K-THEORY OF C^*-ALGEBRAS OF B-PSEUDODIFFERENTIAL OPERATORS

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Abstract. We compute K-theory invariants of algebras of pseudodifferential operators on manifolds with corners and prove an equivariant index theorem for operators invariant with respect to an action of $\mathbb{R}^k$. We briefly discuss the relation between our results and the η-invariant.

Introduction

In this paper we analyze the K-groups of the norm closure of the algebra $\Psi_b^0(M)$ of b-pseudodifferential (or totally characteristic) operators acting on the compact manifold with corners M. In the case of a compact manifold with boundary this class of operators was introduced in [14], see also [15] and [11]. For the general case of a compact manifold with corners it was described in [20]. There are closely related algebras which have the same completion, see [17].

The algebra $\Psi_b^0(M)$ can be identified with a $*$-closed subalgebra of the bounded operators on $L^2_{\mathrm{b}}(M) = L^2(M, \Omega_{\mathrm{b}})$ (corresponding to a logarithmically divergent measure), and its Fredholm elements can then be characterized by the invertibility of a joint symbol consisting of the principal symbol, in the ordinary sense, and an 'indicial operator' (as for Fuchsian differential operators) at each boundary face, which arises by freezing the coefficients at the boundary face in question. In view of the invariance of the index with respect to small perturbations [9], we consider (as in the case $\partial M = \emptyset$ for the Atiyah-Singer index theorem, [3, 24]) the C^*-algebra obtained by norm closure, which we denote $\mathfrak{A}(M)$. Its K-theory is easier to compute than that of the uncompleted algebra. Just as in the case of a manifold without boundary, the principal symbol map σ has a continuous extension to $\mathfrak{A}(M)$ with values in $C({}^bS^*M)$, where ${}^bS^*M \equiv S^*M$ as manifolds.

The algebra $\mathfrak{A}(M)$ contains the algebra of compact operators on $L^2_{\mathrm{b}}(M)$, denoted $\mathcal{K}(L^2_{\mathrm{b}}(M))$. Let $\mathfrak{Q}(M) = \mathfrak{A}(M)/\mathcal{K}(L^2_{\mathrm{b}}(M))$ be the quotient. If $\partial M = \emptyset$, then $\mathfrak{Q}(M)$ is isomorphic to the algebra $\mathcal{C}(S^*M)$ 'of symbols.' In the general case, we call $\mathfrak{Q}(M)$ the *algebra of joint symbols*, since it involves both the principal symbol and extra morphisms giving the 'indicial operators.' A model space $\overline{N^+H}$ is associated to each boundary hypersurface H of M. As a manifold with corners $\overline{N^+H} \cong [-1,1] \times H$ carries a natural action of $\mathbb{R}^*_+ = (0,\infty)$. This allows us to introduce the (completed) indicial algebra at H, denoted $\mathfrak{A}(H,M)$ and consisting of the $\mathbb{R}^*_+$-invariant elements of $\mathfrak{A}(\overline{N^+H})$. The indicial morphism at H localizes

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$A \in \mathfrak{A}(M)$ to $\mathrm{In}_{H,M}(A) \in \mathfrak{A}(H, M)$. The joint symbol map is the direct sum of the principal symbol and the indicial operators at all boundary hypersurfaces. Its range is subject to compatibility conditions between symbol and indicial operators and between indicial operators at the intersections of boundary hypersurfaces. Using these indicial maps, we construct a composition series for $\mathfrak{A}(M)$:

$$\mathfrak{A}(M) \supset \mathfrak{I}_0 \supset \mathfrak{I}_1 \supset \ldots \supset \mathfrak{I}_n, \quad n = \dim M. \tag{1}$$

The subquotients of this composition series are identified in Theorem 2:

$$\mathfrak{I}_l/\mathfrak{I}_{l+1} \cong \bigoplus_{F \in \mathcal{F}^l(M)} \mathcal{C}_0\left(\mathbb{R}^{n-l}; \mathcal{K}(L^2_{\mathrm{b}}(F))\right), \quad 0 \le l \le n,$$

as the sum over the boundary faces of dimension l of the C^* algebras of continuous functions vanishing at infinity on $\mathbb{R}^{n-l}$ and taking values in the compact operators on an associated Hilbert space (of dimension one when $l = 0$). The end cases are

$$\mathfrak{I}_n \cong \mathcal{K}(L^2_{\mathrm{b}}(M)) \text{ and } \mathfrak{A}(M)/\mathfrak{I}_0 \cong \mathcal{C}({}^bS^*M).$$

The K-theory of each of these subquotients is readily computed, and this leads to a spectral sequence for the K-theory of $\mathfrak{A}(M)$.

To deduce the composition series (1), we first describe joint symbol maps 'at dimension l' in the smooth (i.e. uncompleted) setting; the ideals $\mathfrak{I}_l$ are the completions of the null spaces of these morphisms. To show the appropriate exactness properties for the morphisms obtained by continuous extension, we use lifting properties for the symbol and indicial morphisms.

In the particular case of a compact manifold with boundary, as already noted, the principal symbol map induces an isomorphism of K_0-groups, whereas each component of the boundary contributes an extra copy of $\mathbb{Z}$ to K_1; this can be attributed to "spectral flow" invariants [2]. More precisely, if ∂M has q components, there is a short exact sequence

$$0 \longrightarrow \mathbb{Z}^q \longrightarrow K_1(\mathfrak{Q}(M)) \xrightarrow{\sigma_*} K_1(\mathcal{C}({}^bS^*M)) \longrightarrow 0.$$

If the boundary is connected, the index morphism $\mathrm{Ind} : K_1(\mathfrak{Q}(M)) \to \mathbb{Z}$ provides a splitting of this exact sequence. In a forthcoming note we will discuss the surjectivity of σ_* in relation to boundary conditions of Atiyah-Patodi-Singer type for elliptic operators, and use that discussion as a model for Fredholm boundary conditions on general manifolds with corners.

We also compare the algebraic and topological K-theory of the uncompleted algebra $\Psi^0_b(M)$, and thereby interpret a result in [16] on the η-invariant in this setting. We conclude the paper with some results on the equivariant index of operators on manifolds equipped with a proper action of $\mathbb{R}^k$.

In summary the contents of this paper are as follows. In the first section we recall background material and notation concerning manifolds with corners. In §2 the symbol map and indicial morphisms for the algebra of b-pseudodifferential operators are discussed. In the next section the alternative description of the indicial morphism in terms of indicial families, obtained by taking the Mellin transform, is described. In §4 the continuous extension of the symbol map to the closure of the algebra of the b-pseudodifferential operators in the bounded operators on L^2 is considered.

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1. Manifolds with corners

We shall work in the context of smooth manifolds with corners M. By definition, in such a space every point $p \in M$ has coordinate neighborhoods diffeomorphic to $[0, \infty)^k \times \mathbb{R}^{n-k}$, where n is the dimension of M, $k = k(p)$ is the *codimension* of the face containing p, and p corresponds to 0 under this isomorphism. The transitions between such coordinate neighborhoods must be smooth up to the boundary; this is the same as being extendible smoothly across the boundary. An *open face* is a path component of the set $\partial_k M$ of all points p with a fixed $k = k(p)$. The closure, in M, of an open face will be called a boundary face, or simply a *face.* A boundary face of codimension one may be called specifically a boundary hypersurface. In general, such a boundary face does not have a covering by coordinate neighborhoods of the type described above, because boundary points may be identified. To avoid this problem, we demand, as part of the definition of a manifold with corners, that the boundary hypersurfaces be embedded. More precisely, this means that we assume that, for each boundary hypersurface H of M, there is a smooth function $\rho_H \geq 0$ on M, such that

$$H = \{\rho_H = 0\}, \quad \text{where } d(\rho_H) \neq 0 \text{ at } H. \tag{2}$$

If $p \in F$, a face of codimension k, then exactly k of the functions ρ_H vanish at p. Denoting them $\rho_1, \dots, \rho_k$, the differentials $d\rho_1, \dots, d\rho_k$ must be linearly independent at p; it follows that the addition of some $n - k$ functions (with independent differentials on F at p) gives a coordinate system near p; in fact, this is what we shall mean by a coordinate system at p.

We denote by $\mathcal{F}(M)$ the set of boundary faces of M, by $\mathcal{F}_1(M)$ the set of boundary hypersurfaces $H \in \mathcal{F}(M)$ (i.e. faces of codimension 1) and, more generally, by $\mathcal{F}_l(M)$, for $0 \leq l \leq n = \dim M$, the set of boundary faces of codimension l. It is also convenient to let $\mathcal{F}^l(M) = \mathcal{F}_{n-l}(M)$ denote the set of boundary faces of dimension l. In view of the assumed existence of boundary defining functions, (2), they are all manifolds with corners. Without loss of generality, it can be assumed that M is connected, and hence that there is a unique face of codimension 0, namely M. If $F \in \mathcal{F}_l(M)$ then $\mathcal{F}_k(F) \subset \mathcal{F}_{k+l}(M)$ consists of those $G \in \mathcal{F}_{k+l}(M)$ which are contained in F.

It is useful to make a choice of functions ρ_H as in (2) and fix a metric h which locally at any point p has the form

$$h = (dx_1)^2 + \dots + (dx_k)^2 + h_0(y_1, \dots, y_{n-k}),$$

where $x_1, \dots, x_k$, $y_1, \dots, y_{n-k}$ are some local coordinates at p, and

$$x_1 = \rho_{H_1}, \dots, x_k = \rho_{H_k}$$

are the chosen defining functions. The existence of such a metric is shown in [10], for example.

The choice of the functions ρ_H for all $H \in \mathcal{F}_1(M)$ establishes a trivialization $NF \simeq F \times \mathbb{R}^k$ of the normal bundle to each boundary face. In fact, these bundles are naturally decomposed as sums of trivial (but not canonically so) line bundles; namely the normal bundles to the hypersurfaces containing F

$$NF = \bigoplus_{H \in \mathcal{F}_1(M),\ H \supset F} N_F H. \tag{3}$$

We denote by $N^+F \subset NF$ the closed set of normal vectors that point into the manifold M. They are exactly those vectors which have non-negative x-components.

The group $(0,\infty)^k = \mathbb{R}_+^{*k}$ acts naturally on N^+F by dilations. Consider the projective compactification of the closed half-line (as in (54) in the appendix)

$$[0,\infty) \ni s \longrightarrow \frac{s-1}{s+1} \in [-1,1]. \tag{4}$$

The multiplicative action of $(0,\infty)$ on $[0,\infty)$ lifts to be smooth on $[-1,1]$, so the k-fold application of this compactification embeds the inward pointing normal bundle $N^+F = [0,\infty)^k \times F$ to any boundary face of a manifold with corners into $\overline{N^+F} \cong [-1,1]^k \times F$ with the $\mathcal{C}^\infty$ structure on the compactification independent of the choice of boundary defining functions used to produce the trivialization; the action of $\mathbb{R}_+^{*k}$ lifts to be smooth on $\overline{N^+F}$.

These compactified inward-pointing normal bundles to the boundary faces play an important rôle in the 'localization' of operators at the boundary. In particular, the space $\overline{N^+F}$ is a 'model' for M near F. If $G \subset F$ is a pair of boundary faces then the closure in $\overline{N^+F}$ of the union N_G^+F of the fibers over G forms a boundary face $G_{NF} \subset \overline{N^+F}$. Use of the boundary defining functions shows that there is a natural identification of the compactified inward-pointing normal bundle of G_{NF}, as a boundary face of $\overline{N^+F}$, with $\overline{N^+G}$:

$$\overline{N^+G_{NF}} \equiv \overline{N^+G}. \tag{5}$$

Now the action of $(0,\infty)^k$ by dilations lifts to an action on $L^2(\overline{N^+F}, \Omega_b^{\frac{1}{2}})$:

$$\begin{aligned} \lambda_\epsilon(u)(x_1,\dots,x_k,y_1,\dots,y_{n-k}) \\ = u(\epsilon_1^{-1}x_1,\dots,\epsilon_k^{-1}x_k,y_1,\dots,y_{n-k}),\ \epsilon=(\epsilon_1,\dots,\epsilon_k). \end{aligned} \tag{6}$$

This action is independent of the choice of defining functions; here Ω_b is the b-density bundle, with global section $\nu_b = \nu / \prod_{H\in\mathcal{F}_1(M)} \rho_H$.

The exponential map associated to the Levi-Civita connection of a metric of product type, as described above, gives a diffeomorphism from a neighborhood V_F of the zero section in N^+F to an open neighborhood of F in M :

$$\Phi_F = \exp : V_F \longrightarrow M,\ V_F \subset N^+F.$$

Due to the particular choice of the metric h, Φ_F is a diffeomorphism of manifolds with corners, which maps the zero section of NF onto F.

Let φ_F be a smooth function on M, $0 \le \varphi_F \le 1$, supported inside $\Phi_F(V_F)$, and such that $\varphi_F = 1$ in a neighborhood of F. Later we shall later use the maps

$$\begin{aligned} L_F : L^2(N^+F, \Omega_b^{\frac{1}{2}}) = L^2(N^+F, \frac{dx_1\dots dx_k}{x_1\dots x_k}dy_1\dots dy_{n-k}) \longrightarrow L^2(M,\Omega_b^{\frac{1}{2}}), \\ \text{where } L_F(u) = \varphi_F(u\circ\Phi_F^{-1}). \end{aligned} \tag{7}$$

The maps L_F are well defined since $\operatorname{supp}\varphi_F \subset \Phi_F(V_F)$.

