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KÄHLER MANIFOLDS ADMITTING A FLAT COMPLEX CONFORMAL CONNECTION

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Abstract

We prove that any Kähler manifold admitting a flat complex conformal connection is a Bochner-Kähler manifold with special scalar distribution and zero geometric constants. Applying the local structural theorem for such manifolds we obtain a complete description of the Kähler manifolds under consideration.

1. Introduction

Let (M, g, J) ($\dim M = 2n$) be a Kähler manifold with complex structure J, metric g, Levi-Civita connection ∇, curvature tensor R, Ricci tensor ρ and scalar curvature τ. The Bochner curvature tensor $B(R)$ is given by

$$\begin{aligned} B(R)(X,Y)Z = {} & R(X,Y)Z - Q(Y,Z)X + Q(X,Z)Y - g(Y,Z)Q(X) \\ & + g(X,Z)Q(Y) - Q(JY,Z)JX + Q(JX,Z)JY + 2Q(JX,Y)JZ \\ & - g(JY,Z)JQ(X) + g(JX,Z)JQ(Y) \\ & + 2g(JX,Y)JQ(Z), \quad X,Y,Z \in \mathfrak{X}M, \end{aligned}$$

where $Q(X,Y) = \dfrac{1}{2(n+2)}\rho(X,Y) - \dfrac{\tau}{8(n+1)(n+2)}g(X,Y)$ and $Q(X)$ is the corresponding tensor of type (1.1).

The manifold is said to be *Bochner flat* (*Bochner-Kähler*) if its Bochner curvature tensor vanishes identically, i.e.

$$\begin{aligned} R(X,Y)Z = {} & Q(Y,Z)X - Q(X,Z)Y + g(Y,Z)Q(X) - g(X,Z)Q(Y) \\ & + Q(JY,Z)JX - Q(JX,Z)JY - 2Q(JX,Y)JZ \\ & + g(JY,Z)JQ(X) - g(JX,Z)JQ(Y) \\ & - 2g(JX,Y)JQ(Z), \quad X,Y,Z \in \mathfrak{X}M. \end{aligned} \tag{1.1}$$

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For any real $\mathscr{C}^\infty$ function u on M we denote $\omega = du$ and $P = grad\ u$.

In [10] Yano introduced on a Kähler manifold a complex conformal connection and proved

THEOREM A. *If in a 2n-dimensional $(n \geq 2)$ Kähler manifold there exists a scalar function u such that the complex conformal connection*

$$\mathscr{D}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X, Y)P \\ - \omega(JX)JY - \omega(JY)JX - g(JX, Y)JP, \quad X, Y \in \mathfrak{X}M,$$

is of zero curvature, then the Bochner curvature tensor of the manifold vanishes.

In [7] Seino proved the inverse

THEOREM B. *In a Kählerian space with vanishing Bochner curvature tensor if there exists a non-constant function u satisfying the equality*

$$(\nabla_X \omega)(Y) + 2\omega(JX)\omega(JY) + \omega(P)g(X, Y) = 0,$$

then the complex conformal connection is of zero curvature.

In this paper we prove

THEOREM 3.1. *A Kähler manifold (M, g, J) $(\dim M = 2n \geq 6)$ admits a flat complex conformal connection if and only if it is a Bochner-Kähler manifold whose scalar distribution D_τ is a B_0-distribution with function $a + k^2 = 0$ and geometric constants $\mathfrak{B} = \mathfrak{b}_0 = 0$.*

Applying the local structural theorem [2] for Bochner-Kähler manifolds whose scalar distribution is a B_0-distribution, we describe locally all Kähler manifolds admitting a flat complex conformal connection.

2. Preliminaries

Let (M, g, J) $(\dim M = 2n)$ be a Kähler manifold with metric g, complex structure J and Levi-Civita connection ∇. We denote by $\mathfrak{X}M$ the Lie algebra of all $\mathscr{C}^\infty$ vector fields on M. The fundamental Kähler form Ω is defined as follows

$$\Omega(X, Y) = g(JX, Y), \quad X, Y \in \mathfrak{X}M.$$

For any $\mathscr{C}^\infty$ real function u on M we consider the conformal metric $\bar{g} = e^{2u}g$. We denote the 1-form $\omega := du$ and $P := grad\ u$ with respect to the metric g. Then $(M, \bar{g}, J)$ is a locally conformal Kähler manifold, or a W_4-manifold in the classification scheme of [3]. The fundamental Kähler form and the Lee form of the structure $(\bar{g}, J)$ are $\bar{\Omega}(X, Y) = \bar{g}(JX, Y)$, $X, Y \in \mathfrak{X}M$ and $\bar{\varpi} = 2\omega = 2du$, respectively. The Lee vector $\bar{P}$ corresponding to $\bar{\varpi}$ with respect to the metric $\bar{g}$ is $\bar{P} = 2e^{-2u}P$.

