

## $(k, s)$ -Riemann-Liouville fractional integral and applications

Mehmet Zeki SARIKAYA<sup>\*†</sup>, Zoubir DAHMANI<sup>‡</sup>, Mehmet Eyüp KIRIS<sup>§</sup> and Farooq AHMAD<sup>¶</sup>

### Abstract

In this paper, we introduce a new approach on fractional integration, which generalizes the Riemann-Liouville fractional integral. We prove some properties for this new approach. We also establish some new integral inequalities using this new fractional integration.

**Keywords:** Riemann-Liouville fractional integrals, synchronous function, Chebyshev inequality, Hölder inequality.

*2000 AMS Classification:* 26A33, 26A51, 26D15.

*Received :* 21.07.2014 *Accepted :* 24.11.2014 *Doi :* 10.15672/HJMS.20164512484

### 1. Introduction

Fractional calculus and its widely application have recently been paid more and more attentions. For more recent development on fractional calculus, we refer the reader to [7, 12, 15, 16, 19]. There are several known forms of the fractional integrals of which two have been studied extensively for their applications [5, 10, 11, 14, 21]. The first is the Riemann-Liouville fractional integral of  $\alpha \geq 0$  for a continuous function  $f$  on  $[a, b]$  which is defined by

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha \geq 0, \quad a < x \leq b.$$

---

<sup>\*</sup>Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce, Turkey, Email: sarikayamz@gmail.com

<sup>†</sup>Corresponding Author.

<sup>‡</sup>Laboratory of Pure and Applied Mathematics, UMAB, University of Mostaganem, Algeria, Email: zzdahmani@yahoo.fr

<sup>§</sup>Department of Mathematics, Faculty of Science and Arts, Afyon Kocatepe University, Afyon-TURKEY, Email: mkiris@gmail.com, kiris@aku.edu.tr

<sup>¶</sup>Mathematics Department, College of Science, Majmaah University, Al-Zulfi, Kingdom of Saudi Arabia, Email: farooqgujar@gmail.com, f.ishaq@mu.edu.sa

This integral is motivated by the well known Cauchy formula:

$$\int_a^x dt_1 \int_a^{t_1} dt_2 \dots \int_a^{t_{n-1}} f(t_n) dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt, n \in \mathbb{N}^*.$$

The second is the Hadamard fractional integral introduced by Hadamard [9]. It is given by:

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad \alpha > 0, x > a.$$

The Hadamard integral is based on the generalization of the integral

$$\int_a^x \frac{dt_1}{t_1} \int_a^{t_1} \frac{dt_2}{t_2} \dots \int_a^{t_{n-1}} \frac{f(t_n)}{t_n} dt_n = \frac{1}{\Gamma(\alpha)} \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{n-1} f(t) \frac{dt}{t}$$

for  $n \in \mathbb{N}^*$ .

In [10], Katugampola gave a new fractional integration which generalizes both the Riemann-Liouville and Hadamard fractional integrals into a single form. This generalization is based on the observation that, for  $n \in \mathbb{N}^*$ ,

$$\int_a^x t_1^s dt_1 \int_a^{t_1} t_2^s dt_2 \dots \int_a^{t_{n-1}} t_n^s f(t_n) dt_n = \frac{(s+1)^{1-n}}{\Gamma(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{n-1} t^s f(t) dt,$$

which gives the following fractional version

$${}^s J_a^\alpha f(x) = \frac{(s+1)^{1-n}}{\Gamma(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\alpha-1} t^s f(t) dt,$$

where  $\alpha$  and  $s \neq -1$  are real numbers.

Recently, in [6], Diaz and Pariguan have defined new functions called  $k$ -gamma and  $k$ -beta functions and the Pochhammer  $k$ -symbol that is respectively generalization of the classical gamma and beta functions and the classical Pochhammer symbol:

$$\Gamma_k(x) = \lim_{n \rightarrow \infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k > 0,$$

where  $(x)_{n,k}$  is the Pochhammer  $k$ -symbol for factorial function. It has been shown that the Mellin transform of the exponential function  $e^{-\frac{t^k}{k}}$  is the  $k$ -gamma function, explicitly given by

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, x > 0.$$

Clearly,  $\Gamma(x) = \lim_{k \rightarrow 1} \Gamma_k(x)$ ,  $\Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k})$  and  $\Gamma_k(x+k) = x \Gamma_k(x)$ . Furthermore,  $k$ -beta function is defined as follows

$$B_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1} (1-t)^{\frac{y}{k}-1} dt,$$

so that  $B_k(x, y) = \frac{1}{k} B(\frac{x}{k}, \frac{y}{k})$  and  $B_k(x, y) = \frac{\Gamma_k(x) \Gamma_k(y)}{\Gamma_k(x+y)}$ .

