Kac–Moody superalgebras and integrability

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Summary. The first part of this paper is a review and systematization of known results on (infinite-dimensional) contragredient Lie superalgebras and their representations. In the second part, we obtain character formulae for integrable highest weight representations of $\mathrm{sl}\,(1|n)$ and $\mathrm{osp}\,(2|2n)$; these formulae were conjectured by Kac–Wakimoto.

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1 Introduction

The principal goal of this paper is to study a special class of Lie superalgebras which, in our opinion, plays the same role in the theory of Lie superalgebras as the Kac–Moody Lie algebras play in the theory of Lie algebras. Since the terminology is not completely uniform even in the case of Lie algebras, we start with brief discussion of this case.

Given an arbitrary matrix A, one can define a *contragredient Lie algebra* $\mathfrak{g}\,(A)$ by generators and relations (see [2] or Section 2 for precise definition). People are usually interested in the subclass of so called *Kac–Moody Lie algebras*; by definition those are contragredient Lie algebras whose matrices satisfy the conditions: $a_{ii} = 2$, $a_{ij} \in \mathbb{Z}_{\leq 0}$, and $a_{ij} = 0$ implies $a_{ji} = 0$. The main property which distinguishes Kac–Moody Lie algebras among all contragredient Lie algebras is the local nilpotency of generators in the adjoint representation. That allows one to define the Weyl group and the notion of an integrable representation whose character has a large group of symmetries. Finite-dimensional semisimple Lie algebras and affine Lie algebras can be characterized as the only Kac–Moody superalgebras which are of polynomial growth.

The first indication of usefulness of the notion of a *contragredient Lie superalgebra* appeared in [1]: a significant part of the list of finite-dimensional simple superalgebras is contragredient. Many properties of contragredient Lie algebras are shared by contragredient Lie superalgebras; one notable exception is the uniqueness of representation: contragredient Lie superalgebras $\mathfrak{g}(A)$ and $\mathfrak{g}(B)$ might be isomorphic even if A is essentially different from B.

Our understanding of contragredient Lie superalgebras advanced a lot in the last three decades. In [10] Kac classified finite growth contragredient superalgebras whose Cartan matrices do not have zeros on the diagonal. Superalgebras of this class have the Weyl group and nice character formulas. But this class does not cover all finite-dimensional simple contragredient Lie superalgebras, and that makes one to look for further generalization.

Van de Leur in [8] studied a class of contragredient superalgebras with symmetrizable matrices. The condition that A is symmetrizable is equivalent to existence of an even invariant symmetric form on the superalgebra $\mathfrak{g}(A)$. He classified such superalgebras of finite growth; as in the usual case, those superalgebras are either finite-dimensional, or central extensions of (twisted) loop superalgebras (the latter are called affine superalgebras).

Representation theory of affine superalgebras has interesting applications in physics and number theory (see [11] and [12]). (These applications are based on conjectural character formulae for highest-weight representations; we prove these formulae for algebras $\mathrm{sl}(1|n)^{(1)}$ and $\mathrm{osp}(2|n)^{(1)}$.)

In this paper we discuss some aspects of structure theory and representations of contragredient Lie superalgebras. In particular, we suggest a definition of a *Kac–Moody Lie superalgebra* as a contragredient superalgebra with locally nilpotent ad_{X_i} and ad_{Y_i} for any choice of a set of contragredient generators X_i, Y_i, h_i. The observation that all finite-growth contragredient Lie superalgebras are in this class led to a classification of finite-growth contragredient Lie superalgebras ([9]); this gives an evidence that such definition is reasonable. Defined in this way, Kac–Moody superalgebras have a very interesting replacement of the notion of Weyl group; it is not a group but a groupoid which acts transitively on the set of all Cartan matrices describing the given algebra. (This groupoid was the tool used in [9] to classify contragredient superalgebras of finite-growth; also see Section 7.) [9] also incorporates the classification of Kac–Moody superalgebras with indecomposable Cartan matrices satisfying certain nondegeneracy conditions; this classification was obtained by C. Hoyt in her thesis [3].

In the first part of the paper we review the classification results mentioned above, as well as develop the structure theory similar to [2]. The main new result of this part is Theorem 4.14, which provides an algorithm listing all Cartan matrices defining the same Kac–Moody superalgebra. This result is parallel to the transitivity of the Weyl group action on the set of bases (see [2] Proposition 5.9).

In Section 2 we define contragredient Lie superalgebras, introduce the notions of admissibility and type, and apply restrictions of quasisimplicity and regularity. The purpose of the restriction of quasisimplicity is, essentially, to simplify the discussion; on one hand, the regularity is a key restriction required by methods of this paper; on the other hand, in the examples we know, similar results seems to hold without this restriction. In Section 3 odd reflections and Weyl groupoid for a contragredient Lie superalgebra $\mathfrak{g}$ are defined. We use odd reflections to construct a "large contragredient Lie subalgebra" $\mathfrak{g}(B)$ of $\mathfrak{g}$ and to investigate the span of roots of $\mathfrak{g}(B)$, as well as the cone spanned by positive roots of $\mathfrak{g}(B)$. In Section 4 we introduce notions of Kac–Moody superalgebra, Weyl group, real and imaginary roots, and discuss some geometric properties of roots of Kac–Moody superalgebras. Next, we provide useful examples of quasisimple Kac–Moody superalgebras: in Section 5, of rank 2, and in Section 6, what would turn out to be all quasisimple regular Kac–Moody Lie superalgebras. In Section 7 we collect together all known classification results for contragredient Lie superalgebras. Section 8 contains amplifications of several results from Sections 3 and 4. In Section 9 we revisit the examples and provide a more detailed description of their structure.

In the rest of the paper we use the Weyl groupoid to study highest weight representations of Kac–Moody superalgebras. The main results there are Corollary 14.5 and Theorem 14.7, where we prove Kac–Wakimoto conjecture of [11, 12] for completely integrable highest weight representations.

In Section 10 we define highest weight modules and integrable modules, and classify integrable highest weight modules over regular quasisimple Kac–Moody superalgebras. Section 11 collects results about highest weight representations which are true for any quasisimple regular Kac–Moody superalgebra. In section 12 we study in detail highest weight representations for the Lie superalgebras $\mathrm{sl}(1|n)^{(1)}$, $\mathrm{osp}(2|2n)^{(1)}$ and $S(1,2,b)$. Those are the only infinite-dimensional Kac–Moody superalgebras of finite growth which have non-trivial integrable highest weight representations and a non-trivial Weyl groupoid. The invariant symmetric form starts playing its role in Section 13; this excludes $S(1,2,b)$ from the discussion; here we obtain "the non-obvious" results on geometry of odd reflections. Finally, in Section 14 the collected information allows us to obtain character formulae for all highest weight integrable representations over $\mathrm{sl}(1|n)^{(1)}$ and $\mathrm{osp}(2|2n)^{(1)}$.

The first part of the paper is mostly a review of results obtained somewhere else, we skip most technical proofs and refer the reader to the original papers. In the second part all proofs are written down as most of results there are new.

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2 Basic definitions

Our ground field is $\mathbb{C}$. By $p(x)$ we denote the parity of an element x in a vector superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$.

Let I be a finite set of indices, $p : I \to \mathbb{Z}_2$, and $A = (a_{ij})\,, i, j \in I$ be a matrix. Fix an even vector space $\mathfrak{h}$ of dimension $2|I| - \operatorname{rk}(A)$. It was shown in [2] that one can choose linearly independent $\alpha_i \in \mathfrak{h}^*$ and $h_i \in \mathfrak{h}$ such that $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$, and this choice is unique up to a linear transformation of $\mathfrak{h}$. Define a superalgebra $\bar{\mathfrak{g}}(A)$ by generators $X_i, Y_i,\ i \in I$ and $\mathfrak{h}$ with relations

$$[h, X_i] = \alpha_i(h)\, X_i, \quad [h, Y_i] = -\alpha_i(h)\, Y_i, \quad [X_i, Y_j] = \delta_{ij} h_i, \qquad [\mathfrak{h}, \mathfrak{h}] = 0. \tag{1}$$

Here we assume that $h \in \mathfrak{h}$, $p(X_i) = p(Y_i) = p(i)$.

By $\mathfrak{g}(A)$ (or $\mathfrak{g}$ when the Cartan matrix is fixed) denote the quotient of $\bar{\mathfrak{g}}(A)$ by the (unique) maximal ideal which intersects $\mathfrak{h}$ trivially. It is clear that if $B = DA$ for some invertible diagonal D, then $\mathfrak{g}(B) \cong \mathfrak{g}(A)$. Indeed, an isomorphism can be obtained by mapping h_i and X_i to $d_{ii}h_i$ and $d_{ii}X_i$ respectively. Therefore, without loss of generality, we may assume that $a_{ii} = 2$ or 0. We call such matrices *normalized*.

The action of $\mathfrak{h}$ on the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ is diagonalizable and defines a *root* decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha, \qquad \Delta \subset \mathfrak{h}^*;$$

by linear independence of α_i, every root space $\mathfrak{g}_\alpha$ is either purely even or purely odd. Therefore one can define $p : \Delta \to \mathbb{Z}_2$ by putting $p(\alpha) = 0$ or 1 whenever $\mathfrak{g}_\alpha$ is even or odd respectively. By Δ_0 and Δ_1 we denote the set of even and odd roots respectively. The relations (1) imply that every root is a purely positive or purely negative integer linear combination $\sum_{i \in I} m_i \alpha_i$. According to this, we call a root *positive* or *negative*, and have the decomposition $\Delta = \Delta^+ \cup \Delta^-$. The *triangular decomposition* is by definition the decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$, where $\mathfrak{n}^\pm$ are subalgebras generated by $X_i, i \in I$, respectively $Y_i, i \in I$.

The roots $\alpha_i, i \in I$, are called *simple roots*. Sometimes instead of a_{ij} we write $a_{\alpha\beta}$ with $\alpha = \alpha_i$, $\beta = \alpha_j$; likewise for $X_\alpha, Y_\alpha, h_\alpha$ instead of X_i, Y_i, h_i.

Remark 2.1. Obviously, given a simple root $\alpha = \alpha_i$, one of the following possibilities holds:

1. if $a_{\alpha\alpha} = 0$ and $p(\alpha) = 0$, then X_α, Y_α and h_α generate a Heisenberg subalgebra;
2. if $a_{\alpha\alpha} = 0$ and $p(\alpha) = 1$, then $[X_\alpha, X_\alpha] = [Y_\alpha, Y_\alpha] = 0$ and X_α, Y_α and h_α generate a subalgebra isomorphic to $\mathrm{sl}(1|1)$; in this case $X_\alpha^2 = Y_\alpha^2 = 0$ in any representation of $\mathfrak{g}(A)$;
3. if $a_{\alpha\alpha} = 2$ and $p(\alpha) = 0$, then X_α, Y_α and h_α generate a subalgebra isomorphic to $\mathrm{sl}(2)$;
4. if $a_{\alpha\alpha} = 2$ and $p(\alpha) = 1$, then $[X_\alpha, X_\alpha] = [Y_\alpha, Y_\alpha] \neq 0$, and X_α, Y_α and h_α generate a $(3|2)$-dimensional subalgebra isomorphic to $\mathrm{osp}(1|2)$; in this case $2\alpha \in \Delta$.

Consider a simple root α. We call α *isotropic* iff $a_{\alpha\alpha} = 0$, otherwise α is called *non-isotropic*; α is called *regular* if for any other simple root β, $a_{\alpha\beta} = 0$ implies $a_{\beta\alpha} = 0$, otherwise a simple root is called *singular*.

A superalgebra $\mathfrak{g} = \mathfrak{g}(A)$ has a natural $\mathbb{Z}$-grading with 0-component being $\mathfrak{h}$, 1-component being $\bigoplus_{i\in I} \mathfrak{g}_{\alpha_i}$. We say that $\mathfrak{g}$ is of *finite growth* if the dimension of n-component grows not faster than a polynomial in n.

We say that A is *admissible* if ad_{X_i} is locally nilpotent in $\mathfrak{g}(A)$ for all $i \in I$. In this case ad_{Y_i} is also locally nilpotent. One can check (see, for instance, [9]) that A is admissible if and only if, after normalization, A satisfies the following conditions:

1. If $a_{ii} = 0$ and $p(i) = 0$, then $a_{ij} = 0$ for any $j \in I$;
2. If $a_{ii} = 2$ and $p(i) = 0$, then $a_{ij} \in \mathbb{Z}_{\leq 0}$ for any $j \in I$, $i \neq j$ and $a_{ij} = 0 \Rightarrow a_{ji} = 0$;
3. If $a_{ii} = 2$ and $p(i) = 1$, then $a_{ij} \in 2\mathbb{Z}_{\leq 0}$ for any $j \in I$, $i \neq j$ and $a_{ij} = 0 \Rightarrow a_{ji} = 0$;

Note that if $a_{ii} = 0$ and $p(i) = 1$, there is no condition on the entries a_{ij}.

We call $\mathfrak{g}(A)$ *quasisimple* if for any ideal $\mathfrak{i} \subset \mathfrak{g}(A)$ either $\mathfrak{i} \subset \mathfrak{h}$, or $\mathfrak{i} + \mathfrak{h} = \mathfrak{g}(A)$. This is equivalent to saying that every ideal of $\mathfrak{g}(A)$ is either in the center of $\mathfrak{g}(A)$, or contains the commutator $[\mathfrak{g}(A), \mathfrak{g}(A)]$. Recall that for usual Kac–Moody Lie algebras, $\mathfrak{g}(A)$ is quasisimple iff A is indecomposable; $\mathfrak{g}(A)$ is simple if, in addition, A is non-degenerate. For finite-dimensional Lie algebras quasisimplicity is equivalent to simplicity, but in general this is not true. For example, affine Lie algebras are quasisimple but not simple. In supercase even finite-dimensional contragredient superalgebras can be quasisimple but not simple (for example $\mathrm{gl}(n|n)$, see Section 6).

The following theorem is proven in [9].

Theorem 2.2. *(Hoyt, Serganova) If $\mathfrak{g}(A)$ is quasisimple and has finite growth, then A is admissible.*

One can describe an arbitrary contragredient Lie superalgebra as a sequence of extensions of quasisimple contragredient superalgebras and Heisenberg superalgebras 9(see [9] for details).

Lemma 2.3. *If $\mathfrak{g}(A)$ is quasisimple, then A is indecomposable and does not have zero rows. In particular, for admissible A, $a_{ii} = 0$ implies $p(i) = 1$, so all simple isotropic roots are odd.*

Proof. If after suitable permutation of indices $A = B \oplus C$ is the sum of two non-trivial blocks, then $\mathfrak{g}(A)$ is a direct sum of ideals $\mathfrak{g}(B) \oplus \mathfrak{g}(C)$. Hence A must be indecomposable. If A has a zero row, i.e. $a_{ij} = 0$ for some fixed $i \in I$ and all $j \in I$, then it is not hard to see that $[X_i, \mathfrak{b}] + \mathbb{C}h_i$ is an ideal in $\mathfrak{g}(A)$, here $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Hence A can not have zero rows.

We call A *regular* if for any $i, j \in I$, $a_{ij} = 0$ implies $a_{ji} = 0$.

The following is a straightforward generalization of Proposition 1.7 in [2] to the supercase.

Lemma 2.4. *Let A be regular. Then $\mathfrak{g}(A)$ is quasisimple iff A is indecomposable and not zero.*

Denote by Q the lattice generated by Δ, and by Q_0 the lattice generated by even roots Δ_0.

Theorem 2.5. *If $\mathfrak{g}(A)$ is quasisimple and $Q_0 \neq Q$, then either $Q/Q_0 \cong \mathbb{Z}$ or $Q/Q_0 \cong \mathbb{Z}_2$. In the first case, $\mathfrak{g}(A)$ has a $\mathbb{Z}$-grading $\mathfrak{g}(A) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{g}_0$ is the even part of $\mathfrak{g}(A)$.*

Proof. Consider the Q/Q_0 grading $\mathfrak{g}(A) = \bigoplus_{i \in Q/Q_0} \mathfrak{g}_i$, where $\mathfrak{g}_i = \bigoplus_{s(\alpha)=i} \mathfrak{g}_\alpha$, $s : Q \to Q/Q_0$ being the natural projection. Obviously, $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$, and $\mathfrak{g}_i \subset \mathfrak{g}_{\bar{1}}$ for any $i \neq 0$. Choose $i \in Q/Q_0$, $i \neq 0$. Let $\mathfrak{l} = \mathfrak{g}_i \oplus [\mathfrak{g}_i, \mathfrak{g}_{-i}] \oplus \mathfrak{g}_{-i}$. We claim that $\mathfrak{l}$ is an ideal of $\mathfrak{g}$. Indeed, $[\mathfrak{g}_i, \mathfrak{g}_i] = [\mathfrak{g}_{-i}, \mathfrak{g}_{-i}]$ vanish if $2i \neq 0$, and coincide with $[\mathfrak{g}_i, \mathfrak{g}_{-i}]$ if $2i = 0$. For $j \notin \{-i, 0, i\}$, $[\mathfrak{g}_j, \mathfrak{g}_{\pm i}] = 0$, thus $[\mathfrak{g}_j, [\mathfrak{g}_i, \mathfrak{g}_{-i}]] = 0$ by Jacobi identity; $[\mathfrak{g}_0, [\mathfrak{g}_i, \mathfrak{g}_{-i}]] \subset [\mathfrak{g}_i, \mathfrak{g}_{-i}]$, again by Jacobi identity. Now quasisimplicity of $\mathfrak{g}(A)$ implies $\mathfrak{l} + \mathfrak{h} = \mathfrak{g}(A)$. Therefore $\mathfrak{g}(A) = \mathfrak{g}_i \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-i}$; hence i is a generator of Q/Q_0.

If $2i = 0$, then $Q/Q_0 \cong \mathbb{Z}_2$. If $2i \neq 0$, then $\mathfrak{g}(A)$ has $\mathbb{Z}$-grading with $\mathfrak{g}_i$ being degree 1. Hence Q/Q_0 must be $\mathbb{Z}$.

Following [1], we call $\mathfrak{g}(A)$ *of type* I, if $Q/Q_0 \cong \mathbb{Z}$, and *of type* II if $Q/Q_0 \cong \mathbb{Z}_2$. In case when $\mathfrak{g}(A)$ is finite dimensional and quasisimple, $\mathfrak{g}(A)$ is of type I if it is isomorphic to $\mathrm{sl}(m|n), m \neq n$, $\mathrm{gl}(n|n)$, or $\mathrm{osp}(2|2n)$, and of type II if it is isomorphic to $\mathrm{osp}(m|2n)$ with $m \neq 2$, G_3 or F_4. For more examples, see Corollary 7.4.

Remark 2.6. It is useful to note that if $\mathfrak{g}(A)$ is of type I, then all its odd simple roots are isotropic.

3 Odd reflections, Weyl groupoid and principal roots

A linearly independent set Σ of roots of a Lie superalgebra $\mathfrak{g}(A)$ is a *base* if one can find $X_\beta \in \mathfrak{g}_\beta$ and $Y_\beta \in \mathfrak{g}_{-\beta}$ for each $\beta \in \Sigma$ such that X_β, Y_β, $\beta \in \Sigma$, and $\mathfrak{h}$ generate $\mathfrak{g}(A)$, and for any $\beta, \gamma \in \Sigma, \beta \neq \gamma$

$$[X_\beta, Y_\gamma] = 0. \tag{2}$$

If we put $h_\beta = [X_\beta, Y_\beta]$, then X_β, Y_β and h_β satisfy the relations

$$[h, X_\beta] = \beta(h) X_\beta, \qquad [h, Y_\beta] = -\beta(h) Y_\beta, \qquad [X_\beta, Y_\gamma] = \delta_{\beta\gamma} h_\beta. \tag{3}$$

This induces a natural surjective mapping $\bar{\mathfrak{g}}(A_\Sigma) \to \mathfrak{g}(A)$, here $A_\Sigma = (\beta(h_\gamma))$, $\beta, \gamma \in \Sigma$; since $\mathfrak{g}(A_\Sigma)$ and $\mathfrak{g}(A)$ have the same Cartan subalgebra $\mathfrak{h}$, the kernel must coincide with $\mathrm{Ker}\, \bar{\mathfrak{g}}(A_\Sigma) \to \mathfrak{g}(A_\Sigma)$. Therefore, $\mathfrak{g}(A)$ is isomorphic to $\mathfrak{g}(A_\Sigma)$. The matrix A_Σ is called the *Cartan matrix* of a base

Σ. The original set $\Pi = \{\alpha_i\}$ is called a *standard base*. If $\mathfrak{g}$ is a Kac–Moody Lie algebra, then every base can be obtained from the standard one by the Weyl group action (and, maybe, for infinite roots systems, by multiplication by -1). It is crucial for our discussion that this is not true for superalgebras.

Let Σ be a base, $\alpha \in \Sigma$, $a_{\alpha\alpha} = 0$ and $p(\alpha) = 1$. For any $\beta \in \Sigma$, define $r_\alpha(\beta)$ by

$$\begin{aligned} r_\alpha(\alpha) &= -\alpha, \\ r_\alpha(\beta) &= \beta \qquad \text{if } \alpha \neq \beta \text{ and } a_{\alpha\beta} = a_{\beta\alpha} = 0, \\ r_\alpha(\beta) &= \beta + \alpha \qquad \text{if } a_{\alpha\beta} \neq 0 \text{ or } a_{\beta\alpha} \neq 0. \end{aligned}$$

Lemma 3.1. *The set $r_\alpha(\Sigma) = \{r_\alpha(\beta) \mid \beta \in \Sigma\}$ is linearly independent. Moreover, if α is regular, then $r_\alpha(\Sigma)$ is a base.*

Proof. The linear independence of $r_\alpha(\Sigma)$ is obvious.

To prove the second statement, set

$$X_{-\alpha} = Y_\alpha, \qquad Y_{-\alpha} = X_\alpha, \qquad X_{r_\alpha\beta} = [X_\alpha, X_\beta], \qquad Y_{r_\alpha(\beta)} = [Y_\alpha, Y_\beta] \quad (4)$$

for any $\beta \in \Sigma$ such that $r_\alpha(\beta) = \beta + \alpha$. First, we have to check that $\left[X_{r_\alpha(\beta)}, Y_{r_\alpha(\gamma)}\right] = 0$ if $\beta, \gamma \in \Sigma$ and $\beta \neq \gamma$. If $\beta, \gamma \neq \alpha$, then $r_\alpha(\beta) - r_\alpha(\gamma) = \beta - \gamma$ or $\beta - \gamma \pm \alpha \notin \Delta$, hence $\left[X_{r_\alpha(\beta)}, Y_{r_\alpha(\gamma)}\right] = 0$. So assume that $\beta = \alpha$. A simple calculation shows that if $a_{\alpha\gamma} = a_{\gamma\alpha} = 0$, then $[Y_\alpha, Y_\gamma] = [X_\alpha, X_\gamma] = 0$. Therefore if $r_\alpha(\gamma) = \gamma$, then $[X_{-\alpha}, Y_\gamma] = [Y_\alpha, Y_\gamma] = 0$. If $r_\alpha(\gamma) = \gamma + \alpha$, then

$$\left[X_{-\alpha}, Y_{r_\alpha(\gamma)}\right] = [Y_\alpha, [Y_\alpha, Y_\gamma]] = \frac{1}{2}\left[[Y_\alpha, Y_\alpha], Y_\gamma\right] = 0,$$

since $[Y_\alpha, Y_\alpha] = 0$. Similarly, one can deal with the case $\alpha = \gamma$.

Now we have to check that $\mathfrak{h}$, $X_{r_\alpha(\beta)}$ and $Y_{r_\alpha(\beta)}$ generate $\mathfrak{g}(A)$. Note that if $r_\alpha(\beta) = \alpha + \beta$, then

$$X_\beta = \frac{1}{\beta(h_\alpha)}[Y_\alpha, [X_\alpha, X_\beta]] = \frac{1}{\beta(h_\alpha)}\left[X_{-\alpha}, X_{r_\alpha(\beta)}\right].$$

Similarly, $Y_\beta = \frac{1}{\beta(h_\alpha)}\left[Y_{-\alpha}, Y_{r_\alpha(\beta)}\right]$. Hence every generator X_β, Y_β, $\beta \in \Sigma$, can be expressed in terms of $X_{r_\alpha(\beta)}, Y_{r_\alpha(\beta)}$.