The unique linear connection $\mathscr{D}$ with torsion $\mathscr{T}$ satisfying the conditions:

$$\text{(2.1)} \qquad \begin{aligned} &1)\ \mathscr{D}J = 0;\\ &2)\ \mathscr{D}\bar{g} = 0;\\ &3)\ \mathscr{T} = -\bar{\Omega} \otimes J\bar{P} \end{aligned}$$

is said to be a *complex conformal connection* [10].

In terms of the Kähler structure (g, J) $\mathscr{D}$ is given by

$$\text{(2.2)} \qquad \begin{aligned} \mathscr{D}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X, Y)P \\ - \omega(JX)JY - \omega(JY)JX - g(JX, Y)JP, \quad X, Y \in \mathfrak{X}M. \end{aligned}$$

The conditions (2.1) in terms of the Kähler structure (g, J) become

$$\text{(2.3)} \qquad \begin{aligned} &1)\ \mathscr{D}J = 0;\\ &2)\ \mathscr{D}g = -2\omega \otimes g;\\ &3)\ \mathscr{T} = -2\Omega \otimes JP. \end{aligned}$$

Denote by $\mathscr{R}$ the curvature tensor of the complex conformal connection $\mathscr{D}$. Taking into account (2.2) we have the relation between R and $\mathscr{R}$:

$$\text{(2.4)} \qquad \begin{aligned} \mathscr{R}(X, Y)Z = R(X, Y)Z \\ - \left\{(\nabla_Y\omega)(Z) - \omega(Y)\omega(Z) + \omega(JY)\omega(JZ) + \frac{1}{2}\omega(P)g(Y, Z)\right\}X \\ + \left\{(\nabla_X\omega)(Z) - \omega(X)\omega(Z) + \omega(JX)\omega(JZ) + \frac{1}{2}\omega(P)g(X, Z)\right\}Y \\ - g(Y, Z)\left\{\nabla_X P - \omega(X)P - \omega(JX)JP + \frac{1}{2}\omega(P)X\right\} \\ + g(X, Z)\left\{\nabla_Y P - \omega(Y)P - \omega(JY)JP + \frac{1}{2}\omega(P)Y\right\} \\ + \left\{(\nabla_Y\omega)(JZ) - \omega(Y)\omega(JZ) - \omega(JY)\omega(Z) + \frac{1}{2}\omega(P)g(Y, JZ)\right\}JX \\ - \left\{(\nabla_X\omega)(JZ) - \omega(X)\omega(JZ) - \omega(JX)\omega(Z) + \frac{1}{2}\omega(P)g(X, JZ)\right\}JY \\ + g(Y, JZ)\left\{\nabla_X JP - \omega(X)JP + \omega(JX)P + \frac{1}{2}\omega(P)JX\right\} \\ - g(X, JZ)\left\{\nabla_Y JP - \omega(Y)JP + \omega(JY)P + \frac{1}{2}\omega(P)JY\right\} \\ - (\nabla_X\omega)(JY)JZ + (\nabla_Y\omega)(JX)JZ + 2g(X, JY)\{\omega(JZ)P + \omega(Z)JP\} \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}M$.

From (2.4) it follows that the curvature tensor $\mathscr{R}$ satisfies the first Bianchi identity (i.e. $\mathscr{R}$ is a Kähler tensor) if and only if [7]:

$$(2.5)\qquad (\nabla_X\omega)(Y) + 2\omega(JX)\omega(JY) + \omega(P)g(X,Y) = 0, \quad X, Y \in \mathfrak{X}M,$$

which is equivalent to the condition

$$\mathscr{D}_X P = 0, \quad X \in \mathfrak{X}M.$$

If the 1-form ω satisfies (2.5), then (2.4) becomes

$$(2.6)\qquad \begin{aligned} \mathscr{R}(X,Y)Z = {} & R(X,Y)Z + L(Y,Z)X - L(X,Z)Y + g(Y,Z)L(X) \\ & - g(X,Z)L(Y) + L(JY,Z)JX - L(JX,Z)JY \\ & - 2L(JX,Y)JZ + g(JY,Z)JL(X) - g(JX,Z)JL(Y) \\ & - 2g(JX,Y)JL(Z), \quad X, Y, Z \in \mathfrak{X}M, \end{aligned}$$

where $L(X,Y) = \omega(X)\omega(Y) + \omega(JX)\omega(JY) + \frac{1}{2}\omega(P)g(X,Y)$ and $L(X)$ is the corresponding tensor of type $(1,1)$ with respect to the Kähler metric g.