The b-pseudodifferential operators considered here are obtained by a process of 'microlocalization' of the Lie algebra, $\mathcal{V}_b(M)$, of smooth vector fields which are tangent to all the boundary faces. As such, they are closely related to the b-cotangent bundle ${}^bT^*M$. This bundle is naturally defined over any manifold with corners. Over the interior ${}^bT^*M$ is canonically identified with T^*M, but at a boundary point p

its fiber is the space of equivalence class of differentials

$$\sum_{p\in H} a_H \frac{d\rho_H}{\rho_H} + d\phi, \qquad a_H \in \mathbb{R},\ H \in \mathcal{F}_1(M),\ \phi \in \mathcal{C}^\infty(M), \tag{8}$$

modulo the space of smooth differentials which, after being pulled back to $F = \bigcap_{p\in H} H$, vanish at p. It can be defined more naturally as the dual bundle to the bundle ${}^bTM$, with sections consisting precisely of the space $\mathcal{V}_b(M)$.

2. The algebra of b-pseudodifferential operators

Two definitions of the algebra of b-pseudodifferential operators are recalled in the appendix. The most accessible of these starts from an explicit description of the algebra $\Psi_b^*(M)$ for the special, model, case of $M = [-1,1]^n$. The general case is then obtained by localization and $\Psi_b^*(M)$ consists of operators from $\mathcal{C}_c^\infty(M)$ to $\mathcal{C}^\infty(M)$. If M is a manifold without boundary this definition reduces to that of 1-step polyhomogeneous (i.e. classical) pseudodifferential operators in the usual sense. The second approach, readily shown to be equivalent to the first, is to define the appropriate class of kernels directly on a stretched version of M^2. This intrinsically global approach has the virtue of making many of the proofs below transparent.

We shall be concerned mainly here with the algebra $\Psi_b^0(M)$ of (1-step polyhomogeneous) b-pseudodifferential operators of non-positive integral order on M, a given compact manifold with corners. It is a $\star$-closed algebra of bounded operators on $L^2_{\mathrm{b}}(M)$ and is a Frechet space.

As in the boundaryless case, the principal invariant of a pseudodifferential operator is its *principal symbol*, it is a function on the b-cotangent space. Let ${}^bS^*M$ be the quotient of ${}^bT^*M \setminus 0$ by the fiber action of $(0,\infty)$ and let P^m be the bundle over ${}^bS^*M$ with sections which are homogeneous functions of degree m on ${}^bT^*M$.

Proposition 1. *There is a natural short exact sequence*

$$0 \longrightarrow \Psi_b^{m-1}(M) \hookrightarrow \Psi_b^m(M) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^bS^*M; P^m) \longrightarrow 0, \tag{9}$$

which is multiplicative if M is compact, where $\sigma_m(A)$ is determined by 'oscillatory testing' in the sense that if $\psi \in \mathcal{C}_c^\infty(M)$, $\phi \in \mathcal{C}^\infty(M)$ is real valued and $a_H \in \mathbb{R}$ are such that the corresponding section α of ${}^bT^*M$ given by (8) *is non-vanishing over the support of ψ, then*

$$\sigma_m(A;\alpha)\psi = \lim_{\lambda\to\infty} \lambda^{-m} \prod_{p\in H} \rho_H^{i\lambda a_H} e^{i\lambda\phi} A\big(\prod_{p\in H} \rho_H^{-i\lambda a_H} e^{-i\lambda\phi}\psi\big). \tag{10}$$

In case $m = 0$, we simplify the notation and write $\sigma_0 = \sigma$; the bundle P^0 is canonically trivial, so the short exact sequence (9) becomes

$$0 \longrightarrow \Psi_b^{-1}(M) \hookrightarrow \Psi_b^0(M) \xrightarrow{\sigma} \mathcal{C}^\infty({}^bS^*M) \longrightarrow 0. \tag{11}$$

The algebra $\Psi_b^0(M)$ acts as bounded operators on $L^2_{\mathrm{b}}(M)$, and (10) gives

$$\|\sigma(A)\| \le \|A\|.$$

However, when the boundary is non-trivial, the ideal $\Psi_b^{-1}(M)$ does not map into the compact operators. To capture compactness, we need to consider the localization of the operators at boundary faces. To do so, we introduce a subalgebra of $\Psi_b^0(\overline{N^+F})$, where $F \in \mathcal{F}(M)$ and $\overline{N^+}F$ is the compactified inward-pointing normal bundle discussed above.

Definition 1. *If M is a compact manifold with corners then, the indicial algebra $\Psi^*_{b,I}(\overline{N^+}F)$ corresponding to a boundary face $F \subset M$ is the algebra consisting of those b-pseudodifferential operators on $\overline{N^+}F$ which are invariant under the natural $\mathbb{R}^{*k}_+$ action* (6)*.*

The operators $T \in \Psi^*_{b,I}(\overline{N^+}F)$ of order at most m form a subspace denoted $\Psi^m_{b,I}(\overline{N^+}F)$, so $\Psi^*_{b,I}(\overline{N^+}F) = \bigcup_m \Psi^m_{b,I}(\overline{N^+}F)$. These spaces are delineated by the symbol maps σ_m defined above, with M replaced by $\overline{N^+}F$. The $\mathbb{R}^{*k}_+$ action on ${}^bS^*\overline{N^+}F$ makes it a bundle over ${}^bS^*_F M$, the restriction of ${}^bS^*M$ to F, with fiber $[-1,1]^k$. The symbols of elements of $\Psi^0_{b,I}(\overline{N^+}F)$ are invariant under this action, so we define a 'reduced' symbol map

$$\sigma_F : \Psi^0_{b,I}(\overline{N^+}F) \longrightarrow \mathcal{C}^\infty({}^bS^*_F M). \tag{12}$$

It gives rise to a short exact sequence for the indicial operators

$$0 \longrightarrow \Psi^{-1}_{b,I}(\overline{N^+}F) \hookrightarrow \Psi^0_{b,I}(\overline{N^+}F) \xrightarrow{\sigma_F} \mathcal{C}^\infty({}^bS^*_F M) \longrightarrow 0. \tag{13}$$

Every b-pseudodifferential operator has an invariant *indicial operator* at each boundary face. To define it, let L_F be as in (7).

Theorem 1. *For any boundary face $F \in \mathcal{F}(M)$ there is a surjective morphism*

$$\mathrm{In}_{F,M} : \Psi^0_b(M) \longrightarrow \Psi^0_{b,I}(\overline{N^+}F)$$

independent of any choices and uniquely determined by the property

$$\mathrm{In}_{F,M}(T)u = \lim_{\epsilon_i \to 0} (\lambda_{\epsilon^{-1}} L_F^* T L_F \lambda_\epsilon) u \tag{14}$$

for any $u \in \mathcal{C}^\infty_c(N^+F)$, $\epsilon = (\epsilon_1, \dots, \epsilon_k)$, k being the codimension of F.

Although this is a basic result of the calculus, we outline a 'local' proof and then describe the global approach.

Proof. Suppose that $T \in \Psi^0_b(M)$. As discussed in the Appendix, T is locally of the form (55). If $x_1, \dots, x_k$ are defining functions for the face to which p belongs and $y_1, \dots, y_{n-k}$ are additional local coordinates then the defining formula (55) reduces to

$$Tu(x,y) = \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^{n-k}} T(x_1, \dots, x_k, x_1', \dots, x_k', y_1, \dots, y_{n-k}, y_1', \dots, y_{n-k}') \\ u(x_1'x_1, \dots, x_k'x_k, y_1', \dots, y_{n-k}') \frac{dx_1'}{x_1'} \cdots \frac{dx_k'}{x_k'} dy_1' \dots dy_{n-k}', \tag{15}$$

where now $T(s,x,y,y')$ is conormal at $x_i' = 1$, $y = y'$ or smooth as the localizing functions are in the same or different coordinate patches; it is still rapidly decreasing as $x_i' \to 0$ or ∞ and now has compact support in y, y'.

Since the computation is local, we can assume that $T = L_F^* T L_F$, and then

$$\begin{aligned}(16)\quad &(\lambda_{\epsilon^{-1}} T \lambda_\epsilon) u(x,y)\\ &= \int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^{n-k}} T(\epsilon_1 x_1, \dots, \epsilon_k x_k, x_1', \dots, x_k', y_1, \dots, y_{n-k}, y_1', \dots, y_{n-k}')\\ &\qquad u(x_1' x_1, \dots, x_k' x_k, y_1', \dots, y_{n-k}') \frac{dx_1'}{x_1'} \cdots \frac{dx_k'}{x_k'} dy_1' \dots dy_{n-k}'\end{aligned}$$

(after a dilation in the x'-variables). This shows immediately the existence of the limit as $\epsilon \longrightarrow 0$ in the statement, and that the the localized indicial operator is given by

$$\begin{aligned}(17)\quad \mathrm{In}_{F,M}(T)u = &\int_0^\infty \cdots \int_0^\infty \int_{\mathbb{R}^{n-k}} T(0, \dots, 0, x_1', \dots, x_k', y_1, \dots, y_{n-k}, y_1', \dots, y_{n-k}')\\ &\qquad u(x_1' x_1, \dots, x_k' x_k, y_1', \dots, y_{n-k}') \frac{dx_1'}{x_1'} \cdots \frac{dx_k'}{x_k'} dy_1' \dots dy_{n-k}',\end{aligned}$$

for any $u \in \dot{\mathcal{C}}^\infty(N^+F)$, this is an element of $\Psi^0_{b,I}(\overline{N^+F})$.

To see the surjectivity of the indicial morphism for F, it is enough to work locally on F^2, since the invariance properties are preserved under such localization. Thus T' can be assumed to have support in a product of coordinate patches, so takes the form (17). Inserting cut-off factors $\phi(x_j)$ and $\psi(x_j/x_j')$, for $j = 1, \dots, k$, where $\phi, \psi \in \mathcal{C}^\infty(\mathbb{R})$ have supports near 0 and 1, respectively, and satisfy $\phi(0) = 1$ and $\psi(1) = 1$, gives an element $T \in \Psi^0_{\mathrm{b}}(M)$ with $\mathrm{In}_{F,M}(T) = T'$. $\square$

In the global description of the kernels as distributions on M^2_{b}, the stretched product of M with itself, the indicial morphism simply corresponds to the restriction of the kernel to a boundary face of M^2_{b}. Let $H_1, \dots, H_k$ be the boundary hypersurfaces containing F. Each of the boundary faces $H_i \times H_i$ is blown up in the construction of M^2_{b} from M^2 so corresponds to a boundary hypersurface $\mathrm{ff}(H_i) \in \mathcal{F}_1(M^2_{\mathrm{b}})$. Consider the component lying above F of the intersection of these $\mathrm{ff}(H_i)$. It is canonically isomorphic to the corresponding face in the stretched product of the model space at F, $(\overline{N^+F})^2_{\mathrm{b}}$, and $\mathrm{In}_{F,M}(A)$ is the unique element of $\Psi^m_{b,I}(\overline{N^+F})$ with kernel having the same restriction as A to this face.

Recall that if $G \subset F$ are both boundary faces of M, then G determines a boundary face G_F of $\overline{N^+F}$; using the identification (5), the boundary maps can be iterated and identified directly from the formulæ in the proof above.

Corollary 1. *If $G \subset F$ are boundary faces of M, then the indicial maps satisfy*

$$(18)\qquad \mathrm{In}_{G_F, \overline{N^+F}} \circ \mathrm{In}_{F,M} = \mathrm{In}_{G,M}\,.$$

The null space of the indicial map for a boundary hypersurface is easily seen from the local coordinate discussion above, or even more readily from the more direct global definition. Namely, for each $H \in \mathcal{F}_1(M)$, there is a short exact sequence

$$(19)\qquad 0 \longrightarrow \rho_H \Psi^m_b(M) \hookrightarrow \Psi^m_b(M) \xrightarrow{\mathrm{In}_{H,M}} \Psi^m_{b,I}(\overline{N^+H}, M) \longrightarrow 0.$$

This has a useful extension to several hypersurfaces.

Lemma 1. *If $H_i \in \mathcal{F}_1(M)$, for $i = 1, \dots, L$, is a collection of boundary hypersurfaces, the joint null space of the indicial maps $\mathrm{In}_{H_i,M}$ is $\rho_1 \dots \rho_L \Psi_b^0(M)$, where $\rho_i = \rho_{H_i}$ are defining functions for the H_i.*

Proof. Proceed by induction over L. By (19), if $\mathrm{In}_{H_1,M}(T) = 0$, then $T = \rho_1 T_1$. Now, from the fact that $\mathrm{In}_{H_i,M}$ is a morphism:

$$0 = \mathrm{In}_{H_i,M}(T) = \mathrm{In}_{H_i,M}(\rho_1 T_1) = \rho_1|_{H_i} \mathrm{In}_{H_i,M}(T_1) \Longrightarrow \\ \mathrm{In}_{H_i,M}(T_1) = 0, \ \forall\ i > 1.$$

Applying the inductive hypothesis to T_1, for these $L-1$ hypersurfaces, gives the inductive hypothesis for L hypersurfaces. □

The b-pseudodifferential operators of order m define bounded operators on the natural Sobolev spaces $A : H^l_{\mathrm{b}}(M) \longrightarrow H^{l-m}_{\mathrm{b}}(M)$, for any l. As such, an operator A is compact if and only if its symbol vanishes (hence it is in $\Psi^{m-1}_{\mathrm{b}}(M)$) and all its indicial operators vanish, so it is in $\rho\Psi^m_{\mathrm{b}}(M)$, where $\rho = \prod_{H\in\mathcal{F}_1(M)} \rho_H$, (and so, in fact, is in $\rho\Psi^{m-1}_{\mathrm{b}}(M)$).

