KÄHLER METRICS ON TORIC ORBIFOLDS

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Abstract. A theorem of E. Lerman and S. Tolman, generalizing a result of T. Delzant, states that compact symplectic toric orbifolds are classified by their moment polytopes, together with a positive integer label attached to each of their facets. In this paper we use this result, and the existence of "global" action-angle coordinates, to give an effective parametrization of all compatible toric complex structures on a compact symplectic toric orbifold, by means of smooth functions on the corresponding moment polytope. This is equivalent to parametrizing all toric Kähler metrics and generalizes an analogous result for toric manifolds.

A simple explicit description of interesting families of extremal Kähler metrics, arising from recent work of R. Bryant, is given as an application of the approach in this paper. The fact that in dimension four these metrics are self-dual and conformally Einstein is also discussed. This gives rise in particular to a one parameter family of self-dual Einstein metrics connecting the well known Eguchi-Hanson and Taub-NUT metrics.

1. Introduction

The space of Kähler metrics on a Kähler manifold (or orbifold) can be described in two equivalent ways, reflecting the fact that a Kähler manifold is both a complex and a symplectic manifold.

From the complex point of view, one starts with a fixed complex manifold (M, J_0) and Kähler class $\Omega \in H^{1,1}_{J_0} \cap H^2(M, \mathbb{R})$, and considers the space $\mathcal{S}(J_0, \Omega)$ of all symplectic forms ω on M that are compatible with J_0 and represent the class Ω. Any such form $\omega \in \mathcal{S}(J_0, \Omega)$ gives rise to a Kähler metric $\langle \cdot, \cdot \rangle \equiv \omega(\cdot, J_0 \cdot)$.

The symplectic point of view arises naturally from the observation that any two forms $\omega_0, \omega_1 \in \mathcal{S}(J_0, \Omega)$ define equivalent symplectic structures on M. In fact, the family $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$, for $t \in [0, 1]$, is an isotopy of symplectic forms in the same cohomology class Ω, and so Moser's theorem [M] gives a family of diffeomorphisms $\varphi_t : M \to M$, $t \in [0, 1]$, such that $\varphi_t^*(\omega_t) = \omega_0$. In particular, the Kähler manifold (M, J_0, ω_1) is Kähler isomorphic to (M, J_1, ω_0), where $J_1 = (\varphi_1)_*^{-1} \circ J_0 \circ (\varphi_1)_*$.

This means that one can also describe the space of Kähler metrics starting with a fixed symplectic manifold (M, ω_0) and considering the space $\mathcal{J}(\omega_0, [J_0])$ of all

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complex structures J on M that are compatible with ω_0 and belong to some diffeomorphism class $[J_0]$, determined by a particular compatible complex structure J_0. Any such $J \in \mathcal{J}(\omega_0, [J_0])$ gives rise to a Kähler metric $\langle\cdot,\cdot\rangle \equiv \omega_0(\cdot, J\cdot)$.

The symplectic point of view fits into a general framework, proposed by Donaldson in [D1] and [D2], involving the geometry of infinite dimensional groups and spaces, and the relation between symplectic and complex quotients. Although this framework can be useful as a guiding principle, the symplectic point of view does not seem to be very effective for solving specific problems in Kähler geometry, the reason being that the space $\mathcal{J}(\omega_0, [J_0])$ is non-linear and difficult to parametrize. The complex point of view fairs much better in this regard, since the space $\mathcal{S}(J_0, \Omega)$ can be identified with an open convex subset of the linear space of smooth functions on M. Indeed, the $\partial\overline{\partial}$-lemma asserts that given $\omega_0 \in \mathcal{S}(J_0, \Omega)$ any other $\omega \in \mathcal{S}(J_0, \Omega)$ can be written as

$$\omega = \omega_0 + 2i\partial\overline{\partial}f\ ,\ \text{for some } f \in C^\infty(M)\,. \tag{1.1}$$

Moreover, the set of functions $f \in C^\infty(M)$ for which the form ω defined by (1.1) is in $\mathcal{S}(J_0, \Omega)$ is open and convex.

There are however particular situations in which the space $\mathcal{J}(\omega_0, [J_0])$ admits a parametrization similar to the one just described for $\mathcal{S}(J_0, \Omega)$, and the symplectic point of view can then be used very effectively. In [A2] this was shown to be the case for Kähler toric manifolds. In this paper we show that this can also be done for all Kähler toric orbifolds, and describe an application of the effectiveness of the symplectic approach in this context.

Let (M, ω) be a symplectic toric orbifold of dimension $2n$, equipped with an effective Hamiltonian action $\tau : \mathbb{T}^n \to \mathrm{Diff}(M, \omega)$ of the standard (real) n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$, i.e. (M, ω, τ) is a symplectic toric orbifold. Denote by $\phi : M \to (\mathbb{R}^n)^*$ the moment map of such an action. The image $P \equiv \phi(M) \subset (\mathbb{R}^n)^*$ is a convex rational simple polytope (see Definition 2.3). When M is a manifold, a theorem of Delzant [Del] says that, up to equivariant symplectomorphism, the polytope P completely determines the symplectic toric manifold (M, ω, τ). In [LT] Lerman and Tolman generalize Delzant's theorem to orbifolds. The result is that the polytope P, together with a positive integer label attached to each of its facets, completely determines the symplectic toric orbifold (see Theorem 2.5).

The proof, in both manifold and orbifold cases, gives an explicit construction of a canonical model for each symplectic toric orbifold, i.e. it associates to each labeled polytope P an explicit symplectic toric orbifold (M_P, ω_P, τ_P) with moment map $\phi_P : M_P \to P$ (see §2.2). Moreover, it follows from the construction that M_P has a canonical $\mathbb{T}^n$-invariant complex structure J_P compatible with ω_P (see Remark 2.7). In other words, associated to each labeled polytope $P \subset (\mathbb{R}^n)^*$ one has a canonical Kähler toric orbifold $(M_P, \omega_P, J_P, \tau_P)$ with moment map $\phi_P : M_P \to P$.