If (M,g,J) admits a flat complex conformal connection (2.2), then $\mathscr{R}$ satisfies the first Bianchi identity, i.e. (2.5) holds good. Then (2.6) implies that the Kähler manifold is Bochner flat.

Conversely, if (M,g,J) admits a 1-form ω satisfying (2.5), then (2.4) becomes (2.6). The condition (M,g,J) is Bochner flat implies that $\mathscr{R} = 0$, i.e. the complex conformal connection (2.2) is flat.

3. A Curvature characterization of Kähler manifolds admitting flat complex conformal connection

For any Bochner-Kähler manifold (M,g,J) in [2] we proved that

$$(3.1)\qquad \begin{aligned} (\nabla_X\rho)(Y,Z) = {} & \frac{1}{4(n+1)}\{2d\tau(X)g(Y,Z) + d\tau(Y)g(X,Z) + d\tau(Z)g(X,Y) \\ & + d\tau(JY)g(X,JZ) + d\tau(JZ)g(X,JY)\}, \quad X, Y, Z \in \mathfrak{X}M. \end{aligned}$$

This equality shows that the conditions $\tau = const$ and $\nabla\rho = 0$ are equivalent on a Bochner-Kähler manifold. Because of the structural theorem in [8] the case $B(R) = 0$, $d\tau = 0$, can be considered as well-studied.

We consider Bochner-Kähler manifolds satisfying the condition $d\tau \neq 0$ for all points $p \in M$. This condition allows us to introduce the frame field

$$\left\{\xi = \frac{grad\ \tau}{\|d\tau\|}, J\xi = \frac{J\ grad\ \tau}{\|d\tau\|}\right\}$$

and the J-invariant distributions D_τ and $D_\tau^\perp = span\{\xi, J\xi\}$.

Thus our approach to the local theory of Bochner-Kähler manifolds is to treat them as Kähler manifolds (M,g,J,D_τ) endowed with a J-invariant distribution D_τ generated by the Kähler structure (g, J). We call this distribution *the scalar distribution* of the manifold [2].

A J-invariant distribution D_τ, $(D_\tau^\perp = span\{\xi, J\xi\})$ is said to be a B_0-distribution [1] if dim $M = 2n \geqq 6$ and

$$\begin{aligned} &\text{i)} \quad \nabla_{x_0}\xi = \frac{k}{2}x_0, \quad k \neq 0,\ x_0 \in D_\tau, \\ &\text{ii)} \quad \nabla_{J\xi}\xi = -p^* J\xi, \\ &\text{iii)} \quad \nabla_\xi \xi = 0, \end{aligned}$$

where k and p^* are functions on M.

The above conditions are equivalent to the equalities

$$\nabla_X \xi = \frac{k}{2}\{X - \eta(X)\xi + \eta(JX)J\xi\} + p^*\eta(JX)J\xi, \quad X \in \mathfrak{X}M, \tag{3.2}$$

$$dk = \xi(k)\eta, \quad p^* = -\frac{\xi(k) + k^2}{k}.$$

In [2] we have shown that

$$\mathfrak{B} = \|\rho\|^2 - \frac{\tau^2}{2(n+1)} + \frac{\Delta\tau}{n+1} \tag{3.3}$$

is a constant on any Bochner-Kähler manifold. We call this constant *the Bochner constant* of the manifold.

Let us denote

$$\begin{aligned} 4\pi(X, Y)Z :=\ & g(Y, Z)X - g(X, Z)Y - 2g(JX, Y)JZ \\ & + g(JY, Z)JX - g(JX, Z)JY, \\ 8\Phi(X, Y)Z :=\ & g(Y, Z)(\eta(X)\xi - \eta(JX)J\xi) - g(X, Z)(\eta(Y)\xi - \eta(JY)J\xi) \\ & + g(JY, Z)(\eta(X)J\xi + \eta(JX)\xi) - g(JX, Z)(\eta(Y)J\xi + \eta(JY)\xi) \\ & - 2g(JX, Y)(\eta(Z)J\xi + \eta(JZ)\xi) \\ & + (\eta(Y)\eta(Z) + \eta(JY)\eta(JZ))X - (\eta(X)\eta(Z) + \eta(JX)\eta(JZ))Y \\ & - (\eta(Y)\eta(JZ) - \eta(JY)\eta(Z))JX + (\eta(X)\eta(JZ) - \eta(JX)\eta(Z))JY \\ & + 2(\eta(X)\eta(JY) - \eta(JX)\eta(Y))JZ, \quad X, Y, Z \in \mathfrak{X}M. \end{aligned}$$