Remark 3.2. It is useful to write down the generators $h_{r_\alpha(\beta)}$ of the reflected base $r_\alpha(\Sigma)$

$$h_{r_\alpha(\beta)} = h_\beta \quad \text{if } a_{\alpha\beta} = a_{\beta\alpha} = 0, \qquad h_{r_\alpha(\beta)} = a_{\alpha\beta}h_\beta + a_{\beta\alpha}h_\alpha \quad \text{if } a_{\alpha\beta} \neq 0 \text{ or } a_{\beta\alpha} \neq 0.$$

If α is singular, then $r_\alpha(\Sigma)$ generates a subalgebra in $\mathfrak{g}(A)$ (see Section 6, examples $D(2,1;0)$, $S(1,2,\pm 1)$).

We see from Lemma 3.1 that, given a base Σ with the Cartan matrix A_Σ and a regular isotropic odd root $\alpha \in \Sigma$, one can construct another base Σ' with the Cartan matrix $A_{\Sigma'}$ such that $\mathfrak{g}(A_\Sigma) \cong \mathfrak{g}(A_{\Sigma'})$. Using (4), one can construct an isomorphism $\mathfrak{g}(A_\Sigma) \to \mathfrak{g}(A_{\Sigma'})$; denote it by the same symbol r_α. We say that $A_{\Sigma'}$ is obtained from A_Σ and a base Σ' is obtained from Σ by an *odd reflection* with respect to α. If $\Sigma' = r_\alpha(\Sigma)$ and $\Sigma'' = r_\beta(\Sigma')$ one can define a composition $r_\beta r_\alpha : \mathfrak{g}(A_\Sigma) \to \mathfrak{g}(A_{\Sigma''})$. It is clear from definition of r_α that $r_{-\alpha} r_\alpha(\Sigma) = \Sigma$.

For any base Π denote by $\Delta^\pm(\Pi)$ the set of positive (negative) roots with respect to the base Π. If $\alpha \in \Pi$, then $-\alpha \in r_\alpha(\Pi) \subset \Delta^+(r_\alpha(\Pi))$.

Lemma 3.3. *Let $\Sigma = r_\alpha(\Pi)$, where $\alpha \in \Pi$ is an odd isotropic regular root. Then $\Delta^+(\Pi) \setminus \{\alpha\} = \Delta^+(\Sigma) \setminus \{-\alpha\}$.*

Proof. Assume $\gamma \in \Delta^+(\Pi) \setminus \{\alpha\}$. Then $\gamma = \sum_{\beta\in\Pi} m_\beta \beta$ for some non-negative integers m_β, moreover, $m_\beta > 0$ for at least one $\beta \neq \alpha$. Note that $r_\alpha(\beta) = \beta + n_\beta \alpha$ for some integer n_β; hence

$$\gamma = \sum_{\beta\neq\alpha} m_\beta r_\alpha(\beta) + \left(m_\alpha - \sum_{\beta\neq\alpha} m_\beta n_\beta \right) \alpha = \sum_{\beta\neq\alpha} m_\beta r_\alpha(\beta) + \left(\sum_{\beta\neq\alpha} m_\beta n_\beta - m_\alpha \right) r_\alpha(\alpha).$$

At least one of m_β is positive, $\beta \neq \alpha$, therefore $\gamma \in \Delta^+(\Sigma)$. Hence $\Delta^+(\Pi) \setminus \{\alpha\} \subset \Delta^+(\Sigma) \setminus \{-\alpha\}$, and lemma follows by symmetry.

Definition 3.4. $\mathfrak{g}(A)$ *is called a* regular contragredient superalgebra *if any A' obtained from A by odd reflections is regular.*

Consider a category $\mathcal{C}$, which has an object $\mathfrak{g}(A)$ for every square matrix A. A $\mathcal{C}$-morphism $f : \mathfrak{g}(A) \to \mathfrak{g}(A')$ is by definition an isomorphism of superalgebras which maps a Cartan subalgebra of $\mathfrak{g}(A)$ to the Cartan subalgebra of $\mathfrak{g}(A')$. *A Weyl groupoid* $\mathcal{C}(A)$ of a contragredient superalgebra $\mathfrak{g}(A)$ is a connected component of $\mathcal{C}$ which contains $\mathfrak{g}(A)$.

Let $\mathfrak{g}(A)$ be a regular superalgebra. With each base Σ of $\mathfrak{g}(A)$ we associate a Dynkin graph Γ_Σ whose vertices are elements of Σ, and α and β are connected iff $a_{\alpha\beta} \neq 0$. If one denotes even, odd non-isotropic and odd isotropic roots by white, black and gray circles respectively, and writes $a_{\alpha\beta}$ and $a_{\beta\alpha}$ above and under the edge joining α and β, then one can reconstruct the Cartan matrix of A_Σ from Γ_Σ.

Definition 3.5. *Call an even root $\alpha \in \Delta_0$* principal *if there exists a base Π' obtained from a standard base Π by odd reflections such that α or $\alpha/2$ belongs to Π'. Let $\mathcal{B}$ denote the set of all principal roots. For each $\beta \in \mathcal{B}$ choose $X_\beta \in \mathfrak{g}_\beta$, $Y_\beta \in \mathfrak{g}_{-\beta}$, set $h_\beta = [X_\beta, Y_\beta]$, and define the matrix $B = b_{\alpha\beta} = \beta(h_\alpha)$, $\alpha, \beta \in \mathcal{B}$.*

Note that by Lemma 3.3 a positive even root remains positive for any base obtained from Π by odd reflections. Hence all principal roots are positive.

Lemma 3.6. $[X_\alpha, Y_\beta] = \delta_{\alpha\beta} h_\beta$ *for any* $\alpha, \beta \in \mathcal{B}$.

Proof. It is sufficient to check that if $\alpha \neq \beta$, then $\alpha - \beta$ is not a root. Assume that $\gamma = \alpha - \beta$ is a root. Without loss of generality, one may assume that γ is positive. Then $\alpha = \beta + \gamma$. Choose a base Π' such that α or $\alpha/2$ belongs to Π', then α can not be a sum of two positive roots.

Generally speaking, $\mathcal{B}$ may be infinite. We define a contragredient Lie algebra with infinite Cartan matrix as an inductive limit of usual contragredient Lie algebras.

Let $\mathfrak{g}'$ denote the Lie subalgebra of $\mathfrak{g}_{\bar 0}(A)$ generated by X_β, Y_β for all principal β.

Lemma 3.7. *There exist a subalgebra* $\mathfrak{c}$ *in the center of* $\mathfrak{g}(B)$ *and a homomorphism* $q\colon \mathfrak{g}' \to \mathfrak{g}(B)/\mathfrak{c}$ *which maps* X_β *to the corresponding generator of* $\mathfrak{g}(B)$. $\operatorname{Im} q = [\mathfrak{g}(B), \mathfrak{g}(B)]/\mathfrak{c}$.

Proof. Let $\bar X_\beta$, $\bar Y_\beta$ be the generators of $\bar{\mathfrak{g}}(B)$. Lemma 3.6 implies that there exists a surjective homomorphism $s\colon [\bar{\mathfrak{g}}(B), \bar{\mathfrak{g}}(B)] \to \mathfrak{g}'$ defined by $s\left(\bar X_\beta\right) = X_\beta$, $s\left(\bar Y_\beta\right) = Y_\beta$. Since all principal roots are positive, one can find h in the Cartan subalgebra of $\mathfrak{g}(A)$ such that $\beta(h) > 0$ for all $\beta \in \mathcal{B}$. Hence $\mathfrak{g}'$ has an $\mathbb{R}$-grading such that $\mathfrak{g}'_0 = \mathfrak{h} \cap \mathfrak{g}'$, $[\bar{\mathfrak{g}}(B), \bar{\mathfrak{g}}(B)]$ has the similar grading and s is a homomorphism of $\mathbb{R}$-graded Lie algebras. Then $\operatorname{Ker} s$ is a graded ideal, in particular, $\operatorname{Ker} s = \mathfrak{m}^- \oplus \mathfrak{m}^0 \oplus \mathfrak{m}^+$, where $\mathfrak{m}^0$ lies in the Cartan subalgebra of $\bar{\mathfrak{g}}(B)$, and $\mathfrak{m}^\pm$ are the positive (negative) graded components. Since simple roots are not weights of $\mathfrak{m}^+$, $\mathfrak{m}^0$ is central and $\left[\bar Y_\beta, \mathfrak{m}^+\right] \subset \mathfrak{m}^+$; likewise $\left[\bar X_\beta, \mathfrak{m}^-\right] \subset \mathfrak{m}^-$. Therefore, $\mathfrak{m}^- \oplus \mathfrak{m}^+$ generates the ideal in $\bar{\mathfrak{g}}(B)$ which intersects the Cartan subalgebra trivially. If $p\colon \bar{\mathfrak{g}}(B) \to \mathfrak{g}(B)$ denotes the natural projection, then $\mathfrak{m}^- \oplus \mathfrak{m}^+ \subset \operatorname{Ker} p$. Let $\mathfrak{c} = p\left(\mathfrak{m}^0\right)$, and p' : $\bar{\mathfrak{g}}(B)/\mathfrak{m}_0 \to \mathfrak{g}(B)/\mathfrak{c}$ be the surjection induced by p. Then $q\colon \mathfrak{g}' \to \mathfrak{g}(B)/\mathfrak{c}$ given by

$$q(X_\beta) = p'\left(s^{-1} X_\beta\right), \qquad q(Y_\beta) = p'\left(s^{-1} Y_\beta\right)$$

is well defined. It is straightforward to check that q satisfies the conditions of the lemma.

Remark 3.8. The subalgebra $\mathfrak{g}'$ is the best approximation to "the largest almost contragredient Lie subalgebra" of $\mathfrak{g}_{\bar 0}$ we can construct explicitly (compare with Lemma 3.10). In many examples q is an embedding and we suspect that is so in general. It would be nice to prove it.

For any base Σ consider a closed convex cone

$$C^+(\Sigma) = \left\{ \sum_{\alpha \in \Sigma} a_\alpha \alpha \mid a_\alpha \geq 0 \right\}.$$

Denote by C^+_Π the intersection of $C^+(\Sigma)$ for all bases Σ' obtained from Π by odd reflections. By Lemma 3.3, C^+_Π contains all positive even roots.

Remark 3.9. Note that Lemma 3.1 used only the following relation of $\mathfrak{g}(A)$ which is not a relation in $\bar{\mathfrak{g}}(A)$: if $a_{\alpha\gamma} = a_{\gamma\alpha} = 0$, then $[X_\alpha, X_\gamma] = 0$. Thus one could redo our discussion for any quotient of $\bar{\mathfrak{g}}(A)$ in which these relations hold.

Lemma 3.10. *Assume that there are only finitely many Σ obtained from Π by odd reflections. Then C^+_Π is the convex cone generated by principal roots.*

Proof. We prove the statement by induction on the number of simple roots n. Note that C^+_Π is an intersection of simplicial cones, hence a closed convex set. Let $C^+(\mathcal{B})$ be the cone generated by principal roots; obviously $C^+(\mathcal{B}) \subset C^+_\Pi$. To prove inverse inclusion it suffices to show that the boundary of C^+_Π belongs to $C^+(\mathcal{B})$. If γ is a point on the boundary of C^+_Π, then there is Σ such that γ lies on the boundary of $C^+(\Sigma)$; in other words, γ belongs to one of the faces of $C^+(\Sigma)$. Hence there exists $\alpha \in \Sigma$ such that $\gamma = \sum_{\beta \in \Sigma \setminus \{\alpha\}} m_\beta \beta$ for some non-negative m_β. Let $\mathfrak{k}$ be the contragredient subalgebra of $\mathfrak{g}$ with simple roots $\Sigma' = \Sigma \setminus \{\alpha\}$ (as proven in [9], the corresponding X_γ, Y_γ, h_γ generate a subalgebra $\mathfrak{k}$ which is contragredient; alternatively, one could use Remark 3.9, and work with $\mathfrak{k}$ directly). Since odd reflections of Σ' are induced by ones of Σ, $\gamma \in C^+_{\Sigma'}$. By inductive assumption γ is a non-negative linear combination of principal roots of $\mathfrak{k}$. Clearly, any principal root of $\mathfrak{k}$ is a principal root of $\mathfrak{g}$. Thus $\gamma \in C^+(\mathcal{B})$.

Remark 3.11. We suspect that Lemma 3.10 is true without any finiteness assumption.

Denote by Q' the sublattice of Q_0 generated by all principal roots.

Lemma 3.12. *For any regular contragredient superalgebra $\mathfrak{g}(A)$ with n simple roots and indecomposable A, the rank of Q' is at least $n-1$.*

Proof. Choose a base Π of $\mathfrak{g}(A)$, let Γ be its graph. We say that a subgraph $\alpha_1, \dots, \alpha_k$ is a *string* if it is connected, the vertices α_1 and α_k are of valence 1 and odd, and all vertices $\alpha_2, \dots, \alpha_{k-1}$ are of valence 2 and even. We claim that, for each string $\alpha_1, \dots, \alpha_k$, either $2(\alpha_1 + \dots + \alpha_s)$ is principal for some $s < k$, or $\alpha_1 + \dots + \alpha_k$ is principal. Indeed, note that either α_1 is non-isotropic, then $2\alpha_1$ is principal, or α_1 is isotropic, then $r_{\alpha_1}(\alpha_2) = \alpha_1 + \alpha_2$ is an odd root. Again either $\alpha_1 + \alpha_2$ is non-isotropic, then $2(\alpha_1 + \alpha_2)$ is principal, or $\alpha_1 + \alpha_2$ is isotropic, then $r_{\alpha_1+\alpha_2}(\alpha_3) = \alpha_1 + \alpha_2 + \alpha_3$ is an odd root. Proceeding in this manner, one gets either a principal root $2(\alpha_1 + \dots + \alpha_s)$ for some $s < k$, or a principal root $\alpha_1 + \dots + \alpha_k$. Since all even simple roots are principal, one can see that for any odd roots α and $\beta \in \Pi$ joined by a string, at least one of 2α and $\alpha + \beta$ belongs to Q'. Consider all odd roots α such that $2\alpha \notin Q'$. Since Γ is connected, the sum or difference of any two of them is in Q'. Therefore Q' generates a subspace of dimension at least $n-1$. Hence the rank of Q' is at least $n-1$.

4 Kac–Moody superalgebras

In this section we, finally, introduce the main character of our story.

Definition 4.1. $\mathfrak{g}(A)$ *is a* Kac–Moody superalgebra *if any A' obtained from A by odd reflections is admissible.*

Lemma 4.2. *Let $\mathfrak{g}(A)$ be a regular Kac–Moody superalgebra. Then $\mathfrak{g}(A)$ is a direct sum of quasisimple regular Kac–Moody superalgebras and even or odd Heisenberg superalgebras.*

Proof. Follows from Lemma 2.4.

Remark 4.3. Note that the theory of highest-weight representations of Heisenberg (super)algebras can be formulated in a very similar way to what we do in this paper. For "typical" representations, the Weyl character formula (22) holds. For "atypical" representations, the character can be easily described. Therefore, although this paper concentrates on regular quasisimple Kac–Moody superalgebras, one could easily rewrite the theory for all regular Kac–Moody superalgebras.

We start with defining the Weyl group. Let $\mathfrak{g}$ be a quasisimple Kac–Moody superalgebra. Then an even simple root is not isotropic. Therefore, for any principal root α, there exists an $\mathrm{sl}(2)$-triple $\{X_\alpha \in \mathfrak{g}_\alpha, h_\alpha \in \mathfrak{h}, Y_\alpha \in \mathfrak{g}_{-\alpha}\}$. Define an *even reflection* $r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ by

$$r_\alpha(\mu) = \mu - \mu(h_\alpha)\alpha.$$

Since the $\mathrm{sl}(2)$–subalgebra spanned by $\{X_\alpha, h_\alpha, Y_\alpha\}$ acts locally finitely on $\mathfrak{g}(A)$, r_α is dual to the restriction of the automorphism $\exp \mathrm{ad}_{X_\alpha} \exp \mathrm{ad}_{-Y_\alpha} \exp \mathrm{ad}_{X_\alpha}$ to $\mathfrak{h}$. Therefore r_α permutes the roots and maps a base to a base. If β is an odd root such that 2β is principal, then we put $r_\beta = r_\alpha$.

The group W generated by all even reflections is called the *Weyl group* of $\mathfrak{g}(A)$. The main difference between even and odd reflections is that an even reflection depends only on a root and does not depend on a base to which the root belongs. The following straightforward identity is very important:

$$wr_\alpha(\Pi) = r_{w(\alpha)}w(\Pi) \tag{5}$$

for a base Π, a regular isotropic $\alpha \in \Pi$ and $w \in W$.

Note that one can define $\exp \mathrm{ad}_{X_\alpha} \exp \mathrm{ad}_{-Y_\alpha} \exp \mathrm{ad}_{X_\alpha}$ for any even α such that ad_{X_α} and ad_{Y_α} are locally nilpotent. We do not know any example where α is not in the W-orbit of a simple root (in which case the corresponding automorphism of $\mathfrak{h}^*$ is, obviously, in W).

Lemma 4.4. *Let $\Sigma = r_\alpha(\Pi)$, where $\alpha \in \Pi$ is any root (even or odd). Then $\Delta^+(\Pi) \smallsetminus \{\alpha, 2\alpha\} = \Delta^+(\Sigma) \smallsetminus \{-\alpha, -2\alpha\}$.*

Proof. The same as for Lemma 3.3.

Corollary 4.5. *Let Δ be finite, Π and Π' be bases. Then Π' can be obtained from Π by even and odd reflections.*

Proof. Assume $\Pi \neq \Pi'$. There exists $\alpha \in \Pi \cap \Delta^-(\Pi')$. Then

$$|\Delta^+(r_\alpha(\Pi)) \cap \Delta^-(\Pi')| < |\Delta^+(\Pi) \cap \Delta^-(\Pi')|.$$

Repeating this process, one can get Π'' such that $\Delta^+(\Pi'') \cap \Delta^-(\Pi') = \varnothing$. But then $\Pi'' = \Pi$.

Recall notations from Definition 3.5.

Lemma 4.6. *If $\mathfrak{g}$ is a quasisimple Kac–Moody Lie algebra, then B is a Cartan matrix of a Kac–Moody Lie algebra.*

Proof. By quasisimplicity of $\mathfrak{g}$, $\alpha(h_\alpha) \neq 0$ for any $\alpha \in \mathcal{B}$. We normalize h_α so that $\alpha(h_\alpha) = 2$. First, we have to show that $\alpha(h_\beta) \in \mathbb{Z}_{\leq 0}$ for any $\alpha, \beta \in \mathcal{B}$, $\alpha \neq \beta$. By definition of a Kac–Moody superalgebra, the adjoint action of sl_2-subalgebra spanned by $X_\beta \in \mathfrak{g}_\beta$, $h_\beta \in \mathfrak{h}$, $Y_\beta \in \mathfrak{g}_{-\beta}$ on $\mathfrak{g}$ is locally finite, and X_α is a lowest vector, hence $\alpha(h_\beta) \in \mathbb{Z}_{\leq 0}$.

Finally, $\alpha(h_\beta) = 0$ implies $[X_\beta, X_\alpha] = 0$, therefore $\beta(h_\alpha) = 0$.

Remark 4.7. Recall that the *Serre's relations* in a Kac–Moody Lie algebra $\mathfrak{g}(B)$ are the relations

$$\left(\mathrm{ad}_{X_\beta}\right)^{1-b_{\beta\gamma}} X_\gamma = 0, \qquad \left(\mathrm{ad}_{Y_\beta}\right)^{1-b_{\beta\gamma}} Y_\gamma = 0.$$

If B is symmetrizable, then Serre's relations together with the "contragredient" relations

$$[h, X_\beta] = \beta(h) X_\beta, \qquad [h, Y_\beta] = \beta(h) Y_\beta, \qquad [X_\beta, Y_\gamma] = \delta_{\beta\gamma} h_\beta, \qquad [\mathfrak{h}, \mathfrak{h}] = 0$$

define the Kac–Moody Lie algebra $\mathfrak{g}(B)$.

Since ad_{X_β}, ad_{Y_β}, $\beta \in \mathcal{B}$, are locally nilpotent operators in $\mathfrak{g}$, Serre's relations hold in $\mathfrak{g}'$. Thus, if B is symmetrizable, the homomorphism q constructed in Lemma 3.7 is injective.

Remark 4.8. Later, in Section 9, we will see that for all quasisimple regular Kac–Moody superalgebras the set $\mathcal{B}$ of principal roots is finite. It would be interesting to obtain a proof avoiding use of the classification.

Remark 4.9. As the preceding remark shows, the group W is manifestly finitely generated. Later the fact that we restrict our attention to (a finite set of) principal roots leads to an effective criterion of integrability of a weight (see Theorem 10.5).

On the other hand, the group W is also "sufficiently large". Indeed, there are the following indications:

1. it induces the "even" Weyl group when reduced to the "even contragredient part" $\mathfrak{g}(B)$ of $\mathfrak{g}$ (Corollary 4.10);
2. when character formula for integrable infinite-dimensional highest weight irreducible representations is known for weights in general position, it coincides with Weyl character formula (Corollary 14.5).

Corollary 4.10. *Let $\mathfrak{g}$ be a quasisimple Kac–Moody superalgebra. Then the Weyl group W is isomorphic to the Weyl group of $\mathfrak{g}(B)$.*

Define real and imaginary roots following [2]. A root α is *real* iff there exists a base Π' obtained from Π by even and odd reflections such that one of α or $\alpha/2$ belongs to Π'. By definition, a principal root is real. If α is not real, then we call it *imaginary.*

Lemma 4.11. *(a) The set of real roots is W-invariant.*
(b) If α is real, then $\mathfrak{g}_\alpha$ is one-dimensional.
(c) If α is a real odd isotropic root, then $m\alpha \in \Delta$ implies $m = \pm 1$.
(d) If α is a real odd non-isotropic root, then $m\alpha \in \Delta$ implies $m = \pm 1, \pm 2$.
(e) If α is an even real root, then $m\alpha \in \Delta$ implies $m = \pm 1, \pm 1/2$.

Proof. W-invariance follows from the fact that if Π is a base, then $w(\Pi)$ is a base. The other statements easily follow from the fact that α belongs to some base Π and Remark 2.1 .

Denote by D^+_Π the intersection of $C^+(\Sigma)$ for all Σ obtained from Π by even and odd reflections. It follows from (5) that

$$D^+_\Pi = \bigcap_{w\in W} w\left(C^+_\Pi\right).$$

Corollary 4.12. *If α is an imaginary root, then α or $-\alpha$ belongs to D^+_Π.*

Proof. Assume that α is a positive root, and $\alpha \notin D^+_\Pi$. By definition $\alpha \in C^+(\Pi)$, but since $\alpha \notin D^+_\Pi$ there exists some base Π' obtained from Π by even and odd reflection such that $\alpha \notin C^+(\Pi')$. Write $\Pi' = r_1 \dots r_s(\Pi)$, and choose i such that $\alpha \in C^+(r_{i+1}\dots r_s(\Pi))$ and $\alpha \notin C^+(r_i \dots r_s(\Pi))$. Then Lemma 4.4 implies $r_i = r_\alpha$, hence α or $\alpha/2 \in r_{i+1}\dots r_s(\Pi)$. Thus α is real. Similarly for α being a negative root.

Corollary 4.13. *If Δ is finite, then every root is real; therefore $\mathfrak{g}$ is finite-dimensional.*

Proof. By Corollary 4.5, every base can be obtained from a standard base Π by even and odd reflections. In particular, $-\Pi$ can be obtained from Π in this way. Therefore $D^+_\Pi = \{0\}$. Now the statement follows from Corollary 4.12.

Theorem 4.14. *Let $\mathfrak{g}(A)$ and $\mathfrak{g}(A')$ be regular quasisimple Kac–Moody superalgebras which belong to the same connected component of $\mathcal{C}$. Then A' is obtained from A by a composition of odd reflections, permutation of indices, and multiplication by an invertible diagonal matrix.*

This theorem will be proven in Section 8. (See Remark 8.4.)

5 Regular Kac–Moody superalgebras with two simple roots

It is easy to classify rank 2 regular quasisimple Kac–Moody superalgebras. We assume that a superalgebra $\mathfrak{g}(A)$ has at least one simple isotropic root; otherwise, by definition, there is no restrictions on a matrix A (except the admissibility conditions).

Theorem 5.1. *([9]) Let $\mathfrak{g}(A)$ be a regular quasisimple Kac–Moody superalgebra with 2 simple roots such that at least one root is isotropic. Then $\mathfrak{g}$ is isomorphic to one of the following*

1. $\mathfrak{g} \cong \mathrm{sl}(1|2)$, $A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ *with* $p(1) = 0$, *or* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;
2. $\mathfrak{g} \cong \mathrm{osp}(3|2)$, $A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$ *with* $p(1) = 0$, *or* $p(1) = 1$.