The symplectic description of compatible toric complex structures and Kähler metrics is based on the following set-up (see [A2] for details). Let $\breve{P}$ denote the interior of P, and consider $\breve{M}_P \subset M_P$ defined by $\breve{M}_P = \phi_P^{-1}(\breve{P})$. One can easily check that $\breve{M}_P$ is a smooth open dense subset of M_P, consisting of all the points where the $\mathbb{T}^n$-action is free. It can be described as

$$\breve{M}_P \cong \breve{P} \times \mathbb{T}^n = \left\{(x, \theta) : x \in \breve{P} \subset (\mathbb{R}^n)^*\ ,\ \theta \in \mathbb{R}^n/2\pi\mathbb{Z}^n\right\}\,,$$

where (x,θ) are symplectic (or action-angle) coordinates for ω_P, i.e.

$$\omega_P = dx \wedge d\theta = \sum_{j=1}^{n} dx_j \wedge d\theta_j \ .$$

If J is any ω_P-compatible toric complex structure on M_P, the symplectic (x,θ)-coordinates on $\breve{M}_P$ can be chosen so that the matrix that represents J in these coordinates has the form

$$\begin{bmatrix} 0 & \vdots & -G^{-1} \\ \cdots & \cdots & \cdots \\ G & \vdots & 0 \end{bmatrix} \tag{1.2}$$

where $G = G(x) = [g_{jk}(x)]_{j,k=1}^{n,n}$ is a symmetric and positive-definite real matrix. The integrability condition for the complex structure J is equivalent to G being the Hessian of a smooth function $g \in C^\infty(\breve{P})$, i.e.

$$G = \mathrm{Hess}_x(g)\,,\ g_{jk}(x) = \frac{\partial^2 g}{\partial x_j \partial x_k}(x)\,,\ 1 \le j,k \le n\,. \tag{1.3}$$

Holomorphic coordinates for J are given in this case by

$$z(x,\theta) = u(x,\theta) + iv(x,\theta) = \frac{\partial g}{\partial x}(x) + i\theta \ .$$

We will call g the **potential** of the compatible toric complex structure J. Note that the Kähler metric $\langle \cdot,\cdot \rangle = \omega_P(\cdot, J\cdot)$ is given in these (x,θ)-coordinates by the matrix

$$\begin{bmatrix} G & \vdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \vdots & G^{-1} \end{bmatrix} \tag{1.4}$$

In particular, the induced metric on any slice of the form $\breve{P} \times \{\text{point}\} \subset \breve{M}_P$ is given by the matrix G.

Every convex rational simple polytope $P \subset (\mathbb{R}^n)^*$ can be described by a set of inequalities of the form

$$\langle x, \mu_r \rangle \ge \rho_r\,,\ r = 1, \ldots, d,$$

where d is the number of facets of P, each μ_r is a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (the inward-pointing normal to the r-th facet of P), and each ρ_r is a real number. The labels $m_r \in \mathbb{N}$ attached to the facets can be incorporated in the description of P by considering the affine functions $\ell_r : (\mathbb{R}^n)^* \to \mathbb{R}$ defined by

$$\ell_r(x) = \langle x, m_r\mu_r \rangle - \lambda_r \ \text{ where } \lambda_r = m_r\rho_r \text{ and } r = 1, \ldots, d\,.$$

Then x belongs to the r-th facet of P iff $\ell_r(x) = 0$, and $x \in \breve{P}$ iff $\ell_r(x) > 0$ for all $r = 1, \ldots, d$.

We are now ready to state the main results of this paper. The first is a straightforward generalization to toric orbifolds of a result of Guillemin [G1].

Theorem 1. *Let (M_P, ω_P, τ_P) be the symplectic toric orbifold associated to a labeled polytope $P \subset (\mathbb{R}^n)^*$. Then, in suitable symplectic (x,θ)-coordinates on $\breve{M}_P \cong$*

$\breve{P} \times \mathbb{T}^n$, the canonical compatible toric complex structure J_P is of the form (1.2)-(1.3) for a potential $g_P \in C^\infty(\breve{P})$ given by

$$g_P(x) = \frac{1}{2} \sum_{r=1}^{d} \ell_r(x) \log \ell_r(x) \ .$$

The second result provides the symplectic version of (1.1) in this toric orbifold context, generalizing an analogous result for toric manifolds proved in [A2].

Theorem 2. *Let J be any compatible toric complex structure on the symplectic toric orbifold (M_P, ω_P, τ_P). Then, in suitable symplectic (x,θ)-coordinates on $\breve{M}_P \cong \breve{P} \times \mathbb{T}^n$, J is given by (1.2)-(1.3) for a potential $g \in C^\infty(\breve{P})$ of the form*

$$g(x) = g_P(x) + h(x) \,,$$

where g_P is given by Theorem 1, h is smooth on the whole P, and the matrix $G = \mathrm{Hess}(g)$ is positive definite on $\breve{P}$ and has determinant of the form

$$\mathrm{Det}(G) = \left(\delta \prod_{r=1}^{d} \ell_r \right)^{-1} \,,$$

with δ being a smooth and strictly positive function on the whole P.

Conversely, any such potential g determines by (1.2)-(1.3) a complex structure on $\breve{M}_P \cong \breve{P} \times \mathbb{T}^n$, that compactifies to a well-defined compatible toric complex structure J on the symplectic toric orbifold (M_P, ω_P, τ_P).

Note that there is no imposed condition of J being in the same diffeomorphism class as J_P. The reason is that, by Theorem 9.4 in [LT], any compatible toric J on (M_P, ω_P, τ_P) is equivariantly biholomorphic to J_P.

