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K-RINGS OF SMOOTH COMPLETE TORIC VARIETIES AND RELATED SPACES

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Abstract. The K-rings of non-singular complex projective varieties as well as quasi-toric manifolds were described in terms of generators and relations in earlier work of the author with V. Uma. In this paper we obtain a similar description for the more general class of torus manifolds with locally standard torus action and orbit space a homology polytope, which includes the class of all smooth complete complex toric varieties.

1. Introduction. This paper consists of two parts, the first of which gives a description of K-ring of a non-singular complete toric variety in terms of generators and relations. In the second part, which subsumes the first, we obtain the same result for the class of torus manifolds with locally standard action whose orbit space is a homology polytope. Although the proofs in both cases involve the same steps, they are easier to establish and well-known in the context of toric varieties and helps to focus ideas. Besides, the language used in the two proofs are different. For these reasons, we have separated out the algebraic geometric case from the topological one, taking care to avoid unnecessary repetitions.

Let $N \cong \boldsymbol{Z}^n$ and let $N_{\boldsymbol{R}} := N \otimes_{\boldsymbol{Z}} \boldsymbol{R}$. We shall denote by M the dual lattice $\mathrm{Hom}_{\boldsymbol{Z}}(N, \boldsymbol{Z})$ and by $\boldsymbol{T}$ the n-dimensional complex algebraic torus whose coordinate ring is the group algebra $\boldsymbol{C}[M]$. The lattice M is identified with the group of characters of $\boldsymbol{T}$ and N with the group of 1-parameter subgroups of $\boldsymbol{T}$. Let Δ be a fan in N and let $X(\Delta)$ be the corresponding complex $\boldsymbol{T}$-toric variety. When Δ is complete the variety $X(\Delta)$ is complete. When Δ is regular, (i.e., the cones of Δ are generated by part of a $\boldsymbol{Z}$-basis for N) the variety $X(\Delta)$ is smooth. We refer the reader to [4] for an introduction to toric varieties.

Let Δ be a complete regular fan in N and let $X := X(\Delta)$ be the corresponding non-singular complete variety. Recall that one has an inclusion-reversing correspondence between cones of Δ and $\boldsymbol{T}$-orbit closures of X. In fact, the orbit closure $V(\sigma)$ of a cone $\sigma \in \Delta$ equals the union of $\boldsymbol{T}$-orbits of certain (closed) points $p_\gamma \in X$ as γ varies over those cones of Δ which contain σ. In particular, if $\sigma, \sigma' \in \Delta$ are not faces of a cone $\gamma \in \Delta$, then $V(\sigma) \cap V(\sigma') = \emptyset$. The dimension of $V(\sigma)$ equals the co-dimension of σ in $N_{\boldsymbol{R}}$. The orbit closures are themselves toric varieties (under the action of suitable quotients of $\boldsymbol{T}$) and are smooth as we assumed X to be smooth.

We shall denote by $\Delta(k)$ the set of all k-dimensional cones of Δ.

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We shall now recall the description of the Chow ring $A^*(X)$ of X. The ring $A^*(X)$ is generated by the classes of divisors $[V(\rho)]$ as ρ varies over the edges of Δ. Also, one has the following relations among these classes:

(i) If $\rho_1, \dots, \rho_k \in \Delta(1)$ do not span a cone in Δ, then

$$[V(\rho_1)] \cdots [V(\rho_k)] = 0 \,. \tag{1}$$

(ii) Denoting by $v_\rho \in N$ the primitive vector on any edge $\rho \in \Delta(1)$, one has the following relation for each $u \in M$:

$$\sum_{\rho \in \Delta(1)} \langle u, v_\rho \rangle [V(\rho)] = 0 \,. \tag{2}$$

Danilov's theorem [2, Theorem 10.8] asserts that these are no further generating relations and that $A^*(X)$ has no torsion. It turns out that the cycle class map $A^*(X) \to H^*(X)$ is an isomorphism that doubles the gradation. So Danilov's result also yields a description of the singular cohomology (with $\boldsymbol{Z}$-coefficients). In the case of non-singular projective toric varieties these results are due to Jurkiewicz [7].

In this paper we give a description of the 'topological' K-ring of $X(\Delta)$, denoted as $K(X(\Delta))$, in terms of generators and relations. Denoting the Grothendieck ring of algebraic vector bundle by $\mathcal{K}(X(\Delta))$, it turns out that the forgetful homomorphism $\mathcal{K}(X(\Delta)) \to K(X(\Delta))$ is an isomorphism and hence we obtain a similar description of $\mathcal{K}(X(\Delta))$ as well. (See Theorem 4.1.) We shall also establish a similar description of the K-ring of a torus manifolds with locally standard action whose orbit space is a homology polytope. These results were established for non-singular projective toric varieties in [13] and for quasi-toric manifolds in [14]. Recently, V. Uma [15] has extended the results of [14] to the case of torus manifolds under an additional shellability hypothesis on the orbit space which is assumed to be locally standard with quotient a homology polytope. The method of proof adopted here, which involves only elementary considerations, is quite different in spirit and applies equally well to previously established cases.