Proof. If $\mathfrak{g}$ has a base with two isotropic roots, then, without loss of generality, one may assume that $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (since multiplication by a diagonal matrix does not change the algebra). The reflection with respect to α_2 gives $\alpha_1' = \alpha_1 + \alpha_2$, $\alpha_2' = -\alpha_2$, $h_1' = h_1 + h_2$, $h_2' = h_2$ (see Remark 3.2). Hence $A' = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$.

Now assume that $\mathfrak{g}$ has a base with one non-isotropic odd root α_1 and one isotropic odd root α_2. One may assume without loss of generality that $A = \begin{pmatrix} 2 & 2b \\ 1 & 0 \end{pmatrix}$ for some negative integer b. Again by Remark 3.2, the reflection r_{α_2} gives $\alpha_1' = \alpha_1 + \alpha_2$, $\alpha_2' = -\alpha_2$, $h_1' = \frac{1}{1+2b} h_1 + \frac{2b}{1+2b} h_2$ (after normalization), $h_2' = h_2$. Since $\alpha_2'(h_1') \in \mathbb{Z}_{\leq 0}$, we obtain $\frac{-2b}{1+2b} \in \mathbb{Z}_{\leq 0}$, which is possible only if $b = -1$.

Finally assume that $\mathfrak{g}$ has a base with one even root and one isotropic root. In this case the odd reflection will move the base to one of the above cases. Hence theorem is proven.

The following hereditary principle is obvious but very important. If A is a Cartan matrix of some regular Kac–Moody superalgebra, then any main minor of A is also a Cartan matrix of some regular Kac–Moody superalgebra.

Corollary 5.2. *Let A be a Cartan matrix of a quasisimple regular Kac–Moody superalgebra. If α is an even simple root and β is an isotropic root, then $a_{\alpha\beta} = 0, -1$ or -2. If α is an odd non-isotropic root and β is an isotropic root, then $a_{\alpha\beta} = 0$ or -2.*

6 Examples of regular quasisimple superalgebras

Finite-dimensional superalgebras. Finite-dimensional quasisimple Kac–Moody superalgebras were classified in [1]. In most cases, they have non-degenerate

Cartan matrices. However the superalgebra $\mathrm{gl}\,(n|n)$ is quasisimple, but not simple, and its Cartan matrix has corank 1. (To get the simple superalgebra $\mathrm{psl}\,(n|n)$ one should factor the commutator $\mathrm{sl}\,(n|n) = [\mathrm{gl}\,(n|n)\,, \mathrm{gl}\,(n|n)]$ by the center consisting of scalar matrices). For example, the superalgebra $\mathrm{gl}\,(2|2)$ has the following Cartan matrices

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

Another unusual example in this class is a family $D\,(2,1;a)$ depending on the parameter a. If we start with the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 0 & a \\ 0 & -1 & 2 \end{pmatrix}$$

with $a \neq 0, -1$, then reflection r_{α_2} transforms it to the matrix

$$\begin{pmatrix} 0 & 1 & -1-a \\ 1 & 0 & a \\ -1-\frac{1}{a} & 1 & 0 \end{pmatrix},$$

and the odd reflections of the latter matrix give the matrices of $D\,(2,1;-1-a)$ and $D\left(2,1;-1-\frac{1}{a}\right)$ (after suitable permutations of indices). Thus, the group S_3 generated by $a \to \frac{1}{a}$, and $a \to -1-a$ acts on the space of parameters, and the points on the same orbit of this action correspond to isomorphic superalgebras. Hence one can describe the moduli space of such algebras as $\mathbb{CP}^1 \setminus \{0,-1,\infty\}$ modulo the above S_3-action. The cases $a = 0, -1, \infty$ correspond to a non-regular Kac–Moody superalgebra. For instance, if $a = 0$, then the singular odd reflection r_{α_2} maps the generators into an ideal isomorphic to $\mathrm{psl}\,(2|2)$. It is not hard to show that $D\,(2,1;0)$ is isomorphic to the algebra of all derivations of $\mathrm{psl}\,(2|2)$, and $D\,(2,1;0)/\,\mathrm{psl}\,(2|2) \cong \mathrm{sl}\,(2)$. Another 3-element S_3-orbit $\left\{a = 1, -2, -\frac{1}{2}\right\}$ corresponds to the algebra $\mathrm{osp}\,(4|2)$.

All Cartan matrices of regular finite-dimensional quasisimple Kac–Moody superalgebras are symmetrizable, hence they have non-degenerate invariant bilinear symmetric forms. Furthermore, Δ is a finite set, hence any root is real. Principal roots are the simple roots of $\mathfrak{g}' = [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$.

Affine Kac–Moody superalgebras. Let $\mathfrak{s}$ be a finite-dimensional simple Lie superalgebra from the previous class (so $\mathfrak{s} \neq \mathrm{gl}\,(n|n)$), $(\cdot,\cdot)$ be a non-degenerate invariant symmetric form on $\mathfrak{s}$. Define an infinite dimensional superalgebra $\mathfrak{s}^{(1)}$ as

$$\mathfrak{s} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}D \oplus \mathbb{C}K,$$

here D, K are even elements and the bracket is defined by

$$\begin{gathered} \left[X \otimes t^k, Y \otimes t^l\right] = [X,Y] \otimes t^{k+l} + k\delta_{k,-l}\,(X,Y)\,K, \\ [D,K] = 0, \qquad \left[D, X \otimes t^k\right] = kX \otimes t^k. \end{gathered} \tag{6}$$

It is not difficult to show (as in the Lie algebra case) that $\mathfrak{s}^{(1)}$ is a regular quasisimple Kac–Moody superalgebra. To construct the set of generators choose some generators $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ of $\mathfrak{s}$, and add $X_0 = x \otimes t$, $Y_0 = y \otimes t^{-1}$, where x is a lowest weight vector, and y is a highest weight vector in the adjoint module.

Quasisimplicity follows from simplicity of $\mathfrak{s}$. Regularity follows from existence of an invariant symmetric form defined by

$$\begin{gathered} \left(X \otimes t^k, Y \otimes t^l\right) = (X, Y)\, \delta_{k,-l}, \qquad (D, K) = 1, \\ \left(K, X \otimes t^k\right) = \left(D, X \otimes t^k\right) = (D, D) = (K, K) = 0. \end{gathered} \tag{7}$$

Indeed, existence of the form implies that every Cartan matrix is symmetrizable, therefore there is no singular roots.

Now let us define the affinization of $\mathrm{gl}\,(n|n)$. Let $\mathfrak{s} = \mathrm{psl}\,(n|n)$ be the quotient of $\mathrm{sl}\,(n|n)$ by the center, for simplicity of $\mathfrak{s}$ we need $n \geq 2$. Then $\mathfrak{s}^{(1)}$ is defined as

$$\mathfrak{s} \otimes \mathbb{C}\left[t, t^{-1}\right] \oplus \mathbb{C}D \oplus \mathbb{C}K \oplus \mathbb{C}D' \oplus \mathbb{C}K',$$

the bracket with additional even elements D' and K' is given by

$$\begin{gathered} \left[D', X \otimes t^k\right] = \left(1 - (-1)^{p(X)}\right) X \otimes t^k, \qquad \left[K', X \otimes t^k\right] = 0, \\ \left[X \otimes t^k, Y \otimes t^l\right] = [X, Y] \otimes t^{k+l} + k\delta_{k,-l}\,(X, Y)\,K + \delta_{k,-l}\,\mathrm{tr}\,[X, Y]\,K', \end{gathered}$$

here tr is the usual trace (not the supertrace), which is not zero only if both X and Y are odd. The Cartan matrix of affine superalgebra $\mathrm{psl}\,(n|n)^{(1)}$ has corank 2, therefore $\mathrm{psl}\,(n|n)^{(1)}$ is a regular quasisimple Kac–Moody superalgebra with two dimensional center. For example, one of Cartan matrices of $\mathrm{psl}\,(2|2)^{(1)}$ is

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

What follows is applicable for any finite-dimensional simple $\mathfrak{s}$. To describe the roots of $\mathfrak{s}^{(1)}$, define $\delta \in \mathfrak{h}^*$ by conditions $\delta\,(h_i) = 0$, $\delta\,(D) = 1$. Denote by Δ° the roots of $\mathfrak{s}$. Then the roots of $\mathfrak{g}$ are of the form $\alpha + k\delta$ with $k \in \mathbb{Z}$, $\alpha \in \Delta^\circ$, or $k\delta$ with $k \in \mathbb{Z} \setminus \{0\}$. The standard base $\Pi = \{\alpha_0, \dots, \alpha_n\}$ is obtained from the standard base $\{\alpha_1, \dots, \alpha_n\}$ of $\mathfrak{s}$ by adding the root $\alpha_0 = \theta + \delta$, where θ is the lowest root of $\mathfrak{s}$.

Twisted affine superalgebras. Choose an automorphism ϕ of $\mathfrak{s}$ of finite order p, which preserves the invariant form on $\mathfrak{s}$. Let ε be a p-th primitive root of 1. One can extend the action of ϕ on $\mathfrak{s}^{(1)}$ by

$$\phi\left(X \otimes t^k\right) = \varepsilon^k \phi\,(X) \otimes t^k, \quad \phi\,(D) = D, \quad \phi\,(K) = K.$$

Then $\mathfrak{s}^\phi$ is defined as the set of elements fixed by ϕ. It was proven in [5] that the construction does not depend on ε, and if two automorphisms are in the same connected component of $\operatorname{Aut}\mathfrak{s}$, then the corresponding algebras are isomorphic. Up to isomorphism, there are the following twisted affine superalgebras: $(p)\,\mathrm{sl}\,(m|n)^{(2)}$ if mn is even, $(m,n) \neq (2,2)$; $(p)\,\mathrm{sl}\,(m|n)^{(4)}$ if mn is odd; $\mathrm{osp}\,(2m|2n)^{(2)}$; the upper index denotes the minimal possible order of an automorphism. In the case of twisting of $\mathrm{psl}\,(n|n)$ the corank of a Cartan matrix is 1, and the center is one-dimensional. For example,

$$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$$

with parity functions (0,1,0) or (1,1,1) is a Cartan matrix of $\mathrm{psl}\,(3|3)^{(4)}$. All twisted affine superalgebras are regular quasisimple Kac–Moody superalgebras.

A base in a twisted affine superalgebra can be obtained by the following procedure. Define a $\mathbb{Z}_p$-grading $\mathfrak{s} = \mathfrak{s}^0 \oplus \mathfrak{s}^1 \oplus \cdots \oplus \mathfrak{s}^{p-1}$ by

$$\mathfrak{s}^k = \left\{ s \in \mathfrak{s} \mid \phi\,(s) = \varepsilon^k s \right\}.$$

In all cases ϕ can be chosen so that $\mathfrak{s}^0$ is a simple finite-dimensional superalgebra, $\mathfrak{s}^1$ is an irreducible $\mathfrak{s}^0$-module and $\mathfrak{s}^0 + \mathfrak{s}^1$ generates $\mathfrak{s}$. For example, for $(p)\,\mathrm{sl}\,(m|2n)^{(2)}$, $\mathfrak{s}^0 = \mathrm{osp}\,(m|2n)$, $\mathfrak{s}^1$ is a unique non-trivial irreducible subquotient of the symmetric square of the standard $(m|2n)$-dimensional $\mathfrak{s}^0$-module; for $(p)\,\mathrm{sl}\,(2m+1|2n+1)^{(4)}$, $\mathfrak{s}^0 = \mathrm{osp}\,(2m+1|2n)$, and $\mathfrak{s}^1$ is the standard module with inverted parity; for $\mathrm{osp}\,(2m|2n)^{(2)}$, $\mathfrak{s}^0 = \mathrm{osp}\,(2m-1|2n)$, $\mathfrak{s}^1$ is the standard module.

Let Δ^0 be the set of roots of $\mathfrak{s}^0$ and Δ^j denote the set of weights of $\mathfrak{s}^j$ with respect to a Cartan subalgebra of $\mathfrak{s}^0$. The roots of $\mathfrak{s}^{(p)}$ are of the form $\alpha + k\delta$, with $\alpha \in \Delta^j$, $k \in j + p\mathbb{Z}$, and $k\delta$ with $k \in p\mathbb{Z} \smallsetminus \{0\}$. The standard base $\Pi = \{\alpha_0, \ldots, \alpha_n\}$ can be obtained from a base $\{\alpha_1, \ldots, \alpha_n\}$ of $\mathfrak{s}$ by adding the weight $\alpha_0 = \theta + \delta$, where θ is the lowest weight of $\mathfrak{s}^1$.

Finally, let us mention that the condition that ϕ preserves the invariant form on $\mathfrak{s}$ is non-trivial. For example, an automorphism π of $\mathrm{psl}\,(n|n)$ such that $\pi \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$ does not preserve the invariant form on $\mathrm{psl}\,(n|n)$, and $\mathrm{psl}\,(n|n)^{\pi}$ is not contragredient. It also explains why the Lie superalgebra $\mathrm{psl}\,(2|2)$ does not have a twisted affinization. The subgroup of automorphisms of $\mathrm{psl}\,(2|2)$ preserving the invariant form is connected due to existence of non-trivial derivations, see above example for $D\,(2,1;0)$.

A strange twisted affine superalgebra. The Lie superalgebra $\mathfrak{s} = q\,(n) \subset \mathrm{psl}\,(n|n)$ is the subalgebra of all elements fixed by the automorphism π defined above. The superalgebra $q\,(n)$ is simple for $n \geq 3$. The involution ϕ such that $\phi\,(x) = (-1)^{p(x)}\,x$ does not belong to the connected component of unity in $\operatorname{Aut} q\,(n)$. Although the Lie superalgebras $q\,(n)$ and $q\,(n)^{(1)}$ are not contragredient, twisting by ϕ gives a regular quasisimple Kac–Moody superalgebra which we denote by $q(n)^{(2)}$. As a vector space $q(n)^{(2)}$ is isomorphic to

$$\mathfrak{s}_{\bar{0}} \otimes \mathbb{C}\left[t^2, t^{-2}\right] \oplus \mathfrak{s}_{\bar{1}} \otimes t\mathbb{C}\left[t^2, t^{-2}\right] \oplus \mathbb{C}K \oplus \mathbb{C}D$$

with $D = t\frac{\partial}{\partial t}$ and K being a central element. For any $x, y \in \mathfrak{s}$, the commutator is defined by the formula

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \delta_{m,-n}\left(1 - (-1)^m\right) \operatorname{tr}(xy) K.$$

A Cartan matrix of $q(n)^{(2)}$ has size $n \times n$. Identify the set of indices with the abelian group $\mathbb{Z}_n$. Fix a parity function $p\colon \mathbb{Z}_n \to \{0, 1\}$ so that the number of odd indices is odd. Set $a_{ij} = 0$ if $j \neq i$ or $i \pm 1$ (modulo n); set

$$a_{ii} = 2, \quad a_{i,i\pm 1} = -1 \quad \text{if } p(i) = 0; \qquad a_{ii} = 0, \quad a_{i,i+1} = -1, \quad a_{i,i-1} = 1 \quad \text{if } p(i) = 1.$$

The graph of this Cartan matrix is a cycle with odd number of isotropic vertices. Any two such Cartan matrices are related by a chain of odd reflections; it is a simple exercise to check that they are not symmetrizable. The corank of a Cartan matrix is 1.

Roots of $\mathfrak{g} = q(n)^{(2)}$ are of the form $\alpha + m\delta$, where α is a root of $\operatorname{sl}(n)$ and $m \in \mathbb{Z}$ or $m\delta$ with $m \in \mathbb{Z} \setminus \{0\}$. The parity of a root equals the parity of m.

Non-symmetrizable superalgebra $S(1, 2; b)$. This superalgebra appears in the list of conformal superalgebras classified by Kac and van de Leur [7]. By definition, it is the Kac–Moody superalgebra with Cartan matrix

$$\begin{pmatrix} 0 & b & 1-b \\ -b & 0 & 1+b \\ -1 & -1 & 2 \end{pmatrix}$$

and parity (1,1,0). Obviously, $S(1, 2; b) \cong S(1, 2; -b)$. If $b = 0$, then the matrix is a matrix of $\operatorname{gl}(2|2)$; if $b = -1, 1$, the matrix has singular roots. In all other cases the odd reflection r_{α_2} transforms the matrix of $S(1, 2; b)$ to the matrix of $S(1, 2; 1 + b)$. Hence an isomorphism $S(1, 2; b) \cong S(1, 2; 1 + b)$. The moduli space of $S(1, 2; b)$ can be identified with $\mathbb{CP}^1/G$ where G is the subgroup of $\operatorname{PGL}(2, \mathbb{Z})$ generated by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. A Cartan matrix of $S(1, 2; b)$ has corank 1, hence the algebra has one-dimensional center. The Lie superalgebra $S(1, 2; b)$ is a quasisimple Kac–Moody superalgebra. It is regular if $b \notin \mathbb{Z}$.

Realization of $S(1, 2; b)$. Consider a supercommutative algebra $R = \mathbb{C}\left[t, t^{-1}, \xi_1, \xi_2\right]$ with an even generator t and two odd generators ξ_1, ξ_2. Denote by $\mathcal{W}(1, 2)$ the Lie superalgebra of derivations of R, in other words $\mathcal{W}(1, 2)$ is the superspace of all linear maps $d : R \to R$ such that

$$d(fg) = d(f)\, g + (-1)^{p(d)p(f)} f d(g).$$

An element $d \in \mathcal{W}(1,2)$ can be written as

$$d = f\frac{\partial}{\partial t} + f_1\frac{\partial}{\partial \xi_1} + f_2\frac{\partial}{\partial \xi_2}$$

for some $f, f_1, f_2 \in R$. It is easy to see that the subset of all $d \in \mathcal{W}(1,2)$ satisfying the condition

$$bft^{-1} + \frac{\partial f}{\partial t} - (-1)^{p(d)}\left(\frac{\partial f_1}{\partial \xi_1} + \frac{\partial f_2}{\partial \xi_2}\right) \equiv \text{const},$$

form a subalgebra $\bar{\mathcal{S}}_b$ of $\mathcal{W}(1,2)$. Let $E = \xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2}$. Then ad_E is a diagonalizable operator in $\bar{\mathcal{S}}_b$ which defines the grading

$$\bar{\mathcal{S}}_b = \left(\bar{\mathcal{S}}_b\right)_{-1} \oplus \left(\bar{\mathcal{S}}_b\right)_0 \oplus \left(\bar{\mathcal{S}}_b\right)_1 \oplus \left(\bar{\mathcal{S}}_b\right)_2,$$

such that $\left(\bar{\mathcal{S}}_b\right)_2 = 0$ for $b \notin \mathbb{Z}$, $\left(\bar{\mathcal{S}}_b\right)_2 = \mathbb{C}\xi_1\xi_2 t^{-b}\frac{\partial}{\partial t}$ for $b \in \mathbb{Z}$. Define

$$\mathcal{S}_b = \left(\bar{\mathcal{S}}_b\right)_{-1} \oplus \left(\bar{\mathcal{S}}_b\right)_0 \oplus \left(\bar{\mathcal{S}}_b\right)_1.$$

One can see that $\mathcal{S}_b$ is an ideal in $\bar{\mathcal{S}}_b$ when $b \in \mathbb{Z}$.

The commutator $[\mathcal{S}_b, \mathcal{S}_b]$ is simple and has codimension 1 in $\mathcal{S}_b$. It consists of all derivations $d \in \mathcal{S}_b$ satisfying the condition

$$bft^{-1} + \frac{\partial f}{\partial t} - (-1)^{p(d)}\left(\frac{\partial f_1}{\partial \xi_1} + \frac{\partial f_2}{\partial \xi_2}\right) = 0.$$

Set

$$\begin{array}{lll} X_1 = \frac{\partial}{\partial \xi_1}, & X_2 = -b\xi_1\xi_2 t^{-1}\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial t}, & X_3 = \xi_1\frac{\partial}{\partial \xi_2}, \\ Y_1 = (b+1)\,\xi_1\xi_2\frac{\partial}{\partial \xi_2} + \xi_1 t\frac{\partial}{\partial t}, & Y_2 = t\frac{\partial}{\partial \xi_2}, & Y_3 = \xi_2\frac{\partial}{\partial \xi_1}, \\ h_1 = (b+1)\,\xi_2\frac{\partial}{\partial \xi_2} + t\frac{\partial}{\partial t}, & h_2 = b\xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2} + t\frac{\partial}{\partial t}, & h_3 = \xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2}. \end{array}$$

Then X_i, Y_i, h_i, $i = 1,2,3$, generate $[\mathcal{S}_b, \mathcal{S}_b]$ if $b \neq 0$. They satisfy the relations (1) with Cartan matrix $S(1,2;b)$. The contragredient Lie superalgebra $S(1,2;b)$ can be obtained from $\mathcal{S}_b$ by a suitable central extension.

Divide the fist row of the Cartan matrix $S(1,2;b)$ by b and the second by $-b$, let $a = \frac{1}{b}$. Then the renormalized matrix

$$\begin{pmatrix} 0 & 1 & -1-a \\ 1 & 0 & -1+a \\ -1 & -1 & 2 \end{pmatrix}$$

clearly is a deformation of a Cartan matrix of the affine superalgebra $\mathrm{sl}\,(1|2)^{(1)}$. Thus, $S(1,2,\infty) \cong \mathrm{sl}\,(1|2)^{(1)}$. The roots and the root multiplicities of $S(1,2;b)$ are the same as the roots and the root multiplicities of $\mathrm{sl}\,(1|2)^{(1)}$.

The family $Q^{\pm}(l,m,n)$. This family was discovered by C. Hoyt (see [3]). To define it let us classify 3×3 matrices with zeros on the diagonal and non-zeros anywhere else, which define regular quasisimple Kac–Moody superalgebras. Using multiplication by a diagonal matrix, any such matrix can be reduced to the form

$$\begin{pmatrix} 0 & a & 1 \\ 1 & 0 & b \\ c & 1 & 0 \end{pmatrix}.$$

The odd reflection r_{α_1} transforms it to the matrix (parity$=(1,0,0)$)

$$\begin{pmatrix} 0 & a & 1 \\ -1 & 2 & 1+b+\frac{1}{a} \\ -1 & 1+a+\frac{1}{c} & 2 \end{pmatrix}.$$

The latter matrix must be admissible, therefore $a+\frac{1}{c}, b+\frac{1}{a} \in \mathbb{Z}_{<0}$. Application of two other odd reflections produces one more condition $c+\frac{1}{b} \in \mathbb{Z}_{<0}$. The matrix B is

$$\begin{pmatrix} 2 & 1+b+\frac{1}{a} & 1+b+\frac{1}{a} \\ 1+a+\frac{1}{c} & 2 & 1+a+\frac{1}{c} \\ 1+c+\frac{1}{b} & 1+c+\frac{1}{b} & 2 \end{pmatrix}.$$

If all non-diagonal entries of B are zero, $\mathfrak{g}(A)$ belongs to the family $D(2,1;a)$ of finite-dimensional superalgebras defined above. If all of them equal -1, A is of type $q(3)^{(2)}$. It was shown in [3] that for any three $k,l,m \in \mathbb{Z}_{\geq 1}$, $klm > 1$, there exist two solutions of the system

$$1+a+\frac{1}{c} = -k, \qquad 1+b+\frac{1}{a} = -l, \qquad 1+c+\frac{1}{b} = -m$$

one with $-1 < a,b,c < 0$ and one with $a,b,c \leq -1$. We denote the corresponding superalgebras $Q^{+}(k,l,m)$ and $Q^{-}(k,l,m)$ respectively.

In conclusion, let us note that all superalgebras listed above are quasisimple and regular except $D(2,1;0)$, which is not regular and not quasisimple, and $S(1,2;b)$ with $b \in \mathbb{Z}$, which is quasisimple but not regular.

7 Classification results

Quasisimple finite-dimensional contragredient superalgebras were listed by V. Kac in [1]. The reader can find there the description of root systems and Cartan matrices.

The Kac–Moody superalgebras without isotropic simple roots and of finite growth were also classified by V. Kac in [10]. They are all affine or twisted affine superalgebras (automatically symmetrizable). This result was generalized by Van de Leur [8] as follows.

Theorem 7.1. *(Van de Leur) Any finite-growth contragredient superalgebra with an indecomposable symmetrizable Cartan matrix is isomorphic to a finite-dimensional superalgebra, its affinization, or its twisted affinization.*

It is not difficult to see that the property of a contragredient superalgebra to have finite growth does not depend on a choice of a base (Cartan matrix). Thus, by Theorem 2.2, any finite growth contragredient superalgebra is a Kac–Moody superalgebra; therefore hereditary principle (Corollary 5.2) is applicable to the Cartan matrices (the original one, and those obtained by odd reflections). That was used in [9] to obtain the following result.

