KAC-DYNKIN DIAGRAMS AND SUPERTABLEAUX

I. Bars

CERN - Geneva
and
J.W. Gibbs Lab., Dept. of Physics,
Yale Univ., New Haven, CT 06520

B. Morel[*] and H. Ruegg[*]

Département de Physique Théorique,
Université de Genève, 1211 Genève 4

A B S T R A C T

We show the relation between Kac-
Dynkin diagrams and supertableaux.

1. INTRODUCTION

There exist presently two approaches to labelling representations of the super-groups SU(N/M). Kac [1] has proposed a unique labelling of irreducible representations in analogy to Dynkin diagrams. We shall refer to these as Kac-Dynkin diagrams. Balantekin and Bars [2-4] introduced supertableaux in analogy to Young tableaux, which rely on the properties of the permutation group, to arrive at irreducible supertensors which provide a basis for supergroup representations. The relation between these approaches has been found [5]. Our aim is to elaborate further on this connection, add new insights and clarifications and establish a useful dictionary.

A Kac-Dynkin diagram provides the highest weight Λ. The remaining weights are in principle obtained by applying lowering operators. This requires lengthy (but straightforward) calculations [7], which yield the eigenvalues of the generators belonging to the Cartan subalgebra. With this method necessary and sufficient conditions as well as dimension formulas for "typical" representations have been given [1,7]. Also branching rules for supersubalgebras, especially irregular ones, have been computed [7].

The supertableaux, and the associated supertensors, provide all the states in a representation and the content of the states is immediately obvious. This makes them very useful in physical applications [8,9]. Typical and atypical representations are not distinguished in this approach and the supertableau methods apply to both. In supertableaux one uses the concept of supersymmetrization [2], which means that when bosonic indices corresponding to a row are symmetrized, the fermionic indices are antisymmetrized. This can be done by an efficient method [2] which keeps close analogy to representations constructed via ordinary Young tableaux. These analogies can be applied as follows

$$SU(N) \longleftrightarrow SU(N/M)$$
$$SO(N) \longleftrightarrow Osp(N/M)$$
$$Sp(2N) \longleftrightarrow P(2N)$$

Through these analogies many practical and useful properties have been computed for the supergroups indicated above for all supertableau representations:

  i) Matrix representations of the supergroup in tensor space [2,7].

 ii) Character formulas [2,4].

iii) Dimension formulas [2,4].

 iv) Eigenvalues of Casimir operators [2,3,4].

  v) Branching rules for

$$SU(M/N) \rightarrow SU(M) \times SU(N) \times U(1)$$
$$SU(M_1 + M_2 / N_1 + N_2) \rightarrow SU(M_1/N_1) \times SU(M_2/N_2) \times U(1)$$
$$SU(M_1 M_2 + N_1 N_2 / M_1 N_2 + M_2 N_1) \rightarrow SU(M_1/N_1) \times SU(M_2/N_2)$$

 vi) Harmonic oscillator representations [4,6,8].

vii) Analytic unitary representations of noncompact $SL(N/M)$ in a harmonic oscillator basis [10] and in a superspace Z-basis [10].

The connection to Kac-Dynkin diagrams [5] for $SU(M/N)$ can be seen by computing the highest weight through the aid of the $SU(M/N) \rightarrow SU(M) \times SU(N) \times U(1)$ decomposition. In this paper after reviewing this procedure and giving a translation dictionary to Kac-Dynkin diagrams, and several examples, we will be able to establish the following statements for $SU(M/N)$:

a) Supertableaux containing only covariant (undotted) or only contravariant (dotted) boxes correspond to irreducible representations.

b) Supertableaux containing mixed dotted and undotted boxes correspond to irreducible representations provided M,N are sufficiently large compared to the number of boxes.

c) Mixed supertableaux with too many boxes compared to M,N are reducible but indecomposable.

d) All atypical representations are described by supertableaux.

e) Typical representations with a_M = integer (defined below) are naturally described. a_M = arbitrary real number is described with the additional concept of an over-all $U(1)$ phase of the representation in addition to the tableau.

f) To a given Kac-Dynkin diagram one can find many corresponding supertableaux.

g) One can usefully employ supertableaux to compute the decomposition of direct product representations, provided indecomposable supertableaux are reduced via Kac-Dynkin diagrams.

2. THE SUPERALGEBRA $SU(M/N)$

In the classification of Kac [1,11], this is a classical superalgebra of type I, called $A(M-1,N-1)$. $SU(M/N)$ is simple for $M \neq N$. For $M = N$, one has to divide by $U(1)$. It consists of an even ("Bosonic") part, the subalgebra $SU(M) \times SU(N) \times U(1)$, and an odd ("Fermionic") part, which transforms as the representation $(M,N*) + (M*,N)$ of the even part. The Cartan subalgebra consists of the $M + N - 1$ mutually commuting generators H_i, the $M - 1$ first ones belonging to $SU(M)$, the $N - 1$ last ones to $SU(N)$, H_M playing a special role. The generator Q of $U(1)$ is a linear combination of H_i (see Eq. (2.10) below). To each H_i corresponds a simple root α_i, a "raising" operator E_i^+ and a "lowering" operator E_i^-. We shall need the commutation relations :

$$\left[H_i , E_j^{\pm} \right] = \pm a_{ij} E_j^{\pm} \qquad i,j = 1 \cdots M+N-1 \tag{2.1}$$

where a_{ij} are the elements of the Cartan matrix of SU(M/N) given by Kac[1]

$$\{ a_{ij} \} = \begin{array}{c} \qquad\qquad M \\ \left| \begin{array}{ccccccccc} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & \ddots & & & & & & \\ & & & -1 & 2 & -1 & & & \\ & & & & -1 & 0 & +1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & \ddots & \ddots & -1 \\ & & & & & & & -1 & 2 \end{array} \right| \end{array} \quad M \tag{2.2}$$

Notice that $a_{MM} = 0$ and $a_{MM+1} = +1$, otherwise we recognize the Cartan matrices of SU(M) and SU(N).