The Grothendieck ring $\mathcal{K}(X(\Delta))$ of a smooth complete complex toric variety $X(\Delta)$ seems to be well-known and has first been computed by Klyachko [8]. Vezzosi and Vistoli [16] have obtained deep results concerning equivariant $\mathcal{K}$-theory in a very general setting where a smooth separated noetherian scheme X over a perfect field $\mathbf{k}$ is acted on by a diagonalizable group scheme over $\mathbf{k}$. In particular, they obtain, in [16, §6], a description of the $\boldsymbol{T}$-equivariant $\mathcal{K}$-ring $\mathcal{K}_*(X; \boldsymbol{T})$ as an algebra over the higher algebraic K-ring $\mathcal{K}_*(\mathbf{k})$ of $\mathbf{k}$, in the case when X is *any* smooth toric variety. They show that the Merkurjev spectral sequence [11] collapses when X is a smooth complete toric variety leading to a description of $\mathcal{K}_*(X)$. Thus, our description of $\mathcal{K}(X(\Delta))$ can also be deduced from [16, Theorem 6.4, Corollary 6.10]. See §4 for further details.

2. K-theory of smooth complete toric varieties. Let Δ be a complete regular fan in $N \cong \boldsymbol{Z}^n$ and let $X = X(\Delta)$. Let ρ be any edge in Δ. As X is smooth, the Weil divisor $V(\rho)$ determines a $\boldsymbol{T}$-equivariant line bundle $\mathcal{O}(V(\rho))$, which will be denoted by L_ρ. The bundle L_ρ admits a ($\boldsymbol{T}$-equivariant) algebraic cross-section $s_\rho\colon X \to L_\rho$ which vanishes to order 1 along $V(\rho)$, that is, the zero scheme of s equals the variety $V(\rho)$. (This is a general fact concerning the line bundle $\mathcal{O}(D)$ associated to an effective Weil divisor D on a smooth variety. Any line bundle can be regarded as a subsheaf of κ_X, the sheaf of 'total quotient rings' of X. When D is effective, $\mathcal{O}(-D) \hookrightarrow \kappa_X$ is the ideal sheaf of the subscheme D. See [6, Ch. II, Proposition 6.18].) The inclusion $\mathcal{O}(-D) \hookrightarrow \mathcal{O}_X$ yields, upon taking duals, a morphism $s_D\colon \mathcal{O}_X \to \mathcal{O}(D)$ which defines the required global section with zero scheme D. Applying these considerations to L_ρ we obtain the required section s_ρ. Furthermore, $c_1(L_\rho) \in A^1(X)$ equals the divisor class $[V(\rho)]$. Topologically, the first Chern class $c_1(L_\rho) \in H^2(X)$ can also be described as the cohomology class—also denoted by $[V(\rho)] \in H^2(X)$—dual to the submanifold $V(\rho) \subset X$, which is the image of $[V(\rho)] \in A^1(X)$ under the isomorphism $A^*(X) \cong H^*(X)$.

If $\rho_1, \dots, \rho_k$ are edges of Δ which do not span a cone of Δ, then the section $s\colon X \to L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}$, defined as $s(x) = (s_{\rho_1}(x), \dots, s_{\rho_k}(x))$, is nowhere vanishing, since $V(\rho_1) \cap \cdots \cap V(\rho_k) = \emptyset$. Therefore, s defines a monomorphism $\tilde{s}\colon \mathcal{O}_X \to L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}$ of vector bundles. Denote by E the quotient of $L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}$ by the image of $\tilde{s}$. Applying the γ-operation in $K(X)$, we see that $\gamma^k([L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}] - k) = \lambda^k([E]) = 0$ as rank of E equals $k-1$. On the other hand, $\gamma^k([L_{\rho_1} \oplus \cdots \oplus L_{\rho_k}] - k) = \prod_{1 \le i \le k} \gamma^1([L_{\rho_i}] - 1)$. That is,

$$\prod_{1 \le i \le k} ([L_{\rho_i}] - 1) = 0\,. \tag{3}$$

Let $u \in M$. Set $L_u := \prod_{\rho \in \Delta(1)} L_\rho^{\langle u, v_\rho \rangle}$. The first Chern class of the line bundle L_u can be readily calculated to be $\sum_{\rho \in \Delta(1)} \langle u, v_\rho \rangle c_1(L_\rho) = \sum \langle u, r_\rho \rangle [V(\rho)] = 0$ in view of Equation (2) above. Since the isomorphism class of a line bundle is determined by its first Chern class, we have the following equation in $K(X)$:

$$\prod_{\rho \in \Delta(1)} [L_\rho]^{\langle u, v_\rho \rangle} = 1\,. \tag{4}$$

DEFINITION 2.1. Let Δ be a complete regular fan in N. Let $R(\Delta)$ denote the ring $\boldsymbol{Z}[x_\rho \mid \rho \in \Delta(1)]/\mathfrak{I}$, where $\mathfrak{I}$ is the ideal generated by the elements:

(i) $x_{\rho_1} \cdots x_{\rho_k} = 0$, whenever $\rho_1, \dots, \rho_k \in \Delta(1)$ do not span a cone of Δ.

(ii) $z_u := \prod_{\rho \in \Delta(1), \langle u, v_\rho \rangle > 0} (1 - x_\rho)^{\langle u, v_\rho \rangle} - \prod_{\rho \in \Delta(1), \langle u, v_\rho \rangle < 0} (1 - x_\rho)^{-\langle u, v_\rho \rangle}$ for $u \in M$.

We are now ready to state our main theorem.

THEOREM 2.2. *The K-ring of a complete non-singular toric variety $X(\Delta)$ is isomorphic to $R(\Delta)$ under the isomorphism ψ which sends x_ρ to* $(1 - [L_\rho])$, $\rho \in \Delta(1)$.