Theorem 7.2. *(Hoyt, Serganova) A non-symmetrizable quasisimple superalgebra of finite growth is isomorphic to* $q(n)^{(2)}$ *or* $S(1,2;b)$.

In fact, [9] contains classification of all finite-growth contragredient superalgebras (without assumption of quasisimplicity), we do not formulate it here, since it is rather technical.

In [3] C. Hoyt classified all quasisimple regular Kac–Moody superalgebras.

Theorem 7.3. *(C. Hoyt) Any quasisimple regular Kac–Moody superalgebra with at least one isotropic simple root is isomorphic to a finite-dimensional Kac–Moody superalgebra, its affinization or twisted affinization,* $q(n)^{(2)}$, $S(1,2;b)$ *with* $b \notin \mathbb{Z}$, *or* $Q^{\pm}(k,l,m)$.

Corollary 7.4. *A quasisimple regular Kac–Moody superalgebra of type I is isomorphic to one of the following list:*

1. $\mathrm{sl}(m|n)$ *with* $1 \leq m < n$;
2. $\mathrm{gl}(n|n)$ *with* $n \geq 2$;
3. $\mathrm{osp}(2|2n)$ *with* $n \geq 2$;
4. $\mathrm{sl}(m|n)^{(1)}$ *with* $1 \leq m < n$;
5. $\mathrm{psl}(n|n)^{(1)}$ *with* $n \geq 2$;
6. $\mathrm{osp}(2|2n)^{(1)}$ *with* $n \geq 2$;
7. $S(1,2;b)$ *with* $b \notin \mathbb{Z}$.

8 Applications of classification results

In this section we prove some general results about regular quasisimple Kac–Moody superalgebras. We use the classification (see Theorem 7.3). It would be very desirable to obtain proofs without use of classification; unfortunately we were unable to do so. First, we can "improve" Lemma 3.12.

Lemma 8.1. *For any quasisimple regular Kac–Moody superalgebra* $\mathfrak{g}$, *the lattice* Q' *has a finite index in* Q_0.

Proof. As follows from the proof of Lemma 3.12, for any odd roots α and $\beta \in \Pi$ joined by a string, $2\alpha + 2\beta \in Q'$. If Π contains an odd root α such that $2\alpha \in Q'$ (for example, if α is non-isotropic), then $2\beta \in Q'$ for any root β, hence the rank of Q' is n. If Γ contains a cycle with exactly three isotropic vertices $\alpha_1, \alpha_2, \alpha_3$, then $2\alpha_i + 2\alpha_j \in Q'$ implies $4\alpha_i \in Q'$, hence the rank of Q' is also n.

One can see from the classification that the only regular quasisimple Kac–Moody superalgebras which have no base with non-isotropic odd roots are $\mathrm{sl}(m|n)$, $\mathrm{gl}(m|m)$, $\mathrm{osp}(2m|2n)$, $D(2,1;a)$, $F(4)$, their affinizations, $S(1,2;a)$, $q(m)^{(1)}$, or $Q^{\pm}(k,l,m)$. The superalgebras of type II in this list are $\mathrm{osp}(2m|2n)$ with $m \geq 2$, $D(2,1;a)$, $F(4)$, their affinizations, $q(m)^{(2)}$, or $Q^{\pm}(k,l,m)$; they all have a graph Γ with a 3-cycle of isotropic vertices. Therefore if $\mathfrak{g}$ has type II, the rank of Q' is n (the same as the rank of Q_0). Thus $[Q_0 : Q']$ is finite. If $\mathfrak{g}$ is of type I the rank of Q' is $n-1$ (again the same as the Q_0). Hence again $[Q_0 : Q']$ is finite. The lemma is proven.

Lemma 8.2. *Let $\mathfrak{g}(A)$ be a regular quasisimple Kac–Moody superalgebra. Then at least one of the conditions below is true:*

(a) A does not have zeros on the diagonal;
(b) $\mathfrak{g}(A)$ is finite-dimensional;
(c) There exists a finite set $S \subset \mathfrak{h}^*$ and $\delta \in \Delta$ such that $\Delta \subset \mathbb{Z}\delta + S$;
(d) $\mathfrak{g}(A)$ is isomorphic to $Q^{\pm}(k,l,m)$.

Proof. Suppose (a) and (b) do not hold. Then, by Theorem 7.1 and Theorem 7.3, $\mathfrak{g}(A)$ is (twisted) affine, $S(1,2;b)$, $q(n)^{(2)}$, or $Q^{\pm}(k,l,m)$. In the first three cases (c) holds, as follows from the description of roots in Section 6.

Theorem 8.3. *Let Π and Π' be two bases of a quasisimple regular Kac–Moody superalgebra $\mathfrak{g}$. Then Π' or $-\Pi'$ can be obtained from Π by even and odd reflections.*

Proof. We already proved the statement for finite-dimensional $\mathfrak{g}$ in Corollary 4.5. In fact, its proof implies that in general, it suffices to prove that either $\Delta^+(\Pi) \cap \Delta^-(\Pi')$ or $\Delta^+(\Pi) \cap \Delta^+(\Pi')$ is finite.

Assume first that $\mathfrak{g}$ satisfies (c) from Lemma 8.2. Choose $h, h' \in \mathfrak{h}$ such that $\alpha(h) = 1$ for any $\alpha \in \Pi$, and $\beta(h') = 1$ for any $\beta \in \Pi'$. Then $\gamma(h), \gamma(h') \neq 0$ for any root $\gamma \in \Delta$, in particular, $\delta(h), \delta(h') \neq 0$. Suppose $\delta(h) > 0$ and $\delta(h') > 0$. We claim that $\Delta^+(\Pi) \cap \Delta^-(\Pi')$ is finite. Indeed, let C be the maximum of $\{|s(h)|, |s(h')| \mid s \in S\}$. Let $\alpha \in \Delta^+(\Pi) \cap \Delta^-(\Pi')$. Since $\alpha = m\delta + s$ for some $s \in S$, the conditions $\alpha(h) > 0$, $\alpha(h') < 0$ imply $|m| \leq C$. Thus there are only finitely many α in $\Delta^+(\Pi) \cap \Delta^-(\Pi')$. If $\delta(h) > 0$, $\delta(h') < 0$, then $\Delta^+(\Pi) \cap \Delta^-(\Pi')$ is finite by similar argument.

Now let $\mathfrak{g}$ satisfy (a) or (d). Then the matrix B is indecomposable, and the conditions of Lemma 3.10 hold (in the first case B is obtained from A by dividing all odd rows by 2 and multiplying all odd columns by 2, in the second case it follows from the direct computation done in Section 6). Moreover, Lemma 3.10 implies that $D^+_\Pi = \bigcap_{w \in W} w\left(C^+(\mathcal{B})\right)$. It was shown in [2], Lemma 5.8, that the set of rays spanned by positive imaginary roots of $\mathfrak{g}(B)$ is dense in D^+_Π in any metric topology. We choose this metric so that the roots of Π' form an orthonormal basis in Q.

First, we claim that for any base Π', either D^+_Π, or $-D^+_\Pi$ is contained in $C^+(\Pi')$. Indeed, if this is not true, there exists $h \in \mathfrak{h}$ such that $\alpha(h) \geq 1$ for all $\alpha \in \Pi'$ and $\gamma(h) = 0$ for some $\gamma \in D^+_\Pi$. Since the set of rays spanned by positive imaginary roots of $\mathfrak{g}(B)$ is dense in D^+_Π, one can choose a sequence β_i of positive imaginary roots of $\mathfrak{g}(B)$ such that $\frac{\beta_i}{|\beta_i|}$ approaches $\frac{\gamma}{|\gamma|}$. But each β_i is a root of $\mathfrak{g}$, hence $\beta_i = \sum_{\alpha \in \Pi'} m_{i\alpha}\alpha$ where all $m_{i\alpha}$ are non-negative (or non-positive). Therefore $\frac{\beta_i(h)}{|\beta_i|} \geq 1$, or $\frac{\beta_i(h)}{|\beta_i|} \leq -1$, and we obtain contradiction with the assumption $\gamma(h) = 0$.

Without loss of generality, one may assume that $D^+_\Pi \subset C^+(\Pi')$ (if necessary one can use $-\Pi'$ instead of Π'). Let us prove that in this case $\Delta^+(\Pi) \cap \Delta^-(\Pi')$ is finite. Assume the opposite. Then one can find an infinite sequence of roots $\alpha_1, \dots, \alpha_n, \dots \in \Delta^+(\Pi) \cap \Delta^-(\Pi')$. Let γ be a limit point of the sequence $\frac{\alpha_i}{|\alpha_i|}$. By Lemma 4.4, after any chain of reflections $r_1 \dots r_k$ infinitely many of α_i remain in $\Delta^+(r_1 \dots r_k(\Pi))$. Hence $\gamma \in D^+_\Pi$. On the other hand, $\gamma \in -C^+(\Pi')$. Contradiction.

Remark 8.4. Theorem 8.3 implies Theorem 4.14.

9 Description of $\mathfrak{g}(B)$ and $\mathfrak{g}'$ in examples

Let us recall (see [1]) that if $\mathfrak{s}$ is a simple classical Lie superalgebra with a non-degenerate even invariant symmetric form, then $\mathfrak{s}_{\bar{0}}$ is either semi-simple, or reductive with one-dimensional center. Every such $\mathfrak{s}$ is contragredient.

Now let us consider the case of (twisted) affine superalgebras.

Theorem 9.1. *Let $\mathfrak{s}$ be a simple classical Lie superalgebra with non-degenerate even invariant symmetric form $(\cdot,\cdot)$, $\mathfrak{g} = \mathfrak{s}^{(1)}$, $[\mathfrak{s}_{\bar{0}}, \mathfrak{s}_{\bar{0}}] = \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_k$ be the sum of simple ideals $\mathfrak{s}_1, \dots, \mathfrak{s}_k$, $q\colon \mathfrak{g}' \to \mathfrak{g}(B)/\mathfrak{c}$ be as defined in Lemma 3.7. Then*

(a) $\mathfrak{g}(B) = \mathfrak{s}_1^{(1)} \oplus \dots \oplus \mathfrak{s}_k^{(1)}$*;*

(b) $\dim \mathfrak{c} = k - 1$, $\operatorname{Ker} q = 0$*;*

(c) If $\mathfrak{s}_{\bar{0}}$ is semisimple, then $\mathfrak{g}_{\bar{0}} = \mathfrak{g}' \oplus \mathbb{C}D$. If $\mathfrak{s}_{\bar{0}}$ has one-dimensional center $\mathbb{C}c$, then $\mathfrak{g}_{\bar{0}} = (\mathfrak{g}' + \mathfrak{i}) \oplus \mathbb{C}D$ and $\mathfrak{g}' \cap \mathfrak{i} = \mathbb{C}K$, where $\mathfrak{i}$ is the subalgebra generated by $c \otimes t^k$. In particular, $\Delta_0(\mathfrak{g}) = \Delta(\mathfrak{g}(B))$.

(d) Let $K_1, \dots, K_k$ denote the canonical central elements of $\mathfrak{s}_1^{(1)}, \dots, \mathfrak{s}_k^{(1)}$, $(\cdot,\cdot)_i$ denote the Killing form on $\mathfrak{s}_i$ and $(x,y) = b_i(x,y)_i$ for $x, y \in \mathfrak{s}_i$, then

$$\mathfrak{c} = \left\{ a_1K_1 + \dots + a_kK_k \mid \sum a_ib_i = 0 \right\}. \tag{8}$$

(e) $\Delta_0(\mathfrak{g}) = \Delta(\mathfrak{g}') = \Delta(\mathfrak{g}(B))$, and imaginary roots are roots of the form $m\delta$, where $m \in \mathbb{Z} \setminus \{0\}$.

Proof. (a) follows from the fact that even roots of $\mathfrak{g}$ are the same as the roots of $\mathfrak{s}_1^{(1)} \oplus \cdots \oplus \mathfrak{s}_k^{(1)}$. This fact can be also used to prove (c). Indeed, $(\mathfrak{g}_{\bar{0}})_\alpha = \mathfrak{g}'_\alpha$ for any even root α of the form $\alpha^\circ + m\alpha$ with $\alpha^\circ \in \Delta_0(\mathfrak{s})$, and

$$(\mathfrak{g}_{\bar{0}})_{m\delta} = \mathfrak{h}_{\mathfrak{s}} \otimes t^m, \qquad \mathfrak{g}'_{m\delta} = \mathfrak{h}_{[\mathfrak{s},\mathfrak{s}]} \otimes t^m,$$

where $\mathfrak{h}_{\mathfrak{s}}$, $\mathfrak{h}_{[\mathfrak{s},\mathfrak{s}]}$ are Cartan subalgebras of $\mathfrak{s}$ and $[\mathfrak{s},\mathfrak{s}]$ respectively. If $\mathfrak{s}_{\bar{0}}$ is semisimple, then $\mathfrak{h}_{\mathfrak{s}} = \mathfrak{h}_{[\mathfrak{s},\mathfrak{s}]}$ and $\mathfrak{g}'_{m\delta} = (\mathfrak{g}_{\bar{0}})_{m\delta}$. If $\mathfrak{s}_{\bar{0}}$ is not semisimple, then

$$(\mathfrak{g}_{\bar{0}})_{m\delta} = \mathfrak{g}'_{m\delta} \oplus \mathbb{C}(c \otimes t^m),$$

and hence $\mathfrak{g}_{\bar{0}} = \mathfrak{g}' + \mathfrak{i}$.

Next we will prove (d). Pick up elements $x_1 \otimes t, \ldots, x_k \otimes t$, $y_1 \otimes t^{-1}, \ldots, y_k \otimes t^{-1}$ such that x_i, y_i belong to the Cartan subalgebra of $\mathfrak{s}_i$, $(x_i, y_i) = 1$. Let $\bar{K}_i$ be the images of K_i under the natural projection $\mathfrak{g}(B) \to \mathfrak{g}(B)/\mathfrak{c}$. Then

$$q(K) = q\left(\left[x_i \otimes t, y_i \otimes t^{-1}\right]\right) = (x_i, y_i)_i \bar{K}_i = \frac{1}{b_i}\bar{K}_i.$$

Thus, $\mathfrak{c}$ is generated by $\frac{1}{b_i}K_i - \frac{1}{b_j}K_j$, that implies (d).

To prove (b) note that $\dim \mathfrak{c} = k - 1$ by (d) and $\operatorname{Ker} q = 0$ since $\mathfrak{g}(B)$ is symmetizable (see Remark 4.7); this implies $\Delta(\mathfrak{g}') = \Delta(\mathfrak{g}(B))$.

Roots of $\mathfrak{i}$ coincide with roots of $\mathfrak{h} \otimes t^{\mathbb{Z}}$; this implies $\Delta_0(\mathfrak{g}) = \Delta(\mathfrak{g}')$. Since Weyl groups of $\mathfrak{g}(B)$ and $\mathfrak{g}$ coincide, even real roots of $\mathfrak{g}$ coincide with real roots of $\mathfrak{g}(B)$. Thus the required description of real even roots follows from the theory of affine Lie algebras, see [2]. The absence of odd imaginary roots is not used in this paper; note only that it follows easily from the fact that the Weyl group of $\mathfrak{g}(B)$ acts transitively on the subset of isotropic and on the subset of non-isotropic odd roots in the set $\Delta_1^\circ + \mathbb{Z}\delta$ of odd roots of $\mathfrak{g}$. The latter fact can be checked case by case (see [11] for details); therefore all odd roots are real.

Remark 9.2. Observe that if $\mathfrak{g} = \mathfrak{s}^{(1)}$, then $k = 1$ only for $\mathfrak{s} = \mathrm{sl}(1|n)$, $\mathrm{osp}(1|2n)$ or $\mathrm{osp}(2|2n)$.

Example 9.3. Let $\mathfrak{s} = D(2,1;a)$. Then

$$\mathfrak{s}_{\bar{0}} \cong \mathrm{sl}(2) \oplus \mathrm{sl}(2) \oplus \mathrm{sl}(2), \qquad \mathfrak{g}(B) \cong \mathrm{sl}(2)^{(1)} \oplus \mathrm{sl}(2)^{(1)} \oplus \mathrm{sl}(2)^{(1)},$$

and $\mathfrak{c}$ is spanned by $K_2 - aK_1$ and $(1+a)K_1 + K_3$, where K_1, K_2 and K_3 are the standard central elements of the components in $\mathfrak{g}(B)$ isomorphic to $\mathrm{sl}(2)^{(1)}$. In this case $\mathfrak{g}_{\bar{0}} = \mathfrak{g}' \oplus \mathbb{C}D$.

Corollary 9.4. *In notations of Theorem 9.1 let $k = 2$. Then $\mathfrak{c}$ is spanned by $K_1 + uK_2$ for some positive rational u.*

Proof. Follows from the fact that b_1, b_2 are rational and have different signs. The latter fact can be found in [1].

Below we list $\mathfrak{g}(B)$ for all twisted affine superalgebras.

$\mathfrak{g}$		$\mathfrak{g}(B)$
$\mathrm{sl}(1\|2n)^{(2)}$,	$n \geq 2$	$\mathrm{sl}(2n)^{(2)}$
$\mathrm{sl}(2\|2n)^{(2)}$,	$n \geq 2$	$\mathrm{sl}(2)^{(1)} \oplus \mathrm{sl}(2n)^{(2)}$
$(p)\,\mathrm{sl}(m\|2n)^{(2)}$,	$m \geq 3, n \geq 2$	$\mathrm{sl}(m)^{(2)} \oplus \mathrm{sl}(2n)^{(2)}$
$\mathrm{sl}(2m+1\|2)^{(2)}$,	$m \geq 1$	$\mathrm{sl}(2m+1)^{(2)} \oplus \mathrm{sl}(2)^{(1)}$
$\mathrm{sl}(1\|2n+1)^{(4)}$,	$n \geq 1$	$\mathrm{sl}(2n+1)^{(2)}$
$(p)\,\mathrm{sl}(2m+1\|2n+1)^{(4)}$,	$n \geq m \geq 1$	$\mathrm{sl}(2m+1)^{(2)} \oplus \mathrm{sl}(2n+1)^{(2)}$
$\mathrm{osp}(2\|2n)^{(2)}$,	$n \geq 1$	$\mathrm{sp}(2n)^{(1)}$
$\mathrm{osp}(4\|2n)^{(2)}$,	$n \geq 1$	$\mathrm{sl}(2)^{(1)} \oplus \mathrm{sp}(2n)^{(1)}$
$\mathrm{osp}(2m\|2n)^{(2)}$,	$m \geq 3, n \geq 1$	$o(2m)^{(2)} \oplus \mathrm{sp}(2n)^{(1)}$

Remark 9.5. From this table one can see that $\mathfrak{g}(B)$ is either (twisted) affine Lie algebra, or a direct sum of two (twisted) affine Lie algebras. In the former case $\mathfrak{c} = 0$, and in the latter case $\mathfrak{c}$ is one dimensional, and generated by $K_1 + uK_2$; here by K_1 and K_2 we denote the canonical central elements of (twisted) affine superalgebras which appear as direct summands of $\mathfrak{g}(B)$, and u is an appropriate positive rational number.

It is not difficult to see that $\mathfrak{g}_{\bar{0}} = \mathfrak{g}' \oplus \mathbb{C}D$ for the twisted affinization of $\mathrm{osp}(2m|2n)$, $m \geq 2$, and of $\mathrm{psl}(n|n)$, $n \geq 3$. If $\mathfrak{g}$ is the twisted affinization of $\mathrm{sl}(m|n)$ with $m \neq n$, or $\mathrm{osp}(2|2n)$, then $\mathfrak{g}_{\bar{0}} = (\mathfrak{g}' + \mathfrak{i}) \oplus \mathbb{C}D$, $\mathfrak{i} \cap \mathfrak{g}' = \mathbb{C}K$, here $\mathfrak{i}$ is generated by $c \otimes t^{2k+1}$ for $k \in \mathbb{Z}$ if the order of the twisting automorphism is 2, and by $c \otimes t^{4k+2}$ for $k \in \mathbb{Z}$ if the order of the twisting automorphism is 4 (as before, c denotes a central element of $\mathfrak{s}_{\bar{0}}$).

Strange twisted affine superalgebra $q(n)^{(2)}$. $\mathfrak{g}(B) \cong \mathrm{sl}(n)^{(1)}$, $\mathfrak{c}$ coincides with the center of $\mathrm{sl}(n)^{(1)}$, the homomorphism $q : \mathfrak{g}' \to \mathfrak{g}(B)/\mathfrak{c}$ is injective, and

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}' \oplus \mathbb{C}D \oplus \mathbb{C}K. \tag{9}$$

Non-symmetrizable superalgebra $S(1,2;b)$, $b \notin \mathbb{Z}$. To describe $\mathfrak{g}_{\bar{0}}$, one has to use the realization $\mathcal{S}_b = S(1,2;b)/\mathbb{C}K$, here K denotes a central element of $S(1,2;b)$. It is a straightforward calculation that

$$\mathcal{S}_b = \mathbb{C}t\frac{\partial}{\partial t} \oplus [\mathcal{S}_b, \mathcal{S}_b],$$

and $[\mathcal{S}_b, \mathcal{S}_b]_{\bar{0}}$ is a semidirect sum of the subalgebra $\mathfrak{L}$ spanned by $t^m \frac{\partial}{\partial t} + (m+b)\,t^{m-1}E$, and the ideal $\mathfrak{g}'/\mathbb{C}K$ spanned by $t^m \sum_{i,j=1,2} c_{ij}\xi_i \frac{\partial}{\partial \xi_j}$ with $c_{11} + c_{22} = 0$. Note that $\mathfrak{L}$ is isomorphic to the algebra of derivations of $\mathbb{C}\left[t, t^{-1}\right]$, and $\mathfrak{g}'/\mathbb{C}K$ is isomorphic to the loop algebra $\mathrm{sl}(2) \otimes \mathbb{C}\left[t, t^{-1}\right]$. Going to the central extension $\mathfrak{g} = S(1,2;b)$ of $\mathcal{S}_b$, one obtains

$$\mathfrak{g}_{\bar{0}} = \widehat{\mathcal{L}} + \mathfrak{g}' \oplus \mathbb{C}t\frac{\partial}{\partial t}, \qquad \mathfrak{g}' \oplus \mathbb{C}t\frac{\partial}{\partial t} \cong \mathrm{sl}(2)^{(1)}, \quad \widehat{\mathcal{L}} \cap \mathfrak{g}' = \mathbb{C}K,$$

here $\widehat{\mathcal{L}}$ is isomorphic to the Virasoro algebra.

The family $Q^{\pm}(l,m,n)$. Due to the lack of an explicit description, we can say only few things about the roots and the structure of $\mathfrak{g}$ and $\mathfrak{g}(B)$ for $\mathfrak{g} = Q^{\pm}(l,m,n)$. For example, we would like to find out what are multiplicities of imaginary roots of these algebras, but we did not succeed yet.

Let $\alpha_1, \alpha_2, \alpha_3$ be the base with three isotropic roots, then there are three linearly independent principal roots $\beta_1 = \alpha_2 + \alpha_3$, $\beta_2 = \alpha_1 + \alpha_3$, $\beta_3 = \alpha_1 + \alpha_2$. Since $[Q : Q'] = 2$ and $[Q : Q_0] = 2$, one has $Q_0 = Q'$. Note that the matrix B

$$\begin{pmatrix} 2 & -k & -k \\ -l & 2 & -l \\ -m & -m & 2 \end{pmatrix}.$$

has a negative determinant (as easily follows from the condition $k,l,m \geq 1$ and $klm > 1$). Therefore $\mathfrak{g}(B)$ is a simple Kac–Moody Lie algebra of indefinite type. In this case we do not know if q is injective.