Let us note some examples of b-pseudodifferential operators. If M is a manifold with corners and $M \hookrightarrow \widetilde{M}$ is embedded in a manifold without boundary, of the same dimension (say by doubling across the boundary hypersurfaces), then the pseudodifferential operators of order m on $\widetilde{M}$ with kernels supported in M^2 are in $\Psi^m_{\mathrm{b}}(M)$. Examples of the indicial operators can be obtained in a similar way. Consider a pseudodifferential A operator of order m on $\mathbb{R}^k \times F$, where $\partial F = \emptyset$, which is invariant under all translations in $\mathbb{R}^k$ and has its convolution kernel (on $\mathbb{R}^k \times F^2$) compactly supported. Then compactifying $\mathbb{R}^k$ to $[-1,1]^k$ by first mapping each component $x_i \in \mathbb{R}$ to $t_i = \exp(x_i) \in (0,\infty)$ and then using the projective compactification, (4), gives an operator in $\Psi^m_{\mathrm{b}}([-1,1]^k \times F)$ which is $(\mathbb{R}^*_+)^k$ invariant. If F is realized as a boundary face of any manifold with corners M, this construction gives many elements of $\Psi^m_{b,I}(\overline{N^+}F)$, enough to span the space modulo $\Psi^{-\infty}_{b,I}(\overline{N^+}F)$.

3. Indicial family

The indicial morphism is closely related to the fact that $\Psi^m_b(M)$ is invariant under conjugation by complex powers of each boundary defining function, i.e.

$$\Psi^m_b(M) \ni A \longmapsto \rho_H^{-z} A \rho_H^z \in \Psi^m_b(M),\ z \in \mathbb{C} \tag{20}$$

is an isomorphism. Taking $z = 1$, it follows that $A\rho_H v = \rho_H(\rho_H^{-1} A \rho_H)v$ vanishes on $H \in \mathcal{F}_1(M)$, for any $v \in \mathcal{C}^\infty(M)$. Thus, if $u \in \mathcal{C}^\infty(H)$, then

$$A_H u = (Aw)|_H, \quad w \in \mathcal{C}^\infty(M),\ w|_H = u \tag{21}$$

defines an operator on $\mathcal{C}^\infty(H)$. This restriction map is a surjective morphism

$$\Psi^m_b(M) \xrightarrow{|_H} \Psi^m_b(H). \tag{22}$$

Using the Mellin transform, the relationship between $\mathrm{In}_{H,M}(A)$ and $A|_H$ is easily seen to be

$$\mathrm{In}_{H,M}(A)(d\rho_H)^z f = (d\rho_H)^z(\rho_H^{-z} A \rho_H^z)|_H f,$$

where $d\rho_H$ is a well-defined function on NH and hence a distribution on $\overline{N^+}H$. This follows directly from the limiting formulæ in the proof of Theorem 1. For a boundary face of codimension k, the analogous result holds using the k defining functions for the boundary hypersurfaces containing F.

It is therefore natural to define the *indicial family* of $A \in \Psi_b^m(M)$ at $F \subset \mathcal{F}_k(M)$ by

$$\widehat{\operatorname{In}}_{F,M}(A; z_1 \dots, z_k) = \left(\rho_1^{-z_1}\rho_2^{-z_2}\cdots\rho_k^{-z_k} A \rho_1^{z_1}\rho_2^{z_2}\cdots\rho_k^{z_k}\right)|_F \in \Psi_b^m(F), \tag{23}$$

where $A \in \Psi_{b,I}^m(\overline{N^+F})$. Note that the definition does depend on the choice of defining functions for the boundary hypersurfaces containing F.

Although it is straightforward to characterize the range of the map in (23), less precise information suffices for our purposes below.

Proposition 2. *The indicial family $\widehat{\operatorname{In}}_{F,M}(A; z)$ determines $A \in \Psi_{b,I}^m(\overline{N^+F})$ in the sense that if $\widehat{\operatorname{In}}_{F,M}(A; z)$ vanishes for all $z \in \mathbb{R}^k$ then $A = 0$. For any $A \in \Psi_{b,I}^m(\overline{N^+F})$, $\widehat{\operatorname{In}}_{F,M}(A; z)$ is an entire function of $z \in \mathbb{C}^k$ with values in $\Psi_b^m(F)$. If $m < 0$, then, as operators on the Sobolev spaces $L_b^2(F) \to H_b^{-m/2}(F)$,*

$$\| \widehat{\operatorname{In}}_{F,M}(A; z)\|_{0,\frac{1}{2}m} \leq C(1 + |z|)^{\frac{1}{2}m}, \quad z \in \mathbb{R}^k. \tag{24}$$

The range of $\widehat{\operatorname{In}}_{F,M}$ includes all entire functions of $g(z)$ with values in the space $\dot{\mathcal{C}}^\infty(F^2; \pi_R^*\Omega_b F)$, of fully smoothing kernels, satisfying the estimates

$$\sup_{|\Im z| \leq C} (1 + |z|)^p \|f(z, \cdot)\| < \infty, \tag{25}$$

for every C, p and seminorm $\| \bullet \|$ on $\dot{\mathcal{C}}^\infty(F^2; \pi_R^*\Omega_b F)$

Proof. Consider the second result first. Fixing a positive global section of Ω_{b}, the elements of $\Psi_{\mathrm{b}}^{-\infty}(M)$ correspond to smooth functions on M_{b}^2 vanishing to infinite order at all boundary hypersurfaces other than the $\operatorname{ff}(H)$, $H \in \mathcal{F}_1(M)$. In the case of $\overline{N^+F}$, the elements of $\Psi_{b,I}^{-\infty}(\overline{N^+F})$ correspond exactly to those elements of $\mathcal{C}^\infty(F_{\mathrm{b}}^2 \times [-1, 1]^k)$ vanishing on all boundary hypersurfaces other than the $\operatorname{ff}(G) \times [-1, 1]^k$, $G \in \mathcal{F}_1(F)$. In particular,

$$\dot{\mathcal{C}}^\infty(F^2 \times [-1, 1]^k) = \dot{\mathcal{C}}^\infty(F_{\mathrm{b}}^2 \times [-1, 1]^k) \subset \Psi_{b,I}^{-\infty}(\overline{N^+F}), \tag{26}$$

since these are the smooth functions vanishing to infinite order at all boundary faces. Since the indicial family is obtained by taking the Mellin transform in each of the variables in $[1, 1]^k$, the Paley-Wiener theorem shows that entire smoothing operators satisfying (25) are in the range of $\widehat{\operatorname{In}}_{F,M}$.

The first part of the statement follows from similar standard estimates for the Mellin transform (and hence the Fourier transform). ☐

4. Joint symbols

By combining the definitions of the symbol map in (10) and of the indicial operator in (14), the compatibility condition between the two is immediately apparent

$$\sigma_F\left(\operatorname{In}_{F,M}(T)\right) = \sigma(T)|_F, \ \forall\, T \in \Psi_b^0(M).$$

These are the only compatibility conditions. This can be formalized by defining the joint symbol

$$j(T) = \sigma(T) \oplus \bigoplus_{H \in \mathcal{F}_1(M)} \operatorname{In}_{H,M}(T) \in \mathcal{C}^\infty({}^bS^*M) \oplus \bigoplus_{H \in \mathcal{F}_1(M)} \Psi_{b,I}^0(\overline{N^+F}).$$

In view of (18), all the indicial operators for boundary faces of codimension greater than 1 can also be extracted from $j(T)$.

Proposition 3. *The joint symbol map has as range the subspace*

$$\big\{(a, S_H); a \in \mathcal{C}^\infty({}^bS^*M),\ S_H \in \Psi^0_{b,I}(\overline{N^+}H) \text{ such that } \sigma_H(S_H) = a|_H \text{ and}$$
$$\operatorname{In}_{G_F,\overline{N^+}H}(S_H) = \operatorname{In}_{G_F,\overline{N^+}H'}(S_{H'}),\ \forall\ F \subset H \cap H',\ F \in \mathcal{F}(M)\big\}.$$

Proof. That these compatibility conditions on the range hold has already been shown. To prove the surjectivity of j it is convenient to prove the more general statement that for any collection of boundary hypersurfaces H_i and and $S_i \in \Psi^0_{b,I}(\overline{N^+}H_i)$ there exists $T \in \Psi^0_b(M)$ with $\operatorname{In}_{H_i,M}(T) = S_i$ proved that the corresponding compatibility conditions are satisfied, that whenever $F \in \mathcal{F}(M)$ and $F \subset H_i \cap H_j$

$$\operatorname{In}_{G_F,\overline{N^+}H_i}(S_i) = \operatorname{In}_{G_F,\overline{N^+}H_j}(S_j).$$

This is the desired result for the set of all boundary hypersurfaces and is already known from the exactness in (19) for one hypersurface.

We proceed by induction over the number L of hypersurfaces. By the surjectivity of the indicial map at H_1 we can choose $T_1 \in \Psi^0_b(M)$ so that $\operatorname{In}_{H_1,M}(T_1) = S_1$; set $S'_i(1) = \operatorname{In}_{H_i,M}(T_1)$, for $i > 1$. Consider the differences, $S_i - S'_i \in \Psi^0_{b,I}(\overline{N^+}H_i)$, for each $i > 1$. By Corollary 1 if $F \in \mathcal{F}(M)$ is a component of $H \cap H_i$ and F_i is the corresponding face of H_i then $\operatorname{In}_{F_i,\overline{N^+}H_i}(S_i - S'_i) = 0$. Since $\rho_H|_{H_i}$ is a product of defining functions for these boundary faces, as boundary hypersurfaces of H_i it follows that $S_i - S'_i = \rho_H|_{H_i} S''_i$, with $S''_i \in \Psi^0_{b,I}(\overline{N^+}H_i)$. Now the S''_i satisfy the compatibility conditions for the remaining $L - 1$ hypersurfaces, therefore, by the inductive hypothesis, there exists $T' \in \Psi^0_b(M)$ with $\operatorname{In}_{H_i,M}(T') = S''_i$, for $i > 1$. Then $T = T_1 + \rho_H T'$ satisfies the requirements of the inductive hypothesis. □

More generally, if F is a boundary face of M, we can define a 'joint symbol morphism' for the indicial algebra at F by

$$j_F(T) = \sigma_F(T) \oplus \bigoplus_{H\in\mathcal{F}_1(F)} \operatorname{In}_{H,F,M}(T) \in \mathcal{C}^\infty({}^bS^*_FM) \oplus \bigoplus_{H\in\mathcal{F}_1(F)} \Psi^0_{b,I}(\overline{N^+}H). \tag{27}$$

The same argument as in the proof of the proposition above identifies the range of this morphism as the set of operators satisfying the 'obvious' compatibility conditions:

$$R_{F,M} = \big\{(f, T_H) \in \mathcal{C}^\infty({}^bS^*_FM) \oplus \bigoplus_{H\in\mathcal{F}_1(F)} \Psi^0_{b,I}(\overline{N^+}H);$$
$$\operatorname{In}_{H,F,M}(T_{H'}) = \operatorname{In}_{H',F,M}(T_H),\ \forall\ H, H' \in \mathcal{F}_1(F) \text{ and } \sigma_H(T_H) = f|_H\big\}.$$

Proposition 4. *For any boundary face F of M, the joint symbol map at F gives a short exact sequence*

$$0 \longrightarrow \rho_F\Psi^{-1}_{b,I}(\overline{N^+}F) \longrightarrow \Psi^0_{b,I}(\overline{N^+}F) \longrightarrow R_{F,M} \longrightarrow 0, \tag{28}$$

where $\rho_F \in \mathcal{C}^\infty(F)$ is the product of boundary defining functions for the boundary hypersurfaces of F.

Combining the indicial operators at the boundary faces of a given dimension with the symbol, consider

$$(29)\quad j_l : \Psi_b^m(M) \longrightarrow \mathcal{C}^\infty({}^bS^*M) \oplus \bigoplus_{F\in\mathcal{F}^l(M)} \Psi_{b,I}^m(\overline{N^+F}),\ j_l(A) = (\sigma_m(A), \mathrm{In}_{F,M}).$$

The range of this map is the subspace satisfying the appropriate compatibility conditions on a and $A_F \in \Psi_{b,I}^m(\overline{N^+F})$, $F \in \mathcal{F}^l(M)$:

(30)

$$\sigma_m(A_F) = a|_{{}^bS^*_FM},\ \mathrm{In}_{G,F,M}(A_F) = \mathrm{In}_{G,F',M}(A_{F'}),\ \forall\ F' \in \mathcal{F}^{l-1}(M),\ G \subset F \cap F'.$$

The null space is simply

$$(31)\qquad \ker(j_l) = \{A \in \Psi_b^{-1}(M); \mathrm{In}_{G,M} = 0,\ \forall\ G \in \mathcal{F}^l(M)\},$$

with j_F given in (27).