In view of Equations (3) and (4) it is clear that there is a homomorphism of rings $\psi\colon R(\Delta) \to K(X(\Delta))$. In §3 we shall show that ψ is onto. In §4 we complete the proof by showing that both the abelian groups $R(\Delta)$ and $K(X(\Delta))$ are free of the same rank.

3. Line bundles and K-theory. In this section X denotes a path connected finite CW complex. Assume that $H^*(X)$ is generated by $H^2(X)$. Then the K-ring of X is generated (as a ring) by the classes of line bundles on X. This was proved in [13] under the hypothesis that X has cells only in even dimensions. However, essentially the same proof works under our weaker hypothesis. Indeed, suppose that $\dim(X) \leq 2n$ and that $H^2(X)$ is generated as an abelian group by k elements. Then one has a continuous map $f\colon X \to (\boldsymbol{CP}^n)^k$ which induces a surjection in cohomology in dimension 2 and hence, by our hypothesis on X, in all dimensions. Since the cohomology of X vanishes in odd dimensions, it follows that the Atiyah-Hirzebruch sequence collapses. Since f^* induces a surjection in cohomology, the naturality of the spectral sequence implies that f^* induces a surjection in K-theory. Since $K(\boldsymbol{CP}^n)$ is generated by line bundles, it follows by the Künneth theorem for K-theory that $K((\boldsymbol{CP}^n)^k)$ is generated by line bundles. Hence $K(X)$ is also generated by line bundles. As a consequence we obtain

PROPOSITION 3.1. *Let $X(\Delta)$ be a complete non-singular toric variety. With notation as in Theorem 2.2, the ring homomorphism $\psi\colon R(\Delta) \to K(X(\Delta))$ is a surjection.*

REMARK 3.2. Suppose that $H^*(X) = H^{\mathrm{ev}}(X)$ is a free abelian group. A straightforward argument involving the Atiyah-Hirzebruch spectral sequence shows that, as far as the additive structure is concerned, $K(X)$ is a free abelian group of rank equal to the Euler characteristic $\chi(X)$ of X. It is well-known that $\chi(X(\Delta))$ equals $\#\Delta(n)$, the number of n-dimensional cones in Δ [4]. Since ψ is a surjection, we conclude that as an abelian group, the rank of R is at least $\#\Delta(n)$.

4. Proof of Theorem 2.2. Let Δ be a complete regular fan in N and let $R(\Delta)$ be the ring defined in §1. It is clear that the set of all monomials $x(\sigma) := x_{\rho_1} \cdots x_{\rho_k}, \sigma \in \Delta$, where $\sigma \in \Delta(k)$ is spanned by edges $\rho_1, \ldots, \rho_k$, forms a generating set for $R(\Delta)$. We shall show that $R(\Delta)$ is a free abelian group and describe a monomial basis for it.

Filtered rings. Let S be the polynomial ring $\mathbf{Z}[x_1, \ldots, x_d]$ and let $\mathfrak{I} \subset S$ be an ideal generated by elements $f_1, \ldots, f_m$, where each f_j has constant term zero. The ring S is graded, where we set $\deg(x_i) = 1$. We denote the abelian group of all homogeneous polynomials of degree j by $S_{(j)}$. Consider the multiplicative filtration $S = S_0 \supset S_1 \supset \cdots \supset S_r \supset \cdots$, where $S_r = \bigoplus_{j \geq r} S_{(j)}$. In view of our assumption about the generators of $\mathfrak{I}$, this is an $\mathfrak{I}$-filtration, i.e., $\mathfrak{I}S_r \subset S_{r+1}$ for $r \geq 1$. Let R denote the quotient ring $S/\mathfrak{I}$. The filtration of S induces a decreasing multiplicative filtration $R_0 \supset R_1 \supset \cdots$ of R. Let $\mathrm{gr}(R)$ denote the associated graded ring. The group of all homogeneous element of degree j in $\mathrm{gr}(R)$ is denoted by $\mathrm{gr}(R)_{(j)}$. Clearly, $\mathrm{gr}(R)_{(j)} = R_j/R_{j+1} = S_j/(\mathfrak{I} \cap S_j + S_{j+1}) \cong (S_j/S_{j+1})/(\mathfrak{I} \cap S_j/S_{j+1})$ for $j \geq 0$.

Suppose that $\mathcal{I}$ is the ideal of S generated by the initial forms of $f_1, \ldots, f_m$. (By the initial form of $f \in S_r \setminus S_{r+1}$ we mean the homogeneous polynomial $\mathrm{in}(f)$ of degree $r \geq 1$, where $f - \mathrm{in}(f) \in S_{r+1}$.) Set $\tilde{R} := S/\mathcal{I}$. Since $\mathcal{I}$ is a homogeneous ideal, the ring $\tilde{R}$ inherits a grading from S. The group $\tilde{R}_{(j)}$ of all homogeneous elements of degree j is isomorphic to $S_{(j)}/\mathcal{I} \cap S_{(j)} \cong (S_j/S_{j+1})/((\mathcal{I} \cap S_j + S_{j+1})/S_{j+1})$. Note that $\mathcal{I} \cap S_j + S_{j+1} \subset \mathfrak{I} \cap S_j + S_{j+1}$.

Hence there exists a natural epimorphism $\eta_j\colon \tilde{R}_{(j)} \to \mathrm{gr}(R)_{(j)}$. We let $\eta\colon \mathrm{gr}(\tilde{R}) \to \mathrm{gr}(R)$ be the direct sum of η_j, $j \geq 0$.