10 Integrable modules and highest weight modules

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a regular Kac–Moody superalgebra with a standard base Π. A $\mathfrak{g}$-module M is called a *weight* module if $\mathfrak{h}$ acts semisimply on M, in other words M has a weight decomposition

$$M = \bigoplus_{\mu \in \mathfrak{h}^*} M_\mu, \qquad M_\mu = \{m \in M \mid hm = \mu(h)\, m, \forall h \in \mathfrak{h}\},$$

and $\dim M_\mu < \infty$ for all $\mu \in \mathfrak{h}^*$. The set

$$P(M) = \{\mu \in \mathfrak{h}^* \mid M_\mu \neq 0\}$$

is called the *set of weights* of M. The formal *character* $\operatorname{ch} M$ of M is defined by the formula

$$\operatorname{ch} M = \sum_{\mu \in P(M)} \dim M_\mu \, e^{\mu}.$$

A module M is *integrable* if M is a weight module, and X_β and Y_β act locally nilpotently on M for every principal root β of $\mathfrak{g}$. Note that if α is isotropic, then $X_\alpha^2 = 0$, hence X_α acts locally nilpotently on any module. If β is a principal root, then $\exp X_\beta \exp(-Y_\beta) \exp X_\beta$ is a well defined linear operator on an integrable module M. Thus the Weyl group W of $\mathfrak{g}$ acts on M, therefore $\operatorname{ch} M$ is W-invariant. This implies also that X_α and Y_α act locally nilpotently on M for any real root α. By definition of a Kac–Moody superalgebra, its adjoint module is integrable.

Our next step is to define the category $\mathcal{O}$ of highest weight modules.

Define a *Verma module* $M(\lambda)$ with the *highest weight* λ as the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C(\lambda)$, where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $C(\lambda)$ is a one-dimensional $\mathfrak{b}$-module with generator v such that $hv = \lambda(h)\, v$, $X_\alpha v = 0$ for all simple roots

α. A vector v is called a *highest vector* of $M(\lambda)$. Any quotient V of $M(\lambda)$ is an indecomposable module generated by the image of v under the natural projection $M(\lambda) \to V$.

The following lemma can be proven exactly as in the Lie algebra case.

Lemma 10.1. *Let* $m(\alpha) = \dim \mathfrak{g}_\alpha$.

(a) $\operatorname{ch} M(\lambda) = e^\lambda \frac{\prod_{\alpha\in\Delta_1^+}\left(1+e^{-\alpha}\right)^{m(\alpha)}}{\prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})^{m(\alpha)}}$.

(b) $M(\lambda)$ *has a unique irreducible quotient which we denote by* $L(\lambda)$.

The category $\mathcal{O}$ is a full subcategory of the category of $\mathfrak{g}$-modules, whose objects are weight modules M such that $P(M) \subset \bigcup_{i=1}^s P(M(\mu_i))$ for some finite set $\{\mu_1,\dots,\mu_s\} \in \mathfrak{h}^*$.

In what follows we will often need to change the base Π to a base Σ by odd reflections. So we will write $\mathfrak{n}^+_\Sigma$, $\mathfrak{b}_\Sigma, \mathcal{O}_\Sigma$, $M_\Sigma(\lambda)$ and $L_\Sigma(\lambda)$ if we mean the corresponding object for a non-standard base. If the subindex is omitted, then $\Sigma = \Pi$.

Lemma 10.2. *Let* Σ' *is obtained from* Σ *by an odd reflection* r_α.

(a) $M_{\Sigma'}(\lambda)$ *and* $L_{\Sigma'}(\lambda)$ *are objects of* $\mathcal{O}_\Sigma$.

(b) If $\lambda(h_\alpha) \neq 0$, *then* $M_{\Sigma'}(\lambda-\alpha) \cong M_\Sigma(\lambda)$, *and* $L_{\Sigma'}(\lambda-\alpha) \cong L_\Sigma(\lambda)$.

(c) Let $\lambda(h_\alpha) = 0$, *then* $L_{\Sigma'}(\lambda) \cong L_\Sigma(\lambda)$. *If* $\mathfrak{p} = \mathbb{C}Y_\alpha \oplus \mathfrak{b}$, *then the* $\mathfrak{b}$*-module structure on* $C(\lambda)$ *and* $C(\lambda-\alpha)$ *extends uniquely to a* $\mathfrak{p}$*-module structure, and the following exact sequences hold*

$$0 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C(\lambda-\alpha) \to M_\Sigma(\lambda) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C(\lambda) \to 0,$$
$$0 \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C(\lambda) \to M_{\Sigma'}(\lambda-\alpha) \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C(\lambda-\alpha) \to 0.$$

Proof. The first statement of the lemma follows from the identity

$$\operatorname{ch} M_{\Sigma'}(\lambda-\alpha) = \operatorname{ch} M_\Sigma(\lambda),$$

which is a straightforward consequence of Lemma 10.1 (a).

Now prove (b). Let v be a highest vector of $M_\Sigma(\lambda)$. Slightly abusing notations, we denote the image of v in $L_\Sigma(\lambda)$ by v. A simple calculation shows that $Y_\alpha v$ is $\mathfrak{n}^+_{\Sigma'}$-invariant. If $\lambda(h_\alpha) \neq 0$, then $X_\alpha Y_\alpha v$ is proportional to v with a non-zero coefficient. Hence

$$M_\Sigma(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_{\Sigma'})} (\mathbb{C}Y_\alpha v) \cong M_{\Sigma'}(\lambda-\alpha), \text{ and } L_{\Sigma'}(\lambda-\alpha) \cong L_\Sigma(\lambda).$$

To show (c), assume that $\lambda(h_\alpha) = 0$. Then $Y_\alpha v$ generates a proper submodule in $M_\Sigma(\lambda)$. Therefore $Y_\alpha v = 0$ in $L_\Sigma(\lambda)$, thus v is a highest vector with respect to both Σ and Σ'. Hence $L_{\Sigma'}(\lambda) \cong L_\Sigma(\lambda)$. Finally, note that the submodule of $M_\Sigma(\lambda)$ generated by $Y_\alpha v$ is isomorphic to $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C(\lambda-\alpha)$ (indeed, choose a subalgebra $\mathfrak{m} \subset \mathfrak{n}^-$ of codimension one, such that $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{p}$. Then $U(\mathfrak{n}^-) = U(\mathfrak{m}) \oplus U(\mathfrak{m}) Y_\alpha$, hence $U(\mathfrak{g}) Y_\alpha v = U(\mathfrak{m}) Y_\alpha v$). That implies the first exact sequence. The second follows by symmetry.

Corollary 10.3. *If Σ is obtained from Π by odd reflections, then $\mathcal{O}_\Sigma = \mathcal{O}_\Pi$.*

Remark 10.4. Corollary 10.3 guarantees that the category $\mathcal{O}$ does not change if we change a base by odd reflection. In what follows we omit subindex in the notation for category $\mathcal{O}$.

A weight λ is called *integrable* if $L(\lambda)$ is an integrable module. One can use Lemma 10.2 to describe integrable weights: it implies that for any Σ obtained from Π by odd reflections there is exactly one weight λ_Σ such that $L_\Pi(\lambda) \cong L_\Sigma(\lambda_\Sigma)$.

Theorem 10.5. *Let $\mathfrak{g}$ be a regular quasisimple Kac–Moody superalgebra. A weight λ is integrable iff for any principal root β and any Σ obtained from Π by odd reflections such that $\beta \in \Sigma$, the condition $\lambda_\Sigma(h_\beta) \in \mathbb{Z}_{\geq 0}$ holds. If it holds for one Σ such that $\beta \in \Sigma$, then it holds for any Σ such that $\beta \in \Sigma$.*

Proof. If λ is integrable, then Y_β acts locally nilpotently on $L_\Sigma(\lambda_\Sigma) \cong L(\lambda)$; in particular, sufficiently large power of Y_β annihilates a highest vector v_Σ of $L_\Sigma(\lambda_\Sigma)$. The standard $\mathrm{sl}(2)$-calculation implies that $\lambda_\Sigma(h_\beta) \in \mathbb{Z}_{\geq 0}$.

On the other hand, if $\lambda_\Sigma(h_\beta) \in \mathbb{Z}_{\geq 0}$, then $Y_\beta^{\lambda_\Sigma(h_\beta)+1} v_\Sigma = 0$. Since $L_\Sigma(\lambda) = U(\mathfrak{g}) v_\Sigma$, and ad_{Y_β} is locally nilpotent on $U(\mathfrak{g})$, Y_β acts locally nilpotently on $L_\Sigma(\lambda)$.

If the condition $\lambda_\Sigma(h_\beta) \in \mathbb{Z}_{\geq 0}$ holds for one Σ containing β, then Y_β is locally nilpotent; hence the condition holds for any other Σ containing β.

Hence, in absence of odd reflections,

Corollary 10.6. *Let $\mathfrak{g}$ be a regular Kac–Moody superalgebra without isotropic simple roots. Then λ is integrable iff $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0}$ for any even simple root α, and $\lambda(h_\alpha) \in 2\mathbb{Z}_{\geq 0}$ for any odd simple root α.*

Indeed, if α is odd, then $h_\alpha/2$ is h_β for $\beta = 2\alpha \in \mathcal{B}$.

Since $\mathcal{B}$ is finite (see Remark 4.8), Theorem 10.5 is an explicit test for integrability. Using it, one can recover the description of integrable weights given in [1] for finite-dimensional $\mathfrak{g}$ (see appendix in [13]). Let us illustrate the method with examples.

Example 10.7. Let $\mathfrak{g} = D(2,1;a)$; choose the base $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ such that all simple roots are isotropic. All principle roots are $\beta_1 = \alpha_2 + \alpha_3$, $\beta_2 = \alpha_1 + \alpha_3$ and $\beta_3 = \alpha_1 + \alpha_2$. They are linearly independent. Therefore one can parameterize λ by setting $\lambda = (c_1, c_2, c_3)$, where $c_1 = \lambda(h_{\beta_1})$, $c_2 = \lambda(h_{\beta_2})$ and $c_3 = \lambda(h_{\beta_3})$. Using Remark 3.2 one gets

$$h_{\beta_1} = \frac{h_2 + ah_3}{a}, \quad h_{\beta_2} = -\frac{h_1 + ah_3}{a+1}, \quad h_{\beta_3} = h_1 + h_2. \tag{10}$$

Let $\Sigma_k = r_{\alpha_k}(\Pi)$, $k = 1,2,3$. If $\lambda(h_1) \neq 0$, then

$$\lambda_{\Sigma_1} = \lambda - \alpha_1 = (c_1 + 1, c_2 - 1, c_3 - 1).$$

Hence $c_2, c_3 \in \mathbb{Z}_{>0}$. If $\lambda(h_2) \neq 0$, then

$$\lambda_{\Sigma_2} = \lambda - \alpha_2 = (c_1 - 1, c_2 + 1, c_3 - 1)$$

and hence $c_1 \in \mathbb{Z}_{>0}$. We do not have to make the third odd reflection r_{α_3} since $\Sigma_1 = \{-\alpha_1, \beta_2, \beta_3\}$ and $\Sigma_2 = \{\beta_1, -\alpha_2, \beta_3\}$ contain already all principal roots. By symmetry, if $\lambda(h_i), \lambda(h_j) \neq 0$ for some $i \neq j$, then $c_1, c_2, c_3 \in \mathbb{Z}_{>0}$. On the other hand if $\lambda(h_i) = \lambda(h_j) = 0$ for some $i \neq j$, then two odd reflections do not change λ and hence $c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}$. Thus, $\lambda = (c_1, c_2, c_3)$ is integrable iff $c_1, c_2, c_3 \in \mathbb{Z}_{\geq 0}$ and, in addition, one of the following conditions hold

1. $c_1, c_2, c_3 > 0$;
2. $c_1 = (a+1) c_2 + c_3 = 0$;
3. $c_2 = -ac_1 + c_3 = 0$;
4. $c_3 = -ac_1 + (a+1) c_2 = 0$.

Note that for $\lambda \neq 0$ the conditions $(2) - (4)$ imply $a \in \mathbb{Q}$. Thus, there are more integrable weights for rational a than for irrational a.

Now, do similar calculations for $Q^{\pm}(k, l, m)$, which also has a base with all isotropic roots.

Example 10.8. Let $\mathfrak{g} = Q^{\pm}(m, k, l)$. Again $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, the principal roots are $\beta_1 = \alpha_2 + \alpha_3$, $\beta_2 = \alpha_1 + \alpha_3$ and $\beta_3 = \alpha_1 + \alpha_2$. Since the principal roots are linearly independent, we again can use parameterization $\lambda = (c_1, c_2, c_3)$, where $c_1 = \lambda(h_{\beta_1})$, $c_2 = \lambda(h_{\beta_2})$ and $c_3 = \lambda(h_{\beta_3})$. It is easy to check that

$$h_{\beta_3} = \frac{h_1}{a} + h_2, \quad h_{\beta_1} = \frac{h_2}{b} + h_3, \quad h_{\beta_2} = \frac{h_3}{c} + h_1. \tag{11}$$

We notice that, as in the previous example, integrability implies $c_i \in \mathbb{Z}_{\geq 0}$ for $i = 1, 2, 3$. Moreover, if at least for two h_i, h_j, $\lambda(h_i)$ and $\lambda(h_j) \neq 0$, then $c_1, c_2, c_3 \in \mathbb{Z}_{>0}$. Assume that $\lambda(h_1) = \lambda(h_2) = 0$. Then by (11), $c_1 = \lambda(h_3)$, $c_2 c = \lambda(h_3)$. It was shown in [3] that $a, b, c \notin \mathbb{Q}$. Hence $c_1 = c_2 = c_3 = 0$. Similarly, $\lambda(h_1) = \lambda(h_3) = 0$ and $\lambda(h_3) = \lambda(h_2) = 0$ imply $\lambda = 0$. Thus, all integrable weights are $(c_1, c_2, c_3) \in \mathbb{Z}^3_{>0}$ and 0.

The following theorem was proven in [11] for untwisted affine superalgebras. Other cases were done in [3].

Theorem 10.9. *Let* $\mathfrak{g} = q(n)^{(2)}$, $(p)\,\mathrm{sl}(m|n)^{(1)}$ *or* $(p)\,\mathrm{sl}(m|n)^{(2)}$ *with* $m, n \geq 2$, $\mathrm{osp}(m|2n)^{(1)}$ *or* $\mathrm{osp}(m|2n)^{(2)}$ *with* $m \geq 3$, $G_3^{(1)}$ *or* $F_4^{(1)}$*; then any integrable module* $L(\lambda)$ *is one-dimensional and trivial over* $[\mathfrak{g}, \mathfrak{g}]$.

Proof. Let $\mathfrak{g} = q(n)^{(2)}$. As follows from Section 9,

$$\mathfrak{g}' \oplus \mathbb{C}D \cong \mathrm{sl}(n)^{(1)} / \mathfrak{c},$$

where $\mathfrak{c}$ is the center of $\mathrm{sl}(n)^{(1)}$. Thus, λ is an integrable highest weight of $\mathrm{sl}(n)^{(1)}$ with zero central charge. A non-trivial integrable highest weight module over an affine Lie algebra must have a positive central charge (see [2]). Therefore $\lambda(h_\beta) = 0$ for all principal β. Recall from (9) that

$$\mathfrak{h} = \bigoplus_{\beta \in \mathcal{B}} \mathbb{C}h_\beta \oplus \mathbb{C}K \oplus \mathbb{C}D, \qquad \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}] = \bigoplus_{\beta \in \mathcal{B}} \mathbb{C}h_\beta \oplus \mathbb{C}K.$$

We claim next that $\lambda(K) = 0$.

Indeed, choose a base $\Pi = \{\alpha_1, \dots, \alpha_n\}$ such that α_1 is odd (automatically isotropic), and α_i is even for any $i \geq 2$ (and therefore principal); the Dynkin diagram is a loop, so α_1 is connected with α_2 and α_n. Let $\Sigma = r_{\alpha_1}(\Pi)$. A direct calculation shows that the subspace of $\mathfrak{h}$ spanned by h_β (for principal β) has codimension 1 in the space spanned by h_α, $\alpha \in \Delta$, and K and h_{α_1} are not in this subspace. (This does not contradict algebra being of type II, since in absence of duality this does not relate to geometry of $\{h_\beta \mid \beta \in \mathcal{B}\} \subset \mathfrak{h}$.) If $\lambda(K) \neq 0$, then $\lambda(h_{\alpha_1}) \neq 0$, therefore $\lambda_\Sigma = \lambda - \alpha_1$. Note that then $\lambda_\Sigma(h_{\alpha_i}) \neq 0$ for α_i connected with α_1 in Γ_Π. But $\lambda_\Sigma(h_\beta) = 0$ for any principal β. Contradiction. Thus $\lambda(\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]) = 0$, hence $L(\lambda)$ is one-dimensional.

For other superalgebras in the list $\mathfrak{g}(B)$ is a sum of two or three components. If there are two components, i.e. $\mathfrak{g} \neq D(2,1;a)^{(1)}$, one can use Corollary 9.4 or Remark 9.5, $\mathfrak{g}' \cong [\mathfrak{g}(B), \mathfrak{g}(B)]/\mathfrak{c}$, where $\mathfrak{c}$ is spanned by $K_1 + uK_2$ for some positive u. Since $L(\lambda)$ is integrable over $\mathfrak{g}'$, the $\mathfrak{g}'$-submodule $L_{\mathfrak{g}'}(\lambda) \subset L(\lambda)$ generated by a highest vector is a highest weight $\mathfrak{g}'$-integrable module. Therefore it is a $\mathfrak{g}(B)$-integrable module, and thus K_1 and K_2 have non-negative eigenvalues (see [2]). But $K_1 + uK_2$ acts by zero on $L(\lambda)$, therefore both K_1 and K_2 are zero. That implies $\lambda(h_\beta) = 0$ for any principal β and, by the same argument as for $q(n)^{(2)}$, $\lambda(\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]) = 0$, and $L(\lambda)$ is one-dimensional.

In the case $D(2,1;a)^{(1)}$ we have again $\mathfrak{g}' \cong [\mathfrak{g}(B), \mathfrak{g}(B)]/\mathfrak{c}$. The submodule $L_{\mathfrak{g}'}(\lambda)$ is the $\mathfrak{g}(B)$-module $L_{\mathfrak{g}(B)}(\mu)$; here μ is an integrable weight for $\mathfrak{g}(B)$ such that $\mu(\mathfrak{c}) = 0$. The description of $\mathfrak{c}$ in Example 9.3 implies

$$\mu(K_2) = a\mu(K_1); \qquad \mu(K_3) = -(1+a)\,\mu(K_1); \qquad \mu(K_i) \in \mathbb{Z}_{\geq 0};$$

therefore $\mu(K_i) = 0$. Apply representations theory of affine Lie algebras again; hence $L_{\mathfrak{g}(B)}(\mu)$ is one-dimensional, and $\lambda(h_\beta) = 0$ for all principal β. Since h_β generate $\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$, as above, $L(\lambda)$ is one-dimensional.

The above theorem, Theorem 7.1 and Theorem 7.3 imply the following corollary.

Corollary 10.10. *Let $\mathfrak{g}$ be an infinite-dimensional quasisimple regular Kac–Moody superalgebra, and $\mathfrak{g}$ has infinite-dimensional integrable highest weight modules. Then $\mathfrak{g}$ has no simple isotropic roots, or $\mathfrak{g}$ is isomorphic to* $\mathrm{sl}\,(1|n)^{(1)}$, $\mathrm{osp}\,(2|2n)^{(1)}$*,* $S\,(1,2;b)$ *or* $Q^{\pm}\,(k,l,m)$.

We describe the set of integrable weights for $\mathrm{sl}\,(1|n)^{(1)}$, $\mathrm{osp}\,(2|2n)^{(1)}$, $S\,(1,2;b)$ in Section 12 ($Q^{\pm}\,(m,k,l)$ is done in Example 10.8).

Remark 10.11. Theorem 10.9 shows that most affine superalgebras do not have interesting integrable highest weight representation. [11] and [12] propose a weaker condition of integrability by requiring integrability over one of the components of $\mathfrak{g}\,(B)$ and a finite-dimensional part of another component. These papers contain a conjecture about character formulae of such representations; these formulae have interesting applications in number theory and combinatorics. Although we do not discuss here partially integrable modules, we will prove the Kac–Wakimoto conjecture for $\mathrm{sl}\,(1|n)^{(1)}$ and $\mathrm{osp}\,(2|2n)^{(1)}$ in Theorem 14.7 (in this case $k=1$, hence their condition of integrability coincides with our condition).

11 General properties of category $\mathcal{O}$

Throughout this section $\mathfrak{g}=\mathfrak{g}\,(A)$ is a regular quasisimple Kac–Moody superalgebra with the standard base Π. Here we investigate properties of the category $\mathcal{O}$ which we need to calculate characters, and which are valid in the context of such $\mathfrak{g}$.

Let $\rho=\rho_{\Pi}$ be any element of $\mathfrak{h}^*$ satisfying the condition

$$\rho\,(h_\alpha)=\frac{a_{\alpha\alpha}}{2}$$

for all $\alpha\in\Pi$; the choice of ρ is unique unless A is degenerate. When Σ is obtained from Π by odd reflections, define

$$\rho_\Sigma=\rho+\sum_{\beta\in\Delta^+(\Pi)\cap\Delta^-(\Sigma)}\beta.$$

Lemma 11.1. *Let Σ be a base obtained from Π by odd reflections. Then $\rho_\Sigma\,(h_\alpha)=\frac{a_{\alpha\alpha}}{2}$ for any $\alpha\in\Sigma$.*

Proof. It is sufficient to check that if $\Sigma'=r_\beta\,(\Sigma)$ for some isotropic $\beta\in\Sigma$, then $\rho_\Sigma\,(h_\alpha)=\frac{\alpha(h_\alpha)}{2}$ for any $\alpha\in\Sigma$ implies $\rho_{\Sigma'}\left(h_{r_\beta(\alpha)}\right)=\frac{r_\beta(\alpha)\left(h_{r_\beta(\alpha)}\right)}{2}$. Note that, by Lemma 4.4, $\rho_{\Sigma'}=\rho_\Sigma+\alpha$. The statement can be checked by a direct calculation, which is not trivial only if $r_\beta\,(\alpha)=\alpha+\beta$. Then, by Remark 3.2, $h_{\alpha+\beta}=\alpha\,(h_\beta)\,h_\alpha+\beta\,(h_\alpha)\,h_\beta$, so

$$\rho_{\Sigma'}\,(h_{\alpha+\beta})=\frac{\alpha\,(h_\beta)\,\alpha\,(h_\alpha)}{2}+\alpha\,(h_\beta)\,\beta\,(h_\alpha)=\frac{\langle\alpha+\beta,h_{\alpha+\beta}\rangle}{2}.$$

Remark 11.2. It is convenient to reformulate Lemma 10.2 in the following way. Let $\Sigma' = r_\alpha(\Sigma)$ for an odd isotropic $\alpha \in \Sigma$. If $\langle \lambda + \rho, h_\alpha \rangle \neq 0$, then

$$\lambda_{\Sigma'} + \rho_{\Sigma'} = \lambda_\Sigma + \rho_\Sigma. \tag{12}$$

If $\langle \lambda + \rho, h_\alpha \rangle = 0$, then

$$\lambda_{\Sigma'} + \rho_{\Sigma'} = \lambda_\Sigma + \rho_\Sigma + \alpha. \tag{13}$$

A weight $\lambda \in \mathfrak{h}^*$ is called *typical* if $\langle \lambda + \rho, h_\alpha \rangle \neq 0$ for any real isotropic root α. (12) implies that if λ is typical, then $\lambda_\Sigma + \rho_\Sigma$ is invariant with respect to odd reflections. Hence the notion of typicality can be defined for an irreducible highest weight module $L(\lambda)$ independently of a choice of a base, i.e., if λ is typical, then λ_Σ is typical.

If A is symmetrizable, then $\mathfrak{g}$ admits an invariant symmetric non-degenerate even form $(\cdot, \cdot)$; this form induces an isomorphism $\eta : \mathfrak{h} \to \mathfrak{h}^*$ such that $\eta(h_\alpha) = \frac{2\alpha}{(\alpha,\alpha)}$ for every real even root α. If α is a real isotropic root, then $\eta(h_\alpha)$ is proportional to α. In this case the typicality condition can be re-written as $(\lambda + \rho, \alpha) \neq 0$ for any real isotropic root α. Note also that in this case (12) and (13) imply that for any Σ' obtained from Σ by odd reflections

$$(\lambda_{\Sigma'} + \rho_{\Sigma'}, \lambda_{\Sigma'} + \rho_{\Sigma'}) = (\lambda_\Sigma + \rho_\Sigma, \lambda_\Sigma + \rho_\Sigma). \tag{14}$$

The following statement can be found in [11]; it is a straightforward generalization of the corresponding statement for Lie algebras.