Later, under the above definitions, in [13], Mubeen and Habibullah have introduced the  $k$ -fractional integral of the Riemann-Liouville type as follows:

$${}_k J^\alpha f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt, \quad \alpha > 0, x > 0.$$

Note that when  $k \rightarrow 1$ , then it reduces to the classical Riemann-Liouville fractional integral.

## 2. $(k, s)$ -Riemann-Liouville fractional integral

In this section, we present the  $(k, s)$  fractional integration which generalizes all of the above Riemann-Liouville fractional integrals as follows:

**2.1. Definition.** Let  $f$  be a continuous function on on a the finite real interval  $[a, b]$ . Then  $(k, s)$ -Riemann-Liouville fractional integral of  $f$  of order  $\alpha > 0$  is defined by:

$$(2.1) \quad {}_k^s J_a^\alpha f(x) := \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s f(t) dt, \quad x \in [a, b],$$

where  $k > 0, s \in \mathbb{R} \setminus \{-1\}$ .

In the following theorem, we prove that the  $(k, s)$  fractional integral is well defined:

**2.2. Theorem.** Let  $f \in L_1[a, b], s \in \mathbb{R} \setminus \{-1\}$  and  $k > 0$ . Then  ${}_k^s J_a^\alpha f(x)$  exists for any  $x \in [a, b], \alpha > 0$ .

*Proof.* Let  $\Delta := [a, b] \times [a, b]$  and  $P : \Delta \rightarrow \mathbb{R}; P(x, t) = \left[ (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s \right]$ . It clear to see that  $P = P_+ + P_-$ , where

$$P_+(x, t) := \begin{cases} (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s & , a \leq t \leq x \leq b \\ 0 & , a \leq x \leq t \leq b \end{cases}$$

and

$$P_-(x, t) := \begin{cases} (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s & , a \leq t \leq x \leq b \\ 0 & , a \leq x \leq t \leq b. \end{cases}$$

Since  $P$  is measurable on  $\Delta$ , then we can write

$$\int_a^b P(x, t) dt = \int_a^x P(x, t) dt = \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s dt = \frac{k}{\alpha} (x^{s+1} - a^{s+1})^{\frac{\alpha}{k}}.$$

By using the repeated integral, we obtain

$$\begin{aligned} \int_a^b \left( \int_a^b P(x, t) |f(x)| dt \right) dx &= \int_a^b |f(x)| \left( \int_a^b P(x, t) dt \right) dx \\ &= \frac{k}{\alpha} \int_a^b (x^{s+1} - a^{s+1})^{\frac{\alpha}{k}} |f(x)| dx \\ &\leq \frac{k}{\alpha} (b^{s+1} - a^{s+1})^{\frac{\alpha}{k}} \int_a^b |f(x)| dx. \end{aligned}$$

That is

$$\begin{aligned} \int_a^b \left( \int_a^b P(x, t) |f(x)| dt \right) dx &= \int_a^b |f(x)| \left( \int_a^b P(x, t) dt \right) dx \\ &\leq \frac{k}{\alpha} (b^{s+1} - a^{s+1})^{\frac{\alpha}{k}} \|f(x)\|_{L_1[a, b]} < \infty. \end{aligned}$$

Therefore, the function  $Q : \Delta \rightarrow \mathbb{R}; Q(x, t) := P(x, t)f(x)$  is integrable over  $\Delta$  by Tonelli's theorem. Hence, by Fubini's theorem  $\int_a^b P(x, t)f(x)dx$  is an integrable function on  $[a, b]$ , as a function of  $t \in [a, b]$ . That is,  ${}_k^s J_a^\alpha f(x)$  exists.  $\square$

Now, we prove the commutativity and the semigroup properties of the  $(k, s)$ -Riemann-Liouville fractional integral. We have:

**2.3. Theorem.** *Let  $f$  be continuous on  $[a, b]$ ,  $k > 0$  and  $s \in \mathbb{R} \setminus \{-1\}$ . Then,*

$${}_k^s J_a^\alpha \left[ {}_k^s J_a^\beta f(x) \right] = {}_k^s J_a^{\alpha+\beta} f(x) = {}_k^s J_a^\beta \left[ {}_k^s J_a^\alpha f(x) \right],$$

for all  $\alpha > 0, \beta > 0, x \in [a, b]$ .