5. The norm closure, $\mathfrak{A}(M)$

Next we discuss the properties of the algebra, $\mathfrak{A}(M)$, obtained by taking the norm closure of $\Psi_b^0(M)$ as an algebra of bounded operators on $L^2_{\mathrm{b}}(M)$. The compactified normal space of a boundary face is a special case and then we also denote by $\mathfrak{A}(F,M) \subset \mathfrak{A}(\overline{N^+F})$ the closure in norm of the invariant subalgebra $\Psi_{b,I}^0(\overline{N^+F}) \subset \Psi_b^0(\overline{N^+F})$. The closure in norm of $\Psi_{b,I}^{-1}(\overline{N^+F}) \subset \Psi_{b,I}^0(\overline{N^+M})$ will be denoted $\mathfrak{A}^-(F,M) \subset \mathfrak{A}(F,M)$. Thus $\mathfrak{A}(M) = \mathfrak{A}(M,M)$ and $\mathfrak{A}^-(M,M)$ will be similarly denoted $\mathfrak{A}^-(M)$. Notice that $\mathfrak{A}^-(F,M)$ is also the closure of $\Psi_{b,I}^{-\infty}(\overline{N^+F}) \subset \Psi_{b,I}^{-1}(\overline{N^+F})$, since by standard properties of conormal distributions, $\Psi_{b,I}^{-\infty}(\overline{N^+F})$ is dense in $\Psi_{b,I}^{-1}(\overline{N^+F})$ in the topology of bounded operators on $L^2_{\mathrm{b}}(M)$. The same argument shows that $\mathfrak{A}^-(F,M)$ is the closure of $\Psi_{b,I}^{-\epsilon}(\overline{N^+F})$ for any $\epsilon > 0$.

Each of these norm closed algebras of operators on a Hilbert space is closed under conjugation, so by the theorem of Gelfand and Naimark they are all C^*-algebras. Below we will use the fact that any *algebraic* morphism of C^*-algebra is continuous and has closed range [8]. In particular, as in the case of a compact manifold without boundary the symbol map extends by continuity.

For a locally compact space X, we shall denote by $\mathcal{C}_0(X)$ the C^*-algebra of those continuous functions on X that vanish at infinity. It is the norm closure in supremum norm of the algebra $\mathcal{C}_c(X)$ of continuous compactly supported functions on X. If X is a smooth manifold the set of compactly supported smooth functions will be denoted by $\mathcal{C}_c^\infty(X)$; it is also dense in $\mathcal{C}_0(X)$.

Proposition 5. *The symbol maps in* (13) *and* (12) *extend by continuity to surjective maps*

$$(32)\qquad \sigma : \mathfrak{A}(M) \longrightarrow \mathcal{C}({}^bS^*M) \text{ and } \sigma_F : \mathfrak{A}(F,M) \longrightarrow \mathcal{C}({}^bS^*_FM).$$

Proof. Consider first the full algebra $\mathfrak{A}(M)$. From the oscillatory testing property of the principal symbol map, Proposition 1, it follows that $\|\sigma_F(T)\| \le \|T\|$, for all $T \in \Psi_b^0(M)$. Moreover, the principal symbol morphism σ_F is a $*$-morphism, i.e. it satisfies

$$\sigma_F(T^*) = \overline{\sigma_F(T)}.$$

Consequently its range is closed [8]. The same is true for the indicial algebras, just replacing M by $\overline{N^+F}$. Since the range of σ_F contains $\mathcal{C}^\infty({}^bS^*_FM)$, which is dense in $\mathcal{C}({}^bS^*_FM)$, the maps in (32) are surjective. □

Essentially the same proof shows that the indicial morphisms also extend to the norm closed algebras introduced above.

Proposition 6. *For any boundary faces F of M, the indicial morphisms extend to surjective maps*

$$\mathrm{In}_{F,M} : \mathfrak{A}(M) \longrightarrow \mathfrak{A}(F, M),$$

and for any pair of boundary faces $G \subset F$

$$\mathrm{In}_{G,F,M} : \mathfrak{A}(F, M) \longrightarrow \mathfrak{A}(G, M) \tag{33}$$

is defined by continuous extension of $\mathrm{In}_{G_F,\overline{N^+F}}$, and hence satisfies

$$\mathrm{In}_{F,F,M} = \mathrm{In}_{F,M,M} = \mathrm{In}_{F,M} \ \ \textit{and}\ \mathrm{In}_{F'',F',M} \circ \mathrm{In}_{F',F,M} = \mathrm{In}_{F'',F,M}$$

for any triple of boundary faces $F'' \subset F' \subset F$.

Proof. It follows from the definition of the indicial morphisms $\mathrm{In}_{F,M}$ given in Theorem 1 that they satisfy

$$\| \mathrm{In}_{F,M}(T)u\| = \| \lim_{\epsilon_i \to 0} (\lambda_{\epsilon^{-1}} L^*_F T L_F \lambda_\epsilon)u\| \leq \|T\|\,\|u\|$$

and hence $\| \mathrm{In}_{F,M}(T)\| \leq \|T\|$. This show that $\mathrm{In}_{F,M}$ extends by continuity to the norm closure. The surjectivity follows from the corresponding surjectivity of the indicial maps in Theorem 1; the remainder of the proof now follows from Corollary 1. □

6. Cross-sections

In order to analyze the null spaces of the symbol map, (32), and of the indicial morphism, (33), we construct a cross-section for $\mathrm{In}_{G,F,M}$.

Proposition 7. *For each $F \in \mathcal{F}(M)$ and each $G \in \mathcal{F}_1(F)$, there is a linear map $\lambda_{F,G} : \mathfrak{A}(G, M) \longrightarrow \mathfrak{A}(F, M)$ with the following properties:*

$$\lambda_{F,G}(\Psi^0_{b,I}(\overline{N^+G})) \subset \Psi^0_{b,I}(\overline{N^+F}), \tag{34}$$

$$\mathrm{In}_{G,F,M} \circ \lambda_{F,G}(T) = T,\ \forall\ T \in \mathfrak{A}(G, M), \tag{35}$$

$$\|\lambda_{F,G}(T)\| \leq \|T\|, \tag{36}$$

and, whenever $G' \in \mathcal{F}(F)$ is another face with $G' \not\subset G$, then

$$\begin{gathered} \mathrm{In}_{G',F,M} \circ \lambda_{F,G}(T) = 0, \qquad \textit{if } G \cap G' = \emptyset \\ \mathrm{In}_{G',F,M} \circ \lambda_{F,G}(T) = \lambda_{G',K} \circ \mathrm{In}_{K,G,M}(T), \qquad \textit{if } K \textit{ is a component of } G \cap G'. \end{gathered} \tag{37}$$

Note that in (37), $G \cap G'$ is either empty or else is a non-trivial union of boundary hypersurfaces of G'.

Proof. Initially, we define $\lambda_{H,M}$, for every $H \in \mathcal{F}_1(M)$, by the formula

$$\lambda_{M,H}(T) = L_H T L^*_H,$$

where L_H is as in (7).

For an arbitrary pair (F, G), as in the statement, there exists a unique $H \in \mathcal{F}_1(M)$ such that G is a component of $F \cap H$; let the other components be G_i,

$i = 1, \dots, L$. Using the local coordinate representations above we see that the indicial operator $\mathrm{In}_{F,M} \circ \lambda_{M,H}(T)$ depends only on $\mathrm{In}_{G,H}(T)$ and the $\mathrm{In}_{G_i,H}(T)$. Hence we can define the linear section $\lambda_{F,G}$ by the requirement

$$\lambda_{F,G} \circ \mathrm{In}_{G,H}(T) = \mathrm{In}_{F,M} \circ \lambda_{M,H}(T), \qquad \forall\, T \in \mathfrak{A}(H,M) \text{ with } \mathrm{In}_{G_i,H,M}(T) = 0,\ i = 1, \dots, L. \tag{38}$$

Here we use the disjointness of these boundary hypersurfaces to conclude that $\mathrm{In}_{G,H.M}$ is still surjective onto $\mathfrak{A}(G,M)$ when the domain is restricted as in (38).

Consider now three faces F, G and G', $G \in \mathcal{F}_1(F)$, $G' \subset F$, as in the statement of the proposition. Let $H \in \mathcal{F}_1(M)$ be such that G is a component of $F \cap H$, with the G_i as above. Then, for all $T \in \mathfrak{A}(H,M)$ with $\mathrm{In}_{G_i,H}(T) = 0$

$$\mathrm{In}_{G',F} \circ \lambda_{F,G} \circ \mathrm{In}_{G,H}(T) = \mathrm{In}_{G',F} \circ \mathrm{In}_{F,M} \circ \lambda_{M,H}(T) = \mathrm{In}_{G',F} \circ \lambda_{M,H}.$$

Thus

$$\mathrm{In}_{G',F} \circ \lambda_{F,G} \circ \mathrm{In}_{G,H}(T) = 0, \quad \text{ if } G' \cap H = \emptyset.$$

This is enough to conclude the proof in the case $G' \cap G = \emptyset$.

On the other hand, if $G' \cap H \neq \emptyset$ and K is one of its components, then it is necessarily a boundary hypersurface of G', so $\lambda_{G',K}$ is defined and

$$\mathrm{In}_{G',F} \circ \lambda_{F,G} \circ \mathrm{In}_{G,H}(T) = \lambda_{G',K} \circ \mathrm{In}_{K,G} \circ \mathrm{In}_{F',F_0}(T),$$

where we have used the definition (38), the properties of the indicial morphisms proved in the previous proposition. This completes the proof of the Proposition. $\square$

By placing extra conditions on the functions ϕ_F it is actually possible to define $\lambda_{G,F}$ satisfying $\mathrm{In}_{G,F} \circ \lambda_{G,F} = \mathrm{Id}$ and $\lambda_{G',G} \circ \lambda_{G,F} = \lambda_{G',F}$.

Corollary 2. *Let $T_{F'} \in \mathfrak{A}(F',M)$, respectively $T_{F'} \in \Psi^0_{b,I}(\overline{N^+}F')$, $F' \in \mathcal{F}_1(F)$, satisfy the compatibility condition*

$$\mathrm{In}_{G,F'}(T_{F'}) = \mathrm{In}_{G,F''}(T_{F''})$$

for all pairs F', F'' and any connected component G of $F' \cap F''$. Then we can find $T \in \mathfrak{A}(F,M)$, respectively $T \in \Psi^0_{b,I}(\overline{N^+}F)$, such that $T_{F'} = \mathrm{In}_{F',F}(T)$ and $\|T\| \leq C \max \|T_{F'}\|$, where the constant $C > 0$ depends only on the face F.

Proof. Let $\mathcal{F}_1(F) = \{F_1, F_2, \dots, F_m\}$ and define $T_1 = \lambda_{F,F_1}(T_{F_1}) \in \mathfrak{A}(F,M)$ (respectively $T_1 \in \Psi^0_{b,I}(\overline{N^+}F)$) and

$$T_{l+1} = T_l + \lambda_{F,F_{l+1}}(T_{F_{l+1}} - \mathrm{In}_{F_{l+1},F}(T_l)).$$

We will prove by induction on l that $\mathrm{In}_{F_j,F}(T_l) = T_{F_j}$ for all indices $j \leq l$. Indeed, for $l = 1$, this is the basic property of the sections λ. We first prove the inductive statement for $l+1$ and $j \leq l$.

If $F_j \cap F_{l+1} = \emptyset$, then $\mathrm{In}_{F_j,F} \circ \lambda_{F,F_{l+1}} = 0$. If $F_j \cap F_{l+1} \neq \varnothing$, we have

$$\mathrm{In}_{F_j,F} \circ \lambda_{F,F_{l+1}} = \lambda_{F_j,F_j \cap F_{l+1}} \mathrm{In}_{F_j \cap F_{l+1},F_{l+1}}.$$

Using the inductive hypothesis for l and the compatibility relation from the assumptions of the Corollary, we have

$$\begin{aligned}\mathrm{In}_{F_j\cap F_{l+1},F_{l+1}}(T_{F_{l+1}} - \mathrm{In}_{F_{l+1},F}(T_l))&\\ = \mathrm{In}_{F_j\cap F_{l+1},F_{l+1}}(T_{F_{l+1}}) - \mathrm{In}_{F_j\cap F_{l+1},F}(T_l)&\\ = \mathrm{In}_{F_j\cap F_{l+1},F_{l+1}}(T_{F_{l+1}}) - \mathrm{In}_{F_j\cap F_{l+1},F_j}\circ \mathrm{In}_{F_j,F}(T_l)&\\ = \mathrm{In}_{F_j\cap F_{l+1},F_{l+1}}(T_{F_{l+1}}) - \mathrm{In}_{F_j\cap F_{l+1},F_j}(T_{F_j}) = 0.&\end{aligned}$$

Thus in both cases $\mathrm{In}_{F_j,F}(T_{l+1}) = \mathrm{In}_{F_j,F}(T_l) = T_{F_j}$. Finally, for $j = l+1$, $\mathrm{In}_{F_j,F}\circ\lambda_{F,F_{l+1}} = \mathrm{Id}$ and

$$\mathrm{In}_{F_{l+1},F}(T_{l+1}) = \mathrm{In}_{F_{l+1},F}(T_l) - T_{F_{l+1}} - \mathrm{In}_{F_{l+1},F}(T_l) = T_{F_{l+1}}.$$

From this construction we may take $C = 3^{m-1}$. □

It is also possible to construct a cross-section for the symbol map; however, we content ourselves with the existence of suitable liftings.

Proposition 8. *For any compact manifold with corners M there is a constant C such that for any $a \in \mathcal{C}^\infty({}^bS^*M)$ there exists $A \in \Psi^0_b(M)$ with*

$$\sigma(A) = a \text{ and } \|A\|_{L^2_b(M)} \le C\|a\|_{L^\infty({}^bS^*M)}. \tag{39}$$

In the case of a normal space $\overline{N^+}F$, if $a \in \mathcal{C}^\infty({}^bS^*_FM)$, then A can be chosen in $\Psi^0_{b,I}(\overline{N^+}F)$.