We also note the commutation relations

$$\left[E_i^+ , E_j^- \right] = \delta_{ij} H_j \qquad \forall i,j \neq M \tag{2.3}$$

and the anticommutation relations of the two odd generators corresponding to the simple root α_M :

$$\left[E_M^+ , E_M^- \right]_+ = H_M \tag{2.4}$$

The full system of commutation (anticommutation) relations can either be obtained from (2.1) to (2.4) which characterize "simple" generators, plus the generalized Jacobi identity [1,11], or by the explicit realization of the fundamental representation of dimension M + N. This will now be done. The generators are the matrices X :

$$X = \begin{array}{c} \overbrace{}^{M} \overbrace{}^{N} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \begin{array}{l} \Big\} M \\ \Big\} N \end{array} \end{array} \qquad (2.5)$$

with the restriction for the supertrace :

$$Str \; X \equiv tr \, A - tr \, D = 0 \qquad (2.6)$$

Introducing the matrices E^i_j with matrix elements :

$$\left(E^i_j \right)^a_\ell = \delta^{ia} \delta_{j\ell} \qquad (2.7)$$

$$i, j, a, \ell = 1 \cdots \cdots M + N$$

one gets for the Cartan subalgebra

$$H_i = E^i_i - E^{i+1}_{i+1} \qquad i = 1 \cdots M-1, \; M+1 \cdots N-1$$

$$H_M = E^M_M + E^{M+1}_{M+1} \qquad (2.8)$$

all H_i have zero supertrace.

The raising operators are E^i_j, $i < j$, the lowering operators E^i_j, $i > j$, the "simple" generators of Eq. (2.1) correspond to $j = i + 1$, resp. $i - 1$. For the odd generators, $i \leq M$, $j > M$ or $i > M$, $j \leq M$. Notice that the anticommutator of two odd raising or two odd lowering operators is zero.

Finally, the generator Q of $U(1)$ is given, up to a constant, by

$$Q = \quad$$ (2.9)

which corresponds to

$$Q = \sum_{k=1}^{M-1} \frac{k\, H_k}{M} + H_M - \sum_{\ell=1}^{N-1} \frac{(N-\ell)}{N}\, H_{M+\ell}$$ (2.10)

From (2.1) and (2.2) one gets :

$$[Q, E_M^\pm] = \pm \left(\frac{M-1}{M} a_{M-1,M} - \frac{N-1}{N} a_{M+1,M} \right) E_M^\pm$$ (2.11)

$$= \pm \left(\frac{1}{M} - \frac{1}{N} \right) E_M^\pm$$

3. THE KAC-DYNKIN DIAGRAM

According to Kac [1], the irreducible representations (IR) of the superalgebra SU(M/N) are characterized in a similar way as IR of Lie algebras. They are uniquely determined by the highest weight Λ, which is a vector in the root space. The state in the representation space corresponding to Λ is defined by :

$$E_i^+ |\Lambda\rangle = 0 \qquad i = 1 \cdots M+N-1$$ (3.1)

$$H_i |\Lambda\rangle = a_i |\Lambda\rangle \qquad i = 1 \cdots M+N-1$$ (3.2)

The numbers a_i are non negative integers for $i \neq M$. a_M may be any real number.

An IR of SU(M/N) is thus defined by the values a_i of the highest weight, which can be noted on a Kac-Dynkin diagram

$$\underset{\text{O-O-}\cdots\cdots\text{O-}\otimes\text{-O}\cdots\cdots\text{O}}{a_1 \quad a_2 \qquad a_{M-1} \; a_M \; a_{M+1} \qquad a_{M+N-1}} \qquad (3.3)$$

The part without $\otimes$ decomposes into ordinary Dynkin diagrams for SU(M) and SU(N). $\otimes$ corresponds to the odd root α_M (whose length is zero !), or to the special generator H_M.

One distinguishes typical and atypical IR. For the latter, <u>one</u> of the following conditions must be satisfied : (Kac, Ref. (1), Hurni and Morel, Ref. (7))

$$a_M = \sum_{t=M+1}^{j} a_t - \sum_{t=1}^{M-1} a_t - 2M + i + j$$

$$1 \leq i \leq M \leq j \leq M+N-1 \qquad (3.4)$$

For the typical representations, <u>none</u> of these relations is satisfied. Their interpretation is the following. One gets all the weights of a given IR by starting with the highest weight and applying lowering operators. The action of the even operators is well known. There are MN odd generators which anticommute. Hence, each one can be applied at most once, and the state obtained by applying two different odd generators is antisymmetric. If $|\psi\rangle$ is some state in the representation space, it may happen that $E_M^+ E_M^- |\psi\rangle = 0$. This is just the case when one of the relations (3.4) is satisfied. For example, if $a_M = 0$, $E_M^+ E_M^- |\Lambda\rangle = 0$ where Λ is the highest weight. This means that the state $E_M^- |\psi\rangle$ does not belong to the same representation : either the representation starting with $|\Lambda\rangle$ is not irreducible, or we must put $E_M^- |\psi\rangle = 0$. This is the atypical case.

For typical representations, one can apply each odd generator exactly once. If d is the dimension of the IR of $SU(M) \times SU(N) \times U(1)$ corresponding to the highest weight Λ, the dimension D of the corresponding IR of $SU(M/N)$ is

$$D = 2^{MN} d \tag{3.5}$$

For atypical representations, the dimension will always be lower.

The fundamental representation $(D = M + N)$ is given by :

$$\tag{3.6}$$

Since $a_M = 0$, it is atypical (put $i = j = M$ into Eq. (3.4)). For $SU(1/N)$, we have :

$$\tag{3.7}$$

Here $a_M = a_1 = a_2 + 1$, which satisfies again (3.4) (put $i = M = 1$, $j = M + 1 = 2$).

The conjugate representation $(D = M + N)$ is

$$\tag{3.8}$$

and is again atypical, also for $SU(M/1)$.

The adjoint representation is

$$\tag{3.9}$$

This is atypical, except for $SU(1/2)$:

$$\otimes\!\!-\!\!\bigcirc \qquad\qquad (3.10)$$

whose dimension is $2 \cdot 2^2 = 8$.