Our next results describe an application of the parametrization of compatible toric complex structures given by Theorem 2. In a recent paper [Br] R. Bryant studies and classifies Bochner-Kähler metrics, i.e. Kähler metrics with vanishing Bochner curvature. He shows in particular that these metrics always have a very high degree of symmetry, the least symmetric ones being of toric type. It turns out that the models for these least symmetric Bochner-Kähler metrics, given by Theorem 9 in [Br], have a very simple explicit description in the above symplectic framework.

For us, the most relevant geometric property of these metrics is that of being **extremal** in the sense of Calabi (see §4.1), and we will construct them only as such. However, the reader should keep in mind that these are indeed the same metrics given by Theorem 9 in [Br], and hence the word "extremal" can be replaced by "Bochner-Kähler" in the statements that follow.

Let $P^n_{\mathbf{m}}$ denote the labeled simplex in $(\mathbb{R}^n)^*$ defined by the affine functions

$$\ell_r(x) = m_r(1+x_r), r = 1, \ldots, n \,,\ \ell_{n+1}(x) = m_{n+1}(1-\psi) \,,\ \psi = \sum_{j=1}^{n} x_j \,, \tag{1.5}$$

where $\mathbf{m} = (m_1, \ldots, m_{n+1}) \in \mathbb{N}^{n+1}$ is a vector of positive integer labels. The associated symplectic toric orbifold will be called a **labeled projective space** and denoted by $(\mathbb{SP}^n_{\mathbf{m}}, \omega_{\mathbf{m}}, \tau_{\mathbf{m}})$ (the "$\mathbb{S}$" is supposed to emphasize its $\mathbb{S}$ymplectic nature).

Theorem 3. *For any vector of labels* $\mathbf{m} \in \mathbb{N}^{n+1}$*, the potential* $g \in C^\infty(\breve{P}^n_{\mathbf{m}})$ *defined by*

$$g(x) = \frac{1}{2}\left(\sum_{r=1}^{n+1} \ell_r(x)\log\ell_r(x) - \ell_\Sigma(x)\log\ell_\Sigma(x)\right),$$

where the ℓ_r*'s are given by (1.5) and*

$$\ell_\Sigma(x) = \sum_{r=1}^{n+1} \ell_r(x),$$

gives rise to an extremal compatible toric complex structure on $(\mathbb{SP}^n_{\mathbf{m}}, \omega_{\mathbf{m}}, \tau_{\mathbf{m}})$*. In other words, the metric defined by (1.4) is an extremal Kähler metric.*

As we will see in §2.3, there is a close relation between labeled projective spaces $\mathbb{SP}^n_{\mathbf{m}}$ and the more common weighted projective spaces $\mathbb{CP}^n_{\mathbf{a}}$. These are defined for a given vector of positive integer weights $\mathbf{a} = (a_1, \ldots, a_{n+1}) \in \mathbb{N}^{n+1}$ as

$$\mathbb{CP}^n_{\mathbf{a}} \equiv \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^*,$$

where the action of $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on $\mathbb{C}^{n+1}$ is given by

$$(z_1, \ldots, z_{n+1}) \stackrel{t}{\mapsto} (t^{a_1}z_1, \ldots, t^{a_{n+1}}z_{n+1}),\ t \in \mathbb{C}^*.$$

The relation between $\mathbb{SP}^n_{\mathbf{m}}$ and $\mathbb{CP}^n_{\mathbf{m}}$ implies the following corollary to Theorem 3 (see also Theorem 11 in [Br]).

Corollary 1. *Every weighted projective space* $\mathbb{CP}^n_{\mathbf{a}}$ *has an extremal Kähler metric.*

The potential g of Theorem 3 defines an extremal Kähler metric on $\breve{P}^n_{\mathbf{m}} \times \mathbb{T}^n$ for any positive **real** vector of labels $\mathbf{m} \in \mathbb{R}^{n+1}_+$. Although these do not correspond in general to compact orbifold metrics, they do admit a natural compactification as metrics with simple conical singularities.

Theorem 4. *Consider the smooth symplectic toric manifold* $(\mathbb{SP}^n_{\mathbf{1}} \cong \mathbb{CP}^n, \omega_{\mathbf{1}}, \tau_{\mathbf{1}})$ *associated to the simplex* $P^n_{\mathbf{1}} \subset (\mathbb{R}^n)^*$*. Denote by* $\phi_{\mathbf{1}} : \mathbb{SP}^n_{\mathbf{1}} \to P^n_{\mathbf{1}}$ *the corresponding moment map. Then, for any* $\mathbf{m} \in \mathbb{R}^{n+1}_+$*, the extremal Kähler metric (1.4) defined on* $\breve{P}^n_{\mathbf{1}} \times \mathbb{T}^n$ *by the potential* g *of Theorem 3, corresponds to an extremal Kähler metric on* $\mathbb{SP}^n_{\mathbf{1}}$ *with conical singularities of angles* $2\pi/m_r$ *around the pre-images* $N_r \equiv \phi^{-1}_{\mathbf{1}}(F_r)$ *of each facet* $F_r \subset P^n_{\mathbf{1}}$*,* $r = 1, \ldots, n+1$ *(note that* $N_r \cong \mathbb{SP}^{n-1}_{\mathbf{1}} \cong \mathbb{CP}^{n-1}$*).*

As noted before, all extremal Kähler metrics of Theorem 4 are in fact Bochner-Kähler. In dimension four ($n = 2$) the Bochner tensor is the same as the anti-self-dual part of the Weyl tensor, and so in this case "Bochner-Kähler" is the same as "self-dual Kähler". A local study and classification of these metrics in this later context was also obtained in recent work of Apostolov and Gauduchon [AG]. They show in particular that, whenever the scalar curvature S is nonzero, a self-dual Kähler metric is conformally Einstein with conformal factor given by S^{-2}. In Section 5 we consider a particularly interesting family of such metrics provided by Theorem 4 when $n = 2$ and $\mathbf{m} = (1, 1, m),\ m \in \mathbb{R}_+$. We will see in particular that this family gives rise to a one-parameter family of $U(2)$-invariant self-dual Einstein metrics of positive scalar curvature, with end points the Ricci-flat Eguchi-Hanson metric on $T\,\mathbb{CP}^1$ ($m = 1/2$) and the also Ricci-flat Taub-NUT metric on $\mathbb{R}^4$ ($m = +\infty$). We will also point out how, for a particular discrete set of values of the parameter m, these metrics are related to the ones constructed by Galicki-Lawson in [GL] using quaternionic reduction.