*Proof.* Thanks to Definition 1 and by Dirichlet's formula, we have

$$\begin{aligned} {}_k^s J_a^\alpha \left[ {}_k^s J_a^\beta f(x) \right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s {}_k^s J_a^\beta f(t) dt \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s \left[ \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1} \tau^s f(\tau) d\tau \right] dt \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s \left[ \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1} \tau^s f(\tau) d\tau \right] dt. \end{aligned}$$

That is

(2.2)

$${}_k^s J_a^\alpha \left[ {}_k^s J_a^\beta f(x) \right] = \frac{(s+1)^{2-\frac{\alpha+\beta}{k}}}{k^2\Gamma_k(\alpha)\Gamma_k(\beta)} \int_a^x \tau^s f(\tau) \left[ \int_\tau^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s (t^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1} dt \right] d\tau.$$

Using the change of variable  $y = (t^{s+1} - \tau^{s+1}) / (x^{s+1} - \tau^{s+1})$ , we can write

(2.3)

$$\begin{aligned} \int_\tau^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} (t^{s+1} - \tau^{s+1})^{\frac{\beta}{k}-1} t^s dt &= \frac{(x^{s+1} - \tau^{s+1})^{\frac{\alpha+\beta}{k}-1}}{s+1} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy \\ &= \frac{(x^{s+1} - \tau^{s+1})^{\frac{\alpha+\beta}{k}-1}}{s+1} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy = \frac{(x^{s+1} - \tau^{s+1})^{\frac{\alpha+\beta}{k}-1}}{s+1} k B_k(\alpha, \beta). \end{aligned}$$

According to the  $k$ -beta function and by (2.2) and (2.3), we obtain

$$\begin{aligned} {}_k^s J_a^\alpha \left[ {}_k^s J_a^\beta f(x) \right] &= \frac{(s+1)^{1-\frac{\alpha+\beta}{k}}}{k\Gamma_k(\alpha+\beta)} \int_a^x (x^{s+1} - \tau^{s+1})^{\frac{\alpha+\beta}{k}-1} \tau^s f(\tau) d\tau \\ &= {}_k^s J_a^{\alpha+\beta} f(x). \end{aligned}$$

This completes the proof of the Theorem 2.3. □

**2.4. Theorem.** *Let  $\alpha, \beta > 0, k > 0$  and  $s \in \mathbb{R} \setminus \{-1\}$ . Then, we have*

$$(2.4) \quad {}_k^s J_a^\alpha \left[ (x^{s+1} - a^{s+1})^{\frac{\beta}{k}-1} \right] = \frac{\Gamma_k(\beta)}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+\beta)} (x^{s+1} - a^{s+1})^{\frac{\alpha+\beta}{k}-1},$$

where  $\Gamma_k$  denotes the  $k$ -gamma function.

*Proof.* By Definition 1 and using the change of variable  $y = (x^{s+1} - t^{s+1}) / (x^{s+1} - a^{s+1})$ ;  $x \in ]a, b]$ , we get

$$\begin{aligned} {}_k^s J_a^\alpha \left[ (x^{s+1} - a^{s+1})^{\frac{\beta}{k}-1} \right] &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s (t^{s+1} - a^{s+1})^{\frac{\alpha+\beta}{k}-1} dt \\ &= \frac{(s+1)^{-\frac{\alpha}{k}} (x^{s+1} - a^{s+1})^{\frac{\alpha+\beta}{k}-1}}{k\Gamma_k(\alpha)} \int_0^1 (1-y)^{\frac{\alpha}{k}-1} y^{\frac{\beta}{k}-1} dy \\ &= \frac{(x^{s+1} - a^{s+1})^{\frac{\alpha+\beta}{k}-1}}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha)} B_k(\alpha, \beta). \end{aligned}$$

The case  $a = x$  is trivial. The proof of Theorem 2.4 is complete.  $\square$

**2.5. Remark.** (i :) Taking  $s = 0, k > 0$  in (2.4), we obtain

$$(2.5) \quad {}_k J_a^\alpha \left[ (x-a)^{\frac{\beta}{k}-1} \right] = \frac{\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)} (x-a)^{\frac{\alpha+\beta}{k}-1}.$$

(ii :) The formula (2.4) for  $s = 0, k = 1$  becomes

$$J_a^\alpha \left[ (x-a)^{\beta-1} \right] = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\alpha+\beta-1}.$$

**2.6. Corollary.** Let  $k > 0$  and  $s \in \mathbb{R} \setminus \{-1\}$ . Then the formula

$$(2.6) \quad {}_k^s J_a^\alpha(1) = \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (x^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2}$$

is valid for any  $\alpha > 0$ .