Another convenient characterization of the highest weight Λ is obtained by considering the eigenvalue q of the $U(1)$ generator Q :

$$Q |\Lambda\rangle = q_\Lambda |\Lambda\rangle \qquad\qquad (3.11)$$

Using (2.10), (3.2) one gets

$$q_\Lambda = \sum_{k=1}^{M-1} \frac{k a_k}{M} + a_M - \sum_{\ell=1}^{N-1} \frac{(N-\ell) a_{M+\ell}}{N} \qquad\qquad (3.12)$$

Applying odd lowering operators, one gets the other $SU(M) \times SU(N) \times U(1)$ multiplets. Each odd operator is obtained by the commutator of E_M^- with "simple" even generators. Since the latter commute with Q, it is enough to consider the commutator given by Eq. (2.11), applied on a state $|\psi\rangle$:

$$\begin{aligned} [Q, E_M^-] |\psi\rangle &= Q E_M^- |\psi\rangle - q_\psi E_M^- |\psi\rangle \\ &= -\left(\frac{1}{M} - \frac{1}{N}\right) E_M^- |\psi\rangle \end{aligned} \qquad (3.13)$$

Thus, for $M < N$, resp. $M > N$, the odd generators E_M^- lowers, resp. raises the value of q_ψ. Hence, the "highest" weight Λ corresponds to

$$\begin{aligned} q_\Lambda &= maximum \quad for \quad M < N \\ q_\Lambda &= minimum \quad for \quad M > N \end{aligned} \qquad (3.14)$$

For typical representations, one has to apply the MN odd generators in a completely antisymmetric way to get the lowest weight λ. Since such an antisymmetric combination is a singlet under $SU(M) \times SU(N)$, the state $|\lambda>$ belongs to the same IR of this subalgebra. The eigenvalue q_λ is given, using (3.13), by :

$$q_\lambda = q_\Lambda + M - N \qquad (3.15)$$

where

$$Q|\lambda> = q_\lambda|\lambda>$$
$$E_i^-|\lambda> = 0 \qquad \forall i \qquad (3.16)$$

For atypical representations, q_λ will be different from the expression (3.15), namely larger if $M < N$ and smaller if $M > N$.

In conclusion, the Kac-Dynkin diagram characterizes uniquely all IR of SU(M/N). It gives immediately the eigenvalues of H_i and Q of the highest weight. It allows a usually lengthy but straightforward computation of all states of the representation. It gives immediately the dimension and $SU(M) \times SU(N) \times U(1)$ content of typical representations, but not those of atypical representations.

4. SUPERYOUNG TABLEAUX FOR SU(M/N)

Young tableaux for Lie algebras are very convenient for computing branching rules for representations of subalgebras and for establishing the Clebsch-Gordan series of tensor products of IR. They are very useful in practical physics applications because it is possible to describe states in tensor notation with the symmetries of Young tableaux.

Balantekin and Bars [2,3,4] have introduced Young supertableaux for SU(M/N)

and showed that these in addition to providing a very convenient labelling of re-presentations, are useful in calculating many properties of super-representations.

For SU(M), Young tableaux give the symmetry of the indices of covariant tensors $t_{ABC\cdots}$. One can also introduce contravariant tensors $t^{A'B'C'}\cdots$. They are related to the former ones by the completely antisymmetric symbol $\varepsilon_{A_1,\cdots,A_M}$, which is invariant due to the determinant of SU(M) group elements being one. Although this is not necessary, King[12] has introduced Young tableaux for contravariant tensors (distinguished graphically by a point in the box). A next step is to introduce traceless mixed tensors $t^{A'B'\cdots}_{AB\cdots}$.

For SU(M/N), the ε symbol is not invariant. Thus both co- and contravariant tensors are necessary. These correspond to mixed supertableaux. Furthermore, it is possible to have tensors corresponding to long columns in the supertableaux with more than M + N dotted or undotted boxes.

BB[2,3] assign to the covariant tensor $t_{AB\cdots}$ the Young supertableaux :

$$(4.1)$$

b_i (i = 1,...,m) counts the boxes in the row i,

c_j (j = 1,...,n) counts the boxes in the column j, with the conditions :

$$b_1 \geqslant b_2 \geqslant \cdots\cdots \geqslant b_m > 0$$
$$c_1 \geqslant c_2 \geqslant \cdots\cdots \geqslant c_n > 0 \qquad (4.2)$$

The conjugate tableau is obtained by interchanging rows and columns :

$$
\begin{array}{c}
b_1 \quad \cdots \quad b_m \\
\text{[tableau diagram]} \quad c_1 \\
\quad \vdots \\
\quad c_n
\end{array}
\tag{4.3}
$$

The supersymmetry property of $t_{AB\ldots}$ under interchange of the indices $A,B\ldots$ is analogous to $SU(M+N)$ except that when bosonic indices in a row are symmetrized, fermionic indices are antisymmetrized. This is the meaning of super-symmetrization.

Consider now the IR of the subalgebra $SU(M) \times SU(N) \times U(1)$ contained in an IR of $SU(M/N)$. The procedure to get these IR is the same as for $SU(M+N)$, with the essential difference that the tableau one would obtain for an IR of the second algebra $SU(N)$ has to be replaced by the conjugate tableau. This follows from supersymmetrization.

Starting from fundamental representation (dimension $M + N$)

$$
SU(M/N) \qquad SU(M) \quad SU(N)
$$
$$
\square = \left(\square, 1 \right) + \left(1, \square \right)
\tag{4.4}
$$

the rule is shown in the following example where the decomposition of an IR of $SU(M+N)$ is compared to the decomposition of an IR of $SU(M/N)$:

$$
SU(M+N) \qquad SU(M) \quad SU(N)
$$
$$
\begin{aligned}
\text{[tableau]} = &\left(\text{[tableau]}, 1 \right) + \left(\text{[tableau]}, \square \right) + \left(\square\square\square, \square \right) \\
&+ \left(\text{[tableau]}, \square\square \right) + \left(\square\square, \text{[tableau]} \right) + \left(\square, \text{[tableau]} \right) + \left(\square, \square\square\square \right) + \\
&+ \left(1, \text{[tableau]} \right)
\end{aligned}
\tag{4.5}
$$

$$SU(M/N) \qquad SU(M) \quad SU(N)$$

$$\yng = \left(\yng, 1\right) + \left(\yng, \Box\right) + \left(\yng, \Box\right) + \left(\yng, \yng\right)$$

$$+ \left(\yng, \yng\right) + \left(\Box, \yng\right) + \left(\Box, \yng\right) + \left(1, \yng\right) \qquad (4.6)$$

This rule is easy to understand in tensor notation [2]. Also from the point of view of the algebra, each time one replaces an SU(M) index by an SU(N) index, one has to apply an odd generator. Since the product of odd generators is anti-symmetric, rows (symmetric) are changed in to columns (antisymmetric) and vice versa.