A general discussion of the usefulness of the symplectic approach to the construction of $U(n)$-invariant extremal Kähler metrics will be given in [A3].

The rest of the paper is organized as follows. In Section 2, after some necessary preliminaries on orbifolds, we give the definition and combinatorial characterization of symplectic toric orbifolds in terms of labeled polytopes, due to Lerman and Tolman. Labeled projective spaces and their relation to weighted projective spaces is discussed in §2.3. Theorems 1 and 2 are proved in Section 3, while Theorems 3 and 4 are proved in Section 4.

2. Symplectic toric orbifolds

In this section, after some necessary preliminaries on orbifolds, we give the definition and combinatorial characterization of a symplectic toric orbifold, and discuss the family of examples given by weighted and labeled projective spaces. Good references for this material are Satake [S] (for general orbifolds) and Lerman-Tolman [LT] (for symplectic orbifolds).

2.1. **Preliminaries on orbifolds.**

Definition 2.1. An **orbifold** M is a singular real manifold of dimension n, whose singularities are locally isomorphic to quotient singularities of the form $\mathbb{R}^n/\Gamma$, where Γ is a finite subgroup of $GL(n,\mathbb{R})$ such that, for any $1 \neq \gamma \in \Gamma$, the subspace $V_\gamma \subset \mathbb{R}^n$ fixed by γ has $\dim V_\gamma \leq n-2$.

For each singular point $p \in M$ there is a finite subgroup $\Gamma_p \subset GL(n,\mathbb{R})$, unique up to conjugation, such that open neighborhoods of p in M and 0 in $\mathbb{R}^n/\Gamma_p$ are homeomorphic. Such a point p is called an **orbifold point** of M, and Γ_p the **orbifold structure group** of p.

The condition on each nontrivial $\gamma \in \Gamma$ means that the singularities of the orbifold have codimension at least two, and this makes it behave much like a manifold. The usual definitions of vector fields, differential forms, metrics, group actions, etc, extend naturally to orbifolds. In particular, a **symplectic orbifold** can be defined as an orbifold M equipped with a closed non-degenerate 2-form ω, while a **complex orbifold** can be defined as an orbifold M equipped with an integrable complex structure J. A **Kähler orbifold** (M,ω,J) is a symplectic and complex orbifold, with ω and J **compatible** in the sense that the bilinear form $\langle\cdot,\cdot\rangle \equiv \omega(\cdot, J\cdot)$ is symmetric and positive definite, thus defining a **Kähler metric** on M.

All orbifolds we will consider in this paper (underlying a symplectic toric orbifold) arise through the following natural construction. Let Z be an oriented manifold and K an abelian group acting smoothly, properly and effectively on Z, preserving the orientation and such that the stabilizers of points in Z are always finite subgroups of K. Then the quotient $M = Z/K$ is an orbifold (the orientation preserving condition eliminates the possibility of codimension one singularities). Its orbifold points $[p] \in M$ correspond to points $p \in Z$ with nontrivial stabilizer $\Gamma_p \subset K$, and Γ_p is then the orbifold structure group of $[p]$.

Let (M,ω) be a symplectic orbifold, and G a Lie group acting smoothly on M. This group action induces an infinitesimal action of the Lie algebra $\mathfrak{g}$ on M, and for each $\xi \in \mathfrak{g}$ we denote by ξ_M the induced vector field on M. The G-action is said to be symplectic if it preserves ω, and **Hamiltonian** if it has a **moment map**

$\phi : M \to \mathfrak{g}^*$, i.e. a G-equivariant map from M to the dual of the Lie algebra of G such that

$$\iota(\xi_M)\omega = d\langle \xi, \phi\rangle, \text{ for all } \xi \in \mathfrak{g}\,.$$

When $G = S^1 = \mathbb{R}/2\pi\mathbb{Z}$, a moment map is simply given by a Hamiltonian function $H : M \to \mathbb{R}^* \cong \mathfrak{g}^*$, whose Hamiltonian vector field X_H, defined by $\iota(X_H)\omega = dH$, generates the S^1-action. Note that H is uniquely defined up to addition by a constant.

2.2. **Symplectic toric orbifolds.**

Definition 2.2. A **symplectic toric orbifold** is a connected $2n$-dimensional symplectic orbifold (M, ω) equipped with an effective Hamiltonian action $\tau : \mathbb{T}^n \to \mathrm{Diff}(M, \omega)$ of the standard (real) n-torus $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$.

Denote by $\phi : M \to (\mathbb{R}^n)^*$ the moment map of such an action (well-defined up to addition by a constant). When M is a compact smooth manifold, the convexity theorem of Atiyah [At] and Guillemin-Sternberg [GS1] states that the image $P = \phi(M) \subset (\mathbb{R}^n)^*$ of the moment map ϕ is the convex hull of the image of the points in M fixed by $\mathbb{T}^n$, i.e. a convex polytope in $(\mathbb{R}^n)^*$. A theorem of Delzant [Del] then says that the convex polytope $P \subset (\mathbb{R}^n)^*$ completely determines the symplectic toric manifold, up to equivariant symplectomorphisms.