Proof. It suffices to assume that a is real-valued. We will prove the general statement following (39) using induction over the maximal codimension of boundary faces for F (not M, the manifold of which it is a boundary face.) Note that we already know the symbol map to be surjective, it is the norm estimate on an element in the preimage of a that we need.

The basic case where M is compact without boundary is well known. Indeed, for any $A \in \Psi^0(M)$ with symbol a, the spectrum of A outside the disk of radius $\|a\|_{L^\infty(S^*M)}$ is discrete and consists of finite rank smooth eigenspaces. Since A can be replaced by its self-adjoint part, it splits as a sum of the orthogonal actions on the eigenspaces corresponding to eigenvalues in $|z| \le 2\|a\|_{L^\infty}$ and those outside this disk. The latter part is a smoothing operator, so subtracting it gives (39) with $C = 2$.

To complete the initial step in the induction we need the more general, indicial, case with F a manifold without boundary which is a boundary face of M. Thus, given $a \in \mathcal{C}^\infty({}^bS^*_FM)$, we need to find $A \in \Psi^0_{b,I}(\overline{N^+}F)$ with symbol a and satisfying (39). We can replace M by $\overline{N^+}F \cong [-1,1]^k \times F$ and then a can be interpreted as a smooth function on the sphere bundle of $\mathbb{R}^k \times T^*F$. Consider the compact manifold $\tilde{F} = \mathbb{T}^k \times F$, $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ being the standard torus. Now, $T^*\tilde{F} = \mathbb{T}^k \times (\mathbb{R}^k \times T^*F)$ under the standard $\mathbb{R}^k$ action. Thus a can be interpreted as an $\mathbb{R}^k$ invariant function on $S^*\tilde{F}$. As such the discussion for a compact manifold applies, and gives $A_1 \in \Psi^0(\tilde{F})$ with symbol a satisfying (39). In fact, A_1 can be taken to have kernel supported in any preassigned neighborhood of the diagonal in $\tilde{F}^2$; it is only necessary to take a sufficiently fine partition of unity, ϕ_i on $\tilde{F}$ and discard all terms $\phi_iA_1\phi_j$, where the supports of ϕ_i and ϕ_j are disjoint. Furthermore, if this neighborhood is invariant under the diagonal $\mathbb{R}^k$ action, it can be assumed that A is invariant, by averaging (over the dual torus). Now such a sufficiently small neighborhood of the diagonal in $\tilde{F}^2$ can be identified unambiguously as the image under projection on both factors of a neighborhood of the diagonal in $(\mathbb{R}^k \times F)^2$ which is invariant under the diagonal $\mathbb{R}^k$ action. The kernel A_1 lifts to a unique $\mathbb{R}^k$ invariant kernel A with support in this neighborhood of the diagonal. As noted in § 2 operators of this type are in $\Psi^0_b([-1,1]^k \times F)$. Certainly, A has symbol a, and the estimate (39) holds, for a larger C.

Proceeding by induction suppose that the result is known for all faces F themselves having boundary faces only up to codimension $k-1$ in any compact manifold with corners. Suppose F has boundary faces up to dimension k and $a \in \mathcal{C}^\infty({}^bS^*_FM)$ is given. Order the boundary hypersurfaces of F as $H_1, H_2, \dots, H_L$, and let H'_i be the corresponding boundary faces of M. The inductive hypothesis applies to $a|_{H'_1}$ giving $A_1 \in \Psi^0_{b,I}(\overline{N^+H_i})$. Using the section for indicial operators, choose $A'_1 \in \Psi^0_{b,I}(\overline{N^+F})$ with $\mathrm{In}_{H_i,F,M}(A'_1) = A_1$. Subtracting the symbol of A'_1 from the given a, we can now assume that $a|_{H_1} = 0$. Proceeding successively with the boundary faces, we can assume that $a|_{H_i} = 0$, for $i < j$, provided we can then construct $A'_j \in \Psi^0_{b,I}(\overline{N^+F})$ with symbol a_j such that $a_j = a$ on H_i, $i \le j$ (so vanishes for $i < j$). Simply choose $A_j \in \Psi^0_{b,I}(\overline{N^+F})$ as above, for $j = 1$, by extension of $A'_j \in \Psi^0_{b,I}(\overline{N^+H_j})$ with symbol $A|_{H_j}$. By construction and the properties of the section for indicial operators, the symbols of all the $\mathrm{In}_{H_i,F,M}(A_j)$, for $i < j$, vanish. Proceeding in this way, and then summing the A_i over the boundary hypersurfaces of F gives an element $A' \in \Psi^0_{b,I}(\overline{N^+F})$ which satisfies the norm estimate (39) and has all $\mathrm{In}_{H_i,H,M}(A')$ with the correct symbols.

Thus we are reduced to the case that $a \in \mathcal{C}^\infty({}^bS^*_FM)$ vanishes when restricted to each of the ${}^bS^*_HM$ with H a boundary hypersurface of F, i.e. vanishes at the boundary of F. Let ρ be the product of defining functions for the boundary hypersurfaces of F. Thus we can choose $B \in \Psi^0_{b,I}(\overline{N^+F})$ with $\sigma_F(B\rho) = a$. Select $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with $0 \le \phi(r) \le 1$, $\phi(0) = 1$ and $\rho(r) = 0$, $r > \frac{1}{2}$. Then the function $\phi(\rho/\delta) \in \mathcal{C}^\infty(F)$ is 1 on the boundary but with support in $\rho < 1$. In this case we can cut off close to the boundary of F. Thus

$$\|B\rho\phi(\rho/\delta)\| \le \delta\|B\|$$

and $\sigma_F(B\rho\phi(\rho/\delta) = a\phi(\rho/\delta)$. Choosing δ small we are finally reduced to the case that the symbol takes the form $a' = (1-\phi(\rho/\delta))a$, so vanishes identically near the boundary of F. Returning to the beginning of the induction, we can simply double F across all its boundary hypersurfaces to a manifold without boundary and choose an appropriately bounded A' with symbol a'. Again cutting off the kernel near the boundary of F, in both factors, does not change the symbol and gives an element of $\Psi^0_{b,I}(\overline{N^+F})$. This completes the inductive step. □

7. Symbol sequences

Using these cross-sections, we can now analyze the short exact sequences for the symbol maps on the completed algebras.

Proposition 9. *The symbol map* (12) *gives a short exact sequence*

$$0 \longrightarrow \mathfrak{A}^-(F,M) \longrightarrow \mathfrak{A}(F,M) \xrightarrow{\sigma_F} C({}^bS^*_FM) \longrightarrow 0, \tag{40}$$

where $\mathfrak{A}^-(F,M)$ is the norm closure of $\Psi^{-1}_{b,I}(\overline{N^+F})$.

Proof. Only the exactness at $\mathfrak{A}(F,M)$ remains to be shown. By continuity of the symbol map, the algebra $\mathfrak{A}^-(F,M)$ is contained in the null space of σ_F, so suppose $A \in \mathfrak{A}(F,M)$ and $\sigma_F(A) = 0$. By definition, there is a sequence $B_n \in \Psi^0_{b,I}(\overline{N^+F})$ with $B_n \to A$ in norm. Continuity of the symbol map shows that $a_n = \sigma_F(B_n) \to 0$ in L^∞. Using Proposition 8 we can choose $A_n \in \Psi^0_{b,I}(\overline{N^+F})$ with $\sigma(A_n) = a_n$ and $\|A_n\| \le C\|a_n\| \to 0$. Then $B_n - A_n \to A$ in norm and $\sigma_F(B_n - A_n) = 0$, so $A \in \mathfrak{A}^-(F,M)$. ☐

For a given face F of M we are particularly interested in the joint symbol morphism j_F and the replacement for (28) for the completed algebras. Denote by $\mathcal{K}(\mathcal{H})$ the algebra of compact operators on a Hilbert space $\mathcal{H}$,

Proposition 10. *For any boundary face F of codimension k in M the (continuous extension of) the joint symbol map at F gives a short exact sequence*

$$0 \longrightarrow \mathcal{K}_{F,M} \longrightarrow \mathfrak{A}(F,M) \longrightarrow \mathcal{R}_{F,M} \longrightarrow 0, \tag{41}$$

where there is an isomorphism of C^* algebras

$$\mathcal{K}_{F,M} \equiv \mathcal{C}_0(\mathbb{R}^k; \mathcal{K}(L^2_b(F)) \tag{42}$$

and

$$\mathcal{R}_{F,M} = \Big\{(f, T_H) \in \mathcal{C}({}^bS^*_FM) \oplus \bigoplus_{H\in\mathcal{F}_1(F)} \mathfrak{A}(H,M); \mathrm{In}_{G,H',M}(T_{H'}) = \mathrm{In}_{G,H,M}(T_H),$$
$$\forall\ G,H,H' \in \mathcal{F}_1(F), G \subset H\cap H' \text{ and } \sigma_H(T_H) = f|_H\Big\}.$$

Proof. Use of the sections for the indicial morphisms and the lifting property for the symbol map, as above, shows that the norm completion of $R_{F,M}$ in (28) is precisely $\mathcal{R}_{F,M}$ as defined above. Similarly, as in the proof above, the sequence (41) is exact if $\mathcal{K}_{F,M}$ is interpreted as the norm completion of the null space, $\rho_F\Psi^{-1}_{b,I}(\overline{N^+F})$ in (28). Thus the significant part of the proposition is the identification of the null space, equation (42). This identification follows from Proposition 2. ☐

Similar considerations apply to the maps j_l in (29).

Proposition 11. *For each l, the map j_l extends by continuity to a morphism defining a short exact sequence*

$$0 \longrightarrow \mathfrak{I}_l \longrightarrow \mathfrak{A}(M) \longrightarrow \mathfrak{B}_{l,M} \longrightarrow 0, \tag{43}$$

where $\mathfrak{B}_{l,M} \subset \mathcal{C}({}^bS^*M) \oplus \bigoplus_{F\in\mathcal{F}^l(M)} \Psi^0_{b,I}(\overline{N^+F})$ is the subalgebra fixed by the compatibility conditions in (30)*, and where the null space $\mathfrak{I}_l$ if given by*

$$\mathfrak{I}_l = \{A \in \mathfrak{A}^-(M); \mathrm{In}_{F,M}(A) = 0,\ \forall\ F \in \mathcal{F}^l(M)\}, \tag{44}$$

just the closure of the space in (31).

8. Composition series

Using these results on the joint symbols we can now see that the null spaces of the morphisms j_l give a composition series for the completed algebra.

Theorem 2. *The norm closure $\mathfrak{A}(M)$ of the algebra of b-pseudodifferential operators of order zero on the compact connected manifold with corners M has a composition series*

$$\mathfrak{A}(M) \supset \mathfrak{J}_0 \supset \mathfrak{J}_1 \supset \ldots \supset \mathfrak{J}_n\,, \quad n = \dim(M),$$

consisting of the closed ideals in (44)*; the partial quotients are*

$$\sigma_0 : \mathfrak{A}(M)/\mathfrak{J}_0 \xrightarrow{\sim} \mathcal{C}_0(S^*M),$$

and

$$\mathfrak{J}_l/\mathfrak{J}_{l+1} \simeq \bigoplus_{F\in\mathcal{F}^l(M)} \mathcal{C}_0(\mathbb{R}^{n-l}, \mathcal{K}(L^2_b(F))), \quad 0 \leq l \leq n. \tag{45}$$

The composition series and the isomorphisms are natural with respect to maps of manifolds with corners which are local diffeomorphisms.

The last isomorphism reduces to $\mathfrak{J}_n = \mathcal{K}(L^2_{\mathrm{b}}(M))$. Also

$$\mathfrak{J}_0/\mathfrak{J}_1 = \bigoplus_{F\in\mathcal{F}^0(M)} \mathcal{C}_0(N^*F) \simeq \bigoplus_{F\in\mathcal{F}^0(M)} \mathcal{C}_0(\mathbb{R}^n),$$

since $\mathcal{K}(L^2_{\mathrm{b}}(F)) = \mathbb{C}$ if F has dimension 0.

Proof. The fact that the principal symbol induces an isomorphism $\sigma : \mathfrak{A}(M)/\mathfrak{J}_0 \simeq \mathcal{C}({}^bS^*M)$ was proved in Proposition 9.

That the ideals $\mathfrak{J}_l$ form a composition series for $\mathfrak{A}(M)$ follows directly from their definition in (43). To examine the partial quotients consider j_{l+1} acting on $\mathfrak{J}_l$. Essentially by definition its null space is $\mathfrak{J}_{l+1}$. Since the symbol and the indicial operators on faces of dimension less than l already vanish on $\mathfrak{J}_l$ the map j_l can be replaced by the direct sum of the symbol maps at faces of dimension $l+1$. In fact, this gives a short exact sequence

$$0 \longrightarrow \mathfrak{J}_{l+1} \longrightarrow \mathfrak{J}_l \xrightarrow{\tilde{j}_{l+1}} \bigoplus_{F\in\mathcal{F}^{l+1}(M)} \mathcal{K}_{F,M} \longrightarrow 0, \quad \tilde{j}_l = \bigoplus_{F\in\mathcal{F}^{l+1}(M)} \mathrm{In}_{F,M}\,.$$

The surjectivity here follows from Proposition 10. The identification in (42) of $\mathcal{K}_{F,M}$ now leads immediately to the isomorphisms in (45).

The naturality of the composition series follows from the naturality of the principal symbol and of the indicial maps. □

The indicial algebras $\mathfrak{A}(F, M)$ have similar composition series which are compatible with the indicial morphisms.