Eqs (4.4) and (4.5) are independent of M and N, except if M or (and) N are too small. For example, for SU(1/2), the following terms are illegal, applying the rules for SU(M) × SU(N) :

$$\left(\yng, 1\right) \left(\yng, \Box\right) \left(\yng, \yng\right), \left(\Box, \yng\right), \left(1, \yng\right) \qquad (4.7)$$

The eigenvalue q of the U(1) generator Q is obtained from Eq. (2.9). Thus for the fundamental representation :

$$\boxed{\diagup} = \left(\Box, 1\right)_{q=\frac{1}{M}} + \left(1, \Box\right)_{q=\frac{1}{N}} \qquad (4.8)$$

Hence the q value of some SU(M) × SU(N) IR is given by $\frac{1}{M}$ times the number of SU(M) boxes plus $\frac{1}{N}$ times the number of SU(N) boxes. For example, for the first two terms of (4.6) one gets :

$$\left(\yng, 1\right)_{q=\frac{4}{M}} \qquad \left(\yng, \Box\right)_{q=\frac{3}{M}+\frac{1}{N}} \qquad (4.9)$$

Notice that the difference of these two q values agrees with (3.13).

Contravariant tensors $t^{AB\ldots}$ correspond to conjugate representations of SU(M/N), as well as for the subalgebra $SU(M) \times SU(N) \times U(1)$. The fundamental conjugate IR is denoted by

$$SU(M/N) \qquad SU(M) \quad SU(N)$$
$$\boxed{\diagdown} = \left(\boxed{\,} , 1 \right)_{q=-\frac{1}{M}} + \left(1, \boxed{\,} \right)_{q=-\frac{1}{N}} \qquad (4.10)$$

Both q values are negative because the supertrace of Q must be zero (Eq. (2.6)).

Apart from this, the rules are similar as for the covariant tensor. For example :

$$\boxed{\diagdown\diagdown} = \left(\boxed{\,\,} , 1 \right)_{q=-\frac{2}{M}} + \left(\boxed{\,}, \boxed{\,} \right)_{q=-\frac{1}{M}-\frac{1}{N}} + \left(1, \boxed{\square} \right)_{q=-\frac{2}{N}} \qquad (4.11)$$

Finally, mixed tensors $t^{A'B'\ldots}_{AB\ldots}$ correspond to IR only if the supertrace $\sum_x (-)^{g(x)} t^{XB'\ldots}_{XB\ldots} = 0$, where $g(x) = 0,1$ for even, odd components is zero. They are necessary if an IR of SU(M/N) contains IR of the subalgebra with q = 0. This is the case for the adjoint representation

$$\boxed{\diagdown\diagdown} = \left(\boxed{\cdot\,} , 1 \right)_{q=0} + \left(\boxed{\,}, \boxed{\,} \right)_{q=-\frac{1}{M}+\frac{1}{N}} +$$
$$+ \left(\boxed{\,}, \boxed{\,} \right)_{q=\frac{1}{M}-\frac{1}{N}} + \left(1, \boxed{\cdot\square} \right)_{q=0} + \left(1, 1 \right)_{q=0} \qquad (4.12)$$

The notation for the general supertableau for SU(M/N) will be [3] :

5. RELATIONS BETWEEN KAC-DYNKIN DIAGRAMS AND SUPERYOUNG TABLEAUX

Kac [1] has used the highest weight to uniquely determine an IR of SU(M/N). What is the relation between Kac-Dynkin diagrams and supertableaux ?

From (3.14) we know the properties of the eigenvalue q_Λ of Q for the state corresponding to the highest weight Λ :

$$q_\Lambda = maximum \quad for \quad M < N$$
$$q_\Lambda = minimum \quad for \quad M > N \tag{5.1}$$

The case M = N will not be considered.

From (3.12) we know the relation to the Dynkin labels a_i :

$$q_\Lambda = \sum_{k=1}^{M-1} \frac{k a_k}{M} + a_M - \sum_{\ell=1}^{N-1} \frac{N-\ell}{N} a_{M+\ell} \tag{5.2}$$

We now need only the corresponding information for supertableaux.

Let us start with tableaux corresponding to covariant tensors (see Tableau 4.1).

From (4.8) we know for the fundamental representation :

$$SU(M/N) \qquad SU(M) \times SU(N)$$
$$\boxed{/} \quad = \quad \left(\Box, 1 \right)_{q=\frac{1}{M}} + \left(1, \Box \right)_{q=\frac{1}{N}} \tag{5.3}$$

From (5.1) we see that the highest weight belongs to $(\Box, 1)$. From (5.2) it is clear that

$$a_1 = 1 \qquad a_i = 0 \qquad i \neq 1 \tag{5.4}$$

Hence

$$\boxed{/} \;=\; \overset{1}{\underset{0}{0}}\!-\!\overset{0}{0} \cdots\cdots \overset{0}{\underset{\otimes}{}} \cdots\cdots \overset{0}{0} \tag{5.5}$$

in agreement with (3.6).
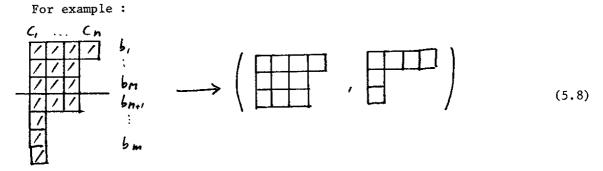
For a general covariant tableau, one gets q by counting boxes. From (5.1) and (5.3) one sees that q_Λ is obtained with the maximum number of $SU(M)$ boxes. If c_1, the number of rows, does not exceed M, the number of $SU(M)$ boxes can be taken to be equal to the number of $SU(M/N)$ boxes. The Dynkin labels are given by the familiar formula for $SU(M)$, while a_M is fixed by (5.2), remembering that for $SU(M)$, a_K is the number of columns with K boxes :

$$
\begin{aligned}
a_i &= b_i - b_{i+1} & i=1 \cdots M-1 \\
a_M &= b_M \\
a_{M+j} &= 0 & j=1 \cdots N-1 \\
c_1 &\leq M
\end{aligned}
\tag{5.6}
$$

