remaining vertices from A' and B', respectively. Similarly to P_3, using the $k-1$ vertices in total, we can obtain a new S-path.

Therefore, by the above four steps, we construct $\left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor + 2$ S-paths, which are obviously internally disjoint.

Moreover, since $1 \le a \le b \le c$, $k-1 = a+b+c-1 \ge c+1$. Hence,

$$\left\lfloor \frac{2(n-c-1)}{k-1} \right\rfloor + 2 \ge \left\lfloor \frac{2(n-k+1)}{k-1} \right\rfloor + 2 = \left\lfloor \frac{2n}{k-1} \right\rfloor.$$

It follows that we can obtain at least $\left\lfloor \frac{2n}{k-1} \right\rfloor$ internally disjoint S-paths in this case; that is, $\pi(S) \ge \left\lfloor \frac{2n}{k-1} \right\rfloor$.

Case 3: $a = 0$ and $1 \le b \le c$.

In this case, $S = (B \cup C)$. We will also construct at least $\left\lfloor \frac{2n}{k-1} \right\rfloor$ internally disjoint S-paths. We divide the construction process into four steps, as follows.

Step 1: Construct the first S-path P_1.

By using $b-1$ vertices $z_{c+1}, \ldots, z_{c+b-1}$ in $Z \setminus C$, connect all the vertices $y_1, \ldots, y_b$ of B into a path, denoted by P_{11}, i.e., $P_{11} = y_1 z_{c+1} y_2 z_{c+2} \ldots y_{b-1} z_{c+b-1} y_b$.

By using $c-1$ vertices $x_1, \ldots, x_{c-1}$ in X, connect all the vertices $z_1, \ldots, z_c$ of C into a path, denoted by P_{12}, i.e., $P_{12} = z_1 x_1 z_2 x_2 \ldots z_{c-1} x_{c-1} z_c$.

Finally, using the vertex x_c to connect y_b and z_1, we obtain the first S-path P_1, i.e., $P_1 = P_{11} \cup \{y_b x_c z_1\} \cup P_{12}$.

Step 2: Construct the second S-path P_2.

Let $P_2 = z_1 y_1 z_2 \ldots y_{c-1} z_c y_c$. Since $c \ge b$, $B \subseteq \{y_1, \ldots, y_c\}$. Hence, P_2 is a path connecting all the vertices of $B \cup C$, and so is an S-path.

Remark. *After the first two steps, we have found two S-paths, which are obviously internally disjoint. Moreover, there are $n-c$ unused vertices in $X \backslash A$ (namely, $x_{c+1}, \ldots, x_n$), $n-c$ unused vertices in $Y \backslash B$ (namely, $y_{c+1}, \ldots, y_n$) and $n-c-(b-1) = n-k+1$ unused vertices in $Z \backslash C$ (namely, $z_{c+b}, \ldots, z_n$). Set $A' = [x_{c+1}, x_n]$, $B' = [y_{c+1}, y_n]$, and $C' = [z_k, z_n]$.*

Step 3: Construct the next $2l$ S-paths, where $l = \left\lfloor \frac{n-c}{k-1} \right\rfloor$.

The method is similar to case 2. If $l = 0$, proceed directly to Step 4. thus, we assume that $l \ge 1$. In general, by $k-1$ vertices in A' and $k-1$ vertices in B', we can obtain S-paths in pairs: use $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices in A' and $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices in B' to construct an S-path; next, use $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices in A' and $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices in B' to construct another S-path; by repeating this process, we can construct $l = \left\lfloor \frac{n-c}{k-1} \right\rfloor$ pairs of S-paths.

However, when $b = c$, $\left\lfloor \frac{k-1}{2} \right\rfloor = \left\lfloor \frac{2b-1}{2} \right\rfloor = b-1$ and $\left\lceil \frac{k-1}{2} \right\rceil = b$. If we only use $b-1$ vertices in A' and b vertices in B' and do not use any other vertex and edge in $E(G[B \cup C])$, we cannot connect all the vertices of $B \cup C$ into a path. Thus, we distinguish two subcases:

Subcase 3.1: $1 \le b < c$.