In [LT] Lerman and Tolman generalize these two theorems to orbifolds. While the convexity theorem generalizes word for word, one needs more information than just the convex polytope P to generalize Delzant's classification theorem.

Definition 2.3. A convex polytope P in $(\mathbb{R}^n)^*$ is called **simple** and **rational** if:

- **(1)** there are n edges meeting at each vertex p;
- **(2)** the edges meeting at the vertex p are rational, i.e. each edge is of the form $p + tv_i,\ 0 \le t \le \infty,$ where $v_i \in (\mathbb{Z}^n)^*$;
- **(3)** the $v_1, \ldots, v_n$ in (2) can be chosen to be a $\mathbb{Q}$-basis of the lattice $(\mathbb{Z}^n)^*$.

A **facet** is a face of P of codimension one. Following Lerman-Tolman, we will say that a **labeled polytope** is a rational simple convex polytope $P \subset (\mathbb{R}^n)^*$, plus a positive integer (**label**) attached to each of its facets.

Two labeled polytopes are **isomorphic** if one can be mapped to the other by a translation, and the corresponding facets have the same integer labels.

Remark 2.4. In Delzant's classification theorem for compact symplectic toric manifolds, there are no labels (or equivalently, all labels are equal to 1) and the polytopes that arise are slightly more restrictive: the "$\mathbb{Q}$" in (3) is replaced by "$\mathbb{Z}$".

Theorem 2.5 (Lerman-Tolman)**.** *Let (M, ω, τ) be a compact symplectic toric orbifold, with moment map $\phi : M \to (\mathbb{R}^n)^*$. Then $P \equiv \phi(M)$ is a rational simple convex polytope. For every facet F of P, there exists a positive integer m_F, the label of F, such that the structure group of every $p \in \phi^{-1}(\check{F})$ is $\mathbb{Z}/m_F\mathbb{Z}$ (here $\check{F}$ is the relative interior of F).*

Two compact symplectic toric orbifolds are equivariant symplectomorphic (with respect to a fixed torus acting on both) if and only if their associated labeled polytopes are isomorphic. Moreover, every labeled polytope arises from some compact symplectic toric orbifold.

The proof of the last claim of this theorem is important for our purposes. It associates to every labeled polytope P a compact symplectic toric orbifold (M_P, ω_P, τ_P), with moment map $\phi_P : M_P \to P \subset (\mathbb{R}^n)^*$. The construction, generalizing Delzant's for the case of symplectic toric manifolds, consists of a very explicit symplectic reduction.

Every labeled polytope $P \subset (\mathbb{R}^n)^*$ can be written uniquely as

$$P = \bigcap_{r=1}^{d} \left\{ x \in (\mathbb{R}^n)^* \ : \ \ell_r(x) \stackrel{\text{def}}{=} \langle x, m_r \mu_r \rangle - \lambda_r \geq 0 \right\} , \tag{2.1}$$

where d is the number of facets, each μ_r is a primitive element of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (the inward-pointing normal to the r-th facet of P), each $m_r \in \mathbb{N}$ is the label attached to the r-th facet of P, and each λ_r is a real number.

Let $(e_1, \ldots , e_d)$ denote the standard basis of $\mathbb{R}^d$, and define a linear map

$$\beta : \mathbb{R}^d \to \mathbb{R}^n \ \text{ by } \ \beta(e_r) = m_r \mu_r \, , \ r = 1, \ldots , d \, . \tag{2.2}$$

Condition (3) of Definition 2.3 implies that β is surjective. Denoting by $\mathfrak{k}$ its kernel, we have short exact sequences

$$0 \to \mathfrak{k} \xrightarrow{\iota} \mathbb{R}^d \xrightarrow{\beta} \mathbb{R}^n \to 0 \quad \text{and its dual} \quad 0 \to (\mathbb{R}^n)^* \xrightarrow{\beta^*} (\mathbb{R}^d)^* \xrightarrow{\iota^*} \mathfrak{k}^* \to 0 \ .$$

Let K denote the kernel of the map from $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$ to $\mathbb{T}^n = \mathbb{R}^n / 2\pi\mathbb{Z}^n$ induced by β. More precisely,

$$K = \left\{ [\theta] \in \mathbb{T}^d \ : \ \sum_{r=1}^{d} \theta_r m_r \mu_r \in 2\pi\mathbb{Z}^n \right\} . \tag{2.3}$$

The Lie algebra of K is $\mathfrak{k} = \mathrm{Ker}(\beta)$.

Consider $\mathbb{R}^{2d}$ with its standard symplectic form

$$\omega_0 = du \wedge dv = \sum_{r=1}^{d} du_r \wedge dv_r \ .$$

We identify $\mathbb{R}^{2d}$ with $\mathbb{C}^d$ via $z_r = u_r + i v_r \, , \ r = 1, \ldots , d$. The standard action of $\mathbb{T}^d$ on $\mathbb{R}^{2d} \cong \mathbb{C}^d$ is given by

$$\theta \cdot z = \left(e^{i\theta_1} z_1, \ldots , e^{i\theta_d} z_d \right)$$

and has moment map

$$\phi_{\mathbb{T}^d}(z_1, \ldots , z_d) = \sum_{r=1}^{d} \frac{|z_r|^2}{2} \, e_r^* + \lambda \in (\mathbb{R}^d)^* \, ,$$

where $\lambda \in (\mathbb{R}^d)^*$ is an arbitrary constant. We set $\lambda = \sum_r \lambda_r e_r^*$ and so

$$\phi_{\mathbb{T}^d}(z_1, \ldots , z_d) = \sum_{r=1}^{d} \left(\frac{|z_r|^2}{2} + \lambda_r \right) e_r^* \in (\mathbb{R}^d)^* \ . \tag{2.4}$$

Since K is a subgroup of $\mathbb{T}^d$, K acts on $\mathbb{C}^d$ with moment map

$$\phi_K = \iota^* \circ \phi_{\mathbb{T}^d} = \sum_{r=1}^{d} \left(\frac{|z_r|^2}{2} + \lambda_r \right) \iota^*(e_r^*) \in \mathfrak{k}^* \ . \tag{2.5}$$

The symplectic toric orbifold (M_P, ω_P) associated to the labeled polytope P is the symplectic reduction of $\mathbb{C}^d$ with respect to the K-action. As an orbifold it is

$$M_P = Z/K \ \text{ where } \ Z = \phi_K^{-1}(0) \equiv \text{ zero level set of moment map,} \tag{2.6}$$

the symplectic structure comes from the standard one in $\mathbb{R}^{2d}$ (via symplectic reduction), while the action of $\mathbb{T}^n \cong \mathbb{T}^d/K$ comes from from the reduction of the action of T^d on Z.