In pictures :

$$\tag{5.7a}$$

where in the $SU(M) \times SU(N) \times U(1)$ decomposition we have shown just the component with the value q corresponding to the highest weight. This immediately yields

$$
\begin{array}{ccccc}
b_1-b_2 & b_2-b_3 & & b_n & o & o \\
\bigcirc\!\!-\!\!\bigcirc\!\!-\!\!\bigcirc & \cdots\cdots & \otimes & \cdots\cdots\bigcirc & \cdots\cdots\bigcirc
\end{array}
\tag{5.7b}
$$

$$
\text{if} \quad c_1 = m \leq M
$$

If c_1 exceeds M, to determine the highest state one writes first the step of Eq. (5.7a), then one has to cut the supertableau in two pieces. The first piece, which contains the first M rows is assigned to $SU(M)$, the remaining rows are assigned to $SU(N)$, after conjugating them. The $SU(M) \times SU(N) \times U(1)$ Young tableau thus defined is the first non-vanishing component in the decomposition of $SU(M/N) \to SU(M) \times SU(N) \times U(1)$ which will have the right value q corresponding to the highest state.

For example :

$$\tag{5.8}$$

On the r.h.s. we have shown the IR of $SU(M) \times SU(N)$ corresponding to the highest weight. The value of q_Λ is :

$$
q_\Lambda = \sum_{i=1}^{M} \frac{b_i}{M} + \sum_{i=1}^{b_{M+1}} \frac{c_i - M}{N} \, \theta(c_i - M)
\tag{5.9}
$$

where only $c_i > M$ contributes.

As far as the $SU(M)$ content is concerned, one can subtract $SU(M)$ singlets

for each column with M boxes. On the other hand, a supertableau is illegal unless

$$\ell_{M+1} \le N \tag{5.10}$$

because otherwise every component in the decomposition vanishes.

With (5.2), (5.9) and the rules for SU(M) and SU(N), we get the generalization of (5.6) :

$$a_i = \ell_i - \ell_{i+1} \qquad i = 1 \cdots M-1$$

$$a_M = \ell_M + c'_1$$

$$a_{M+j} = c'_j - c'_{j+1} \qquad j = 1 \cdots N-1 \tag{5.11}$$

$$c'_j = (c_j - M)\,\theta(c_j - M)$$

$$\ell_{M+1} \le N$$

These values must be put on the Dynkin diagram

$$\overset{a_1}{\circ}\!\!-\!\!\overset{a_2}{\circ}\cdots\cdot\overset{a_M}{\otimes}\cdot\cdots\cdot\overset{a_{M+N-1}}{\circ}$$

One should not forget that the b_i and c_j are not independent.

For conjugate representations, the procedure is similar except for the sign changes. For the fundamental representation, Eq. (4.1) is

$$\overset{SU(M/N)}{\boxed{\cdot}} = \overset{SU(M) \times SU(N)}{\left(\boxed{\cdot}, 1\right)_{q=-\frac{1}{M}} + \left(1, \boxed{\cdot}\right)_{q=-\frac{1}{N}}} \tag{5.12}$$

From (5.1), we see that the highest weight has

$$q_\Lambda = -\frac{1}{N} \tag{5.13}$$

Hence, comparing (3.8) and (5.12)

$$\tag{5.14}$$

For a general tableau, we search for q_Λ using (5.1) and (5.13). If $\bar{b}_1$, the number of columns, is smaller or equal N, we can fullfil (5.1) with $SU(N)$ boxes only. Using again (5.2) and the rules for $SU(N)$, not forgetting to conjugate the $SU(N)$ tableau, we get :

$$
\begin{aligned}
a_i &= 0 & i &= 1 \cdots M-1 \\
a_M &= -\bar{c}_N \\
a_{M+N-j} &= \bar{c}_j - \bar{c}_{j+1} & j &= 1 \cdots N-1 \\
n &= \bar{b}_1 \le N
\end{aligned}
\tag{5.15}
$$

$$\tag{5.16}$$

if $n = \bar{b}_I \le N$.

Next consider the case where $\bar{b}_1$ exceeds N. We now must cut the super-tableau in two pieces by a vertical line. If j is the index of $\bar{c}_j$, the contribution for $j \leq N$ is as before. For $j > N$, one gets an SU(M) IR. Thus,

$$ \tag{5.17} $$

The tableau on the r.h.s. again corresponds to the highest weight of the SU(M/N) IR. The value of q_Λ is :

$$ q_\Lambda = - \sum_{i=1}^{N} \frac{\bar{c}_i}{N} - \sum_{i=1}^{\bar{c}_{N+1}} \frac{(\bar{c}_i - N)}{M} \theta(\bar{c}_i - N) \tag{5.18} $$

A supertableau is illegal unless

$$ \bar{c}_{N+1} \leq M \tag{5.19} $$

Using (5.2) and (5.18), we get for the Dynkin labels a_i :

$$ a_{M-i} = \bar{c}_i' - \bar{c}_{i+1}' \qquad i = 1 \cdots M-1 $$

$$ \bar{c}_i' = (\bar{c}_i - N)\theta(\bar{c}_i - N) $$

$$ a_M = -\bar{c}_N - \bar{c}_1' = -\bar{c}_N - (\bar{c}_1 - N)\theta(\bar{c}_1 - N)\theta(\bar{c}_1 - N) \tag{5.20} $$

$$ a_{M+N-j} = \bar{c}_j - \bar{c}_{j+1} \qquad j = 1 \cdots N-1 $$

$$ \bar{c}_{N+1} \leq M $$

6. DISCUSSION

We have shown that to each covariant tensor (with a corresponding legal super-tableau) one can assign a Kac-Dynkin diagram. The latter, we know, specifies an IR of SU(M/N), for $M \neq N$. The same is true for contravariant tensors. The case of mixed tensors will be considered in the next section.

We now show that this correspondence is not one to one. Take for example

SU(2/3). Consider the following two supertableaux and their highest weight :

$$(6.1)$$

They clearly correspond to the same Kac-Dynkin diagrams.

For SU(M/N), we find the rule : two supertableaux correspond to the same

Kac-Dynkin diagram if $(N + 1)$ columns of M boxes are replaced by N columns

of $M + 1$ boxes. This amounts to replace an SU(M) singlet with $q = \frac{M}{M}$ by an

SU(N) singlet with $q = \frac{N}{N}$, provided there are enough boxes to start with.