Now, let $S = \boldsymbol{Z}[x_\rho \mid \rho \in \Delta(1)]$ and let the ideal $\mathfrak{I}$ and the set of generators of $\mathfrak{I}$ be as in Definition 2.1. The initial form of z_u is seen to be $\mathrm{in}(z_u) = \sum_{\rho\in\Delta(1)} \langle u, v_\rho\rangle x_\rho =: h_u$. Thus the ideal $\mathcal{I}$ is generated by the set of monomials listed in 2.1 (i) and the elements h_u, $u \in M$. By [2, Theorem 10.8] we see that the ring $\tilde{R}(\Delta) = S/\mathcal{I}$ is isomorphic to the Chow ring $A^*(X(\Delta))$ under an isomorphism which maps x_ρ to $[V(\rho)]$. We shall identify $\tilde{R}(\Delta)$ with $A^*(X(\Delta))$. Thus we obtain a surjective homomorphism of graded abelian groups $\eta\colon A^*(X(\Delta)) \to \mathrm{gr}(R(\Delta))$.

PROOF OF THEOREM 2.2. We shall abbreviate $X(\Delta)$ to X etc. Recall that $A^*(X) \cong H^*(X)$ is a free abelian group, its rank being equal to $\#\Delta(n)$, the number of n-dimensional cones of Δ. Since η is a surjection, $\mathrm{gr}(R(\Delta))$ is an abelian group generated by at most $\#\Delta(n)$ elements. Since R and $\mathrm{gr}(R)$ have equal rank, it follows that R must be free abelian of rank *at most* $\#\Delta(n)$. In view of Remark 3.2, we conclude that the rank of R *equals* $\#\Delta(n)$.

Since $\psi\colon R \to K(X)$ is a surjection between free abelian groups of same rank, it follows that ψ is an isomorphism. This completes the proof. $\square$

Let X be a smooth complete variety over $\boldsymbol{C}$. Consider the 'forgetful' homomorphism $f\colon \mathcal{K}(X) \to K(X)$, where $\mathcal{K}(X)$ denotes the Grothendieck K-ring of algebraic vector bundles over X. Since X is smooth, $\mathcal{K}(X)$ is isomorphic to the Grothendieck group $\mathcal{K}'(X)$ of coherent sheaves over X. One has a 'topological filtration' on $\mathcal{K}'(X) \cong \mathcal{K}(X)$ and one has a well-defined surjective homomorphism $\phi\colon A_*(X) \to \mathrm{gr}(\mathcal{K}(X))$ of graded abelian groups. When $X = X(\Delta)$, it follows that $\mathrm{gr}(\mathcal{K}(X))$ is a quotient of the free abelian group $A_*(X) = A^{n-*}(X)$. It follows that $\mathcal{K}(X)$ is generated by at most $\chi(X)$ elements. Since $K(X)$ has rank equal to $\chi(X)$, it follows that $f\colon \mathcal{K}(X) \to K(X)$ is an isomorphism of rings. We record this as

THEOREM 4.1. *Let $X = X(\Delta)$ be a smooth complete toric variety. Then, the forgetful morphism of rings $f\colon \mathcal{K}(X) \to K(X)$ is an isomorphism. In particular, $\mathcal{K}(X(\Delta))$ is isomorphic to $R(\Delta)$.*

REMARK 4.2. It is immediate from the main theorem that $x_\rho \in R$ is nilpotent, since $[L_\rho]$ is invertible. Indeed, the surjectivity of η already implies that $R_j = R_{j+1}$ for all $j > n$. Since $\bigcap_{j\geq 0} R_j = 0$, it follows that $R_j = 0$ for all $j > n$. Setting $y_\rho := [L_\rho] = 1 - x_\rho$, $\rho \in \Delta(1)$, the elements $y_\rho^{\pm 1}$, $\rho \in \Delta(1)$, can be used as generators of R. Thus, we obtain the following alternative description of $K(X(\Delta)) \cong R$ as a quotient of the Laurent polynomial ring: $R = \boldsymbol{Z}[y_\rho^{\pm 1}; \rho \in \Delta(1)]/\mathfrak{J}$ where the ideal $\mathfrak{J}$ is generated by the following elements:

(i) $\prod_{1\leq j\leq k}(1 - y_{\rho_j})$ whenever $\rho_1, \ldots, \rho_k$ do not span a cone of Δ,

(ii) $\zeta_u := \prod_{\rho\in\Delta(1)} y_\rho^{\langle u, v_\rho\rangle} - 1$, for $u \in M$.

(Cf. Definition 2.1.)

REMARK 4.3. It follows from our proof that $\eta\colon A^*(X)\to \mathrm{gr}(R(\Delta))$ is an isomorphism. Recall that if $\sigma \in \Delta(k)$ is spanned by edges $\rho_1, \dots, \rho_k$, then $[V(\sigma)] = [V(\rho_1)]\cdots[V(\rho_k)] \in A^{n-k}(X)$. If $\Gamma \subset \Delta$ is a collection of cones such that $\{[V(\sigma)]\}_{\sigma\in\Gamma}$ is $\boldsymbol{Z}$-basis for $A^*(X)$, then $\{x(\sigma)\}_{\sigma\in\Gamma}$ is a monomial basis for $R(\Delta)$.