Theorem 3. *The algebra $\mathfrak{A}(F, M)$ has a composition series*

$$\mathfrak{A}(F, M) \supset \mathfrak{J}_0 \supset \mathfrak{J}_1 \supset \ldots \supset \mathfrak{J}_n, \quad n = \dim(F),$$

where $\mathfrak{J}_0 = \overline{\Psi^{-1}_{b,I}}(N^+F) = \ker \sigma_F$ and $\mathfrak{J}_l$ is the closure of the ideal of order -1, $b-$pseudodifferential operators whose indicial parts vanish on all faces $F' \subset F$ of dimension less than l. The partial quotients are determined by the natural isomorphisms $\sigma : \mathfrak{A}(F, M)/\mathfrak{J}_0 \xrightarrow{\sim} \mathcal{C}_0({}^bS^*_FM)$, and

$$\mathfrak{J}_l/\mathfrak{J}_{l+1} \simeq \bigoplus_{F'\in\mathcal{F}^l(F)} \mathcal{C}_0(\mathbb{R}^{n-l}, \mathcal{K}(L^2_b(F')))\,, \quad 0 \leq l \leq n.$$

Proof. The proof consists of a repetition of the arguments in the proof of the preceding theorem, replacing M by F. □

We have the following generalization of Proposition 10.

Corollary 3. *If F is a face of codimension k in M, then passage to indicial families gives an isomorphism $\mathfrak{A}^-(F,M) \simeq \mathcal{C}_0(\mathbb{R}^k, \mathfrak{A}^-(F))$.*

Proof. The map

$$\Psi_{b,I}^{(-n-1)}(\overline{N^+F}) \longrightarrow C^\infty(\mathbb{R}^k, \Psi_{b,I}^{(-n-1)}(\overline{N^+F_0})).$$

is compatible with the composition series of $\mathfrak{A}(F,M)$ and $\mathfrak{A}(F_0,M_0)$ of the above Theorem and induces an isomorphism on the partial quotients (after completing in norm). The density property in the above corollary completes the proof. □

Corollary 4. *If M_1 and M_2 are two manifolds with corners then*

$$\mathfrak{A}^-(F_1 \times F_2, M_1 \times M_2) \simeq \mathfrak{A}^-(F_1, M_1) \otimes_{min} \mathfrak{A}^-(F_2, M_2).$$

The tensor product $\otimes_{min}$ is the minimal tensor product of two C^*-algebras and is defined as the completion in norm of $\mathfrak{A}^-(M_1) \otimes \mathfrak{A}^-(M_2)$ acting on $L^2(M_1 \times M_2)$. (The space $L^2(M_1 \times M_2)$ is the Hilbert space tensor product $L^2(M_1)\hat{\otimes}L^2(M_2)$, it is the completion of the algebraic tensor product $L^2(M_1) \otimes L^2(M_2)$ in the natural Hilbert space norm.)

Proof. We will assume that $F_i = M_i$, the general case being proved similarly. We have $\Psi_b^{-\infty}(M_1) \otimes \Psi_b^{-\infty}(M_2) \subset \Psi_b^{-\infty}(M_1 \times M_2)$. From the density of $\Psi_{b,I}^{-\infty}(\overline{N^+F})$ in $\mathfrak{A}^-(F,M)$, discussed at the beginning of § 5, we conclude the existence of a morphism $\chi : \mathfrak{A}^-(M_1) \otimes_{min} \mathfrak{A}^-(M_2) \longrightarrow \mathfrak{A}^-(M_1 \times M_2)$ which preserves the composition series. Moreover, by direct inspection the morphisms induced by χ on the subquotients are isomorphisms. If follows that χ is an isomorphism as well. □

For a compact manifold with boundary the results can be made even more explicit. The theorem below was also obtained by Lauter [13].

Theorem 4. *If M is a compact manifold with boundary then*

$$\mathfrak{I}_0 = \mathfrak{I}_{n-1},\ \mathfrak{I}_{n-1}/\mathfrak{I}_n \simeq \mathcal{C}_0(\mathbb{R}, \mathcal{K}_{\partial M})$$

and $\mathfrak{A}(M)/\mathfrak{I}_0 = \mathcal{C}_0({}^bS^*M)$. The algebra $\mathfrak{Q}(M) = \mathfrak{A}(M)/\mathfrak{I}_n$ has the following fibered product structure $\mathfrak{Q}(M) \simeq \mathfrak{Q} \subset \mathcal{C}_0({}^bS^*M) \oplus \mathfrak{A}(\partial M)$,

$$\mathfrak{Q} = \{(f,T),\ f|_{\partial M} = \sigma_{\partial M}(T)\}.$$

The indicial algebra of the boundary, $\mathfrak{A}(\partial M)$, fits into an exact sequence

$$0 \longrightarrow \oplus_G \mathcal{C}_0(\mathbb{R}, \mathcal{K}(L_b^2(G)) \longrightarrow \mathfrak{A}(\partial M) \xrightarrow{\sigma_{\partial M}} \mathcal{C}_0({}^bS_{\partial M}^*M) \to 0\,,$$

where G ranges through the connected components of ∂M.

9. Computation of the K-groups

Our starting point for the computation of the K-groups of the algebras discussed in the previous section is the short exact sequence, of C^*-algebras,

$$0 \longrightarrow \mathcal{K}(L_{\mathrm{b}}^2(M)) \longrightarrow \mathfrak{A}(M) \longrightarrow \mathfrak{Q}(M) \longrightarrow 0.$$

This exact sequence gives rise to the fundamental six-term exact sequence in K-theory (see [5])

$$\begin{array}{ccccc} K_0(\mathcal{K}(L^2_{\mathrm{b}}(M))) & \longrightarrow & K_0(\mathfrak{A}(M)) & \longrightarrow & K_0(\mathfrak{Q}(M)) \\ \partial\Big\uparrow & & & & \Big\downarrow 0 \\ K_1(\mathfrak{Q}(M)) & \longleftarrow & K_1(\mathfrak{A}(M)) & \longleftarrow & K_1(\mathcal{K}(L^2_{\mathrm{b}}(M))), \end{array} \tag{46}$$

where we are particularly interested in the K-groups of $\mathfrak{Q}(M)$. Now

$$K_0(\mathcal{K}(L^2_{\mathrm{b}}(M))) \simeq \mathbb{Z},$$

and $K_1(\mathcal{K}(L^2_{\mathrm{b}}(M))) \simeq 0$, so the right vertical map is zero.

The left vertical arrow represents "the index map." Consider an $m \times m$ matrix P with values in the b-pseudodifferential operators on M. If P is fully elliptic, in the sense that its image $j(P)$ in $M_m(\mathfrak{Q}(M))$ is invertible, and hence defines an element $[j(P)] \in K_1(\mathfrak{Q}(M))$, then

$$\partial[j(P)] = \mathrm{Ind}(P) = \dim\ker P - \dim\ker P^* \in \mathbb{Z} \simeq K_0(\mathcal{K}(L^2_{\mathrm{b}}(M))), \tag{47}$$

see [5, 6, 12]. We proceed to study the exact sequences

$$0 \longrightarrow \mathfrak{I}_l/\mathfrak{I}_{l+1} \longrightarrow \mathfrak{I}_{l-1}/\mathfrak{I}_{l+1} \longrightarrow \mathfrak{I}_{l-1}/\mathfrak{I}_l \to 0$$

corresponding to the composition series described in Theorem 2. We know that

$$K_i(\mathcal{C}_0(\mathbb{R}^j, \mathcal{K})) \simeq \begin{cases} \mathbb{Z} & \text{if } i+j \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{48}$$

We shall fix these isomorphisms uniquely as follows. For $j = 0$,

$$K_0(\mathcal{K}(L^2_{\mathrm{b}}(M))) \xrightarrow{\sim} \mathbb{Z}$$

will be the dimension function; it is induced by the trace. For $j > 0$ we define the isomorphisms in (48) by induction to be compatible with the isomorphisms

$$\mathbb{Z} \simeq K_{2l-j+1}(\mathcal{C}_0(\mathbb{R}^{j-1}, \mathcal{K})) \xrightarrow{\partial} \\ \longrightarrow K_{2l-j}(\mathcal{C}_0((0,\infty) \times \mathbb{R}^{j-1}, \mathcal{K})) \xrightarrow{\sim} K_{2l-j}(\mathcal{C}_0(\mathbb{R}^j, \mathcal{K})),$$

where the boundary map corresponds to the exact sequence of C^*-algebras

$$0 \longrightarrow \mathcal{C}_0((0,\infty) \times \mathbb{R}^{j-1}, \mathcal{K}) \longrightarrow \mathcal{C}_0([0,\infty) \times \mathbb{R}^{j-1}, \mathcal{K}) \longrightarrow \mathcal{C}_0(\mathbb{R}^{j-1}, \mathcal{K}) \to 0.$$

For any C^*-algebra A, set $SA = \mathcal{C}_0(\mathbb{R}, A) = \mathcal{C}_0(\mathbb{R}) \otimes_{\min} A$, $S^k A = \mathcal{C}_0(\mathbb{R}^k, A)$. Define

$$F_0' = \{(0, \dots, 0, 0)\} \times \mathbb{R}^{l-1}, \ F_0 = \{(0, \dots, 0)\} \times [0,1) \times \mathbb{R}^{l-1},$$

and $M_0 = [0,1)^{n-l+1} \times \mathbb{R}^{l-1}$, $F_0' \subset F_0 \subset M_0$. Also, let $H = [0,1)$ and $0 \subset \mathcal{L}_1 \subset \mathcal{L}_0 \subset \mathfrak{A}(H)$ be the canonical composition series of $\mathfrak{A}(H)$, such that $\mathcal{L}_0 = \mathfrak{A}^-(H)$, $\mathcal{L}_0/\mathcal{L}_1 \simeq \mathcal{C}_0(\mathbb{R})$, $\mathcal{L}_1 \simeq \mathcal{K}$, see Theorem 2.

Lemma 2. *With $\mathcal{K}_1 = \mathcal{K}(L^2(\mathbb{R}^{l-1}))$ there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker(\mathrm{In}_{F_0', F_0}) & \longrightarrow & \mathfrak{A}^-(F_0, M_0) & \longrightarrow & \mathfrak{A}^-(F_0', M_0) & \longrightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & S^{n-l}\mathcal{L}_1 \otimes \mathcal{K}_1 & \longrightarrow & S^{n-l}\mathcal{L}_0 \otimes \mathcal{K}_1 & \longrightarrow & S^{n-l+1}\mathcal{K}_1 & \longrightarrow & 0 \end{array}$$

in which all vertical arrows are isomorphisms, the bottom exact sequence is obtained from $0 \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{C}_0(\mathbb{R}) \to 0$ *by tensoring with* $\mathcal{C}_0(\mathbb{R}^{n-l}, \mathcal{K}_1)$*, and the boundary map*

$$\partial : K_{n-l+1}(\mathcal{C}_0(\mathbb{R}^{n-l+1}, \mathcal{K}_1)) \simeq K_{n-l+1}(\mathfrak{A}^-(F_0', M_0)) \longrightarrow$$
$$K_{n-l}(\ker(\mathrm{In}_{F_0', F_0})) \simeq K_{n-l}(\mathcal{C}_0(\mathbb{R}^{n-l}, \mathcal{K}_1))$$

is (the inverse of) the canonical isomorphism.

Proof. Let $H^{l-1} = [0,1)^{l-1}$ and $F_1 = \{(0, \dots, 0)\} \times H$, $F_1 \subset H^{l-1}$. It follows from the corollary 4 that the algebra $\mathfrak{A}^-(F_0, M_0)$ is isomorphic to $\mathfrak{A}^-(F_1, H^{l-1}) \otimes \mathfrak{A}^-(\mathbb{R}^{l-1})$. Moreover, $\mathfrak{A}^-(\mathbb{R}^{l-1}) = \mathcal{K}(L^2(\mathbb{R}^{l-1}))$. The corollary 3 further gives $\mathfrak{A}^-(F_1, H^{l-1}) \simeq \mathcal{C}_0(\mathbb{R}^{n-l}, \mathfrak{A}^-(F_1, F_1)) = \mathcal{C}_0(\mathbb{R}^{n-l}, \mathfrak{A}^-(H))$. The first commutative diagram then is just an expression of the composition series of $\mathfrak{A}^-(H)$, Theorem 2. Then an easy argument reduces the computation of the connecting morphism ∂ to that of the connecting morphism of the Wiener-Hopf exact sequence (i.e. the Wiener-Hopf extension). This is a well known and easy computation. It amounts to the fact that the multiplication by z has index -1 on the Hardy space $H^2(S^1)$ of the unit circle S^1. See [5] for more details. □

From Theorem 2 we then know that

$$K_i(\mathfrak{I}_l/\mathfrak{I}_{l+1}) \simeq \begin{cases} \bigoplus_{F \in \mathcal{F}_l(M)} \mathbb{Z} & \text{if } n - l + i \quad \text{is even} \\ 0 & \text{otherwise.} \end{cases}$$

Here $n = \dim M$.

Fix from now on an orientation of the normal bundle NF to each face F of M, including M itself. No compatibilities are required. This uniquely determines the above isomorphisms. This choice of orientations fixes an incidence relation $[F : G]$ between boundary faces. If $F \notin \mathcal{F}_1(G)$ and $G \notin \mathcal{F}_1(F)$, then we set $[F : G] = 0$. If $F \in \mathcal{F}_1(G)$ then an orientation of NF induces canonically an orientation of NG. If this orientation of G coincides with the given one, then $[F : G] = 1$, if it is the opposite orientation, then $[F : G] = 0$. Finally, $G \in \mathcal{F}_1(F)$ then $[F : G] = [G : F]$.