We have $\left\lfloor \frac{k-1}{2} \right\rfloor \ge b$.

Firstly, by using b vertices in A', connect all vertices $y_1, \ldots, y_b$ of B and vertex z_1 into a $y_1 z_1$-path, denoted by P_{i1}, where $3 \le i \le 2l+2$.

Next, when i is odd (when i is even), take $\left\lceil \frac{k-1}{2} \right\rceil - b$ ($\left\lfloor \frac{k-1}{2} \right\rfloor - b$) unused vertices from A', and take $\left\lfloor \frac{k-1}{2} \right\rfloor$ ($\left\lceil \frac{k-1}{2} \right\rceil$) unused vertices from B'. Using the $k-1-b = c-1$ vertices in total, we can connect all vertices $z_1, \ldots, z_c$ of C into a $z_1 z_c$-path P_{i2}.

Combining these two paths, we obtain an S-path P_i, i.e., $P_i = P_{i1} \cup P_{i2}$, where $3 \le i \le 2l+2$.

Clearly, when i is odd (when i is even), then the path P_i uses $\left\lceil \frac{k-1}{2} \right\rceil$ ($\left\lfloor \frac{k-1}{2} \right\rfloor$) vertices in A' and $\left\lfloor \frac{k-1}{2} \right\rfloor$ ($\left\lceil \frac{k-1}{2} \right\rceil$) vertices in B', respectively.

Subcase 3.2: $b = c$.

We have $\left\lceil \frac{k-1}{2} \right\rceil = b$ and $\left\lfloor \frac{k-1}{2} \right\rfloor = b-1$.

When i is odd ($3 \leq i \leq 2l+2$), since $\left\lceil \frac{k-1}{2} \right\rceil - b \geq 0$, by the same method as Subcase 3.1, we can construct P_i by $\left\lceil \frac{k-1}{2} \right\rceil$ unused vertices in A' and $\left\lfloor \frac{k-1}{2} \right\rfloor$ unused vertices in B'.

However, when i is even, as noted above, $\left\lfloor \frac{k-1}{2} \right\rfloor$ vertices in A' and $\left\lceil \frac{k-1}{2} \right\rceil$ vertices in B' are not enough to obtain an S-path. We will complete the construction with the help of a vertex in C', as follows.

Firstly, by using $\left\lfloor \frac{k-1}{2} \right\rfloor = b-1$ vertices in A', connect all the vertices $y_1, \ldots, y_b$ of B into a $y_1 y_b$-path, denoted by P_{i1}.

Then, by using $\left\lceil \frac{k-1}{2} \right\rceil - 1 = b-1$ vertices in B', connect all vertices $z_1, \ldots, z_c$ of C into a $z_1 z_c$-path, denoted by P_{i2}.

Finally, by one unused vertex $\hat{y}$ in B' and one unused vertex $\hat{z}$ in C', connect vertices y_b and z_1. Then, we obtain an S-path P_i, i.e., $P_i = P_{i1} \bigcup y_b \hat{z} \hat{y} z_1 \bigcup P_{i2}$, where $3 \leq i \leq 2l+2$ and i is even.

Note that, it remains to show that the vertices in C' are enough. Therefore, we will prove that $|C'| \geq l$.

Since $3 \leq k \leq n$ and $c \geq 2$, we obtain the following.

$$\begin{aligned}
|C'| - l &= n - k + 1 - \left\lfloor \frac{n-c}{k-1} \right\rfloor \\
&\geq n - k + 1 - \frac{n-c}{k-1} \\
&= \frac{n(k-2) - (k-1)^2 + c}{k-1} \\
&\geq \frac{k(k-2) - (k-1)^2 + c}{k-1} \\
&= \frac{c-1}{k-1} \geq 0.
\end{aligned}$$

Thus, in either case, we can always obtain $2l = 2\left\lfloor \frac{n-c}{k-1} \right\rfloor$ S-paths in this step.