**2.7. Remark.** (a :) For  $s = 0, k > 0$  in (2.6), we get

$$(2.7) \quad {}_k J_a^\alpha(1) = \frac{1}{\Gamma_k(\alpha+k)} (x-a)^{\frac{\alpha}{k}-2}.$$

(b :) For  $s = 0, k = 1$  we have

$$J_a^\alpha(1) = \frac{1}{\Gamma(\alpha+1)} (x-a)^{\alpha-2}.$$

### 3. Some new $(k, s)$ -Riemann-Liouville fractional integral inequalities

Chebyshev inequalities can be represented in  $(k, s)$ -fractional integral forms as follows:

**3.1. Theorem.** Let  $f$  and  $g$  be two synchronous on  $[0, \infty)$ . Then for all  $t > a \geq 0$ ,  $\alpha > 0$ ,  $\beta > 0$ , the following inequalities for  $(k, s)$ -fractional integrals hold:

$$(3.1) \quad {}_k^s J_a^\alpha f g(t) \geq \frac{1}{J_a^\alpha(1)} {}_k^s J_a^\alpha f(t) {}_k^s J_a^\alpha g(t)$$

$$(3.2) \quad {}_k^s J_a^\alpha f g(t) {}_k^s J_a^\beta(1) + {}_k^s J_a^\beta f g(t) {}_k^s J_a^\alpha(1) \geq {}_k^s J_a^\alpha f(t) {}_k^s J_a^\beta g(t) + {}_k^s J_a^\alpha g(t) {}_k^s J_a^\beta f(t).$$

*Proof.* Since the functions  $f$  and  $g$  are synchronous on  $[0, \infty)$ , then for all  $x, y \geq 0$ , we have

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Therefore

$$(3.3) \quad f(x)g(x) + f(y)g(y) \geq f(x)g(y) + f(y)g(x).$$

Multiplying both sides of (3.3) by  $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s$ , then integrating the resulting inequality with respect to  $x$  over  $(a, t)$ , we obtain

$$\begin{aligned} & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) g(x) dx \\ & + \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(y) g(y) dx \\ \geq & \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) g(y) dx \\ & + \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(y) g(x) dx, \end{aligned}$$

i.e.

$$(3.4) \quad {}^s J_a^\alpha f g(t) + f(y) g(y) {}^s J_a^\alpha(1) \geq g(y) {}^s J_a^\alpha f(t) + f(y) {}^s J_a^\alpha g(t).$$

Multiplying both sides of (3.3) by  $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} (t^{s+1} - y^{s+1})^{\frac{\alpha}{k}-1} y^s$ , then integrating the resulting inequality with respect to  $y$  over  $(a, t)$ , we obtain

$$\begin{aligned} & {}^s J_a^\alpha f g(t) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\alpha}{k}-1} y^s dy \\ & + {}^s J_a^\alpha(1) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\alpha}{k}-1} y^s f(y) g(y) dy \\ \geq & {}^s J_a^\alpha f(t) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\alpha}{k}-1} y^s g(y) dy \\ & + {}^s J_a^\alpha g(t) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\alpha}{k}-1} y^s f(y) dy, \end{aligned}$$

that is

$${}^s J_a^\alpha f g(t) \geq \frac{1}{J_a^\alpha(1)} {}^s J_a^\alpha f(t) {}^s J_a^\alpha g(t).$$

The first inequality is thus proved.

Multiplying both sides of (3.3) by  $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s$ , then integrating the resulting inequality with respect to  $y$  over  $(a, t)$ , we obtain

$$\begin{aligned} & {}^s J_a^\alpha f g(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s dy \\ & + {}^s J_a^\alpha(1) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) g(y) dy \\ \geq & {}^s J_a^\alpha f(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s g(y) dy \\ & + {}^s J_a^\alpha g(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) dy, \end{aligned}$$

that is

$${}^s_k J_a^\alpha f g(t) {}^s_k J_a^\beta (1) + {}^s_k J_a^\beta f g(t) {}^s_k J_a^\alpha (1) \geq {}^s_k J_a^\alpha f(t) {}^s_k J_a^\beta g(t) + {}^s_k J_a^\alpha g(t) {}^s_k J_a^\beta f(t)$$

and the second inequality is proved. The proof is completed.  $\square$

**3.2. Theorem.** *Let  $f$  and  $g$  be two synchronous on  $[0, \infty)$ ,  $h \geq 0$ . Then for all  $t > a \geq 0$ ,  $\alpha > 0$ ,  $\beta > 0$ , the following inequalities hold:*