Equivariant $\mathcal{K}$-ring and the work of Vezzosi and Vistoli [16]. As mentioned in the introduction, an explicit description of $\boldsymbol{T}$-equivariant $\mathcal{K}$-ring, denoted $\mathcal{K}(X;\boldsymbol{T})$, in terms of generators and relations, of complete smooth complex toric variety X has been obtained by Vezzosi and Vistoli [16, Theorem 6.4]. (The setup in [16] is much more general.) The (non-equivariant) Grothendieck ring $\mathcal{K}(X)$ is deduced from this result in [16, Corollary 6.10] to yield the following description: Recall that $\mathcal{K}(X;\boldsymbol{T})$ is an algebra over the representation ring $\boldsymbol{RT} = \boldsymbol{Z}[M] = \bigoplus_{u\in M}\boldsymbol{Z}\chi^u$ of $\boldsymbol{T}$, which is a Laurent polynomial algebra. One has, by [16, Corollary 6.10], $\mathcal{K}(X) \cong \mathcal{K}(X;\boldsymbol{T})\otimes_{\boldsymbol{RT}}\boldsymbol{Z}$, where $\boldsymbol{Z}$ is regarded as an algebra over the representation ring $\boldsymbol{RT}$ of $\boldsymbol{T}$ via the degree map $\deg\colon \boldsymbol{RT}\to\boldsymbol{Z}$. Once a presentation of $\mathcal{K}(X;T)$ *as an $\boldsymbol{RT}$-algebra* is known, a description of $\mathcal{K}(X)$, in terms of generators and relations, can be readily obtained. Note that [16, Theorem 6.4] describes $\mathcal{K}(X;\boldsymbol{T})$ as a $\boldsymbol{Z}$-algebra in terms of generators and relations, whereas such a description as an $\boldsymbol{RT}$-algebra is not given explicitly. (However, $\mathcal{K}(X;\boldsymbol{T})$ is described, in [16, Theorem 6.2], as *subalgebra* of a direct product of $\boldsymbol{RT}$, by showing that the restriction to $\boldsymbol{T}$-fixed points, $\mathcal{K}(X;\boldsymbol{T})\to\mathcal{K}(X^{\boldsymbol{T}};\boldsymbol{T})\cong\bigoplus_{\sigma\in\Delta(n)}\boldsymbol{RT}$, is a monomorphism of $\boldsymbol{RT}$-algebras.) We shall obtain below the isomorphism $\mathcal{K}(X)\cong R(\Delta)$ from the work of Vezzosi and Vistoli.

Observe that the line bundles L_ρ, $\rho\in\Delta$, defined in §2, are $\boldsymbol{T}$-equivariant since the V_ρ are $\boldsymbol{T}$-divisors. It also follows that s_ρ is $\boldsymbol{T}$-equivariant. Therefore the section $\tilde{s}\colon\mathcal{O}_X\to L_{\rho_1}\oplus\cdots\oplus L_{\rho_k}$, is also $\boldsymbol{T}$-equivariant. It follows that Equation (3) holds in $\mathcal{K}(X;\boldsymbol{T})$.

Let $u\in M = \mathrm{Hom}(\boldsymbol{T},\boldsymbol{C}^*)$ and let $\boldsymbol{C}_u$ denote the corresponding $\boldsymbol{T}$-module. One has the $\boldsymbol{T}$-line bundle ξ_u over X whose total space is $X\times\boldsymbol{C}_u$, with diagonal $\boldsymbol{T}$-action. The $\boldsymbol{RT}$-algebra structure on the equivariant $\mathcal{K}$-theory of X is given by $\chi^u\cdot[E] = [\xi_u\otimes E]$. In particular, $[\xi_u] = \chi^u\cdot 1 = \chi^u$ in $\mathcal{K}(X;\boldsymbol{T})$. The $\boldsymbol{T}$-line bundle is isomorphic to $\mathcal{O}(D_u)$, where D_u is the divisor of the rational function χ^u on X. Using this, it can be seen that there is an isomorphism of $\boldsymbol{T}$-bundles $\xi_u\cong\prod_{\rho\in\Delta(1)}L_\rho^{\langle u,v_\rho\rangle} = L_u$. (See [4, §3.3].) It follows that one has a homomorphism $\theta\colon \boldsymbol{RT}[y_\rho^{\pm1};\rho\in\Delta(1)]/I\to\mathcal{K}(X;\boldsymbol{T})$ of $\boldsymbol{RT}$-algebras defined by $\theta(y_\rho) = [L_\rho]$ where the ideal I is generated by the following elements:

(i) $\prod_{1\le j\le k}(y_{\rho_j}-1)$, whenever $\rho_1,\dots,\rho_k\in\Delta(1)$ do not span of cone of Δ,

(ii) $\prod_{\rho\in\Delta(1)}y_\rho^{\langle u,v_\rho\rangle}-\chi^u$, $u\in M$.

Set $\mathcal{R}(\Delta) := \boldsymbol{Z}[y_\rho^{\pm1};\rho\in\Delta(1)]/J$, where J is the ideal generated by elements listed in (i) above. One has a monomorphism of rings $\boldsymbol{RT}\to\mathcal{R}(\Delta)$ defined by $\chi^u\mapsto\prod_{\rho\in\Delta(1)}y_\rho^{\langle u,v_\rho\rangle}$, $u\in M$, which yields an $\boldsymbol{RT}$-algebra structure on $\mathcal{R}(\Delta)$. This extends to a canonical surjection of $\boldsymbol{RT}$-algebras $\boldsymbol{RT}[y_\rho^{\pm1};\rho\in\Delta(1)]/I\to\mathcal{R}(\Delta)$. A straightforward argument shows that this is an *isomorphism* of $\boldsymbol{RT}$-algebras. By [16, Theorem 6.4], $\mathcal{K}(X;\boldsymbol{T})\cong\mathcal{R}(\Delta)$ and θ is an isomorphism of $\boldsymbol{RT}$-algebras. Thus we obtain