In order to verify these claims, several things need to be checked (see §8 of [LT]):

(i) zero is a regular value of ϕ_K and so Z is a smooth submanifold of $\mathbb{R}^{2d}$ of dimension $2d-(d-n)=d+n$;
(ii) with respect to the action of K on Z, the isotropy of any $z\in Z$ is a discrete subroup Γ_z of K. Hence the reduced space $M_P = Z/K$ is a symplectic orbifold of dimension $d+n-(d-n)=2n$;
(iii) the action of $\mathbb{T}^d$ on Z induces an effective Hamiltonian action of $\mathbb{T}^n \cong \mathbb{T}^d/K$ on M_P, whose moment map $\phi_{\mathbb{T}^n} \equiv \phi_P : M_P \to (\mathbb{R}^n)^*$ has image precisely P;
(iv) the orbifold structure group $\Gamma_{[z]}$, for any point $[z]\in M_P$ that gets mapped by ϕ_P to the interior of the r-th facet of P (cut out by the hyperplane $\{x\in(\mathbb{R}^n)^* :\ \ell_r(x)=0\}$), is precisely $\mathbb{Z}/m_r\mathbb{Z}$.

Regarding (iii) above, recall that the moment map is apriori only defined up to a constant. In this construction we can characterize ϕ_P uniquely by requiring that it fits in the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\phi_{\mathbb{T}^d}} & (\mathbb{R}^d)^* \\ {\scriptstyle\pi}\big\downarrow & & \big\uparrow{\scriptstyle\beta^*} \\ M_P & \xrightarrow{\phi_P} & (\mathbb{R}^n)^* \end{array} \tag{2.7}$$

where $\pi : Z \to M_P = Z/K$ is the quotient map. It is with this normalization that $\phi_P(M_P)=P$.

Remark 2.6. The isotropy and orbifold structure groups of (M_P,ω_P,τ_P) can be determined directly from the labeled polytope P (Lemma 6.6 in [LT]). Given $p\in M_P$, let $\mathcal{F}(p)$ be the set of facets that contain $\phi_P(p)$, i.e.

$$\mathcal{F}(p) = \{r\in\{1,\dots,d\} : \ell_r(\phi_P(p))=0\}\ .$$

The isotropy group of p is the subtorus $H_p\subset\mathbb{T}^n$ whose Lie algebra $\mathfrak{h}_p$ is the linear span of the normals $\mu_r\in\mathbb{R}^n$, for $r\in\mathcal{F}(p)$. The orbifold structure group Γ_p is isomorphic to $\Lambda_p/\hat{\Lambda}_p$, where $\Lambda_p\subset\mathfrak{h}_p$ is the lattice of circle subgroups of H_p, and $\hat{\Lambda}_p$ is the sublattice generated by $\{m_r\mu_r\}_{r\in\mathcal{F}(p)}$.

Remark 2.7. Note that because (M_P,ω_P) is the reduction of a Kähler manifold ($\mathbb{C}^d$ with its standard complex structure and symplectic form) by a group action that preserves the Kähler structure ($K\subset U(d)$), it follows that M_P comes equipped with an invariant complex structure J_P compatible with its symplectic form ω_P (see Theorem 3.5 in [GS2]). In other words, (M_P,ω_P,J_P) is a Kähler toric orbifold.

2.3. **Weighted and labeled projective spaces.** We will now discuss the family of examples of symplectic toric manifolds given by weighted and labeled projective spaces. As we will see, these are closely related to each other.

Consider $\mathbb{C}^{n+1}$ with complex coordinates $(z_1,\dots,z_{n+1})$, and define an action of the complex Lie group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ by

$$(z_1,\dots,z_{n+1}) \stackrel{t}{\mapsto} (t^{a_1}z_1,\dots,t^{a_{n+1}}z_{n+1})\,,\ t\in\mathbb{C}^*\,, \tag{2.8}$$

where $a_1, \ldots, a_{n+1}$ are positive integers with highest common divisor 1. The **weighted projective space** $\mathbb{CP}^n_{\mathbf{a}}$ is defined as the complex quotient

$$\mathbb{CP}^n_{\mathbf{a}} = \left(\mathbb{C}^{n+1} \setminus \{0\}\right)/\mathbb{C}^* ,$$

where $\mathbf{a}$ denotes the vector of weights: $\mathbf{a} = (a_1, \ldots, a_{n+1})$. One checks that $\mathbb{CP}^n_{\mathbf{a}}$ is a compact complex orbifold, whose orbifold structure groups are determined in the following way. Let $[z]_{\mathbf{a}} = [z_1, \ldots, z_{n+1}]_{\mathbf{a}}$ be a point in $\mathbb{CP}^n_{\mathbf{a}}$, and let m be the highest common divisor of the set of those a_j for which $z_j \neq 0$. The orbifold structure group $\Gamma_{[z]_{\mathbf{a}}}$ of $[z]_{\mathbf{a}}$ is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. In particular, $[z]_{\mathbf{a}}$ is a smooth point of $\mathbb{CP}^n_{\mathbf{a}}$ if and only if $m = 1$. Since we assumed that the highest common divisor of all the a_j's is 1, this means that any point $[z]_{\mathbf{a}} = [z_1, \ldots, z_{n+1}]_{\mathbf{a}} \in \mathbb{CP}^n_{\mathbf{a}}$, with all $z_j \neq 0$, is a smooth point. Note also that $\mathbb{CP}^n_{\mathbf{1}}$ is the usual complex projective space $\mathbb{CP}^n$, and we will omit the subscript $\mathbf{1}$ when referring to it.