Lemma 11.3. *Let $\mathfrak{g}$ be a Kac–Moody superalgebra with an invariant non-degenerate symmetric even form. For each $\alpha \in \Delta^+$, let $e_\alpha^1, \dots, e_\alpha^{s_\alpha}$ be a basis in $\mathfrak{g}_\alpha$, $f_\alpha^1, \dots, f_\alpha^{s_\alpha}$ be the dual basis in $\mathfrak{g}_{-\alpha}$, $u_1, \dots, u_r$ and $u^1, \dots, u^r$ be dual bases in $\mathfrak{h}$. The operator*

$$\Omega = 2\eta^{-1}(\rho) + 2 \sum_{\alpha \in \Delta^+} \sum_j f_\alpha^j e_\alpha^j$$

is well-defined on any module from the category $\mathcal{O}$ and commutes with the action of $\mathfrak{g}$. Ω acts on $M(\lambda)$ as a scalar operator with eigenvalue $(\lambda + 2\rho, \lambda)$.

For any base Σ introduce a partial order on $\mathfrak{h}^*$ by putting

$$\mu \leq_\Sigma \lambda \qquad \text{iff} \qquad \lambda - \mu = \sum_{\alpha \in \Delta^+(\Sigma)} m_\alpha \alpha$$

for some $m_\alpha \in \mathbb{Z}_{\geq 0}$. For the standard base Π, we omit the subindex in $\leq_\Pi$.

The next three statements below are straightforward generalizations of the similar statements for Kac–Moody Lie algebras (which can be found in [2]). The proofs are omitted.

Lemma 11.4. *Let $\mathfrak{g}$ be an arbitrary Kac–Moody superalgebra, $M \in \mathcal{O}$ and $\nu \in \mathfrak{h}^*$. There exists a filtration*

$$0 = F^0(M) \subset F^1(M) \subset F^2(M) \subset \cdots \subset F^s(M) = M$$

and a subset of indices J such that $F^{i+1}(M)/F^i(M) = L(\lambda_i)$ for every $i \in J$ and for every $\mu \geq \nu$, $\mu \notin \bigcup_{j \notin J} P\left(F^j(M)/F^{j-1}(M)\right)$. For every $\lambda \geq \nu$, define the multiplicity *$[M : L(\lambda)]$ as $\#\{i \mid \lambda_i = \lambda\}$. The multiplicity $[M : L(\lambda)]$ depends neither on a choice of filtration, nor on $\nu \leq \lambda$. Hence $[M : L(\lambda)]$ is well defined for any λ.*

Lemma 11.5. *(a) $[M(\lambda) : L(\lambda)] = 1$;*

(b) $[M(\lambda) : L(\mu)] > 0$ implies $\lambda \geq \mu$;

(c) If $\mathfrak{g}$ has an invariant form $(\cdot,\cdot)$, then $[M(\lambda) : L(\mu)] > 0$ implies $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$.

Corollary 11.6. *Let $V \neq 0$ be a quotient of $M(\lambda)$. There is a unique way to write the character of V as an (infinite) linear combination*

$$\operatorname{ch} V = \sum_{\mu \leq \lambda} a_\mu \operatorname{ch} M(\mu).$$

Furthermore, $a_\lambda = 1$, and if $T = \{\mu \mid a_\mu \neq 0\}$, then for any $\mu \in T$ there are $\nu_1, \dots, \nu_k \in T$ such that

$$[M(\lambda) : L(\nu_1)] > 0, \quad [M(\nu_1) : L(\nu_2)] > 0, \quad \dots, \quad [M(\nu_k) : L(\mu)] > 0.$$

If $\mathfrak{g}$ has an invariant form $(\cdot,\cdot)$, then all $\mu \in T$ satisfy the additional condition $(\lambda + \rho, \lambda + \rho) = (\mu + \rho, \mu + \rho)$.

Lemma 11.7. *Let Σ be obtained from Π by odd reflections. Then $[M(\lambda) : L(\mu)] = [M_\Sigma(\lambda + \rho - \rho_\Sigma) : L(\mu)]$ for any μ.*

Proof. Sufficient to check that $[M_\Sigma(\lambda) : L(\mu)] = [M_{\Sigma'}(\lambda + \rho_{\Sigma'} - \rho_\Sigma) : L(\mu)]$ for any μ if $\Sigma' = r_\alpha(\Sigma)$. In this case it follows directly from Lemma 10.2 (b) and (c).

Corollary 11.8. *Let V be a subquotient of $M(\lambda)$. For any weight μ of V and any Σ obtained from Π by odd reflections, $\mu + \rho_\Sigma \leq_\Sigma \lambda + \rho$.*

Proof. If μ is a weight of $M(\lambda)$, then μ is a weight of $M_\Sigma(\lambda + \rho - \rho_\Sigma)$

Corollary 11.9. *If $[M(\lambda) : L(\mu)] > 0$, then $\mu_\Sigma + \rho_\Sigma \leq_\Sigma \lambda + \rho$ for any Σ obtained from Π by odd reflections.*

Proof. If $[M(\lambda) : L(\mu)] > 0$, then $[M_\Sigma(\lambda + \rho - \rho_\Sigma) : L(\mu)] > 0$. But $L(\mu) = L_\Sigma(\mu_\Sigma)$. Hence μ_Σ is a weight of $M_\Sigma(\lambda + \rho - \rho_\Sigma)$. Now the statement follows from the previous Corollary.

12 Lie superalgebras $\mathrm{sl}\,(1|n)^{(1)}$, $\mathrm{osp}\,(2|2n)^{(1)}$ and $S\,(1,2;b)$

In this section $\mathfrak{g}$ is $\mathrm{sl}\,(1|n)^{(1)}$, $\mathrm{osp}\,(2|2n)^{(1)}$, or $S\,(1,2;b)\,(b \notin \mathbb{Z})$. The importance of these particular Lie superalgebras for our discussion is that that they are the only finite growth regular quasisimple Kac–Moody superalgebras with isotropic roots which have non-trivial integrable modules, and it is easy to describe all bases obtained by odd reflections.

If $\mathfrak{g} = \mathrm{sl}\,(1|n)^{(1)}$ or $S\,(1,2;b)$, a graph of any base is a cycle of length $n+1$ with exactly two isotropic roots (which are neighbors); here $n = 2$ for the case of $S\,(1,2;b)$. If $\mathfrak{g} = \mathrm{osp}\,(2|2n)$, then we will see that all possible Dynkin graphs are

$$\circ \Rightarrow \circ - \cdots - \circ - \otimes - \otimes - \circ - \cdots - \circ \Leftarrow \circ, \qquad \begin{matrix} \otimes \\ | \\ \otimes \end{matrix} \rangle \circ - \cdots - \circ \Leftarrow \circ.$$

Here we write $\otimes$ instead of gray nodes.

For some of our calculation we need an explicit description of roots of $\mathfrak{g}$. One can choose linearly independent $\varepsilon, \varepsilon_1, \dots, \varepsilon_n, \delta \in \mathfrak{h}^*$ so that the even roots of $\mathrm{sl}\,(1|n)^{(1)}$ are $\varepsilon_i - \varepsilon_j + m\delta$, $m \in \mathbb{Z}$, $i \neq j$, and $m\delta$, $m \in \mathbb{Z} \setminus \{0\}$; odd roots of $\mathrm{sl}\,(1|n)^{(1)}$ are $\pm(\varepsilon + \varepsilon_i) + m\delta$, $m \in \mathbb{Z}$. If $\mathfrak{g} = \mathrm{osp}\,(2|2n)^{(1)}$, then the even roots of $\mathfrak{g}$ are $\pm\varepsilon_i \pm \varepsilon_j + m\delta$, $\pm 2\varepsilon_i + m\delta$, $m \in \mathbb{Z}$, and $m\delta$, $m \in \mathbb{Z} \setminus \{0\}$; odd roots of $\mathfrak{g}$ are $\pm\varepsilon \pm \varepsilon_i + m\delta$, $m \in \mathbb{Z}$. Finally, the roots of $S\,(2,1;b)$ are the same as the roots of $\mathrm{sl}\,(1|2)^{(1)}$, as it was explained in Section 6.

For the standard base in case of $\mathrm{sl}\,(1|n)^{(1)}$ (or $S\,(1,2;b)$) choose

$$\Pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n + \varepsilon, -\varepsilon - \varepsilon_1 + \delta\}\,;$$

for $\mathrm{osp}\,(2|2n)^{(1)}$ choose

$$\Pi = \{-2\varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n + \varepsilon, \varepsilon_n - \varepsilon + \delta\}\,.$$

Since $\mathfrak{g}$ is of type I, it has a grading $\mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$; it is

$$\begin{aligned} \Delta\,(\mathfrak{g}_1) &= \{\varepsilon + \varepsilon_i + m\delta\}, \quad \text{if } \mathfrak{g} = \mathrm{sl}\,(1|n)^{(1)} \text{ or } S\,(1,2;b)\,, \\ \Delta\,(\mathfrak{g}_1) &= \{\varepsilon \pm \varepsilon_i + m\delta\}, \quad \text{if } \mathfrak{g} = \mathrm{osp}\,(2|2n)^{(1)}\,, \end{aligned}$$

and $\Delta\,(\mathfrak{g}_{-1}) = -\Delta\,(\mathfrak{g}_1)$. This grading induces a $\mathbb{Z}$-grading on the root lattice

$$Q = \bigoplus_{i \in \mathbb{Z}} Q^i. \tag{15}$$

Set

$$Q^+ = \bigoplus_{i>0} Q^i, \qquad Q^- = \bigoplus_{i<0} Q^i. \tag{16}$$

This grading induces the decomposition

$$\Delta_1^{\pm} = \Delta^{\pm}(\mathfrak{g}_1) \cup \Delta^{\pm}(\mathfrak{g}_{-1}) .$$

The standard base Π has exactly two isotropic roots, one in $\Delta^+(\mathfrak{g}_1)$, and one in $\Delta^+(\mathfrak{g}_{-1})$. The corresponding vertices in Γ_Π are connected, and each of them is connected with one even vertex (the latter might be the same for both isotropic vertices). These properties also hold for any base Σ; indeed, the set of diagrams described in the beginning of the section is invariant w.r.t. any odd reflection, and contains a diagram of Π. In particular, every base Σ has two isotropic roots, moreover

$$|\Sigma \cap \Delta(\mathfrak{g}_{-1})| = |\Sigma \cap \Delta(\mathfrak{g}_1)| = 1. \tag{17}$$

Let

$$\gamma_1 \in \Sigma \cap \Delta(\mathfrak{g}_{-1}), \qquad \alpha_1 \in \Sigma \cap \Delta(\mathfrak{g}_1).$$

The set $\mathcal{B}$ of principal roots coincides with the set of even simple roots of Σ taken together with $\alpha_1 + \gamma_1$; indeed, the latter set is preserved by both odd reflections. In particular, $\mathcal{B}$ is linearly independent. Define an infinite sequence $\{\alpha_k\} \in \Delta(\mathfrak{g}_1)$ by

$$\alpha_2 \in r_{\alpha_1}(\Sigma) \cap \Delta(\mathfrak{g}_1), \quad \dots, \quad \alpha_{k+1} \in r_{\alpha_k} \dots r_{\alpha_1}(\Sigma) \cap \Delta(\mathfrak{g}_1), \quad \dots$$

Similarly, define $\{\gamma_k\} \in \Delta(\mathfrak{g}_{-1})$ by

$$\gamma_2 \in r_{\gamma_1}(\Sigma) \cap \Delta(\mathfrak{g}_{-1}), \quad \dots, \quad \gamma_{k+1} \in r_{\gamma_k} \dots r_{\gamma_1}(\Sigma) \cap \Delta(\mathfrak{g}_{-1}), \quad \dots$$

Let $N = n$ if $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$, $N = 2n$ for $\mathfrak{g} = \mathrm{osp}(2|2n)^{(1)}$. Using the explicit description of roots, one can see that $\alpha_{k+N} - \alpha_k = \gamma_{k+N} - \gamma_k = \delta$; moreover, $\gamma_k = -\alpha_{N+1-k} + \delta$ if $k \le N$. If $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$, and $\Sigma = \Pi$,

$$\alpha_1 = \varepsilon_n + \varepsilon, \quad \alpha_2 = \varepsilon_{n-1} + \varepsilon, \quad \dots, \quad \alpha_n = \varepsilon_1 + \varepsilon;$$

If $\mathfrak{g} = \mathrm{osp}(2|2n)^{(1)}$, $\Sigma = \Pi$, then

$$\alpha_1 = \varepsilon_n + \varepsilon, \quad \dots, \quad \alpha_n = \varepsilon_1 + \varepsilon, \quad \alpha_{n+1} = -\varepsilon_1 + \varepsilon, \quad \dots, \quad \alpha_{2n} = -\varepsilon_n + \varepsilon.$$

Unless stated otherwise α_i and γ_i are for $\Sigma = \Pi$.

It follows from (17) that every base Σ which can be obtained from Π by odd reflection is either $r_{\alpha_k} \dots r_{\alpha_1}(\Pi)$, or $r_{\gamma_k} \dots r_{\gamma_1}(\Pi)$ for some $k > 0$. Moreover, every odd root is real and isotropic, and

$$\Delta^+(\mathfrak{g}_1) = \{\alpha_1, \dots, \alpha_k, \dots\}, \quad \Delta^+(\mathfrak{g}_{-1}) = \{\gamma_1, \dots, \gamma_k, \dots\}.$$

Imaginary even roots are (by Theorem 9.1) $\{m\delta \mid m \in \mathbb{Z} \setminus \{0\}\}$.

Lemma 12.1. *Let α_1 and γ_1 be isotropic roots of Σ, and $\gamma \in Q^i \cap C^+(\Sigma)$. Then*

$$\gamma = i\alpha_1 + \sum_{\beta \in \mathcal{B}} m_\beta \beta, \quad \text{if } i \geq 0, \qquad \gamma = -i\gamma_1 + \sum_{\beta \in \mathcal{B}} m_\beta \beta, \quad \text{if } i \leq 0,$$

for some $m_\beta \in \mathbb{Z}_{\geq 0}$; the representation is unique.

Proof. Uniqueness is obvious. Express γ as a linear combination of simple roots

$$\gamma = n_1\gamma_1 + n_2\alpha_1 + \sum_{\beta \in \mathcal{B} \cap \Sigma} m_\beta \beta.$$

Note that $i = n_2 - n_1$. If $i \geq 0$, one can write it as $\gamma = i\alpha_1 + n_1(\gamma_1 + \alpha_1) + \sum_{\beta \in \mathcal{B} \cap \Sigma} m_\beta \beta$. If $i < 0$, one can write it as $\gamma = -i\gamma_1 + n_2(\gamma_1 + \alpha_1) + \sum_{\beta \in \mathcal{B} \cap \Sigma} m_\beta \beta$.

Since $\gamma_1 + \alpha_1 \in \mathcal{B}$, this finishes the proof.

Lemma 12.2. *Let $k \geq 0$. Then*

$$Q^+ \cap \bigcap_{s \geq k} C^+(r_{\alpha_s} \dots r_{\alpha_1}(\Pi)) = \varnothing, \qquad Q^- \cap \bigcap_{s \geq k} C^+(r_{\gamma_s} \dots r_{\gamma_1}(\Pi)) = \varnothing.$$

Proof. Let $\gamma \in Q^+ \cap \bigcap_{s \geq k} C^+(r_{\alpha_s} \dots r_{\alpha_1}(\Pi))$. By Lemma 12.1 (applied to $\Sigma = r_{\alpha_s} \dots r_{\alpha_1}(\Pi)$)

$$\gamma = i\alpha_s + \sum_{\beta \in \mathcal{B}} m_\beta^s \beta,$$

for every $s \geq k$. Let $\beta_s = \alpha_{s+1} - \alpha_s$; note that $\beta_s \in \mathcal{B}$. Then

$$m_\beta^{s+1} = m_\beta^s \quad \text{if } \beta \neq \beta_s, \qquad m_\beta^{s+1} = m_\beta^s - i \quad \text{if } \beta = \beta_s.$$

Since $m_\beta^s \geq 0$ for all $\beta \in \mathcal{B}$ and all $s \geq k$, $i \leq 0$. Contradiction.

The proof of the second statement is similar.

Lemma 12.3. *Let $[M(\lambda) : L(\mu)] > 0$. Then there exists a base Σ obtained from Π by odd reflections, and numbers $m_\beta \in \mathbb{Z}_{\geq 0}$ such that*

$$\lambda + \rho - \mu_\Sigma - \rho_\Sigma = \sum_{\beta \in \mathcal{B}} m_\beta \beta.$$

Proof. Let $\gamma_\Sigma = \lambda + \rho - \mu_\Sigma - \rho_\Sigma$. By Corollary 11.9, $0 \leq_\Sigma \gamma_\Sigma$ for any Σ obtained from Π by odd reflections. Let $\Sigma' = r_\alpha(\Sigma)$ for some odd $\alpha \in \Sigma$. Then by (12) and (13)

$$\gamma_\Sigma = \gamma_{\Sigma'} \quad \text{if } \langle \mu_\Sigma + \rho_\Sigma, h_\alpha \rangle \neq 0, \qquad \gamma_\Sigma = \gamma_{\Sigma'} - \alpha \quad \text{if } \langle \mu_\Sigma + \rho_\Sigma, h_\alpha \rangle = 0. \tag{18}$$

Suppose that $\gamma_{\Sigma_1} \in Q^+$ for some Σ_1, and $\gamma_{\Sigma_2} \in Q^-$ for some Σ_2. Since Σ_1 and Σ_2 are connected by odd reflections, and every odd root α is in $Q^{\pm 1}$, there exists Σ such that $\gamma_\Sigma \in Q^0$. In this case the statement is true by Lemma 12.1.

Now assume that $\gamma_{\Sigma} \in Q^{+}$ for all Σ. Then one can find $i > 0$ and $k > 0$ such that $\gamma_{r_{\alpha_s}\dots r_{\alpha_1}(\Pi)} \in Q^{i}$ for all $s \geq k$. Hence, by (18),

$$\gamma_{r_{\alpha_s}\dots r_{\alpha_1}(\Pi)} = \gamma_{r_{\alpha_{s'}}\dots r_{\alpha_1}(\Pi)} = \gamma$$

for all $s, s' > k$. Thus $\gamma \in Q^{+} \cap \bigcap_{s \geq k} C^{+}\left(r_{\alpha_s} \dots r_{\alpha_1}(\Pi)\right)$, which is impossible by Lemma 12.2. Similarly, the case when $\gamma_{\Sigma} \in Q^{-}$ for all Σ is impossible.

Lemma 12.4. *Let $\lambda \in \mathfrak{h}^{*}$. $L_{\Sigma}(\lambda)$ is integrable iff $\langle \lambda + \rho_{\Sigma}, h_{\beta} \rangle \in \mathbb{Z}_{>0}$ for any $\beta \in \mathcal{B} \cap \Sigma$, and one of the following two conditions holds:*

1. $\langle \lambda + \rho_{\Sigma}, h_{\alpha_1 + \gamma_1} \rangle \in \mathbb{Z}_{>0}$,
2. $\langle \lambda + \rho_{\Sigma}, h_{\alpha_1} \rangle = \langle \lambda + \rho_{\Sigma}, h_{\gamma_1} \rangle = 0$.

Proof. There is only one $\beta \in \mathcal{B} \setminus \Pi$. To check the conditions of Theorem 10.5 make an odd reflection r_{α_1}. The details are left to the reader.

Remark 12.5. Call $\mu \in \mathfrak{h}^{*}$ *regular* if $\langle \mu, h_{\beta} \rangle \neq 0$ for any even real root β. As follows from the lemma above, if $L_{\Sigma}(\lambda)$ is integrable, then $\lambda + \rho_{\Sigma}$ is regular iff the first condition holds.

Lemma 12.6. *Let $L_{\Sigma}(\lambda)$ be integrable, $\lambda + \rho_{\Sigma}$ be not regular, and $\lambda(h_{\beta}) \neq 0$ at least for one $\beta \in \mathcal{B}$. Then one can find a base Σ' obtained from Σ by odd reflections such that $L_{\Sigma'}(\lambda) \cong L_{\Sigma}(\lambda)$, and $\lambda + \rho_{\Sigma'}$ is regular. Moreover, one can choose this base Σ' equal to $r_{\gamma_k} \dots r_{\gamma_1}(\Sigma)$, and find $w \in W$ such that*

$$\lambda + \rho_{\Sigma'} = \lambda + \rho_{\Sigma} + \gamma_1 + \dots + \gamma_k = w\left(\lambda + \rho - k\alpha_1\right), \qquad (-1)^{w} = (-1)^{k}, \qquad w(\alpha_1) = -\gamma_k.$$

The choice of k is unique.

Proof. If $\lambda + \rho_{\Sigma}$ is not regular, then $\langle \lambda + \rho_{\Sigma}, h_{\alpha_1} \rangle = \langle \lambda + \rho_{\Sigma}, h_{\gamma_1} \rangle = 0$. Let $\Sigma_k = r_{\gamma_k} \dots r_{\gamma_1}(\Sigma)$. We claim that $L_{\Sigma_k}(\lambda)$ is not isomorphic to $L_{\Sigma}(\lambda)$ at least for one k. Indeed, otherwise $(\lambda, \gamma_i) = 0$ for all i, but that would imply $\langle \lambda, h_{\beta} \rangle = 0$ for all $\beta \in \mathcal{B}$. Choose the last k such that $L_{\Sigma_k}(\lambda)$ and $L_{\Sigma}(\lambda)$ are isomorphic. Then

$$\lambda + \rho_{\Sigma_k} = \lambda + \rho_{\Sigma} + \gamma_1 + \dots + \gamma_k, \qquad \left\langle \lambda + \rho_{\Sigma_k}, h_{\gamma_{k+1}} \right\rangle \neq 0.$$

By Remark 12.5 applied to $\Sigma = \Sigma_k$, $\lambda + \rho_{\Sigma_k}$ must be regular. Next consider even roots

$$\beta_1 = \alpha_1 + \gamma_1, \quad \beta_2 = \gamma_2 - \gamma_1, \quad \dots, \quad \beta_k = \gamma_k - \gamma_{k-1}.$$

Note that for any $i < k$, $\beta_{i+1} \in \mathcal{B} \setminus \Sigma_i$, and that

$$\left\langle \lambda + \rho_{\Sigma} + \gamma_1 + \dots + \gamma_i, h_{\beta_{i+1}} \right\rangle = 0.$$

Furthermore, $r_{\beta_{i+1}}(\gamma_{i+1}) = \gamma_i$ (recall $\gamma_0 = -\alpha_1$). Therefore

$$\lambda + \rho_{\Sigma} + \gamma_1 + \dots + \gamma_k = r_{\beta_k}\left(\lambda + \rho_{\Sigma} + \gamma_1 + \dots + \gamma_{k-1} + \gamma_{k-1}\right) = r_{\beta_k}\left(\lambda + \rho_{\Sigma} + \gamma_1 + \dots + 2\gamma_{k-1}\right) =$$
$$r_{\beta_{k-1}} r_{\beta_k}\left(\lambda + \rho_{\Sigma} + \gamma_1 + \dots + 3\gamma_{k-2}\right) = \dots = r_{\beta_1} \dots r_{\beta_k}\left(\lambda + \rho_{\Sigma} + k\gamma_0\right).$$

Remark 12.7. Recall that $\mathfrak{g}' \oplus \mathbb{C}D$ is isomorphic to $\mathfrak{g}(B)$ which is $\mathrm{sl}(n)^{(1)}$ for $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$, $\mathrm{sl}(2)^{(1)}$ for $\mathfrak{g} = S(1,2;b)$, and $\mathrm{sp}(2n)^{(1)}$ for $\mathfrak{g} = \mathrm{osp}(2|2n)^{(1)}$. Principal roots $\mathcal{B}$ form a base of $\mathfrak{g}'$. Let

$$C = \{\mu \in \mathfrak{h}^* \mid \mu(h_\beta) \in \mathbb{Z}_{\geq 0}, \beta \in \mathcal{B}\}.$$

It follows from Lemma 12.4, that if $L_\Sigma(\lambda)$ is integrable, then $\lambda \in C$. We need two following facts about affine Weyl group action, see [2]. If $\lambda \in C$, then $w(\lambda) = \lambda - \sum_{\beta \in \mathcal{B}} m_\beta \beta$ for some $m_\beta \in \mathbb{Z}_{\geq 0}$. If $\mu(h_\beta) \in \mathbb{Z}$ for all $\beta \in \mathcal{B}$ and $\mu(K) > 0$, then $W\mu$ intersects C in exactly one point.