In [2] we have also proved that

If (M, g, J) (dim $M = 2n \geq 6$) *is a Bochner-Kähler manifold whose scalar distribution* D_τ *is a* B_0*-distribution, then*

$$R = a\pi + b\Phi, \quad b \neq 0, \tag{3.4}$$

where a, b *are the following functions on* M

$$a = \frac{\tau}{(n+1)(n+2)} + \frac{2\mathfrak{b}_0}{n+2}, \quad b = \frac{2\tau}{(n+1)(n+2)} - \frac{2n\mathfrak{b}_0}{n+2}, \tag{3.5}$$

and

$$b_0 = \frac{2a-b}{2} = const. \tag{3.6}$$

In [2] we studied three classes of Bochner-Kähler manifolds whose scalar distribution is a B_0-distribution according to the function $a + k^2$:

$$a + k^2 > 0, \quad a + k^2 = 0, \quad a + k^2 < 0.$$

Now we can prove a curvature characterization of Kähler manifolds admitting a flat complex conformal connection.

Theorem 3.1. *A Kähler manifold* (M, g, J) (dim $M = 2n \geq 6$) *admits a flat complex conformal connection if and only if it is a Bochner-Kähler manifold whose scalar distribution* D_τ *is a* B_0*-distribution with function* $a + k^2 = 0$ *and geometric constants* $\mathfrak{B} = \mathfrak{b}_0 = 0$.

Proof. Let u be a $\mathscr{C}^\infty$ function on M, such that the complex conformal connection $\mathscr{D}$, given by (2.2) with $\omega = du \neq 0$, $P = grad\ u$, is flat. Then (2.5) and (2.6) imply that the curvature tensor R of M has the structure (1.1). Comparing the tensor Q from (1.1) and the tensor L from (2.6) we obtain

$$\rho(X, Y) = -2(n+2)\{\omega(X)\omega(Y) + \omega(JX)\omega(JY) + \omega(P)g(X, Y)\}, \quad X, Y \in \mathfrak{X}M \tag{3.7}$$

and

$$\rho(X, P) = -2(n+2)\omega(P)\omega(X), \quad X \in \mathfrak{X}M. \tag{3.8}$$

After taking a trace in (3.7) we also get

$$\tau = -4(n+1)(n+2)\omega(P). \tag{3.9}$$

Taking into account (2.5) we calculate from (3.7)

$$\begin{aligned}(\nabla_X \rho)(Y, Z) = 2(n+2)\omega(P)\{&2\omega(X)g(Y, Z) + \omega(Y)g(X, Z) \\ &+ \omega(Z)g(X, Y) + \omega(JY)g(X, JZ) + \omega(JZ)g(X, JY)\}, \\ &X, Y, Z \in \mathfrak{X}M.\end{aligned} \tag{3.10}$$

Comparing (3.1) and (3.10) in view of (3.9), we obtain

$$\omega = -\frac{d\tau}{2\tau}, \quad P = -\frac{grad\ \tau}{2\tau}, \quad \|d\tau\|^2 = \frac{-\tau^3}{(n+1)(n+2)}. \tag{3.11}$$

The unit vector field $\xi = \dfrac{grad\ \tau}{\|d\tau\|}$ because of (3.11) gets the form

$$\xi = 2\sqrt{\frac{(n+1)(n+2)}{-\tau}}P.$$

From (2.5) and (3.9) we obtain

$$(3.12)\quad \nabla_X \xi = -\frac{1}{2}\sqrt{\frac{-\tau}{(n+1)(n+2)}}\{X - \eta(X)\xi - 2\eta(JX)J\xi\}, \quad X \in \mathfrak{X}M.$$

Now from (3.2) and (3.12) it follows that the scalar distribution D_τ of the manifold is a B_0-distribution with functions

$$(3.13)\qquad k = -\sqrt{\frac{-\tau}{(n+1)(n+2)}}, \quad p^* = \frac{3}{2}\sqrt{\frac{-\tau}{(n+1)(n+2)}}.$$

Then (2.6) and (3.11) give that the curvature tensor R of the manifold has the form

$$R = \frac{\tau}{(n+1)(n+2)}(\pi + 2\Phi)$$

and the functions a and b are

$$(3.14)\qquad a = \frac{\tau}{(n+1)(n+2)}, \quad b = \frac{2\tau}{(n+1)(n+2)}.$$

From (3.13) and (3.14) we find $a + k^2 = 0$. The equalities (3.6) and (3.14) imply that $\mathfrak{b}_0 = 0$.