There is a natural holomorphic map $\pi_{\mathbf{a}} : \mathbb{CP}^n_{\mathbf{a}} \to \mathbb{CP}^n$ defined by

$$\pi_{\mathbf{a}}\left([z_1, \ldots, z_{n+1}]_{\mathbf{a}}\right) \mapsto [z_1^{\hat{a}_1}, \ldots, z_{n+1}^{\hat{a}_{n+1}}] ,$$

where $\hat{a}_j$ denotes the product of all the weights except the j-th one:

$$\hat{a}_r = \prod_{k=1, k\neq r}^{n+1} a_k \ .$$

The map $\pi_{\mathbf{a}}$ factors through the quotient of $\mathbb{CP}^n_{\mathbf{a}}$ by the following finite group action. Let $\hat{a} = \prod_{k=1}^{n+1} a_k$ and consider the finite group $\Gamma_{\mathbf{a}}$ defined by

$$\Gamma_{\mathbf{a}} = \left(\mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}}\right)/\mathbb{Z}_{\hat{a}} ,$$

where

$$\begin{array}{ccc} \mathbb{Z}_{\hat{a}} & \hookrightarrow & \mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}} \\ \zeta & \mapsto & (\zeta^{a_1}, \ \ldots \ , \zeta^{a_{n+1}}) \end{array}$$

(here $\mathbb{Z}_q \equiv \mathbb{Z}/q\mathbb{Z}$ is identified with the group of q-th roots of unity in $\mathbb{C}$). $\Gamma_{\mathbf{a}}$ acts on $\mathbb{CP}^n_{\mathbf{a}}$ via

$$[\eta]\cdot[z]_{\mathbf{a}} = [\eta_1 z_1, \ldots, \eta_{n+1} z_{n+1}]_{\mathbf{a}} \,, \text{ for all } [\eta] \in \Gamma_{\mathbf{a}} \,, \ [z]_{\mathbf{a}} \in \mathbb{CP}^n_{\mathbf{a}} \,,$$

and one checks easily that

$$\pi_{\mathbf{a}}([z]_{\mathbf{a}}) = \pi_{\mathbf{a}}([z']_{\mathbf{a}}) \text{ iff } [z']_{\mathbf{a}} = [\eta]\cdot[z]_{\mathbf{a}} \text{ for some } [\eta] \in \Gamma_{\mathbf{a}} \,.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{CP}^n_{\mathbf{a}} & \xrightarrow{\pi_{\mathbf{a}}} & \mathbb{CP}^n \\ & \searrow \qquad \nearrow_{[\pi_{\mathbf{a}}]} & \\ & \mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}} & \end{array} \tag{2.9}$$

The action of $\Gamma_{\mathbf{a}}$ is free on $\breve{\mathbb{CP}}^n_{\mathbf{a}} = \{[z_1, \ldots, z_{n+1}]_{\mathbf{a}} \in \mathbb{CP}^n_{\mathbf{a}} : z_j \neq 0 \text{ for all } j\}$. In particular, $\pi_{\mathbf{a}}$ has degree $|\Gamma_{\mathbf{a}}| = (\hat{a})^{n-1}$. It is also clear that, if $\mathbf{a}$ is a nontrivial weight vector, the $\Gamma_{\mathbf{a}}$-action has nontrivial isotropy at some points in $\mathbb{CP}^n_{\mathbf{a}} \setminus \breve{\mathbb{CP}}^n_{\mathbf{a}}$, and so $\mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}}$ has a nontrivial orbifold structure. The bijection $[\pi_{\mathbf{a}}]$, although a biholomorphism between $\mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}}$ and the standard $\mathbb{CP}^n$, is obviously not an orbifold isomorphism between $\mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}}$ and the smooth $\mathbb{CP}^n$. We will look at $[\pi_{\mathbf{a}}]$ as inducing an orbifold structure on $\mathbb{CP}^n$ isomorphic to $\mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}}$.

Definition 2.8. The **orbifold projective space** $\mathbb{CP}^n_{[\mathbf{a}]}$ is defined as the finite quotient

$$\mathbb{CP}^n_{[\mathbf{a}]} \stackrel{\text{def}}{=} \mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}} \stackrel{[\pi_{\mathbf{a}}]}{=} \text{“orbifold” } \mathbb{CP}^n \ .$$

Remark 2.9. Once the orbifold structures are taken into account, the map $\pi_{\mathbf{a}} : \mathbb{CP}^n_{\mathbf{a}} \to \mathbb{CP}^n_{[\mathbf{a}]}$ is an orbifold covering map of degree $(\hat{a})^{n-1}$. In particular, any orbifold geometric structure (symplectic, Kähler, etc) on $\mathbb{CP}^n_{[\mathbf{a}]}$ lifts through $\pi_{\mathbf{a}}$ to an orbifold geometric structure on $\mathbb{CP}^n_{\mathbf{a}}$. For our purposes it is then enough, and, as we will see, also more convenient, to work with $\mathbb{CP}^n_{[\mathbf{a}]}$.

In order to better understand $\mathbb{CP}^n_{[\mathbf{a}]}$, in particular its orbifold structure groups and symplectic description in terms of labeled polytopes, it is useful to go back to $\mathbb{C}^{n+1}$ and consider a finite extension of the $\mathbb{C}^*$-action defined by (2.8).