Consider the ring $\mathcal{R}$ of all formal expressions $\sum_{\mu \in P} c_\mu e^\mu$ for all P satisfying the condition

$$\text{there is a finite set } L \subset \mathfrak{h}^* \text{ such that } P \subset L - \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0}\alpha. \tag{19}$$

It is not difficult to check that indeed such $\mathcal{R}$ is a commutative ring without zero divisors. Let $\mathcal{R}'$ be a subring satisfying the additional condition that (19) holds for $w(P)$ for any $w \in W$. Then $\mathcal{R}'$ enjoys the natural action of W; this action preserves multiplication and addition. For an arbitrary $\mu \in \mathfrak{h}^*$ and Σ obtained from Π by odd reflections, define

$$U_\Sigma(\mu) = \frac{e^{-\rho_\Sigma} \prod_{\alpha \in \Delta_1^+(\Sigma)} (1 + e^{-\alpha})^{m(\alpha)}}{\prod_{\alpha \in \Delta_0^+(\Sigma)} (1 - e^{-\alpha})^{m(\alpha)}} \sum_{w \in W} (-1)^w e^{w(\mu)}.$$

It is an immediate calculation that for any Σ' obtained from Σ by odd reflections and for any $w \in W$

$$U_\Sigma(\mu) = U_{\Sigma'}(\mu), \quad U_\Sigma(w(\mu)) = (-1)^w U_\Sigma(\mu). \tag{20}$$

Hence one can drop the index Σ in $U_\Sigma(\mu)$.

Assume now that $\mu \in C$. Then $w(\mu) \leq \mu$ and hence $U_\Sigma(\mu) \in \mathcal{R}$. It easily follows from Lemma 10.1 (a) that

$$U(\mu) = \sum_{w \in W} (-1)^w \operatorname{ch} M(w(\mu) - \rho). \tag{21}$$

If e^ν appears in $\operatorname{ch} M(\kappa)$ with non-zero coefficient, then $\nu \leq \kappa$. Therefore, if e^ν appears in $U(\mu)$ with non-zero coefficient, then $\nu \leq w(\mu) - \rho \leq \mu - \rho$.

Lemma 12.8. *If μ is regular and integrable, then $U(\mu) \in \mathcal{R}'$. Moreover, $w(U(\mu)) = U(\mu)$ for any $w \in W$.*

Proof. Let $\rho_0 \in \mathfrak{h}^*$ be such that $\rho_0(h_\beta) = \frac{a_{\beta\beta}}{2}$ for all $\beta \in \mathcal{B}$, and let $\rho_1 = \rho_0 - \rho$. Introduce the expressions

$$D_0 = e^{\rho_0} \prod_{\alpha\in\Delta_0^+} \left(1 - e^{-\alpha}\right)^{m(\alpha)}, \qquad D_1 = e^{\rho_1} \prod_{\alpha\in\Delta_1^+} \left(1 + e^{-\alpha}\right)^{m(\alpha)}.$$

Then

$$U(\mu) = \frac{D_1}{D_0} \sum_{w\in W} (-1)^w e^{w(\mu)}.$$

Note that $\Delta(\mathfrak{g}') = \Delta_0$, and the multiplicity of $k\delta$ in $\mathfrak{g}'$ is $m(k\delta) - 1$ (see Theorem 9.1 (c), the case of $\mathfrak{s}$ with one-dimensional center)

$$D_0 = D_0' \prod_{n=1}^{\infty} \left(1 - e^{-n\delta}\right),$$

where D_0' the corresponding expression for the Lie subalgebra $\mathfrak{g}' + \mathfrak{h}$. The expression

$$S(\mu) = \frac{\sum_{w\in W} (-1)^w e^{w(\mu)}}{D_0'}$$

gives a character of the simple integrable module over Kac–Moody Lie algebra $\mathfrak{g}' + \mathfrak{h}$ with highest weight $\mu - \rho_0$ (see [2]). (Note that regularity and integrability of μ implies $\mu - \rho_0 \in C$.) Therefore $S(\mu)$ is W-invariant, and $S(\mu) \in \mathcal{R}'$. On the other hand, $\prod_{n=1}^{\infty} 1/\left(1 - e^{-n\delta}\right)$ is W-invariant, since $W\delta = \delta$. Therefore, it is sufficient to show that D_1 is W-invariant. One has to check that $r_\beta(D_1) = D_1$ for all $\beta \in \mathcal{B}$; assume first that $\beta \in \Pi$; then $\rho_1(h_\beta) = 0$, so $r_\beta(\rho_1) = \rho_1$; moreover, by Lemma 4.4, r_β permutes roots of Δ_1^+. Hence $r_\beta(D_1) = D_1$. If $\beta \notin \Pi$, then

$$\beta = \gamma_1 + \alpha_1, \qquad \rho(h_\beta) = 0, \qquad \rho_1(h_\beta) = 1, \qquad r_\beta(\rho_1) = \rho_1 - \beta = \rho_1 - \alpha_1 - \gamma_1.$$

Furthermore, since $\beta = \alpha_1 + \gamma_1$,

$$r_\beta(\alpha_1) = -\gamma_1, \qquad r_\beta(\gamma_1) = -\alpha_1,$$

therefore $r_\beta r_{\gamma_1} = r_{-\alpha_1} r_\beta$ by (5), which implies that r_β permutes the roots of $\Delta_1^+ \setminus \{\alpha_1, \gamma_1\}$. Hence

$$r_\beta\left(e^{\rho_1}\left(1 + e^{-\alpha_1}\right)\left(1 + e^{-\gamma_1}\right)\right) = e^{\rho_1 - \alpha_1 - \gamma_1}\left(1 + e^{\alpha_1}\right)\left(1 + e^{\gamma_1}\right) = e^{\rho_1}\left(1 + e^{-\alpha_1}\right)\left(1 + e^{-\gamma_1}\right),$$

$$r_\beta \prod_{\alpha\in\Delta_1^+\setminus\{\alpha_1,\gamma_1\}} \left(1 + e^{-\alpha}\right)^{m(\alpha)} = \prod_{\alpha\in\Delta_1^+\setminus\{\alpha_1,\gamma_1\}} \left(1 + e^{-\alpha}\right)^{m(\alpha)}.$$

That finishes the proof.

Remark 12.9. It is useful to note that D_1 is not only W-invariant, but also independent of a choice of Σ. In other words,

$$D_1 = e^{\rho_0 - \rho_\Sigma} \prod_{\alpha\in\Delta_1^+(\Sigma)} \left(1 + e^{-\alpha}\right)$$

for any Σ obtained from Π by odd reflections.

13 Lie superalgebras $\mathrm{sl}\,(1|n)^{(1)}$, $\mathrm{osp}\,(2|2n)^{(1)}$

From now on we assume that $\mathfrak{g}$ is $\mathrm{sl}\,(1|n)^{(1)}$ or $\mathrm{osp}\,(2|2n)^{(1)}$.

Lemma 13.1. *There exists an invariant symmetric even form on $\mathfrak{g}$ such that the corresponding form on $Q \times Q$ is semi-positive, takes integer values, and $(\alpha, \alpha) > 0$ for any even real root α.*

Proof. Just use the form in (7). The positivity conditions follow from those on $\mathrm{sl}\,(1|n)$ and $\mathrm{osp}\,(2|2n)$.

Remark 13.2. One can normalize an invariant form on $\mathfrak{g}$ so that the corresponding form on $\mathfrak{h}^*$ satisfies the relations

$$(\varepsilon, \varepsilon) = -1, \quad (\varepsilon_i, \varepsilon_j) = \delta_{ij}, \qquad (\varepsilon, \varepsilon_i) = (\varepsilon, \delta) = (\varepsilon_i, \delta) = 0.$$

One can see that $(\delta, Q) = 0$. On the other hand, $(\lambda, \delta) = \lambda(K)$ for any $\lambda \in \mathfrak{h}^*$. In the case of $\mathrm{sl}\,(1|n)^{(1)}$, all real even roots have the same length. If $\mathfrak{g} = \mathrm{osp}\,(2|2n)^{(1)}$ with $n \geq 2$, then $(\beta, \beta) = 2$ for a short real even root, and $(\beta, \beta) = 4$ for a long real even root. Since $\eta(h_\beta) = \frac{2\beta}{(\beta,\beta)}$, given an integrable weight λ, we have $(\lambda + \rho, \beta) \in \mathbb{Z}$ for any real even short root β, and $(\lambda + \rho, \beta) \in 2\mathbb{Z}$ for any real even long root β. Moreover, $(\rho, \delta) = N - 1$ for $\mathrm{sl}\,(1|n)^{(1)}$, $(\rho, \delta) = N$ for $\mathrm{osp}\,(2|2n)^{(1)}$.

For a convenience of the reader, let us recall the properties of weights we use in what follows (we reformulate them here using the invariant form). λ is *regular* if $(\lambda, \beta) \neq 0$ for any even real β; λ is *typical* if $(\lambda + \rho, \beta) \neq 0$ for any real isotropic β; λ is *integrable* if $L(\lambda)$ is an integrable module.

Lemma 13.3. *Let $\mu \in \mathfrak{h}^*$ be regular and $(\mu, \delta) \neq 0$. Then there is at most one $\alpha \in \Delta_1^+$ such that $(\mu, \alpha) = 0$.*

Proof. Let $\alpha, \gamma \in \Delta_1^+$ and $\alpha \neq \gamma$. By a direct inspection of the list of roots, one can check that either $\alpha + \gamma$, or $\alpha - \gamma$ is an even root β. So if $(\mu, \alpha) = (\mu, \gamma) = 0$, then $(\mu, \beta) = 0$. Contradiction, since every β is either real or $m\delta$.

Lemma 13.4. *Let Π be now an arbitrary base, $\nu \in \mathfrak{h}^*$ be such that $\nu(h_\beta) \in \mathbb{Z}_{\geq 0}$ for all $\beta \in \mathcal{B}$, Σ be obtained from Π by odd reflections. Then $\langle \nu_\Sigma + \rho_\Sigma, h_\beta \rangle \in \mathbb{Z}_{\geq 0}$ for any $\beta \in \mathcal{B} \cap \Pi$, and $\langle \nu_\Sigma + \rho_\Sigma, h_\beta \rangle \in \mathbb{Z}_{\geq -1}$ for $\beta \in \mathcal{B} \smallsetminus \Pi$.*

Proof. First, observe that Y_β is locally nilpotent on $L(\nu)$ for any $\beta \in \mathcal{B} \cap \Pi$. Since $L_\Sigma(\nu_\Sigma) \cong L_\Pi(\nu)$, then $\langle \nu_\Sigma, h_\beta \rangle \in \mathbb{Z}_{\geq 0}$ for any $\beta \in \mathcal{B} \cap \Pi$. Note also that $\rho_\Sigma(h_\beta) = 1$ if $\beta \in \mathcal{B} \cap \Sigma$, and $\rho_\Sigma(h_\beta) = 0$ if $\beta \in \mathcal{B} \smallsetminus \Sigma$. Hence it suffices to consider the case $\beta = \alpha_1 + \gamma_1 \in \mathcal{B} \smallsetminus \Pi$.

Let $\alpha_0 = -\gamma_1$, $\Pi_0 = \Pi$, $\Pi_k = r_{\alpha_k} \dots r_{\alpha_1}(\Pi)$. We will prove the statement for $\Sigma = \Pi_k$ (for $\Sigma = r_{\gamma_k} \dots r_{\gamma_1}(\Pi)$ the proof is similar). Let $\lambda^k = \nu_{\Pi_k} + \rho_{\Pi_k}$, $\lambda_i^k = \left(\lambda^k, \alpha_i\right)$. Let k be such that $\lambda^{k-1}(h_\beta) \geq 0$, $\lambda^k(h_\beta) < 0$; let $s > k$ be the minimal such that $\alpha_{s+1} - \alpha_s = \alpha_{k+1} - \alpha_k$ (it exists since $\alpha_{t+1} - \alpha_t$ is N-periodic). Note that $\beta = \alpha_1 - \alpha_0$.

Lemma 13.5. *One has* $\lambda_k^k = 0$, $\alpha_{k+1} - \alpha_k = \beta$, $\lambda^k = \lambda^{k+1}$, $\lambda^k(h_\beta) = \lambda^{k+1}(h_\beta) = \cdots = \lambda^{s-1}(h_\beta) = -1$.

Proof. By inspection of the formulae for roots, and by (12) and (13),

$$\alpha_i(h_\beta) = -1 \text{ if } \beta = \alpha_{i+1} - \alpha_i, \quad \alpha_i(h_\beta) = 1 \text{ if } \beta = \alpha_i - \alpha_{i-1}, \quad \alpha_i(h_\beta) = 0 \text{ otherwise;}$$
$$\lambda^t = \lambda^{t-1} \quad \text{if } \lambda_t^{t-1} \neq 0, \qquad \lambda^t = \lambda^{t-1} + \alpha_t \quad \text{if } \lambda_t^{t-1} = 0.$$

Therefore, $\lambda_k^k = 0$; $\alpha_{k+1} - \alpha_k = \beta$, since $\alpha_k(h_\beta) < 0$ (by $\lambda^k(h_\beta) < \lambda^{k-1}(h_\beta)$); this implies $\lambda_{k+1}^k < 0$, thus $\lambda^{k+1} = \lambda^k$. Moreover, $\lambda^k(h_\beta) = \lambda^{k+1}(h_\beta) = \cdots = \lambda^{s-1}(h_\beta) = -1$, because $\alpha_t(h_\beta) = 0$, $k+2 \leq t < s$, and $\alpha_k(h_\beta) = -1$.

Since $\alpha_{i+N} = \alpha_i + \delta$, one has $\lambda_{i+N}^k = \lambda_i^k + M$, here $M = (\nu + \rho, \delta)$. Moreover, $(\nu, \delta) \geq 0$ (since δ is a positive root of $\mathfrak{g}(B)$), thus $M \geq (\rho, \delta)$; therefore $M \geq N - 1$ if $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$, $M \geq N$ if $\mathfrak{g} = \mathrm{osp}(2|2n)^{(1)}$.

Lemma 13.6. *One has* $\lambda_s^k - \lambda_k^k \geq s - k - 2 + (\beta, \beta)/2$. *For* $k + 2 \leq i \leq s - 1$, *one has* $(\alpha_i, \alpha_s) = -1$.

Proof. We prove it case-by-case. Let $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$; hence $N = n$. In this case $\alpha_{i+1} - \alpha_i = \alpha_{j+1} - \alpha_j$ iff $i \equiv j \mod N$. Therefore, $s = k + N$. It is enough to note that $(\alpha_i, \alpha_j) = -1$ if $i \not\equiv j \mod N$, $\lambda_s^k - \lambda_k^k = M \geq s - k - 1$, and $(\beta, \beta) = 2$.

Let $\mathfrak{g} = \mathrm{osp}(2|2n)^{(1)}$; hence $N = 2n$. Then $\alpha_{i+1} - \alpha_i = \alpha_{j+1} - \alpha_j$ iff $i \equiv j \mod N$ or $i + j \equiv 2p \mod N$; here p is the smallest $0 \leq p < n$ such that $\alpha_{p+1} - \alpha_p$ is a long root. Moreover, $(\alpha_i, \alpha_j) = -1$ if $i \not\equiv j$, $i + j \not\equiv 2p + 1 \mod N$.

If $p = 0$, then $(\beta, \beta) = 4$, $n \mid k$, $s = k + N$, and $M \geq s - k$ imply what is needed.

Now assume $p > 0$; then

$$k \equiv 0 \mod N, \quad s \equiv 2p \mod N, \quad \text{or} \quad k \equiv 2p \mod N, \quad s \equiv 0 \mod N.$$

Since $(\beta, \beta) = 2$, what remains to prove is $\lambda_s^k - \lambda_k^k \geq s - k - 1$. Note that for t between k and s, the only long root among $\alpha_{t+1} - \alpha_t$ is one with $t = r \stackrel{\mathrm{def}}{=} (k+s)/2$. We already know that $\lambda_{k+1}^k - \lambda_k^k = -(\beta, \beta)/2 = -1$, and $\lambda_{t+1}^k - \lambda_t^k \geq 1$ for $t \not\equiv k, s \mod N$. Note that $\lambda_{r+1}^k - \lambda_r^k \geq 2$; indeed, $\beta' = \alpha_{r+1} - \alpha_r$ is a long root, $\beta' \neq \beta$, thus $\rho_{\Pi_r}(h_{\beta'}) = 1$, and $(\rho_{\Pi_r}, \beta') = 2$. Hence $\lambda_s^k - \lambda_k^k \geq s - k - 1$ indeed.

Since $\lambda^s = \lambda^k + \alpha_{i_1} + \cdots + \alpha_{i_j}$ for some $k + 1 < i_1 < i_2 < \cdots < i_j \leq s$, and $(\alpha_s, \alpha_s) = 0$, at most $s - k - 2$ terms α_{i_t} can contribute to $\left(\lambda^s - \lambda^k, \alpha_s\right)$; hence $\lambda_s^s - \lambda_s^k \geq -(s - k - 2)$. Therefore $\lambda_s^s = \left(\lambda_s^s - \lambda_s^k\right) + \left(\lambda_s^k - \lambda_k^k\right) \geq (\beta, \beta)/2 > 0$.

Either $\lambda^{s-1} = \lambda^s$, or $\lambda^{s-1} = \lambda^s - \alpha_s$; since $(\alpha_s, \alpha_s) = 0$, one can conclude $\lambda_s^{s-1} > 0$. Therefore $\lambda^s = \lambda^{s-1}$, hence $\lambda^s(h_\beta) = -1$.

Lemma 13.7. *If* $\lambda_s^s \geq 1 + (\beta, \beta)/2$, *then* $\lambda^t = \lambda^s$ *for any* $t > s$.

Proof. By induction in t, it is enough to show that $\lambda_t^s \geq 1$ for any $t > s$. Since $\lambda_{i+N}^k = \lambda_i^k + M > \lambda_i^k$, it is enough to consider $t < s + N$. Since $\lambda^t(h_{\beta'}) \geq 0$ for $\beta' \neq \beta$, one has $\lambda_{j+1}^t - \lambda_j^t \geq 0$ unless $\alpha_{j+1} - \alpha_j = \alpha_{k+1} - \alpha_k$; as shown at the beginning of the proof, the inequalities are strict for $t = k, s$. Hence, λ_j^s may decrease only after j such that $\alpha_{j+1} - \alpha_j = \beta$, and the decrease is $(\beta, \beta)/2$.

By inspection, unless $\mathfrak{g} = \operatorname{osp}(2|2n)^{(1)}$ and β is a short root, there are at most one such value of j per an interval of length N. Otherwise, there are two such values, they are not adjacent, and the decrease in each such value is by $(\beta, \beta)/2 = 1$. Now strict increase at other values implies $\lambda_t^s \geq 1$.

Therefore, if $\lambda_s^s \geq 1 + (\beta, \beta)/2$, then $\langle \lambda^t, h_\beta \rangle = -1$ for any $t \geq k$. On the other hand, if $\lambda_s^s = (\beta, \beta)/2$, then $\lambda_{s+1}^s = \lambda_s^s + (\lambda^s, \beta) = (1 + \lambda^s(h_\beta))(\beta, \beta)/2 = 0$; hence $\lambda^{s+1} = \lambda^s + \alpha_{s+1}$. Since $\alpha_{s+1}(h_\beta) = 1$, this implies $\lambda^{s+1}(h_\beta) = 0$. In any case, this implies that if $\lambda^t(h_\beta) < 0$ for $t_0 \leq t \leq t_1$, then $\lambda^t(h_\beta) = -1$ (indeed, taking t_0 the minimal possible with the given t_1, we may assume $t_0 = k$).

This finishes the proof of Lemma 13.4.

Lemma 13.8. *Let $\lambda, \mu \in \mathfrak{h}^*$, $\lambda(h_\beta) > 0$, $\mu(h_\beta) \geq 0$ for any $\beta \in \mathcal{B}$, and $\lambda - \mu = \sum_{\beta \in \mathcal{B}} m_\beta \beta$ for some $m_\beta \in \mathbb{Z}_{\geq 0}$. If $(\lambda, \lambda) = (\mu, \mu)$, then $\lambda = \mu$.*

Proof. We use the fact that all principal roots have positive square (see Remark 13.2).

$$(\lambda, \lambda) - (\mu, \mu) = (\lambda + \mu, \lambda - \mu) = \left(\lambda + \mu, \sum_{\beta \in \mathcal{B}} m_\beta \beta \right) = 0.$$

But $(\lambda + \mu, \beta) = \frac{(\beta, \beta)}{2} \langle \lambda + \mu, h_\beta \rangle$ is positive. Hence all $m_\beta = 0$.

14 On affine character formulae

Let $\mathfrak{g}$ be a regular Kac–Moody superalgebra with a fixed base Π and Cartan matrix A; let λ be an integrable weight. For a symmetrizable matrix A without zeros on diagonal, the character is described by the famous Weyl character formula

$$\operatorname{ch} L(\lambda) = U(\lambda + \rho), \tag{22}$$

which was proven by Kac, see [10]. The proof is a straightforward generalization of his proof in Lie algebra case (see, for example, [2]); it is based on existence of the Casimir element corresponding to the invariant form.

If $\mathfrak{g}$ is a finite-dimensional Kac–Moody superalgebra, (22) holds for a typical weight. It was also proven by Kac [6], but the proof is more complicated. One has to use either Shapovalov form, or a complete description of the center of the universal enveloping algebra $U(\mathfrak{g})$. The reason why a simpler proof from [2] does not work is existence of real roots of non-positive square of length.

In this section we provide a generalization of these results to some infinite-dimensional algebras, as well as to the case of atypical weights: we calculate the characters of all simple integrable highest weight modules over $\mathrm{sl}\,(1|2n)^{(1)}$, and over $\mathrm{osp}\,(2|2n)^{(1)}$. We use the invariant form, odd reflections and the fact that the defect of $\mathrm{sl}\,(1|2n)^{(1)}$ and $\mathrm{osp}\,(2|2n)^{(1)}$ is 1. Recall that the *defect* is the maximal number of linearly independent pairwise orthogonal real isotropic roots; defect is 1 iff for any real isotropic α and γ one of $\alpha \pm \gamma$ is a root (compare 13.3). Our proof is an adaptation of the Bernstein–Leites proof of character formula for $\mathrm{sl}\,(1|n)$, see [14]. Since in our case the defect is 1, our formulae are easier than those for general finite-dimensional superalgebras (see [4, 15]).

In what follows we assume $\mathfrak{g} = \mathrm{sl}\,(1|n)^{(1)}$ or $\mathrm{osp}\,(2|2n)^{(1)}$, and we use the same notations as in the previous section.

Let $L_\Sigma(\lambda)$ be an integrable simple $\mathfrak{g}$-module, and $\lambda + \rho_\Sigma$ be regular. Let v be a highest vector of $M_\Sigma(\lambda)$. For any principal root β put $k(\lambda,\beta) = \langle \lambda + \rho_\Sigma, h_\beta \rangle$. For $\beta \in \Sigma$, define $v_\beta = Y_\beta^{k(\lambda,\beta)} v$. If $\beta \notin \Sigma$, i.e., $\beta = \alpha_1 + \gamma_1$ for isotropic $\alpha_1, \gamma_1 \in \Sigma$, then, by regularity of $\lambda+\rho_\Sigma$, one can choose $\alpha \in \{\alpha_1, \gamma_1\}$ such that $(\lambda, \alpha) = (\lambda + \rho_\Sigma, \alpha) \neq 0$; define v_β as $Y_\beta^{k(\lambda,\beta)} Y_\alpha v$. Let $V_\Sigma(\lambda)$ be the quotient of $M_\Sigma(\lambda)$ by the submodule generated by v_β, $\beta \in \mathcal{B}$.

Lemma 14.1. *Let $L_\Sigma(\lambda)$ be an integrable simple $\mathfrak{g}$-module, and $\lambda + \rho_\Sigma$ be regular. Then $V_\Sigma(\lambda)$ is an integrable $\mathfrak{g}$-module, and $\lambda-\alpha_1, \lambda-\gamma_1 \in P(V_\Sigma(\lambda))$.*

Proof. Let $\bar{v}$ be the image of v under the natural projection $M_\Sigma(\lambda) \to V_\Sigma(\lambda)$. Then $Y_\beta^{k(\lambda,\beta)} \bar{v} = 0$ for any $\beta \in \Sigma \cap \mathcal{B}$. Since $\bar{v}$ generates $V_\Sigma(\lambda)$, Y_β is locally nilpotent on $V_\Sigma(\lambda)$ for any $\beta \in \Sigma \cap \mathcal{B}$. If $\beta \in \mathcal{B} \setminus \Sigma$, we have $Y_\beta^{k(\lambda,\beta)} Y_\alpha \bar{v} = 0$. But $X_\alpha Y_\alpha \bar{v} = \lambda(h_\alpha)\bar{v}$, therefore $Y_\alpha \bar{v}$ generates $V_\Sigma(\lambda)$, and Y_β is also locally nilpotent if $\beta \in \mathcal{B} \setminus \Sigma$. Thus, $V_\Sigma(\lambda)$ is integrable.