Step 4: Construct the last path if necessary.

Let $d = n - c - l(k-1)$. Since $l = \left\lfloor \frac{n-c}{k-1} \right\rfloor$, $0 \leq d < k-1$. Similarly to Case 2, according to the value of d, distinguish two cases.

If $0 \leq d < \left\lceil \frac{k-1}{2} \right\rceil$, then $2l = 2\left\lfloor \frac{n-c}{k-1} \right\rfloor = \left\lfloor \frac{2(n-c)}{k-1} \right\rfloor$. We stop constructing any new S-path.

If $\left\lceil \frac{k-1}{2} \right\rceil \leq d < k-1$, then $2l = 2\left\lfloor \frac{n-c}{k-1} \right\rfloor = \left\lfloor \frac{2(n-c)}{k-1} \right\rfloor - 1$. We can construct one more new S-path by the remaining d vertices in A' and B', respectively.

Therefore, by the above four steps, we construct $\left\lfloor \frac{2(n-c)}{k-1} \right\rfloor + 2$ S-paths, which are obviously internally disjoint.

Moreover, since $a = 0$ and $1 \leq b \leq c$, $k - 1 = b + c - 1 \geq c$. Hence,

$$\left\lfloor \frac{2(n-c)}{k-1} \right\rfloor + 2 \geq \left\lfloor \frac{2(n-k+1)}{k-1} \right\rfloor + 2 = \left\lfloor \frac{2n}{k-1} \right\rfloor.$$

Thus, in this case, we can also obtain at least $\left\lfloor \frac{2n}{k-1} \right\rfloor$ internally disjoint S-paths; that is, $\pi(S) \geq \left\lfloor \frac{2n}{k-1} \right\rfloor$.

From the above discussion, $\pi(S) \geq \left\lfloor \frac{2n}{k-1} \right\rfloor$ in all cases and $\pi(S)$ is exactly $\left\lfloor \frac{2n}{k-1} \right\rfloor$ in Case 1. Thus, we can conclude that $\pi_k(K_{n,n,n}) = \min\{\pi(S) | S \subseteq V(K_{n,n,n}), |S| = k\} = \left\lfloor \frac{2n}{k-1} \right\rfloor$. $\square$

By Steps 3 and 4 of Case 2 in Theorem 1, we can obtain the following corollary, which may be useful for study on complete tripartite graphs.

Corollary 1. *Let a, b, c, and d be positive integers with $1 \leq a \leq b \leq c$, and G be a complete tripartite graph with three parts X, Y, and Z, where $|X| = a + d$, $|Y| = b + d$, and $|Z| = c$. For any k-subset S of $V(G)$, if $|X \cap S| = a$, $|Y \cap S| = b$, and $|Z \cap S| = c$, then there always exist at least $\left\lfloor \frac{2d}{k-1} \right\rfloor$ internally disjoint S-paths in G, where $k = a + b + c$.*

Remark. *Since $\left(\left\lfloor \frac{2n}{k-1} \right\rfloor + 1\right)(k-1) > \kappa(K_{n,n,n}) = 2n \geq \left\lfloor \frac{2n}{k-1} \right\rfloor (k-1)$, Theorem 1 implies that Hager's conjecture is true for $K_{n,n,n}$ and $3 \leq k \leq n$.*

3. Conclusions

k-path-connectivity is a natural generalization of the traditional connectivity. In this paper, we showed that the k-path-connectivity of the complete balanced tripartite graph $K_{n,n,n}$ is $\left\lfloor \frac{2n}{k-1} \right\rfloor$, for $3 \leq k \leq n$. For future work, we will continue to investigate the k-path-connectivity of $K_{n,n,n}$ for $n + 1 \leq k \leq 3n$. It would also be interesting to study the path connectivity of complete r-partite graphs for $r \geq 4$.

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