$$\begin{aligned} & \frac{1}{(s+1)^{\frac{\beta}{k}} \Gamma_k(\beta+k)} (t^{s+1} - a^{s+1})^{\frac{\beta}{k}-2} {}^s_k J_a^\alpha f g h(t) \\ & + \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (t^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2} {}^s_k J_a^\beta f g h(t) \\ \geq & {}^s_k J_a^\alpha f h(t) {}^s_k J_a^\beta g(t) + {}^s_k J_a^\alpha g h(t) {}^s_k J_a^\beta f(t) - {}^s_k J_a^\alpha h(t) {}^s_k J_a^\beta f g(t) - {}^s_k J_a^\alpha f g(t) {}^s_k J_a^\beta h(t) \\ & + {}^s_k J_a^\alpha f(t) {}^s_k J_a^\beta g h(t) + {}^s_k J_a^\alpha g(t) {}^s_k J_a^\beta f h(t). \end{aligned}$$

*Proof.* Since the functions  $f$  and  $g$  are synchronous on  $[0, \infty)$  and  $h \geq 0$ , then for all  $x, y \geq 0$ , we have

$$(f(x) - f(y))(g(x) - g(y))(h(x) + h(y)) \geq 0.$$

Hence,

$$\begin{aligned} (3.5) \quad & f(x)g(x)h(x) + f(y)g(y)h(y) \\ \geq & f(x)g(y)h(x) + f(y)g(x)h(x) - f(y)g(y)h(x) \\ & - f(x)g(x)h(y) + f(x)g(y)h(y) + f(y)g(x)h(y). \end{aligned}$$

Multiplying both sides of (3.5) by  $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s$ , then integrating the resulting inequality with respect to  $x$  over  $(a, t)$ , we obtain

$$\begin{aligned}
& \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) g(x) h(x) dx \\
& + f(y) g(y) h(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s dx \\
\geq & g(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) h(x) dx \\
& + f(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s g(x) h(x) dx \\
& - f(y) g(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s h(x) dx \\
& - h(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) g(x) dx \\
& + g(y) h(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s f(x) dx \\
& + f(y) h(y) \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^t (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s g(x) dx.
\end{aligned}$$

That is

$$\begin{aligned}
(3.6) \quad & {}_k^s J_a^\alpha f g h(t) + f(y) g(y) h(y) {}_k^s J_a^\alpha(1) \\
\geq & g(y) {}_k^s J_a^\alpha f h(t) + f(y) {}_k^s J_a^\alpha g h(t) - f(y) g(y) {}_k^s J_a^\alpha h(t) - h(y) {}_k^s J_a^\alpha f g(t) \\
& + g(y) h(y) {}_k^s J_a^\alpha f(t) + f(y) h(y) {}_k^s J_a^\alpha g(t).
\end{aligned}$$



Multiplying both sides of (3.6) by  $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s$ , then integrating the resulting inequality with respect to  $y$  over  $(a, t)$ , we obtain

$$\begin{aligned}
& {}_k^s J_a^\alpha f g h(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s dy \\
& + {}_k^s J_a^\alpha (1) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) g(y) h(y) dy \\
\geq & {}_k^s J_a^\alpha f h(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s g(y) dy \\
& + {}_k^s J_a^\alpha g h(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) dy \\
& - {}_k^s J_a^\alpha h(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) g(y) dy \\
& - {}_k^s J_a^\alpha f g(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s h(y) dy \\
& + {}_k^s J_a^\alpha f(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s g(y) h(y) dy \\
& + {}_k^s J_a^\alpha g(t) \frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} \int_a^t (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s f(y) h(y) dy.
\end{aligned}$$

Consequently,

$$\begin{aligned}
{}_k^s J_a^\alpha f g h(t) {}_k^s J_a^\beta (1) + {}_k^s J_a^\alpha (1) {}_k^s J_a^\beta f g h(t) & \geq {}_k^s J_a^\alpha f h(t) {}_k^s J_a^\beta g(t) + {}_k^s J_a^\alpha g h(t) {}_k^s J_a^\beta f(t) \\
& - {}_k^s J_a^\alpha h(t) {}_k^s J_a^\beta f g(t) - {}_k^s J_a^\alpha f g(t) {}_k^s J_a^\beta h(t) \\
& + {}_k^s J_a^\alpha f(t) {}_k^s J_a^\beta g h(t) + {}_k^s J_a^\alpha g(t) {}_k^s J_a^\beta f h(t).
\end{aligned}$$