THEOREM 4.4 (Cf. Vezzosi and Vistoli [16, Theorem 6.4]). *One has an isomorphism of $\boldsymbol{RT}$-algebras $\theta\colon \boldsymbol{RT}[y_\rho^{\pm 1};\rho \in \Delta(1)]/I \to \mathcal{K}(X(\Delta);\boldsymbol{T})$ which sends y_ρ to $[L_\rho]$, $\rho \in \Delta(1)$.*

Now, by [16, Corollary 6.10], we obtain that $\mathcal{K}(X) = \boldsymbol{Z} \otimes_{\boldsymbol{RT}} \boldsymbol{RT}[y_\rho^{\pm 1};\rho \in \Delta(1)]/I \cong \boldsymbol{Z} \otimes_{\boldsymbol{RT}} \mathcal{R}(\Delta) \cong \mathcal{R}(\Delta)/\langle \zeta_u; u \in M\rangle = R(\Delta)$, in view of Remark 4.2.

5. K-theory of torus manifolds. The algebraic geometric notion of a non-singular projective toric variety has been brought home to realm of topology by Davis and Januszkiewicz [3], who introduced the class of quasi-toric manifolds. (Davis and Januszkiewicz called them 'toric manifolds'; the present terminology of 'quasi-toric manifolds' is due to Buchstaber and Panov [1].) See also Masuda [9]. Recently, Masuda and Panov [10] introduced a new class of manifolds called *torus manifolds* in their efforts to develop the analogue of a non-singular complete toric variety. The class of torus manifolds, which includes all complete non-singular complex toric varieties, is much more general than that of quasi-toric manfold as there are non-singular complete toric varieties which are not quasi-toric manifolds.

In this section we obtain a description of the K-ring of a torus manifold, assuming that the orbit space is a *homology polytope*. Most of the ingredients needed for the proof of Theorem 5.3 can be found in [10]. We shall recall here the results of [10], concerning the cohomology of toric manifolds with locally standard action whose orbit space is a homology polytope that are needed for our purposes. We recall below the definition and some basic facts relevant for our purposes, referring the reader to [10] for a detailed exposition of torus manifolds. As mentioned in the Introduction, our main result in this section, namely, Theorem 5.3 subsumes Theorem 2.2. In fact, as we shall see, the steps involved in the proof in both cases are the same; we need only to establish those steps whose proofs in the algebraic geometric situation are either unavailable or are not obvious in the topological context of torus manifolds.

We begin by recalling the basic definition and properties of torus manifolds. Let $T := (\boldsymbol{S}^1)^n$ denote the *compact* torus and let X be a smooth compact oriented connected manifold of dimension $2n$ on which T acts effectively with a finite non-empty set of T-fixed points. Such a manifold X is called a torus manifold. The orbit space $Q := X/T$ is a 'manifold with corners' of dimension n. The T-action on X is called *locally standard* if X is covered by T-invariant open sets U such that U is equivariantly diffeomorphic to an invariant open subset contained in a T-representation $\mathcal{U} \cong \boldsymbol{C}^n$ whose characters form a $\boldsymbol{Z}$-basis for $\mathrm{Hom}(T, \boldsymbol{S}^1) \cong \boldsymbol{Z}^n$. For example, $X = X(\Delta)$, a complete non-singular toric variety with action of $\boldsymbol{T}$ restricted to $T \subset \boldsymbol{T}$, is a torus manifold with locally standard T-action. Also, any quasi-toric manifold is a torus manifold. A *characteristic submanifold* of X is a codimension 2 submanifold which is pointwise fixed by a one-dimensional subgroup of T. There are only finitely many characteristic submanifolds; we denote them by $V_1, \dots, V_d$.

In the case when X is a smooth projective toric variety or a quasi-toric manifold, the orbit space Q is a simple polytope. However, for a general torus manifold, this is not the case. Assume that X is a torus manifold with locally standard T-action. Define the boundary of Q, denoted by ∂Q, to be the set of all points which do *not* have a neighourhood homeomorphic to an open set of $\boldsymbol{R}^n$. Then ∂Q is the image under the quotient map $\pi: X \to Q$ of the union of all characteristic submanifolds of X. Denote the image of V_i by $Q_i \subset \partial Q, 1 \leq i \leq d$; these are the *facets* of Q. A non-empty intersection of facets is called a *preface* of Q; a *face* of Q is a connected component of a preface. The space Q is called a *homology polytope* if every preface F, including Q itself, is acyclic, i.e., $\tilde{H}_*(F; \mathbf{Z}) = 0$, in particular, every preface of Q is path connected. One of the main results of [10] is that the integral cohomology of X is generated by degree 2 elements if and only if the action is locally standard and Q is a homology polytope. *We shall only consider torus manifolds with locally standard T-action whose orbit space is a homology polytope.* Note that any characteristic submanifold of such a torus manifold inherits these properties; see [10, Lemma 2.3]. In particular, characteristic submanifolds are orientable and have T-fixed points. We fix an omni-orientation of X, i.e., orientations of X and all its characteristic submanifolds. Thus the normal bundle ν_i to the imbedding $V_i \hookrightarrow X$ is oriented by the requirement that $\nu_i \oplus \mathcal{T} V_i$ be oriented-isomorphic to $\mathcal{T} X_{|V_i}$, the tangent bundle of X restricted to V_i.