Theorem 5. *Suppose* $n - l + i$ *is even. Then the matrix of the boundary map*

$$\partial : K_{i-1}(\mathfrak{I}_{l-1}/\mathfrak{I}_l) \simeq \bigoplus_{F' \in \mathcal{F}_{l-1}(M)} \mathbb{Z} \longrightarrow \bigoplus_{F \in \mathcal{F}_l(M)} \mathbb{Z} \simeq K_i(\mathfrak{I}_l/\mathfrak{I}_{l+1})$$

is given by the incidence matrix. If $n - l + i$ *is odd, then* $\partial = 0$.

Proof. Let $e_{F'} \in K_{i-1}(\mathfrak{I}_{l-1}/\mathfrak{I}_l)$ and $e_F \in K_i(\mathfrak{I}_l/\mathfrak{I}_{l+1})$ be the canonical generators of these groups. We need to show that

$$\partial(e_{F'}) = \sum_{F \in \mathcal{F}_l(M)} [F : F'] e_F.$$

The idea of the proof is to reduce the computation to the case $M = M_0$, $F = F_0$ and $F' = F_0'$ considered in the preceding lemma:

$$M_0 = H^{n-l+1} \times \mathbb{R}^{l-1}, \quad F_0 = \{(0, \dots, 0)\} \times H \times \mathbb{R}^{l-1}, \quad F_0' = \{(0, \dots, 0, 0)\} \times \mathbb{R}^{l-1},$$

and $F' = F_0'$ the face of minimal dimension.

Choose a point $p \in F'$. There exists a diffeomorphism $\varphi : M_0 \longrightarrow M$ of manifolds with corners onto an open neighborhood of p such that $p \in \varphi(F_0')$. Since we considered only pseudodifferential operators with compactly supported Schwartz kernels, the open map φ induces an inclusion

$$\varphi_0 : \mathfrak{A}(M_0) \longrightarrow \mathfrak{A}(M), \tag{49}$$

which commutes with the indicial maps and hence preserves the canonical composition series of the Theorem 2: $\varphi_0(\mathfrak{I}_l^{(0)}) \subset \mathfrak{I}_l$, where $\mathfrak{I}_n^{(0)} \subset \mathfrak{I}_{n-1}^{(0)} \subset \cdots \subset \mathfrak{A}(M_0)$ is the composition series associated to $\mathfrak{A}(M_0)$. This composition series has the following properties

$$\mathfrak{I}_{l-1}^{(0)} = \cdots = \mathfrak{I}_0^{(0)} = \mathfrak{A}^-(M_0), \quad \mathfrak{I}_{l-1}^{(0)}/\mathfrak{I}_l^{(0)} \simeq \mathcal{C}_0\big(\mathbb{R}^{l-1}, \mathcal{K}(L^2(\mathbb{R}^{l-1})\big)$$

and the induced map on K-theory,

$$\mathbb{Z}e_{F_0'} = K_{i-1}(\mathfrak{I}_{l-1}^{(0)}/\mathfrak{I}_l^{(0)}) \longrightarrow K_{i-1}(\mathfrak{I}_{l-1}/\mathfrak{I}_l),$$

maps $e_{F_0'}$ to $e_{F'}$. We will first compute the boundary map

$$\partial_0 : K_{i-1}(\mathfrak{I}_{l-1}^{(0)}/\mathfrak{I}_l^{(0)}) \longrightarrow K_i(\mathfrak{I}_l^{(0)}/\mathfrak{I}_{l+1}^{(0)}) = \oplus \mathbb{Z}e_{F_j^{(0)}},$$

with the generators of the second group being indexed by the faces $F_j^{(0)}$ of M_0 dimension l. Since the boundary map in K-theory is natural, $\varphi_*\partial_0 = \partial\varphi_*$, and $\varphi_{0*}(e_{F_j^{(0)}}) = e_{F_j}$ if $\varphi(F_j^{(0)}) \subset F_j$, the boundary morphism ∂_0 will determine ∂ and thus prove the theorem. Label the faces $F_0^{(0)}, \ldots, F_{n-l}^{(0)}$ in the order given by the additional coordinate (thus $F_0^{(0)} = F_0$). It is enough to compute ∂_0 for an arbitrary choice of orientations, so we can choose the canonical one (given by the order of components). We then need to prove that the coefficients c_j defined by $\partial e_{F_0'} = \sum c_j e_{F_j^{(0)}}$ satisfy $c_j = (-1)^j$. By symmetry it is enough to assume $j = 0$. The indicial map In_{F_0,M_0} restricts to an onto morphism

$$\psi = \mathrm{In}_{F_0,M_0} : \mathfrak{A}^-(M_0) = \mathfrak{I}_{l-1}^{(0)} \longrightarrow \mathfrak{A}^-(F_0, M_0) = \ker \sigma_{0,F_0}$$

such that $\psi(\mathfrak{I}_l^{(0)}) = \ker(\mathrm{In}_{F_0',F_0} : \mathfrak{A}^-(F_0, M_0) \to \mathfrak{A}^-(F_0', M_0)) \simeq \mathcal{C}_0(\mathbb{R}^{n-l+1}, \mathcal{K})$ and $\psi(\mathfrak{I}_{l+1}^{(0)}) = 0$. (Recall that $F_0 = F_0^{(0)}, F_0' \subset F_0$.) The induced morphism

$$\psi_* : K_i(\mathfrak{I}_l^{(0)}/\mathfrak{I}_{l+1}^{(0)}) \longrightarrow \mathbb{Z}$$

is the projection onto the first component (i.e. it gives the coefficient of e_{F_0}). Using again the naturality of the exact sequence in K-theory, we further reduce the proof to the computation of the boundary map in the exact sequence

$$0 \longrightarrow \ker(\mathrm{In}_{F_0',F_0}) \longrightarrow \mathfrak{A}^-(F_0) \longrightarrow \mathfrak{A}^-(F_0') \longrightarrow 0$$

This computation is the content of previous lemma. The proof is complete. ☐

10. Families of manifolds with boundary

The results obtained above on the structure of the norm closure of the algebra of b-pseudodifferential operators on a manifold with corners can be extended to families of operators acting on the fibers of a fibration. For brevity, we state these results only for families of manifolds with boundary.

Let $\pi : Z \longrightarrow X$ be a smooth fibration, with Z a manifold with boundary and fibers modeled on the manifold with boundary F. We consider the algebra $\Psi^0_{b,c,\pi}(Z)$ of families of b-pseudodifferential operators of order 0 on the fibers of π with Schwartz kernels which are globally compactly supported. Denote its norm closure by $\mathfrak{A}_\pi(Z)$. If we denote by $\partial\pi : \partial Z \longrightarrow X$ the fibration with fibers ∂F, then the indicial operator

$$\mathrm{In}_{\partial\pi,\pi} : \Psi^0_{b,c,\pi}(Z) \longrightarrow \Psi^0_{c,\partial\pi}(N^+\partial Z)^{\mathbb{R}^*_+}$$

is defined fiber-by-fiber and extends to the closure $\mathfrak{A}_\pi(Z)$. Denote by ${}^bS^*_f Z$ the b-cosphere bundle along the fibers of $Z \to X$. Set $\mathfrak{I}_{n-1} = \ker\sigma : \mathfrak{A}_\pi(Z) \longrightarrow \mathcal{C}({}^bS^*_f Z)$, and $\mathfrak{I}_n = \ker(\mathrm{In}_{\partial\pi,\pi}) \cap \mathfrak{I}_{n-1}$.

Theorem 6. *For a fibration $\pi : Z \longrightarrow X$ with fibers manifolds with boundary and with $\mathcal{L}_0 = \mathfrak{A}^-(H)$, $H = [0,1)$ as above, there are isomorphisms $\mathfrak{I}_n \simeq \mathcal{C}_0(X,\mathcal{K})$, $\mathfrak{I}_{n-1}/\mathfrak{I}_n \simeq \mathcal{C}_0(X\times\mathbb{R},\mathcal{K})$, and $\mathfrak{A}_\pi(Z)/\mathfrak{I}_{n-1} \simeq \mathcal{C}_0({}^bS^*_f Z)$. Moreover, $\mathfrak{I}_{n-1}$ is the norm closure of $\Psi^{-1}_{b,c,\pi}(Z) \simeq \mathcal{C}_0(X,\mathcal{L}_0)$.*

Proof. This is essentially a repetition of the arguments above, carrying a parameter $x \in X$. □

This theorem allows us to determine explicitly the K-theory groups of the norm-closed algebras associated to families of manifolds with boundary.

Theorem 7. *Let $\pi : Z \to X$ be as above and set $\mathfrak{Q}(Z) = \mathfrak{A}_\pi(Z)$. Then the principal symbol σ induces isomorphisms*

$$\sigma_* : K_i(\mathfrak{A}_\pi(Z)) \simeq K_i(\mathcal{C}_0({}^bS^*_f Z)) \simeq K^i({}^bS^*_f Z), \tag{50}$$

and the boundary map $\partial : K_i(\mathfrak{A}_\pi(Z)/\mathfrak{I}_{n-1}) \longrightarrow K_{i+1}(\mathfrak{I}_{n-1}/\mathfrak{I}_n)$ is zero, so there is a natural short exact sequence

$$0 \longrightarrow K^i(\mathbb{R}\times X) \xrightarrow{j_Z} K_i(\mathfrak{Q}(Z)) \longrightarrow K^i({}^bS^*_f Z) \to 0 \tag{51}$$

Proof. The groups $K_i(C(X,\mathcal{L}_0))$ can be computed using the Künneth formula [5, 23]. Since $K_*(\mathcal{L}_0) = 0$ it follows that $K_*(\mathfrak{I}_{n-1}) = 0$, which, in view of Theorem 6, proves the first part of the theorem.

In order to prove (51), observe that, using (50), the composite map

$$K_i(\mathfrak{A}_\pi(Z)) \longrightarrow K_i(\mathfrak{Q}(Z)) \longrightarrow K^i({}^bS^*_f Z)$$

is surjective, so the map from $K_i(\mathfrak{Q}(Z))$ to $K^i({}^bS^*_f Z)$ in (51) is also surjective. This shows that $\partial = 0$. □

One important problem is to explicitly compute the family index map [4]

$$\mathrm{Ind} = \partial : K_i(\mathfrak{Q}(Z)) \longrightarrow K_{i-1}(\mathcal{C}_0(X,\mathcal{K})) \simeq K^{i-1}(X)$$

corresponding to the exact sequence $0 \to \mathcal{C}_0(X,\mathcal{K}) \to \mathfrak{A}_\pi(Z) \to \mathfrak{Q}(Z) \to 0$ along the lines of [18]. A consequence of our computations is the following corollary.

Corollary 5. *The composition*

$$\mathrm{Ind}\circ j_Z : K^i(\mathbb{R}\times X) \longrightarrow K^{i-1}(X)$$

is the canonical isomorphism.

Proof. Indeed, $\mathrm{Ind}\circ j_Z$ is, by naturality, the boundary map in the six term K-Theory exact sequence associated to the exact sequence

$$0 \longrightarrow \mathfrak{I}_{n-1} \longrightarrow \mathfrak{I}_n \longrightarrow \mathfrak{I}_{n-1}/\mathfrak{I}_n \longrightarrow 0.$$

Since $K_i(\mathfrak{I}_n) = 0$, it follows that the connecting (i.e. boundary) morphism in the above six term exact sequence is an isomorphism. The descriptions of the ideals $\mathfrak{I}_{n-1}$ and $\mathfrak{I}_n$ in the previous theorem then completes the proof. □

11. An $\mathbb{R}^q$-equivariant index theorem

Consider a smooth manifold X endowed with a proper, free action of $\mathbb{R}^q$. We can assume that $X = \mathbb{R}^q \times F$ with $\mathbb{R}^q$ acting by translations. If D is an elliptic differential operator on $\mathbb{R}^q \times F$ which is invariant under the action of $\mathbb{R}^q$ it is natural to look for index type invariants of D. We construct and compute such invariants using results from the previous sections. More generally, we consider elliptic matrices of operators.

Let n be the dimension F. We have already defined the algebra $\Psi^0_{b,I}(\overline{N^+F})$, consisting of $\mathbb{R}^q$-invariant, pseudodifferential operators order 0 on $\mathbb{R}^q \times F$ when we studied the indicial algebra at a boundary face F of codimension q of the noncompact manifold M.

Appealing to the same philosophy as before, we shall consider the closure in norm of this algebra, denoted $\mathfrak{A}(F, M)$, as before. We then know from Proposition 9 that there is an exact sequence

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R}^q, \mathcal{K}) \longrightarrow \mathfrak{A}(F) \xrightarrow{\sigma_{0F}} \mathcal{C}_0(S^*M|_F) \longrightarrow 0.$$

The connecting morphism (boundary map)

$$\partial : K_{q+1}(\mathcal{C}_0({}^bS^*M|_F)) = K^0(S^*M|_F) \longrightarrow K_q(\mathcal{C}_0(\mathbb{R}^q, \mathcal{K})) \simeq \mathbb{Z}$$

can be interpreted as an $\mathbb{R}^q$-equivariant index, and its computation will then be regarded as an $\mathbb{R}^q$-equivariant index theorem. Above we have used the standard isomorphism $K_q(\mathcal{C}_0(\mathbb{R}^q, \mathcal{K})) \simeq \mathbb{Z}$, as explained in Section 9.