The proof is thus complete.  $\square$

**3.3. Corollary.** *Let  $f$  and  $g$  be two synchronous on  $[0, \infty)$ ,  $h \geq 0$ . Then for all  $t > a \geq 0$ ,  $\alpha > 0$ , the following inequalities hold:*

$$\begin{aligned}
& \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (t^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2} {}_k^s J_a^\alpha f g h(t) \\
& \geq {}_k^s J_a^\alpha f h(t) {}_k^s J_a^\alpha g(t) + {}_k^s J_a^\alpha g h(t) {}_k^s J_a^\alpha f(t) - {}_k^s J_a^\alpha h(t) {}_k^s J_a^\alpha f g(t).
\end{aligned}$$

**3.4. Theorem.** *Let  $f, g$  and  $h$  be three monotonic functions defined on  $[0, \infty)$  satisfying the following*

$$(f(x) - f(y))(g(x) - g(y))(h(x) - h(y)) \geq 0$$

for all  $x, y \in [a, t]$ . Then for all  $t > a \geq 0$ ,  $\alpha > 0$ ,  $\beta > 0$ , the following inequalities are valid:

$$\begin{aligned} & \frac{1}{(s+1)^{\frac{\beta}{k}} \Gamma_k(\beta+k)} (t^{s+1} - a^{s+1})^{\frac{\beta}{k}-2} {}_s J_a^\alpha f g h(t) \\ & - \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (t^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2} {}_s J_a^\beta f g h(t) \\ \geq & {}_s J_a^\alpha f h(t) {}_s J_a^\beta g(t) + {}_s J_a^\alpha g h(t) {}_s J_a^\beta f(t) - {}_s J_a^\alpha h(t) {}_s J_a^\beta f g(t) + {}_s J_a^\alpha f g(t) {}_s J_a^\beta h(t) \\ & - {}_s J_a^\alpha f(t) {}_s J_a^\beta g h(t) - {}_s J_a^\alpha g(t) {}_s J_a^\beta f h(t). \end{aligned}$$

*Proof.* We use the same arguments as in the proof of Theorem 3.2.  $\square$

**3.5. Theorem.** Let  $f$  and  $g$  be two functions on  $[0, \infty)$ . Then for all  $t > a \geq 0$ ,  $\alpha > 0$ ,  $\beta > 0$ , the following inequalities for  $(k, s)$ -fractional integrals hold:

$$\begin{aligned} (3.7) \quad & \frac{1}{(s+1)^{\frac{\beta}{k}} \Gamma_k(\beta+k)} (t^{s+1} - a^{s+1})^{\frac{\beta}{k}-2} {}_s J_a^\alpha f^2(t) \\ & + \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (t^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2} {}_s J_a^\beta g^2(t) \\ \geq & 2 {}_s J_a^\alpha f(t) {}_s J_a^\beta g(t) \\ (3.8) \quad & {}_s J_a^\alpha f^2(t) {}_s J_a^\beta g^2(t) + {}_s J_a^\beta f^2(t) {}_s J_a^\alpha g^2(t) \geq 2 {}_s J_a^\alpha f g(t) {}_s J_a^\beta f g(t). \end{aligned}$$

*Proof.* Since

$$(f(x) - g(y))^2 \geq 0,$$

then, we have

$$(3.9) \quad f^2(x) + g^2(y) \geq 2f(x)g(y).$$

Multiplying both sides of (3.9) by  $\frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} (t^{s+1} - x^{s+1})^{\frac{\alpha}{k}-1} x^s$  and  $\frac{(s+1)^{1-\frac{\beta}{k}}}{k\Gamma_k(\beta)} (t^{s+1} - y^{s+1})^{\frac{\beta}{k}-1} y^s$ , then integrating the resulting inequality with respect to  $x$  and  $y$  over  $(a, t)$  respectively, we obtain (3.7).

On the other hand, since

$$(f(x)g(y) - f(y)g(x))^2 \geq 0,$$

then, with the same arguments as before, we obtain (3.8).  $\square$

**3.6. Corollary.** Let  $f$  and  $g$  be two functions on  $[0, \infty)$ , then for all  $t > a \geq 0$ ,  $\alpha > 0$ , the following inequalities are valid:

$$\begin{aligned} & \frac{1}{(s+1)^{\frac{\alpha}{k}} \Gamma_k(\alpha+k)} (t^{s+1} - a^{s+1})^{\frac{\alpha}{k}-2} \left[ {}_s J_a^\alpha f^2(t) + {}_s J_a^\beta g^2(t) \right] \\ \geq & 2 {}_s J_a^\alpha f(t) {}_s J_a^\alpha g(t) \\ & {}_s J_a^\alpha f^2(t) {}_s J_a^\alpha g^2(t) \geq [{}_s J_a^\alpha f g(t)]^2. \end{aligned}$$