Line bundles over X. Let $V \subset X$ be a codimension 2 oriented closed connected submanifold of an *arbitrary* oriented closed connected manifold of dimension m. Let $[V] \in H^2(X; \mathbf{Z})$ denote the cohomology class dual to V and let L denote a complex line bundle over X with $c_1(L) = [V]$. Assume that $H^1(V; \mathbf{Z}) = 0$.

LEMMA 5.1. *Suppose that $c_1(L) = [V] \neq 0$. Then the complex line bundle L admits a section $s: X \to L$ such that the zero locus of s is precisely V.*

PROOF. Assume that X is endowed with a Riemannian metric and let $N \subset X$ denote a tubular neighbourhood of V which is identified with the disk bundle associated to the normal bundle ν to the imbedding $V \hookrightarrow X$. The normal bundle is canonically oriented by the requirement that $\mathcal{T} V \oplus \nu$ is oriented isomorphic to $\mathcal{T} X|_V$. Since ν is an oriented real 2-plane bundle, we regard it as a complex line bundle. We shall denote by $\pi: N \to V$ the projection of the disk bundle. The complex line bundle $\pi^*(\nu)$ over N evidently admits a canonical cross-section $\eta: N \to \pi^*(\nu)$ which vanishes precisely along V. Take the trivial complex line bundle ε over $(X \setminus \mathrm{int}(N))$ and consider the vector bundle map $\tilde{\eta}: \varepsilon|_{\partial N} \to \pi^*(\nu)|_{\partial N}$ defined by $\tilde{\eta}(x, 1) = \eta(x)$ for all $x \in \partial N$. Gluing $\varepsilon|_{\partial N}$ along $\pi^*(\nu)|_{\partial N}$ using the identification $\tilde{\eta}$ yields a vector bundle ξ over X which clearly admits a cross-section s (which restricts to η on N and $x \mapsto (x, 1)$ on $X \setminus \mathrm{int}(N)$) that vanishes precisely along V. Now it suffices to establish that ξ is isomorphic to L.

Both ξ and L restrict to trivial bundles over $X \setminus \mathrm{int}(N)$. The bundle L restricts to the normal bundle ν over V (cf. [12, Theorem 11.3]). Since $\pi: N \to V$ and $V \hookrightarrow N$ are homotopy inverses, it follows that ξ and L restrict to isomorphic bundles over N as well. Let $\sigma: \varepsilon|_{\partial N} \to L|_{\partial N} \cong \pi^*(\nu)|_{\partial N}$ be the gluing data for L. The vector bundle maps σ and $\tilde{\eta}$ are related by a map $\partial N \to GL_1(\boldsymbol{C})$. The homotopy classes of such maps is isomorphic to $H^1(\partial N; \mathbf{Z})$. Using the Serre spectral sequence of the circle bundle $\partial N \to V$ under the hypotheses that $H^1(V) = 0$ and $c_1(L) \neq 0$ we see that $E_3^{0,1} = 0, E_2^{1,0} = 0$. Therefore

$H^1(\partial N; \mathbf{Z}) = 0$. Hence σ and $\tilde{\eta}$ are homotopic via bundle isomorphisms. It follows that L is isomorphic to ξ. $\square$

Now, applying the above lemma to characteristic submanifolds $V_1, \ldots, V_d$ of a torus manifold X with locally standard T-action whose orbit space is a homology polytope, we see that there exist complex line bundles $L_1, \ldots, L_d$ such that $c_1(L_i) = [V_i]$ and each L_i admits a section $s_i : X \to L_i$ which vanishes precisely along V_i, $1 \le i \le d$. We proceed as we did in obtaining Equation (3) in §2, to obtain the following equation in $K(X)$:

$$\prod_{1 \le k \le r} (1 - [L_{i_k}]) = 0\,, \tag{5}$$

whenever $\bigcap_{1 \le k \le r} V_{i_k} = \emptyset$.

The characteristic map. Recall that the characteristic submanifolds V_i are, by definition, fixed by one dimensional subgroups S_i of T. Our assumption on X (local standardness and orbit space Q being a homology polytope) implies that every characteristic submanifold has a T-fixed point. There is a unique 1-parameter subgroup $v_i \in \mathrm{Hom}(\boldsymbol{S}^1, T) \cong \boldsymbol{Z}^n$ by the following requirements, the first of which determines v_i up to sign: (i) v_i is a primitive element in $\mathrm{Hom}(\boldsymbol{S}^1, T)$ with image S_i, and (ii) the sign ($+v_i$ or $-v_i$) is determined by orienting S_i so that at any point $p \in V_i$, the oriented normal plane ν_p is *oriented* isomorphic to the tangent space to S_i at the identity element.

The map $\Lambda : \{Q_1, \ldots, Q_d\} \to \mathrm{Hom}(\boldsymbol{S}^1, T)$, which maps Q_i to v_i, is called the *characteristic map*. Under our hypothesis of local standardness and Q being a homology polytope, the manifold X is determined up to equivariant diffeomorphisms by the pair (Q, Λ). (See [10, Lemma 4.5].)

Local standardness of the T-action implies that if $V_{i_1} \cap \cdots \cap V_{i_r} \neq \emptyset$, then $\{v_{i_1}, \ldots, v_{i_r}\}$ is part of a $\mathbf{Z}$-basis for $\mathrm{Hom}(\boldsymbol{S}^1; T) \cong \boldsymbol{Z}^n$.