This ambiguity is of course due to the fact that (5.11) does not determine

the b_i's and c_j's uniquelly from a_i.

A similar rule applies to contravariant tensors (5.20). Consider for SU(2/3)

the supertableaux

$$(6.2)$$

N columns of $M + 1$ boxes are replaced by $N + 1$ columns of M boxes.

Consider now the inverse problem : given Dynkin labels a_i, calculate super-tableaux labels b_i. There arises a question : for typical representations, a_M can be any real number while the supertableau describes naturally a_M = integer since the value of Q is determined by an integer number of boxes. However, for typical representations it is possible to add any constant to Q, since it remains supertraceless when the number of bosons is equal to the number of fermions. An additional constant in Q corresponds to an over-all U(1) phase of the whole re-presentation. This U(1) commutes with SU(M/N). Thus, up to this over-all phase an arbitrary representation of the group is recovered through the supertableau. The role of this over-all phase and its significance in representation theory of super-groups is not sufficiently clear.

Keeping this in mind, we start from a Kac-Dynkin diagram, and consider first the $SU(M) \times SU(N)$ labels a_i (i = 1,...,M - 1, M + 1,...,M + N - 1) which spe-cify the highest weight Λ of an IR. To each set a_i, we can assign either a covariant, or a contravariant tensor. The general formulae are of course (5.11) and (5.20). To show how they work, it is best to give an example. Consider the algebra SU(2/3), and the diagram

$$
\begin{array}{cccc}
2 & a_2 & 1 & 2 \\
\text{O} & \!\!\!-\!\!\otimes\!\!-\!\! & \text{O} & \!\!-\!\!\text{O}
\end{array}
\tag{6.3}
$$

For the subalgebra $SU(2) \times SU(3)$, this corresponds to covariant tensors with tableaux :

$$\tag{6.4}$$

For the SU(3) part, we also indicate the conjugate tableau. The supertableau is now given up to b_2 SU(2) singlets :

$$ (6.5) $$

Here, $c_1 = 5$ and b_2 determined by a_2, using (5.11) :

$$ a_2 = b_2 + 3 \qquad (6.6) $$

To get a legal diagram, $b_2 \geq 2$. Hence, (6.5) can be fulfilled for a_2 satisfying

$$ a_2 \geq 5 \qquad (6.7) $$

modulo the additional constant mentioned above.

Consider now contravariant tensors which tableaux (compare with (5.20))

$$ (6.8) $$

The supertableau is given up to $\bar{c}_3$ SU(3) singlets :

$$ (6.9) $$

with (5.20), we get

$$ a_2 = -\bar{c}_3 - 2 \qquad (6.10) $$

Here, $\bar{c}_3 \geq 1$, so that a_2 satisfies

$$a_2 \leq -3 \qquad (6.11)$$

up to the constant mentioned above.

We will see that we can also use mixed supertableaux to obtain representa-

tions of type $\overset{2}{0}\!\!-\!\!-\!\!\overset{a_2}{0}\!\!-\!\!-\!\!\overset{1}{0}\!\!-\!\!-\!\!\overset{2}{0}$ for SU(2/3).

Typical representations are those for which a_M is different from the r.h.s.
of (3.4) :

$$a_M \neq \sum_{t=M+1}^{j} a_t - \sum_{t=1}^{M-1} a_t - 2M + i + j$$

$$1 \leq i \leq M \leq j \leq M + N - 1 \qquad (6.12)$$

A necessary and sufficient condition for covariant tensors is :

$$c_M \geq N$$

This follows from the more general discussion of next section. For example, for

SU(2/3), the following are typical IR

$$\text{[tableau]} \quad = \quad \overset{1}{\underset{}{O}} - \overset{4}{\underset{}{\otimes}} - \overset{1}{\underset{}{O}} - \overset{0}{\underset{}{O}} \qquad \textit{etc.}$$

Similarly, for contravariant tensors :

$$\bar{C}_N \geqslant M$$

For example, for SU(2/3) :

$$\text{[tableau]} = \overset{0}{O} - \overset{-2}{\otimes} - \overset{0}{O} - \overset{0}{O} \qquad\qquad \text{[tableau]} = \overset{1}{O} - \overset{-3}{\otimes} - \overset{0}{O} - \overset{0}{O}$$

$$\text{[tableau]} = \overset{0}{O} - \overset{-2}{\otimes} - \overset{0}{O} - \overset{1}{O}$$

$$\text{[tableau]} = \overset{0}{O} - \overset{-3}{\otimes} - \overset{0}{O} - \overset{0}{O}$$

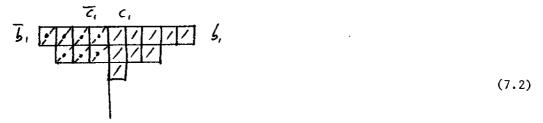$$\text{[tableau]} = \overset{0}{O} - \overset{-2}{\otimes} - \overset{1}{O} - \overset{0}{O}$$

There are, of course, many more atypical representations than typical and the supertableau approach is a convenient tool to describe both.

7. MIXED REPRESENTATIONS

We have seen that for covariant or contravariant tensors, a_M is limited by inequalities of the type (6.7) or (6.11). To get more general situations, one needs mixed, traceless tensors $t_{AB\ldots}^{A'B'\ldots}$. The most important example is the adjoint representation (see (3.9) and (4.12))

$$\text{[tableau]} = \overset{1}{O} - \overset{0}{O} \cdots \cdots \overset{0}{\otimes} \cdots \cdots \overset{1}{O} \qquad\qquad (7.1)$$