**3.7. Theorem.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  with:

$$\bar{f}(x) := \int_a^x t^s f(t) dt, \quad x > a \geq 0, \quad s \in \mathbb{R} \setminus \{-1\}.$$

Then, for  $\alpha \geq k > 0$  we have:

$${}_k^s J_a^\alpha f(x) = \frac{1}{k} {}_k^s J_a^{\alpha-k} \bar{f}(x)$$

*Proof.* By definition of the  $(k, s)$ -fractional integral and by using Dirichlet's formula, we have

$$\begin{aligned} {}_k^s J_a^\alpha \bar{f}(x) &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s \int_a^t u^s f(u) du dt \\ &= \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x u^s f(u) \int_u^x (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1} t^s dt du \\ &= \frac{(s+1)^{-\frac{\alpha}{k}}}{\Gamma_k(\alpha+k)} \int_a^x (x^{s+1} - u^{s+1})^{\frac{\alpha}{k}} u^s f(u) du \\ &= k {}_k^s J_a^{\alpha+k} f(x). \end{aligned}$$

This completes the proof of Theorem 3.7.  $\square$

We give the generalized Cauchy-Buniakovsky-Schwarz inequality as follows:

**3.8. Lemma.** Let  $f, g, h : [a, b] \rightarrow (0, \infty)$  be three functions  $0 \leq a < b$ . Then

(3.10)

$$\left( \int_a^b g^m(t) h^x(t) f(t) dt \right) \left( \int_a^b g^n(t) h^y(t) f(t) dt \right) \geq \left( \int_a^b g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) dt \right)^2,$$

where  $m, n, x, y$  arbitrary real numbers.

*Proof.*

$$\begin{aligned} & \int_a^b \left[ \sqrt{g^m(t) h^x(t) f(t)} \sqrt{\int_a^b g^n(t) h^y(t) f(t) dt} - \sqrt{g^n(t) h^y(t) f(t)} \sqrt{\int_a^b g^m(t) h^x(t) f(t) dt} \right]^2 dt \geq 0 \\ & \int_a^b \left[ g^m(t) h^x(t) f(t) \int_a^b g^n(t) h^y(t) f(t) dt + g^n(t) h^y(t) f(t) \int_a^b g^m(t) h^x(t) f(t) dt \right. \\ & \quad \left. - 2 g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) \sqrt{\int_a^b g^m(t) h^x(t) f(t) dt} \sqrt{\int_a^b g^n(t) h^y(t) f(t) dt} \right] dt \\ & \geq 0 \\ & 2 \left( \int_a^b g^m(t) h^x(t) f(t) dt \right) \left( \int_a^b g^n(t) h^y(t) f(t) dt \right) \\ & \geq 2 \left( \int_a^b g^{\frac{m+n}{2}}(t) h^{\frac{x+y}{2}}(t) f(t) dt \right) \sqrt{\int_a^b g^m(t) h^x(t) f(t) dt} \sqrt{\int_a^b g^n(t) h^y(t) f(t) dt} \end{aligned}$$

which gives the desired inequality.  $\square$

**3.9. Theorem.** Let  $f \in L_1[a, b]$ . Then

$$(3.11) \quad \left( {}_k^s J_a^{m(\frac{\alpha}{k}-1)+1} f^r(x) \right) \left( {}_k^s J_a^{n(\frac{\alpha}{k}-1)+1} f^p(x) \right) \geq \left( {}_k^s J_a^{\frac{m+n}{2}(\frac{\alpha}{k}-1)+1} f^{\frac{r+p}{2}}(x) \right)^2,$$

for  $k, m, n, r, p > 0$  and  $\alpha > 1$ .

*Proof.* By taking  $g(t) = (x^{s+1} - t^{s+1})^{\frac{\alpha}{k}-1}$ ,  $f(t) = \frac{t^{s(s+1)^{1-\frac{\alpha}{k}}}}{k\Gamma_k(\alpha)}$  and  $h(t) = f(t)$  in (3.10), we obtain

$$\begin{aligned} & \left( \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{m(\frac{\alpha}{k}-1)} t^s f^r(t) dt \right) \left( \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{n(\frac{\alpha}{k}-1)} t^s f^p(t) dt \right) \\ & \geq \left( \frac{(s+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{m+n}{2}(\frac{\alpha}{k}-1)} t^s f^{\frac{r+p}{2}}(t) dt \right)^2 \end{aligned}$$

which can be written as (3.11).  $\square$

**3.10. Remark.** For  $k = 1$  in (3.11), we get the following inequalities:

$$\left( J_a^{m(\alpha-1)+1} f^r(x) \right) \left( {}_k J_a^{n(\alpha-1)+1} f^s(x) \right) \geq \left( {}_k J_a^{\frac{m+n}{2}(\alpha-1)+1} f^{\frac{r+s}{2}}(x) \right)^2.$$