Cohomology of X. We now recall from [10, Corollary 7.8] the description of the integral cohomology ring of X.

Let $\mathcal{Q}$ denote the set of all faces of Q. Then the cohomology of X has the following description:

$H^*(X; \mathbf{Z}) \cong \mathbf{Z}[x_F; F \in \mathcal{Q}]/I$, where I is the ideal generated by the following two types of elements:

(i) $x_A x_B - x_{A \vee B} x_{A \cap B}$, where $A \vee B$ denotes the smallest face of Q which contains both A and B,

(ii) $\sum_{1 \le i \le d} \langle u, v_i \rangle x_{Q_i}$, where $u \in \mathrm{Hom}(T, \boldsymbol{S}^1)$ and $v_i = \Lambda(Q_i) \in \mathrm{Hom}(\boldsymbol{S}^1; T)$.

The element x_{Q_i} corresponds, under the isomorphism, to $[V_i] \in H^2(X; \mathbf{Z})$. (It is understood that $x_Q = 1$ and $x_\emptyset = 0$.)

From our hypothesis, $H^*(X; \mathbf{Z})$ is generated by degree two elements. Set $x_i := x_{Q_i}$, for $1 \le i \le d$. Any face F of Q is the intersection of those facets of Q which contain F. If F is of codimension r, then it is contained in exactly r distinct facets, say, $Q_{i_1}, \ldots, Q_{i_r}$ and so the intersection $F = Q_{i_1} \cap \cdots \cap Q_{i_r}$ is transversal. Therefore $x_F = x_{i_1} \cdots x_{i_r}$. Thus, we see that $H^*(X; \mathbf{Z}) = \mathbf{Z}[x_i; 1 \le i \le d]/\mathcal{I}$, where the ideal $\mathcal{I}$ of relations is generated by the elements:

(iii) $x_{i_1} \cdots x_{i_r}$, whenever $V_{i_1} \cap \cdots \cap V_{i_r} = \emptyset$,

(iv) $\sum_{1 \le i \le d} \langle u, v_i \rangle x_i$.

PROPOSITION 5.2 (Masuda-Panov [10]). *Let X be a T-torus manifold with locally standard action whose orbit space is a homology polytope. With the above notation, $H^*(X; \mathbf{Z}) \cong \mathbf{Z}[x_1, \ldots, x_d]/\mathcal{I}$, where x_i represents the cohomology class dual to the characteristic manifold V_i, $1 \le i \le d$. In particular, $H^*(X; \mathbf{Z})$ is a free abelian group.*

PROOF. The only part of the theorem remaining to be established is that $H^*(X; \mathbf{Z})$ is a free abelian group, since other assertions follow from [10, Corollary 7.8] as noted above. By [10, Theorem 7.7], the cohomology of X with coefficients in $\mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}$ for any prime p vanishes in odd dimensions, it follows that $H^*(X; \mathbf{Z})$ is torsion free. As X is a compact manifold, $H^*(X; \mathbf{Z})$ is finitely generated. It follows that $H^*(X; \mathbf{Z})$ is free abelian. □

Recall that L_i is the complex line bundle over X with $c_1(L_i) = [V_i]$. As an immediate consequence of the above description of $H^*(X; \mathbf{Z})$, we obtain the following equality in $K(X)$, analogous to equation (4):

$$\prod_{1 \le i \le d} [L_i]^{\langle u, v_i \rangle} = 1 . \tag{6}$$

We are now ready to state the main result of this section.

THEOREM 5.3. *Let X be any T-torus manifold with locally standard action whose orbit space Q is a homology polytope. Then $K(X)$ is isomorphic to the ring $R(Q, \Lambda) := \mathbf{Z}[y_1, \ldots, y_d]/\mathfrak{J}$, where $\mathfrak{J}$ is the ideal generated by the elements*:

(i) $y_{i_1} \cdots y_{i_r}$, *whenever* $Q_{i_1} \cap \cdots \cap Q_{i_r} = \emptyset$,

(ii) $\prod_{\{i \le d : \langle u, v_i \rangle > 0\}} (1 - y_i)^{\langle u, v_i \rangle} - \prod_{\{j \le d : \langle u, v_j \rangle < 0\}} (1 - y_j)^{-\langle u, v_j \rangle}$ *for each* $u \in \mathrm{Hom}(T; \mathbf{S}^1)$, *where* $v_i := \Lambda(Q_i)$, $1 \le i \le d$.

The isomorphism $R(Q, \Lambda) \to K(X)$ is established by sending y_i to $1 - [L_i]$, $1 \le i \le d$.

PROOF. The proof follows exactly as in the case of non-singular complex toric varieties. In view of Proposition 5.2, $K(X)$ is generated by $[L_i]$, $1 \le i \le d$; see §3. Furthermore, $K(X)$ is a free abelian group of rank $\chi(X)$. Set $R := R(Q, \Lambda)$. From Equations (5) and (6), there is a ring homomorphism $\psi : R \to K(X)$ sending y_i to $1 - [L_i]$, which is *surjective*. Arguing as in §4, we see that there is a decreasing filtration on R and a surjective homomorphism of abelian groups $H^*(X; \mathbf{Z}) \to \mathrm{gr}(R)$, showing that as an abelian group, R is generated by at most $\chi(X)$ many elements. Since ψ is surjective, it follows that R is also a free abelian group of rank $\chi(X)$ and that ψ is an isomorphism of *rings*. □

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