The algebraic rules to go from supertableau labels b_i to Dynkin labels a_i

are obtained again from Eqs (5.11) and (5.20). There is however one essential complication : mixed supertableaux, while being irreducible when N,M are sufficiently large compared to the number of boxes, may not always correspond to irreducible representations of SU(M/N), when N,M are small. But we shall see that they are indecomposable even when they are irreducible. (One of us (I.B.) thanks V. Kac for his comment on this point.) Consider the general supertableau

$$(7.2)$$

Suppose it contains m "covariant" boxes $\boxed{/}$ and n contravariant boxes $\boxed{\bullet}$. Consider now the two SU(M) × SU(N) × U(1) tableaux, obtained from (7.2) :

$$q_1 = \frac{m}{M} - \frac{n}{N} \qquad (7.3)$$

and

$$q_2 = -\frac{m}{M} + \frac{m}{N} \qquad (7.4)$$

$$q_2 = q_1 - (m+n)\left(\frac{1}{M} - \frac{1}{N}\right) \qquad (7.5)$$

Transferring a box ☐ from left to right, or a box ▣ from right to left,

amounts to applying and odd lowering operator. From (3.13)

$$Q\,E^-_M\,|\psi\rangle = \left[\,q_\psi - \left(\tfrac{1}{M} - \tfrac{1}{N}\right)\,\right]\,E^-_M\,|\psi\rangle \tag{7.6}$$

we see that (7.4) is obtained from (7.3) by applying (m + n) different odd

lowering generators. Now, if all tableaux in (7.3) and (7.4) are legal, these cor-

respond to the state $|\Lambda\rangle$, resp. $|\lambda\rangle$ with highest, resp. lowest, weight. But for

an IR, one can apply at most MN odd lowering generators to $|\Lambda\rangle$.

Hence, if m + n > MN, i.e. if there are too many boxes

does not correspond to an IR of SU(M/N).

The simplest example is given by the supertableau of SU(1/2) :

$$m + n = 3 \;;\;\; MN = 2 \tag{7.7}$$

The highest weight belongs to the IR of the bosonic subalgegra SU(2) × U(1)

$$\left(\;\Box\;,\;\begin{array}{c}\boxed{\bullet}\\\boxed{\bullet}\end{array}\;\right)_{q\,=\,0} = \left(\underline{1}\right)_{q\,=\,0} \tag{7.8}$$

The corresponding Kac-Dynkin diagram would be :

$$\tag{7.9}$$

But (7.7) gives rise to series of SU(2) × U(1) IR:

$$\left(\underline{1}\right)_{q=0} + \left(\underline{2}\right)_{q=-\frac{1}{2}} + \left(\underline{2}\right)_{q=-\frac{1}{2}} + \left(\underline{3}\right)_{q=-1}$$

$$+ \left(\underline{1}\right)_{q=-1} + \left(\underline{1}\right)_{q=-1} + \left(\underline{2}\right)_{q=-\frac{3}{2}} \tag{7.10}$$

We see that we need indeed three odd generators to go from $q = 0$ to $q = -\frac{3}{2}$, which is impossible for an IR

Another way to see the reducibility is to try to construct the supertraceless tensor corresponding to the supertableau (7.7), as in Ref. (3), for SU(M/N)

$$t_c^{(AB)} = \varphi_c \, \varphi^{(AB)} - \frac{1}{M-N+1} \left[\delta_c^A (-)^{g(D)} \varphi_D \varphi^{(DB)} + (-)^{g(A)g(B)} \delta_c^B (-)^{g(D)} \varphi_D \varphi^{(DA)} \right] \tag{7.11}$$

such that the supertrace is zero :

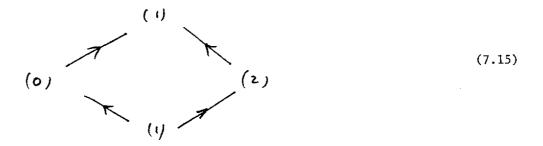$$\sum_{C=1}^{M+N} (-)^{g(C)} t_c^{(CB)} = 0 \; ; \quad \begin{array}{l} g(C) = 0, \; C = 1-M \\ g(C) = 1, \; C = M+1-M+N \end{array} \tag{7.12}$$

However, when $N = M + 1$, e.g. for SU(1/2), the denominator vanishes, so that

$$t_c^{(AB)} = t_c^{(BA)} \tag{7.13}$$

contains an invariant subspace which cannot be subtracted. This means that the tensor is reducible but indecomposable !!

Studying the weight diagram of (7.10) in more detail, one finds the IR of SU(1/2) :

$$\tag{7.14}$$

The second term corresponds to the trace. All these IR are atypical. They are connected together by odd generators of SU(1/2), some of them being lowering, other raising. This can be schematized as follows:

$$(7.15)$$

arrows representing odd generators. A similar example has been given by Scheunert, Nahm and Rittenberg [13].

Such a representation is said to be reducible (it contains an invariant subspace) but not decomposable. Another example is given by the supertableau of SU(2/3) :

$$(7.16)$$

whose highest weight corresponds to the Kac-Dynkin diagram

$$(7.17)$$

and hence is also reducible, although here $m + n < MN$.

On the other hand, the adjoint representation (7.1) is clearly irreducible, as well as the typical representation of SU(2/3) :

$$(7.18)$$

Also, the supertableau (7.7) corresponds to the typical representation of SU(1/3) :

$$\boxed{\cdot / /} \;=\; \overset{\prime}{\otimes}\!\!-\!\!\overset{\prime}{\circ}\!\!-\!\!\overset{0}{\circ} \tag{7.19}$$

and to the atypical IR of SU(2/4)

$$\boxed{\cdot / \cdot /} \;=\; \overset{\prime}{\circ}\!\!-\!\!\overset{0}{\otimes}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ}\!\!-\!\!\overset{0}{\circ} \tag{7.20}$$

Thus, if M,N are sufficiently large the mixed tableau is irreducible.

We can now address the following question : can every IR of SU(M/N), as given by a Kac-Dynkin diagram, be represented by a supertableau ? We have already discussed in Section 6 the problem of typical representations, where one gets naturally integer values for a_M. Allowing the over-all U(1) phase it appears that we recover arbitrary values of a_M.