## References

- [1] Belarbi, S. and Dahmani, Z., *On some new fractional integral inequalities*, J. Ineq. Pure and Appl. Math., 10(3) (2009), Art. 86.
- [2] Dahmani, Z., *New inequalities in fractional integrals*, International Journal of Nonlinear Science, 9(4) (2010), 493-497.
- [3] Dahmani, Z., *On Minkowski and Hermite-Hadamard integral inequalities via fractional integration*, Ann. Funct. Anal. 1(1) (2010), 51-58.
- [4] Dahmani, Z., Tabharit, L. and Taf, S., *Some fractional integral inequalities*, Nonl. Sci. Lett. A, 1(2) (2010), 155-160.
- [5] Dahmani, Z., Tabharit, L. and Taf, S., *New generalizations of Gruss inequality using Riemann-Liouville fractional integrals*, Bull. Math. Anal. Appl., 2(3) (2010), 93-99.
- [6] Diaz, R. and Pariguan, E., *On hypergeometric functions and Pochhammer  $k$ -symbol*, Divulg.Math, 15.(2007),179-192.
- [7] Gorenflo, R. and Mainardi, F., *Fractional calculus: integral and differential equations of fractional order*, Springer Verlag, Wien (1997), 223-276.
- [8] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, 204, Elsevier Sci. B.V., Amsterdam, 2006.
- [9] Hadamard, J., *Essai sur l'étude des fonctions données par leur développement de Taylor*, Jour.PureandAppl.Math4(8),1892,101-186.
- [10] Katugompola, U.N., *New Approach Generalized Fractional Integral*, Applied Math and Comp. 218(2011),860-865.
- [11] Latif, M. A. and Hussain, S., *New inequalities of Ostrowski type for co-ordinated convex functions via fractional integrals*, Journal of Fractional Calculus and Applications, Vol. 2., 2012, No. 9, pp. 1-15.
- [12] Miller, S. and Ross, B., *An introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, USA, 1993, p.2.
- [13] Mubeen, S. and Habibullah, G.M.,  *$k$ -fractional integrals and application*, Int. J. Contemp. Math. Sciences, 7(2), 2012, 89-94.
- [14] Romero, L. G., Luque, L. L., Dorrego, G. A. and Cerutti, R. A., *On the  $k$ -Riemann-Liouville Fractional Derivative*, Int. J. Contemp. Math. Sciences, 8(1), 2013, 41-51.
- [15] Sarikaya, M.Z. and Ogunmez, H., *On new inequalities via Riemann-Liouville fractional integration*, Abstract and Applied Analysis, Volume 2012, Article ID 428983, 10 pages, doi:10.1155/2012/428983.
- [16] Sarikaya, M. Z., Set, E., Yaldiz, H. and Basak, N., *Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 57 (2013) 2403-2407.
- [17] Sarikaya, M. Z., *On the Hermite-Hadamard-type inequalities for co-ordinated convex function via fractional integrals*, Integral Transforms and Special Functions, Volume 25, Issue 2, pp:134-147, 2014.

- [18] Sarikaya, M. Z., *Ostrowski type inequalities involving the right Caputo fractional derivatives belong to  $L_p$* , Facta Universitatis, Series Mathematics and Informatics , Vol. 27 No 2 (2012), 191-197.
- [19] Sarikaya, M. Z. and Yaldiz, H., *On weighted Montgomery identities for Riemann-Liouville fractional integrals*, Konuralp Journal of Mathematics, Volume 1 No. 1 pp. 48-53, 2013.
- [20] Sarikaya, M. Z., and Karaca, A., *On the  $k$ -Riemann-Liouville fractional integral and applications*, International Journal of Statistics and Mathematics, Vol. 1(1), pp. 022-032, August, 2014.
- [21] Tunc, M., *On new inequalities for  $h$ -convex functions via Riemann-Liouville fractional integration*, Filomat 27:4 (2013), 559–565
- [22] Samko, S. G., Kilbas, A. A and Marichev, O. I., *Fractional Integrals and Derivatives Theory and Application*, Gordan and Breach Science, New York, 1993.