Denote by μ_β the weight of v_β. To show that $\lambda - \alpha_1, \lambda - \gamma_1 \in P(V_\Sigma(\lambda))$ it is sufficient to check that $\lambda - \alpha_1$, $\lambda - \gamma_1$ are not weights of a submodule generated by v_β for each $\beta \in \mathcal{B}$. If $\beta \in \Sigma$, then v_β is $\mathfrak{n}_\Sigma^+$-invariant, so $U(\mathfrak{g})\, v_\beta = U\left(\mathfrak{n}_\Sigma^-\right) v_\beta$, and obviously the inequalities $\lambda - \alpha_1 \leq_\Sigma \mu_\beta$, $\lambda - \gamma_1 \leq_\Sigma \mu_\beta$ do not hold. Therefore $\lambda - \alpha_1, \lambda - \gamma_1 \notin P\left(U\left(\mathfrak{n}_\Sigma^-\right) v_\beta\right)$. If $\beta \in \mathcal{B} \setminus \Sigma$, consider $\Sigma' = r_\alpha(\Sigma)$ and note that v_β is $\mathfrak{n}_{\Sigma'}^+$-invariant. In this case $U(\mathfrak{g})\, v_\beta = U\left(\mathfrak{n}_{\Sigma'}^-\right) v_\beta$. Assume without loss of generality that $\alpha = \alpha_1$. Then the inequality $\lambda-\alpha_1 \leq_{\Sigma'} \mu_\beta$ never holds, and $\lambda - \gamma_1 \leq_{\Sigma'} \mu_\beta$ holds only if $k(\lambda,\beta) = 1$. But then $\mu_\beta = \lambda - \alpha_1 - \beta = \lambda - 2\alpha_1 - \gamma_1$, and $\lambda - \gamma_1 = \mu_\beta + 2\alpha_1 \notin P\left(U\left(\mathfrak{n}_{\Sigma'}^-\right) v_\beta\right)$ since $-\alpha_1 \in \Sigma'$ is isotropic. Therefore $\lambda - \alpha_1, \lambda - \gamma_1 \notin P\left(U\left(\mathfrak{n}_{\Sigma'}^-\right) v_\beta\right)$.

One can immediately see that $V_\Sigma(\lambda)$ is the maximal integrable quotient of $M_\Sigma(\lambda)$. The following lemma can be proven exactly as Lemma 10.2 (b). Therefore we omit the proof.

Lemma 14.2. *If $\lambda+\rho$ is regular, Σ is obtained from Π by odd reflections, and $\lambda_\Sigma+\rho_\Sigma=\lambda+\rho$, then $V_\Sigma(\lambda_\Sigma)\cong V_\Pi(\lambda)$.*

Recall that $\lambda_\Sigma+\rho_\Sigma=\lambda+\rho$ holds when λ_Σ is obtained from λ by a chain of "typical" reflections.

Lemma 14.3. *Let $L_\Sigma(\lambda)$ be integrable, $\lambda+\rho_\Sigma$ be regular. If λ is typical, then $V_\Sigma(\lambda)=L_\Sigma(\lambda)$. If $(\lambda+\rho_\Sigma,\alpha)=0$ for a simple isotropic root $\alpha\in\Sigma$, then there is a short exact sequence of $\mathfrak{g}$-modules*

$$0\to L_\Sigma(\lambda-\alpha)\to V_\Sigma(\lambda)\to L_\Sigma(\lambda)\to 0.$$

Proof. Let $[V_\Sigma(\lambda):L_\Sigma(\mu)]>0$. Since $V_\Sigma(\lambda)$ is integrable, $L_\Sigma(\mu)$ is integrable. Hence μ is an integrable weight. By Lemma 12.3, $\lambda+\rho_\Sigma=\mu'+\rho_{\Sigma'}+\sum_{\beta\in\mathcal{B}}m_\beta\beta$ for some $m_\beta\in\mathbb{Z}_{\geq 0}$, some Σ' obtained from Σ by odd reflections, and μ' such that $L_\Sigma(\mu)\cong L_{\Sigma'}(\mu')$. Since

$$(\mu'+\rho_{\Sigma'},\mu'+\rho_{\Sigma'})=(\mu+\rho_\Sigma,\mu+\rho_\Sigma)=(\lambda+\rho_\Sigma,\lambda+\rho_\Sigma),$$

Lemma 13.8 and regularity of $\lambda+\rho_\Sigma$ imply $\lambda+\rho_\Sigma=\mu'+\rho_{\Sigma'}$.

Without loss of generality, one may assume that $\Sigma'=r_{\alpha_k}\dots r_{\alpha_1}(\Sigma)$. Then

$$\mu'+\rho_{\Sigma'}=\mu+\rho_\Sigma+\alpha_{i_1}+\alpha_{i_2}+\dots+\alpha_{i_r}, \tag{23}$$

for some $1\leq i_1<\dots<i_r\leq k$ such $\left(\mu+\rho_\Sigma+\alpha_{i_1}+\dots+\alpha_{i_j},\alpha_{i_{j+1}}\right)=0$ for all $j<r$. In particular, $r=0$ or $(\lambda+\rho_\Sigma,\alpha_{i_r})=0$. If λ is typical, the latter case is impossible; hence $\mu=\lambda$, therefore $V_\Sigma(\lambda)=L_\Sigma(\lambda)$.

Assume that λ is atypical. Since $\lambda+\rho_\Sigma$ is regular, by Lemma 13.3, there is only one α such that $(\lambda+\rho_\Sigma,\alpha)=0$; hence $r=0$, or $\alpha=\alpha_{i_r}$. Since $\alpha\in\Sigma$, and Σ contains only two isotropic simple roots α_1 and γ_1, either $\alpha=\alpha_1$ or $\alpha=\gamma_1$. Therefore, either $\lambda=\mu$, or $\alpha=\alpha_{i_r}=\alpha_1$ and $\lambda=\mu+\alpha$.

Thus $[V_\Sigma(\lambda):L_\Sigma(\mu)]>0$ implies $\mu\in\{\lambda,\lambda-\alpha\}$. It remains to show that $[V_\Sigma(\lambda):L_\Sigma(\lambda-\alpha)]=1$. Since the multiplicity of $\lambda-\alpha$ in $M_\Sigma(\lambda)$ is 1, $[V_\Sigma(\lambda):L_\Sigma(\lambda-\alpha)]\leq 1$. On the other hand, notice that $Y_\alpha\bar{v}\in V_\Sigma(\lambda)$ is not zero (by Lemma 14.1) and generates a submodule with the highest weight $\lambda-\alpha$ (see the proof of Lemma 10.2). Hence $[V_\Sigma(\lambda):L_\Sigma(\lambda-\alpha)]=1$.

The next step is to check that (22) holds if we replace $L_\Sigma(\lambda)$ by $V_\Sigma(\lambda)$.

Theorem 14.4. *Let λ be integrable, $\lambda+\rho_\Sigma$ be regular. Then*

$$\operatorname{ch}V_\Sigma(\lambda)=U(\lambda+\rho_\Sigma).$$

Proof. First, we note that, by regularity, there is only one $\alpha\in\Delta_1^+$ such that $(\lambda+\rho_\Sigma,\alpha)=0$. Without loss of generality, one may assume that $\alpha\in\Sigma$ (if necessary, one can change Σ to Σ' by "typical" odd reflections so that $V_{\Sigma'}(\lambda+\rho_{\Sigma'}-\rho_\Sigma)\cong V_\Sigma(\lambda)$, $\alpha\in\Sigma'$, as in Lemma 14.2).

By Corollary 11.6,

$$\operatorname{ch} V_\Sigma(\lambda) = \sum_{\kappa \in T} a_\kappa \operatorname{ch} M_\Sigma(\kappa), \quad a_\kappa \neq 0.$$

Since $V_\Sigma(\lambda)$ is integrable, $\operatorname{ch} V_\Sigma(\lambda)$ is W-invariant. Therefore

$$\operatorname{ch} V_\Sigma(\lambda) - U(\lambda + \rho_\Sigma) = \sum_{\nu \in F} b_\nu e^\nu = \sum_{\kappa \in T} a_\kappa \operatorname{ch} M_\Sigma(\kappa) - \sum_{w \in W} (-1)^w \operatorname{ch} M_\Sigma(w(\lambda + \rho_\Sigma) - \rho_\Sigma)$$

is W-invariant. We assume that $b_\nu \neq 0$ for any $\nu \in F$. Assume that the Theorem does not hold. Then F is non-empty. Not that F is a W-invariant set. Pick up a maximal $\nu \in F$. First, maximality of ν implies that $\nu \in T \cup (W(\lambda + \rho_\Sigma) - \rho_\Sigma)$. Furthermore, $r_\beta(\nu) \geq \nu$ for any $\beta \in \mathcal{B}$ such that $\nu(h_\beta) < 0$; hence $\nu(h_\beta) \in \mathbb{Z}_{\geq 0}$. In particular, $\nu \neq w(\lambda + \rho_\Sigma) - \rho_\Sigma$, because $w(\lambda + \rho_\Sigma) - \rho_\Sigma \in C$ only when $w = 1$ and, obviously, $\lambda \notin F$ (see Remark 12.7). Recall that by Corollary 11.6, there exist $\nu_1, \ldots, \nu_k \in T$, $\nu_k \neq \nu$, such that

$$[M_\Sigma(\lambda) : L_\Sigma(\nu_1)] > 0, \quad [M_\Sigma(\nu_1) : L_\Sigma(\nu_2)] > 0, \quad \ldots \quad [M_\Sigma(\nu_k) : L_\Sigma(\nu)] > 0.$$

We claim that $\nu_k = w(\lambda + \rho_\Sigma) - \rho_\Sigma$ for some $w \in W$. Indeed, $\nu < \nu_k$, therefore $\nu_k \notin F$; since $\nu_k \in T$ and $\operatorname{ch} V_\Sigma(\lambda) - U(\lambda + \rho_\Sigma)$ does not contain the term $M_\Sigma(\nu_k)$ with non-zero coefficient,

$$\nu_k + \rho_\Sigma = w(\lambda + \rho_\Sigma) = \lambda + \rho_\Sigma - \sum_{\beta \in \mathcal{B}} m''_\beta \beta,$$

for some $m''_\beta \in \mathbb{Z}_{\geq 0}$ (see Remark 12.7). On the other hand, by Lemma 12.3,

$$\nu' + \rho_{\Sigma'} = \nu_k + \rho_\Sigma - \sum_{\beta \in \mathcal{B}} m'_\beta \beta$$

for some base Σ' obtained from Σ by odd reflections, ν' such that $L_{\Sigma'}(\nu') \cong L_\Sigma(\nu)$ and some $m'_\beta \in \mathbb{Z}_{\geq 0}$. As a result

$$\nu' + \rho_{\Sigma'} = \lambda + \rho_\Sigma - \sum_{\beta \in \mathcal{B}} m_\beta \beta, \qquad m_\beta \in \mathbb{Z}_{\geq 0}.$$

By Lemma 13.4, $\langle \nu' + \rho_{\Sigma'}, h_\beta \rangle \geq 0$ for all $\beta \in \mathcal{B} \cap \Sigma$ and $\langle \nu' + \rho_{\Sigma'}, h_\beta \rangle \geq -1$ for $\beta \in \mathcal{B} \setminus \Sigma$. Since $\lambda + \rho_\Sigma$ is regular and λ is integrable, $(\lambda + \rho_\Sigma + \nu' + \rho_{\Sigma'}, \beta) > 0$ for all $\beta \in \mathcal{B} \cap \Sigma$, and $(\lambda + \rho_\Sigma + \nu' + \rho_{\Sigma'}, \beta) \geq 0$ for $\beta \in \mathcal{B} \setminus \Sigma$. The condition $(\lambda + \rho_\Sigma, \lambda + \rho_\Sigma) = (\nu' + \rho_{\Sigma'}, \nu' + \rho_{\Sigma'})$ implies $\left(\lambda + \rho_\Sigma + \nu' + \rho_{\Sigma'}, \sum_{\beta \in \mathcal{B}} m_\beta \beta\right) = 0$.

If $(\lambda + \rho_\Sigma + \nu' + \rho_{\Sigma'}, \beta) > 0$ for $\beta \in \mathcal{B} \setminus \Sigma$, then $\nu' + \rho_{\Sigma'} = \lambda + \rho_\Sigma$. Then, as in the proof of Lemma 14.3, one can show that $\nu = \lambda - \alpha$, but we already proved that the weight $\lambda - \alpha$ has multiplicity one in $V_\Sigma(\lambda)$, as well as in $U(\lambda + \rho_\Sigma)$. Hence $\nu \neq \lambda - \alpha$.

Therefore, $(\lambda + \rho_\Sigma + \nu' + \rho_{\Sigma'}, \beta) = 0$ for $\beta \in \mathcal{B} \setminus \Sigma$; thus $m_{\beta'} = 0$ for $\beta' \in \mathcal{B} \cap \Sigma$, and $\langle \lambda + \rho_\Sigma, h_\beta \rangle = 1$, $\langle \nu' + \rho_{\Sigma'}, h_\beta \rangle = -1$ (hence $m_\beta = 1$) for $\beta \in \mathcal{B} \setminus \Sigma$. Therefore, $\nu' + \rho_{\Sigma'} = r_\beta (\lambda + \rho_\Sigma) = \lambda + \rho_\Sigma - \beta$. Without loss of generality, assume that $\Sigma' = r_{\alpha_k} \dots r_{\alpha_1} (\Sigma)$. Then

$$r_\beta (\lambda + \rho_\Sigma) = \nu' + \rho_{\Sigma'} = \nu + \rho_\Sigma + \alpha_{i_1} + \dots + \alpha_{i_r}.$$

Assume $r \neq 0$; then $(r_\beta (\lambda + \rho_\Sigma), \alpha_{i_r}) = 0$. By our assumption in the beginning of the proof, there are only two possibilities: $(\lambda + \rho_\Sigma, \alpha_1) = 0$ or $(\lambda + \rho_\Sigma, \gamma_1) = 0$. The first case is impossible, because then we have $r_\beta (\alpha_1) = -\gamma_1$, $(r_\beta (\lambda + \rho_\Sigma), \gamma_1) = 0$, and by regularity of $r_\beta (\lambda + \rho_\Sigma)$, $\alpha_{i_r} = \gamma_1$. In the second case, we have $\alpha_{i_r} = \alpha_1$ and

$$\nu = \lambda - \beta - \alpha_1.$$

Since $(\rho_\Sigma, \beta) = 0$,

$$\nu (h_\beta) = \lambda (h_\beta) - \beta (h_\beta) - \alpha_1 (h_\beta) = 1 - 2 - 2 (\alpha_1, \beta)/(\beta, \beta) = -2,$$

and that contradicts the condition $r_\beta (\nu) \leq \nu$.

Corollary 14.5. *For any typical integrable λ the character of $L(\lambda)$ is given by* (22).

Corollary 14.6. *Let $L_\Sigma (\lambda)$ be integrable, and $\lambda + \rho_\Sigma$ be regular. Assume that $(\lambda + \rho_\Sigma, \alpha) = 0$ for some isotropic $\alpha \in \Sigma$. Then*

$$\operatorname{ch} L_\Sigma (\lambda) + \operatorname{ch} L_\Sigma (\lambda - \alpha) = U (\lambda + \rho_\Sigma).$$

Theorem 14.7. *Let λ be an atypical integrable weight, Π be such that $\lambda + \rho$ is regular, and there exists an isotropic $\alpha \in \Pi$ such that $(\lambda + \rho, \alpha) = 0$. Then*

$$\operatorname{ch} L (\lambda) = \frac{\sum_{w \in W} (-1)^w \, w \left(e^{\lambda + \rho} \frac{D_1}{1 + e^{-\alpha}} \right)}{D_0} \tag{24}$$

Proof. Without loss of generality, assume that $\alpha = \alpha_1 = -\gamma_0$. We start with constructing a sequence $(\lambda_i, \Sigma_i, k(i))$, where $\Sigma_i = r_{\gamma_{k(i)}} \dots r_{\gamma_1} \Pi$, λ_i is a weight such that $\left(\lambda_i + \rho_{\Sigma_i}, \gamma_{k(i)}\right) = 0$, and $\lambda_i + \rho_{\Sigma_i}$ is regular. This sequence is defined uniquely by the following rule:

1. $(\lambda_0, \Sigma_0, k(0)) = (\lambda, \Pi, 0)$;
2. $\lambda_{i+1} = \lambda_i + \gamma_{k(i)}$;
3. $\Sigma_{i+1} = \Sigma_i$, $k(i+1) = k(i)$ if $\lambda_{i+1} + \rho_{\Sigma_i}$ is regular;
4. if $\lambda_{i+1} + \rho_{\Sigma_i}$ is not regular, set, using Lemma 12.6, $\Sigma_{i+1} = r_{\gamma_{k(i+1)}} \dots r_{\gamma_{k(i)+1}} (\Sigma_i)$ such that $L_{\Sigma_i} (\lambda_{i+1}) \cong L_{\Sigma_{i+1}} (\lambda_{i+1})$, and $\lambda_{i+1} + \rho_{\Sigma_{i+1}}$ is regular.

These rules ensure that $\left(\lambda_i+\rho_{\Sigma_i},\gamma_{k(i)}\right)=0$ and $\lambda_i+\rho_{\Sigma_i}$ is regular.

Now define

$$\chi_i=\operatorname{ch} L_{\Sigma_i}\left(\lambda_i\right), \qquad \varphi_i=\frac{1}{D_0}\sum_{w\in W}\left(-1\right)^w w\left(e^{\lambda_i+\rho_{\Sigma_i}}\frac{D_1}{1+e^{\gamma_{k(i)}}}\right).$$

(The quotients are taken in $\mathcal{R}'$; obviously, they make sense.) The series converges by the same reasons as for the series for U. We want to show that $\chi_i=\varphi_i$; they are both elements of $\mathcal{R}$.

Our next step is to prove the identity

$$\varphi_i+\varphi_{i+1}=U\left(\lambda_i+\rho_{\Sigma_i}\right).$$

If $\Sigma_{i+1}=\Sigma_i$, then the identity is straightforward. Otherwise, by Lemma 12.6 (with $\Sigma=\Sigma_i$, $\Sigma'=\Sigma_{i+1}$, $\lambda=\lambda_{i+1}$, $k=s-1$)

$$\lambda_{i+1}+\rho_{\Sigma_{i+1}}=g\left(\lambda_{i+1}+\rho_{\Sigma_i}+\left(s-1\right)\gamma_{k(i)}\right)=g\left(\lambda_i+\rho_{\Sigma_i}+s\gamma_{k(i)}\right),$$

where s is the positive integer such that $\lambda_i+\rho_{\Sigma_i}+t\gamma_{k(i)}$ is not regular for any $1\le t<s$; and $g\in W$ is such that $\gamma_{k(i+1)}=g\left(\gamma_{k(i)}\right)$ and $\left(-1\right)^g=\left(-1\right)^{s+1}$. Thus, we obtain

$$\begin{aligned}\varphi_i+\varphi_{i+1}&=\frac{1}{D_0}\sum_{w\in W}\left(-1\right)^w w\left(e^{\lambda_i+\rho_{\Sigma_i}}\frac{D_1}{1+e^{\gamma_{k(i)}}}+e^{g\left(\lambda_i+\rho_{\Sigma_i}+s\gamma_{k(i)}\right)}\frac{D_1}{1+e^{g\left(\gamma_{k(i)}\right)}}\right)\\&=\frac{\sum_{w\in W}\left(-1\right)^w w\left(e^{\lambda_i+\rho_{\Sigma_i}}+\left(-1\right)^{s+1}e^{\lambda_i+\rho_{\Sigma_i}+s\gamma_{k(i)}}\frac{D_1}{1+e^{\gamma_{k(i)}}}\right)}{D_0}\\&=\frac{D_1}{D_0}\sum_{w\in W}\left(-1\right)^w w\left(\sum_{t=0}^{s-1}e^{\lambda_i+\rho_{\Sigma_i}+t\gamma_{k(i)}}\right)=\sum_{t=0}^{s-1}U\left(\lambda_i+\rho_{\Sigma_i}+t\gamma_{k(i)}\right).\end{aligned}$$

However, $\lambda_i+\rho_{\Sigma_i}+t\gamma_{k(i)}$ is not regular for all $1\le t<s$. Hence $U\left(\lambda_i+\rho_{\Sigma_i}+t\gamma_{k(i)}\right)=0$, $t>0$; we obtain the desired identity.

Now recall that by Corollary 14.6 the similar identity holds for χ_i:

$$\chi_i+\chi_{i+1}=U\left(\lambda_i+\rho_{\Sigma_i}\right).$$

Therefore we can conclude that there exists $\Phi\in\mathcal{R}$ such that

$$\chi_i=\varphi_i+\left(-1\right)^i\Phi.$$

We want to show now that $\Phi=0$. It suffices to prove that for every $\nu\in\mathfrak{h}^*$ the monomial e^ν appears with non-zero coefficient only in finitely many φ_i and in finitely many χ_i. We claim that if e^ν appears in χ_i (or φ_i) with non-zero coefficient, then $\nu\le_{\Sigma_i}\lambda_i$. For χ_i it follows from the fact that ν must be a weight of $L_{\Sigma_i}\left(\lambda_i\right)$. To check this in case of φ_i, note that every term which appears in φ_i appears in the expression for $U\left(\lambda_i+\rho_{\Sigma_i}\right)$, so the same argument as for U works. Recall the $\mathbb{Z}$-grading on Q defined in (15), (16). The condition $\nu \leq_{\Sigma_i} \lambda_i$ can be rewritten in the form (see Lemma 12.1 for $\Pi = \Sigma_i$)

$$\lambda_i - \nu = -m_i \gamma_{k(i)} + \sum_{\beta \in \mathcal{B}} m^i_\beta \beta, \quad \text{if } \lambda_i - \nu \in Q^+,$$

$$\lambda_i - \nu = m_i \gamma_{k(i)+1} + \sum_{\beta \in \mathcal{B}} m^i_\beta \beta, \quad \text{if } \lambda_i - \nu \in Q^-,$$

here m_i, $m^i_\beta \in \mathbb{Z}_{\geq 0}$. Note that if $\lambda_i \in Q^j$, then $\lambda_{i+1} \in Q^{j-1}$. Hence for sufficiently large i only the second case is possible. Then

$$\lambda_{i+1} - \nu = m_i \gamma_{k(i)+1} + \gamma_{k(i)} + \sum_{\beta \in \mathcal{B}} m^i_\beta \beta.$$

Now rewrite it in a suitable form

$$\lambda_{i+1} - \nu = (m_i + 1)\, \gamma_{k(i+1)+1} + \sum_{\beta \in \mathcal{B}} m^i_\beta \beta + m_i \left(\gamma_{k(i)+1} - \gamma_{k(i+1)+1}\right) + \left(\gamma_{k(i)} - \gamma_{k(i+1)+1}\right).$$

However, $\gamma_{k(i)+1} - \gamma_{k(i+1)+1}$ is either zero or a negative real root, $\gamma_{k(i)} - \gamma_{k(i+1)+1}$ is always a negative even root. Thus

$$\sum_{\beta \in \mathcal{B}} m^{i+1}_\beta \leq \sum_{\beta \in \mathcal{B}} m^i_\beta - 1.$$

Hence, for sufficiently large i, $\sum_{\beta \in \mathcal{B}} m^i_\beta$ becomes negative, therefore $\nu \leq_{\Sigma_i} \lambda_i$ does not hold. Theorem is proven.

Remark 14.8. Under conditions of Theorem 14.7, it is not hard to show that the complex

$$\cdots \to V_{\Sigma_{i+1}}(\lambda_{i+1}) \to V_{\Sigma_i}(\lambda_i) \to \cdots \to V_{\Sigma_0}(\lambda_0) \to 0$$

with arrows defined by Corollary 14.6 is a resolution of $L(\lambda)$.

Kac and Wakimoto call a representation with highest weight satisfying the condition of Theorem 14.7 *tame*. Character formula (24) coincides with Kac–Wakimoto conjectural formula in [12] in case $\mathfrak{g} = \mathrm{sl}(1|n)^{(1)}$ or $\mathrm{osp}(2|2n)^{(1)}$. On the other hand, using Lemma 12.6, one can construct a chain of odd reflections transforming any atypical integrable weight (except weights of one-dimensional representations) to a weight satisfying the conditions of the last theorem. Therefore, we found character formulae for all integrable simple highest weight modules for $\mathrm{sl}(1|n)^{(1)}$ and $\mathrm{osp}(2|2n)^{(1)}$. It seems possible that using Shapovalov form calculated in [16] as a substitute for a missing invariant symmetric form, one can obtain similar formulae for $S(1,2;b)$.

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