For atypical representations one has to consider Eq. (5.11) for covariant tensors, (5.20) for contravariant tensors, and combine them for mixed tensors. Hence, given the Kac-Dynkin labels for a_i, one has to solve for the supertableau labels b_i and $\bar{b}_i$. Clearly, there are several solutions and in most cases there is a supertableau corresponding to a Kac-Dynkin diagram. But again we sometimes face the difficulty of reducibility when the resulting solution contains too many boxes relative to M,N. For example for SU(2/3) :

$$\overset{3}{\circ}\!\!-\!\!\overset{0}{\otimes}\!\!-\!\!\overset{2}{\circ}\!\!-\!\!\overset{0}{\circ}$$

or $\quad a_1 = 3, \quad a_2 = 0, \quad a_3 = 2 , \quad a_4 = 0 \tag{7.21}$

$$a_1 = b_1 - b_2 + \bar{b}_1' - \bar{b}_2' = 3$$

$$a_2 = b_2 + c_1' - \bar{b}_1' - \bar{c}_3 = 0$$

$$a_3 = c_1' - c_2' + \bar{c}_2 - \bar{c}_3 = 2$$

$$a_4 = c_2' - c_3' + \bar{c}_1 - \bar{c}_2 = 0$$

$$\bar{b}_i' = (\bar{b}_i - 3)\,\theta\,(\bar{b}_i - 3)$$

$$c_j' = (c_j - 2)\,\theta\,(c_j - 2)$$

(7.22)

The solution with the minimal number of boxes is the supertableau :

(7.23)

However, this is reducible and contains not only (7.21), but also IR with lower

weights. Other solutions of (7.22) have more boxes. This means we cannot represent

$$\begin{array}{cccc} 3 & 0 & 2 & 0 \\ \text{0---0---0---0} \end{array}$$ for SU(2/3) with an irreducible tensor.

In conclusion, for each atypical IR one can find a super Young tableau.

Sometimes, this latter is reducible and contains also IR with lower weights (ob

tained by applying odd lowering operators).

8. TENSOR PRODUCTS OF IR

Scheunert, Nahm and Rittenberg [13] have shown that the tensor product of

IR of superalgebras is not always fully reducible. This is due to the fact, men-

tioned in Section 3, that atypical representations are not always fully reducible.

The example they give is for SU(1/2) :

$$\underset{\underset{\textstyle 4}{}}{\overset{a_1 \quad\; 0}{\otimes\!-\!\circ}} \times \underset{\underset{\textstyle 4}{}}{\overset{a_1 \quad\; 0}{\otimes\!-\!\circ}} = \underset{\underset{\textstyle 4}{}}{\overset{2a_1 \quad 0}{\otimes\!-\!\circ}} + \underset{\underset{\textstyle 8}{}}{\overset{2a_1 \quad 1}{\otimes\!-\!\circ}} + \underset{\underset{\textstyle 4}{}}{\overset{2a_1-1 \quad 0}{\otimes\!-\!\circ}} \qquad (8.1)$$

$(a_1 \neq 0,1)$ (8.1) has been obtained by explicitly constructing all sixteen states.

There is a problem for $a_1 = \frac{1}{2}$, because the representations (1,0) and (0,0) are atypical, that is of dimension 3, resp. 1. This means that they hide, in a non reduced form, representations of dimension 1, resp. 3. It can be shown that the complete reduction is not possible.

Another example was shown in Section 7. Keeping this in mind, we can still try to learn from the rules of tensor products for classical Lie algebras, especially SU(M).

There are two main methods : Dynkin diagrams and Young tableaux. If the two IR to be multiplied have highest weights Λ_1 and Λ_2, the decomposition of the product contains the maximal highest weight

$$\Lambda_{max} = \Lambda_1 + \Lambda_2 \qquad (8.2)$$

The next to the maximal Λ is obtained by the method of minimal chain (Ref. (14)). By definition a sequence of simple roots $\alpha_{i_1}, \alpha_{i_2} \ldots \alpha_{i_k}$ is a minimal chain linking Λ_1 and Λ_2 if the following two requirements are fulfilled :

1) $(\Lambda_1, \alpha_{i_1}) \neq 0$, $(\alpha_{i_1}, \alpha_{i_2}) \neq 0, \ldots, (\alpha_{i_k}, \Lambda_2) \neq 0$ and

2) no simple root can be removed from the sequence without violating (1). One now gets the highest weight of an IR contained in the decomposition of the product by subtracting the minimal chain from Λ_{max} :

$$\Lambda = \Lambda_1 + \Lambda_2 - \sum_{j=i_1}^{i_K} \alpha_j \tag{8.3}$$

(8.2) obviously gives the right result for the product (8.1) :

$$\Lambda_{max} = (a_1, 0) + (a_1, 0) = (2a_1, 0) \tag{8.4}$$

Caution is needed to apply (8.3), since $(\alpha_M, \alpha_M) = 0$. So we modify the definition : α_M can be subtracted from a weight, that is belong to the minimal chain, if that weight has a non-zero M^{th} component and was not obtained itself by subtracting α_M.

For $SU(1/2)$, $\alpha_M = \alpha_1 = (0, -1)$, and

$$\Lambda = \Lambda_{max} - \alpha_1 = (2a_1, 1) \tag{8.5}$$

which is the second term in (8.1). The third term is obtained by orthogonality.

Similarly, one may try to apply Young tableau techniques. Again, some changes are necessary. We have shown that supertableaux correspond to integer values of a_M, for atypical representations, as they should, but also for typical representations (modulo the phase, which restores arbitrary values of a_M). Actually, the product does not depend on the value a_M, as long as one stays with typical representations. Thus in the example (8.1) :

$$\tag{8.6}$$

$$\tag{8.7}$$

which, using (5.11), exactly agress with (8.1) for $a_1 = 2$.

For pure covariant or pure contravariant tensors, we have found no example where the usual rules for Young tableaux do not apply.

For mixed tensors, the supertableau give again the correct result if M,N are sufficiently large. But for M,N small compared to the number of boxes the situation is complicated, since one encounters reducible representations. Still, the rules are useful. For example, for SU(1/2)

$$\boxed{/} \quad \times \quad \boxed{\cdot\!\!/\cdot} \quad = \quad \boxed{\cdot\!/\cdot\!/} \tag{8.8}$$

The left-hand side has dimension 3×4. From (7.10), we see that the right-hand side has also dimension 12. Working with Dynkin diagrams we see that the l.h.s. is

$$\overset{\overset{1}{\otimes}\!-\!\overset{0}{\bigcirc}}{} \quad \times \quad \overset{\overset{-1}{\bullet}\!-\!\overset{0}{\bigcirc}}{} \tag{8.9}$$

Using the above rule, we see that it contains an IR with highest weight $\Lambda_1 + \Lambda_2 = (0,0)$, and using the minimal chain, one with $\Lambda = \Lambda_1 + \Lambda_2 - \alpha_1 = (0,1)$. Thus we indeed get two IR contained in (7.14).

The conclusion is that the usual rules seem to work, provided one decomposes the reducible representations, as we have shown above !

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