Denote $Y = S^*M|_F = S^*M/\mathbb{R}^q$, and orient it as the boundary of (the dual of) $\mathbb{R}^q \times TM$, if TM is oriented as an almost complex manifold, as in the Atiyah-Singer index theorem. Also, denote by $\mathcal{T}(F) \in H^{even}(F)$ the Todd class of the complexified cotangent bundle of F and by Ch the Chern character.

Theorem 8. *Let a be an element of $K_{q+1}(\mathcal{C}_0(Y))$ and $Y = S^*M/\mathbb{R}^q$. Then*

$$\partial(a) = (-1)^n \langle Ch(a) p^* \mathcal{T}(F), [Y] \rangle,$$

where $[Y]$ is the fundamental class of Y oriented as above, $p : Y \to F$ is the projection, and $n = \dim F$.

Proof. We will prove the above theorem by induction on q. Consider the manifolds with corners

$$M = [0, 1)^q \times F\ , \quad F_0 = [0, 1) \times \{(0, \dots, 0)\} \times F \subset M$$

and identify F with $\{(0, 0, \dots, 0)\} \times F \subset F_0$. Let $\mathfrak{A}(F_0)$ be as above and consider the following two onto morphisms and their kernels:

$$\sigma_{0,F_0} : \mathfrak{A}(F_0) \longrightarrow \mathcal{C}_0(Y), \quad I = \ker(\sigma_{0,F_0})$$
$$\mathrm{In}_{F,F_0} : \mathfrak{A}(F_0) \longrightarrow \mathfrak{A}(F), \quad J = \ker(\mathrm{In}_{F,F_0})$$

Denote by $\partial_1 = \partial, \partial_2, \partial_3, \partial_4$ the following boundary maps in K-theory:

$$\partial_1 : K_{q+1}(\mathfrak{A}(F_0)/(I + J)) = K_{q+1}(\mathcal{C}_0(Y)) \to K_q(I/I \cap J) = K_q(\mathcal{C}_0(\mathbb{R}^q, \mathcal{K})) \simeq \mathbb{Z},$$
$$\partial_2 : K_{q+1}(\mathfrak{A}(F_0)/(I + J)) \to K_q(J/I \cap J) = K_q(\mathcal{C}_0(Y_0)) = K^q(Y_0),$$
$$\partial_3 : K_q(I/I \cap J) \to K_{q-1}(I \cap J) = K_{q-1}(\mathcal{C}_0(\mathbb{R}^{q-1}, \mathcal{K})) \simeq \mathbb{Z},$$
$$\partial_4 : K_q(J/I \cap J) \to K_{q-1}(I \cap J) \simeq \mathbb{Z},$$

where $Y_0 = S^*M|_{F_0} \setminus S^*M|_F = (0, 1) \times Y$ and we have used the determination of the partial quotients given by Theorem 2.

We have that $\partial_3\partial + \partial_4\partial_2 = 0$, because the composition of connecting morphisms

$$K_{q+1}(\mathfrak{A}(F_0)/(I+J)) \longrightarrow K_q((I+J)/I\cap J) \longrightarrow K_{q-1}(I\cap J)$$

is 0 and $(I+J)/I\cap J \simeq I/I\cap J \oplus J/I\cap J$.

Since ∂_3 is the canonical isomorphism, we obtain that $\partial = -\partial_4\partial_2$, after identifications. We have $\dim F_0 = \dim F + 1 = n+1$, hence the codimension q decreases by 1. By induction, the theorem is true for F_0:

$$\partial_4(a) = (-1)^{n+1}\langle Ch(a)p_0^*\mathcal{T}(F_0), [Y_0]\rangle$$

for all $a \in K_q(\mathcal{C}_0(Y_0))$,. Here we have denoted by $p_0 : Y_0 \to (0,1)\times F$ the projection. Let $\pi : Y_0 = (0,1)\times Y \to Y$ be the projection onto the second component. Then $p_0^*\mathcal{T}(F_0) = \pi^*p^*\mathcal{T}(F)$. Denote by ∂_c the boundary map $H_c^*(Y) \to H_c^{*+1}(Y_0)$. Using the fact that the Chern character is compatible with the boundary maps in K-theory (this fact is proved for algebras in general in [21, 22]) we obtain

$$\begin{aligned}\partial(a) = -\partial_4\partial_2(a) &= -(-1)^{n+1}\langle Ch(\partial_2(a))p_0^*\mathcal{T}(F_0), [Y_0]\rangle\\ &= (-1)^n\langle \partial_c Ch(a)\pi^*p^*\mathcal{T}(F), [Y_0]\rangle\\ &= (-1)^n\langle \partial_c(Ch(a)p^*\mathcal{T}(F)), [Y_0]\rangle\\ &= (-1)^n\langle Ch(a)p^*\mathcal{T}(F), [Y]\rangle,\end{aligned}$$

where the last equality is Stokes' theorem. The theorem is proved. □

Lemma 3. *The connecting morphism ∂ in the previous theorem is onto.*

Proof. Choose a small contractable open subset $U \subset Y$ and choose the class $a \in K_{q+1}(\mathcal{C}_0(Y))$ to come from a generator of $K_{q+1}(\mathcal{C}_0(Y|_U))$. Then $\partial a = \pm 1$. □

The following corollary determines the K-theory groups of the 'higher indicial algebras.' The space ${}^bS^*M|_F$ in the statement of the corollary plays the role of Y in the last theorem.

Corollary 6. *If F is a smooth face of the manifold with corners M, of codimension q, then the K-theory groups of $\mathfrak{A}(F,M)$ are*

$$K_{q+1}(\mathfrak{A}(F,M)) \simeq \ker(\partial : K_{q+1}({}^bS^*M|_F) \to \mathbb{Z})$$

and $K_q(\mathfrak{A}(F,M)) \simeq K_q({}^bS^*M|_F)$.

Proof. The result follows from the K-theory six term exact sequence applied to the short exact sequence in Proposition 9 and the determination of the connecting morphism of that exact sequence obtained in the above lemma. □

One should compare the above theorem with other equivariant index theorems for *noncompact* groups [1, 24] for discrete groups and [7] for connected Lie groups.

12. Final comments

Consider the case $q = 1$ in the preceding theorem, and let, as above, F be a smooth manifold (without corners). The indicial algebra $\mathfrak{A}(F)$ fits into an exact sequence

$$0 \longrightarrow \mathcal{C}_0(\mathbb{R},\mathcal{K}) \longrightarrow \mathfrak{A}(F) \xrightarrow{\sigma_{0,F}} \mathcal{C}_0(S^*M|_F) \to 0.$$

The above results imply that the morphism

$$K_1(\mathcal{C}_0(\mathbb{R},\mathcal{K})) \longrightarrow K_1(\mathfrak{A}(F))$$

vanishes. However at the level of the *algebraic* K_1 and *uncompleted* algebras the morphism

$$i : K_1^{\mathrm{alg}}(\Psi_b^{-\infty}(\mathbb{R}\times F)^{\mathbb{R}}) \longrightarrow K_1^{\mathrm{alg}}(\Psi_b^{\infty}(\mathbb{R}\times F)^{\mathbb{R}})$$

is not zero. This follows from results of [16] who proves the existence of a onto morphism $\eta : K_1^{\mathrm{alg}}(\Psi_b^{\infty}(\mathbb{R}\times F)^{\mathbb{R}}) \longrightarrow \mathbb{C}$, which coincides with the usual η-invariant of [2] for admissible Dirac operators. Moreover, the composition $\eta\circ i$ computes the spectral flow.

In subsequent papers [19, 18] we will use the above observation to exted the result of [16] to families and to study the relation between the η-invariant and cyclic cohomology [6].

Appendix. The structure of b-pseudodifferential operators

For completeness, we give a direction definition of the algebra of b-pseudodifferential operators from which all the basic properties can be deduced. As model space consider the product of intervals $M = M_k = [-1,1]^k$. The space $\Psi_b^m(M)$ may then be defined directly as a space of kernels. They are also defined on a similar model space $M_{\bar b}^2 = [-1,1]^{2k}$, although this should *not* be thought of as the product of M with itself. Rather the smooth map

$$\begin{aligned} &\bar\beta : M_{\bar b}^2 \supset (-1,1)^k\times[-1,1]^k \ni (\tau,R) \longmapsto (x,x')\in M^2\\ &x_i = \frac{2R_i+\tau+R_i\tau_i}{2+\tau_i+R_i\tau_i},\ x_i' = \frac{2R_i-\tau_i-R_i\tau_i}{2-\tau_i-R_i\tau_i}\\ &\Longleftrightarrow R_i = \frac{x_i+x_i'-2x_ix_i'}{2-x_i-x_i'+x_ix_i'},\ \tau_i = \frac{x_i-x_i'}{1-x_ix_i'} \end{aligned} \tag{52}$$

is used to identify the interiors of M^2 and $M_{\bar b}^2$. Consider the 'diagonal' submanifold $\mathrm{Diag}_{\bar b} = \{\tau_i = 0, i = 1,\dots,k\}$ and the boundary hypersurfaces $B_i^{\pm} = \{R_i = \pm1\}$. Then as a linear space

$$\Psi_b^m(M) = \big\{A = A'\nu,\ A\in I^m(M_{\bar b}^2,\mathrm{Diag}_{\bar b}); A\equiv 0 \text{ at } B_i^{\pm},\ \forall\ i\big\}. \tag{53}$$

Here ν is a 'right density' namely

$$\nu = \frac{|dx_1'\dots dx_k'|}{(1-(x_1')^2)\dots(1-(x_k')^2)}$$

and $I^m(M_b^2,\mathrm{Diag}_{\bar b})$ is the space of conormal distributions. If the kernel space is embedded, $M_{\bar b}^2 = [-1,1]^{2k} \hookrightarrow \mathbb{R}^{2k}$, and then rotated so that the linear extension of $\mathrm{Diag}_{\bar b}$ becomes the usual diagonal then this space is precisely the restriction to the image of $M_{\bar b}^2$ of the space of kernels of (polyhomogeneous) pseudodifferential operators on $\mathbb{R}^k$. Since these kernels are smooth away from $\mathrm{Diag}_{\bar b}$ the condition in (53) that the kernels vanish in Taylor series at the boundary faces $B_i^{\pm}$ is meaningful.

The identification $\bar\beta$ in (52) transforms the space $\Psi_b^m(M)$ to a subspace of the space of (extendible) distributional right densities on M^2, so by Schwartz' kernel theorem each element defines an operator. If $u\in\dot{\mathcal{C}}^\infty(M)$, the space of smooth functions vanishing at the boundary in Taylor series, then in principle Au, for $A\in\Psi_b^m(M)$, is a distribution on the interior of M. In fact,

$$A : \dot{\mathcal{C}}^\infty(M) \longrightarrow \dot{\mathcal{C}}^\infty(M),\ \forall\ A\in\Psi_b^m(M).$$

The space $\Psi_b^m(M)$ is a $\mathcal{C}^\infty(M^2)$ module, where the smooth functions on M^2 are lifted under $\bar\beta$ to (generally non-smooth) functions on $M_{\bar b}^2$, it is also invariant under conjugation by a diffeomorphism. As a consequence, for a general manifold with corners, the space $\Psi_b^m(X)$ can be defined by localization.

Definition 2. *If X is any manifold with corners then the space $\Psi_b^m(X)$ consists of those operators $A : \dot{\mathcal{C}}_c^\infty(X) \longrightarrow \dot{\mathcal{C}}^\infty(X)$ such that if $\phi \in \mathcal{C}_c^\infty(X)$ has support in a coordinate patch diffeomorphic to a relatively open subset of M_k then the image of the localized operator $\phi A \phi$ under the diffeomorphism is an element of $\Psi_b^m(M_k)$ and if $\phi, \phi' \in \mathcal{C}_c^\infty(X)$ have disjoint supports in (possibly different) such coordinate patches then the image of $\phi' A \phi$ is in $\Psi_b^{-\infty}(M_k)$.*

The identification of $[-1,1]$ as the projective compactification of $[0,\infty)$

$$[0,\infty) \ni t \longmapsto x = \frac{t-1}{t+1} \in [-1,1] \tag{54}$$

interprets M as a compactification of $[0,\infty)^k$. This induces an action of $(0,\infty)^k$ on M and the invariant elements of $\Psi_b^m(M)$ correspond exactly to the kernels which are independent of the variables R_i. The compactification similarly reduces M_b^2 to a compactification of $[0,\infty)^{2k}$. The action of $A \in \Psi_b^m(M)$ can then be written

$$Au(t_1, \dots, t_k) = \int_0^\infty \cdots \int_0^\infty A(t_1, \dots, t_k, s_1, \dots, s_k) u(s_1 t_1, \dots, s_k t_k) \frac{ds_1}{s_1} \dots \frac{ds_k}{s_k}, \tag{55}$$

where $A(t,s)$ is smooth in t, conormal in s at $s=1$ and vanishes rapidly (uniformly on compact sets) with all derivatives as any $s_i \to 0$ or ∞.

An alternative definition of b-pseudodifferential operators is obtained by considering as in [14] operators of the form

$$(Tu)(x,y) = (2\pi)^{-n} \int e^{i(x-x')\xi + (y-y')\eta} a(x,y,x\xi,\eta) u(x',y') dx' dy' d\xi d\eta,$$

where $a(x,y,\xi,\eta)$ is a classical (i.e. 1-step polyhomogeneous of integral order) symbol satisfying a certain lacunary condition. The operator T is seen to be invariant if and only if it is of the form

$$(Tu)(x,y) = (2\pi)^{-n} \int e^{i(x-x')\xi + (y-y')\eta} a(0,y,x\xi,\eta) u(x',y') dx' dy' d\xi d\eta.$$

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