Taking into account (2.5), (3.11) and (3.7) we find

$$(3.15)\qquad \Delta\tau = \frac{-\tau^2}{n+1}, \quad \|\rho\|^2 = \frac{(n+3)\tau^2}{2(n+1)^2}.$$

Replacing $\Delta\tau$ and $\|\rho\|^2$ in (3.3) we obtain $\mathfrak{B} = 0$.

For the inverse, let (M, g, J) be a Bochner-Kähler manifold whose scalar distribution is a B_0-distribution. Then it follows [2] that (3.2), (3.4) and (3.5) hold good. Under the condition $\mathfrak{b}_0 = 0$ we find that the functions a and b satisfy (3.14).

The condition $a + k^2 = 0$ implies that $k^2 = -a = \dfrac{-\tau}{(n+1)(n+2)}$, i.e. $\tau < 0$. From Theorem 3.5 in [1] it follows that

$$\|d\tau\| = \xi(\tau) = \frac{(n+1)(n+2)}{2}\xi(b) = \frac{(n+1)(n+2)}{2}kb > 0,$$

which gives that the function k is negative and

$$k = -\sqrt{\frac{-\tau}{(n+1)(n+2)}}, \quad \|d\tau\|^2 = \frac{-\tau^3}{(n+1)(n+2)}, \quad p^* = \frac{3}{2}\sqrt{\frac{-\tau}{(n+1)(n+2)}}.$$

Then, from the equality (3.2) for any $X, Y \in \mathfrak{X}M$ we have

$$(\nabla_X\eta)(Y) = -\frac{1}{2}\sqrt{\frac{-\tau}{(n+1)(n+2)}}\{g(X,Y) - \eta(X)\eta(Y) + 2\eta(JX)\eta(JY)\}.$$

Putting $2u := -\ln(-\tau)$ and $\omega := du = -\frac{d\tau}{2\tau} = -\frac{\|d\tau\|}{2\tau}\eta = -\frac{k}{2}\eta$ we prove that $(\nabla_X\omega)(Y)$ satisfies (2.5) and the complex conformal connection (2.2) is flat. **QED**

Let $(Q_0, g_0, \varphi, \tilde{\xi}_0, \tilde{\eta}_0)$ be an α_0-Sasakian space form [4] with constant φ-holomorphic sectional curvatures H_0. In [2] we introduced warped product Kähler manifolds, which are completely determined by the underlying α_0-Sasakian space form Q_0 of type $H_0 + 3\alpha_0^2 \gtreqqless 0$ and the generating function $p(t)$, $t \in I \subset \mathbf{R}$.

In order to obtain a local description of the Kähler manifolds admitting a flat complex conformal connection we apply Theorem 6.1 in [2], which states:

Any Bochner-Kähler manifold whose scalar distribution is a B_0-distribution locally has the structure of a warped product Kähler manifold with generating function $p(t)$ (or $t(p)$) of type 1.–13.

According to Theorem 3.1 any Kähler manifold (M, g, J), $(\dim M = 2n \geqq 6)$ admitting a flat complex conformal connection is locally a Bochner-Kähler manifold whose scalar distribution is a B_0-distribution with function $a + k^2 = 0$ and constants $\mathfrak{B} = \mathfrak{b}_0 = 0$. In terms of [2] the conditions $\mathfrak{B} = \mathfrak{b}_0 = 0$ are equivalent to the conditions $\mathfrak{K} = \mathfrak{b}_0 = 0$.

Hence, (M, g, J) is a warped product Bochner-Kähler manifold whose underlying α_0-Sasakian space form is of type $H_0 + 3\alpha_0^2 = 0$ with metric

$$g = p^2(t)\left\{g_0 + \left(\frac{1}{\alpha_0}\frac{dp}{dt} - 1\right)\tilde{\eta}_0 \otimes \tilde{\eta}_0\right\} + \eta \otimes \eta,$$

generated by the function

$$p(t) = \frac{1}{\sqrt[3]{1 - 3\alpha_0(t - t_0)}}, \qquad t \in \left(-\infty, \frac{1 + 3\alpha_0 t_0}{3\alpha_0}\right)$$

of type 9. [2].

This metric is not complete.

Especially in the case $\alpha_0 = 1$ the underlying manifold is a Sasakian space form with $H_0 = -3$. Sasakian space forms of type $H_0 = -3$ have been studied by Ogiue [5] and Okumura [6]. A classification theorem for Sasakian space forms under the assumption of completeness has been given by Tanno [9].

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