Let $K^{\mathbb{C}}_{\mathbf{a}}$ be the complex Lie group defined by

$$K^{\mathbb{C}}_{\mathbf{a}} = \left(\mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}} \times \mathbb{C}^*\right)/\mathbb{Z}_{\hat{a}} \tag{2.10}$$

where

$$\begin{array}{ccc} \mathbb{Z}_{\hat{a}} & \hookrightarrow & \mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}} \times \mathbb{C}^* \\ \zeta & \mapsto & \left(\zeta^{a_1}\,,\ \ldots\ ,\, \zeta^{a_{n+1}}\,,\ \zeta^{-1}\right) \ . \end{array}$$

$K^{\mathbb{C}}_{\mathbf{a}}$ acts effectively on $\mathbb{C}^{n+1}$ via

$$[(\eta,t)]\cdot z = (\eta_1 t^{a_1} z_1, \ldots, \eta_{n+1} t^{a_{n+1}} z_{n+1})\,,\ \text{for all } [(\eta,t)] \in K^{\mathbb{C}}_{\mathbf{a}}\,,\ z \in \mathbb{C}^{n+1} \,. \tag{2.11}$$

Because of the exact sequence

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}^* & \hookrightarrow & K^{\mathbb{C}}_{\mathbf{a}} & \to & \Gamma_{\mathbf{a}} & \to & 1 \\ & & t & \mapsto & [(\mathbf{1},t)] & & & & \\ & & & & [(\eta,t)] & \mapsto & [\eta] & & \end{array}$$

we have that

$$\left(\mathbb{C}^{n+1}\setminus\{0\}\right)/K^{\mathbb{C}}_{\mathbf{a}} \cong \left[\left(\mathbb{C}^{n+1}\setminus\{0\}\right)/\mathbb{C}^*\right]/\Gamma_{\mathbf{a}} = \mathbb{CP}^n_{\mathbf{a}}/\Gamma_{\mathbf{a}} = \mathbb{CP}^n_{[\mathbf{a}]} \ . \tag{2.12}$$

Hence, the orbifold structure of $\mathbb{CP}^n_{[\mathbf{a}]}$ can be described directly from the different isotropy subgroups of the $K^{\mathbb{C}}_{\mathbf{a}}$-action on $\mathbb{C}^{n+1}$ (see Lemma 2.10 below).

We will now give the symplectic description, in terms of labeled polytopes, for the orbifold projective spaces $\mathbb{CP}^n_{[\mathbf{a}]}$. Recall that the polytope corresponding to $\mathbb{CP}^n$, with symplectic (Kähler) form in the same cohomology class as the first Chern class, is the simplex $P^n_{\mathbf{1}}$ in $(\mathbb{R}^n)^*$ defined by

$$P^n_{\mathbf{1}} = \bigcap_{r=1}^{n+1} \left\{x \in (\mathbb{R}^n)^* : \ell_r(x) \equiv \langle x, m_r\mu_r\rangle - \lambda_r \geq 0\right\}\ , \tag{2.13}$$

where: $m_r = 1 = -\lambda_r\,,\ r = 1,\ldots,n+1$; $\mu_r = e_r\,,\ r = 1,\ldots,n$, and $\mu_{n+1} = -\sum_{j=1}^n e_j$. Here $(e_1,\ldots,e_n)$ denotes the standard basis of $\mathbb{R}^n$.

From §2.2 we know that a facet of a labeled polytope has label $m \in \mathbb{N}$ if and only if the orbifold structure group of the points that are mapped to its relative interior, via the moment map, is $\mathbb{Z}_m \equiv \mathbb{Z}/m\mathbb{Z}$. In the case of $\mathbb{CP}^n$, the pre-image of the r-th facet

$$F_r = \{x \in P^n_{\mathbf{1}} : \ell_r(x) = 0\}$$

is

$$N_r = \{[z_1,\ldots,z_{n+1}] \in \mathbb{CP}^n : z_r = 0\}\ ,$$

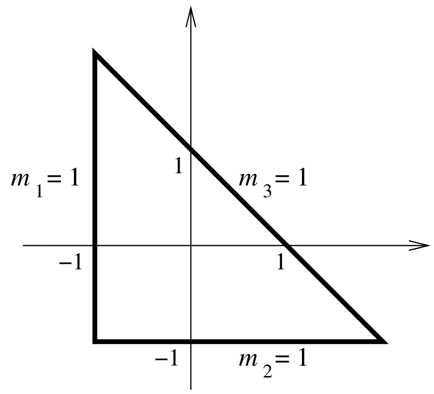

FIGURE 1. The polytope $P^2_{\mathbf{1}}$ corresponding to $\mathbb{CP}^2$.

while the pre-image of its interior $\breve{F}_r$ is

$$\breve{N}_r = \{[z_1, \ldots, z_{n+1}] \in N_r : z_k \neq 0 \text{ for all } k \neq r\} \ .$$

In $\mathbb{CP}^n_{[\mathbf{a}]}$ this corresponds to

$$\breve{N}_{[\mathbf{a}],r} = \left\{ [z_1, \ldots, z_{n+1}]_{[\mathbf{a}]} \in \mathbb{CP}^n_{[\mathbf{a}]} : z_r = 0 \text{ and } z_k \neq 0 \text{ for all } k \neq r \right\} \ .$$

Lemma 2.10. *The orbifold structure group* $\Gamma_{[z]_{[\mathbf{a}]}}$ *of any point* $[z]_{[\mathbf{a}]} \in \breve{N}_{[\mathbf{a}],r} \subset \mathbb{CP}^n_{[\mathbf{a}]}$ *is isomorphic to* $\mathbb{Z}_{m_r}$ *where*

$$m_r = \hat{a}_r = \prod_{k=1, k\neq r}^{n+1} a_k \ .$$

Proof. Because of (2.12), the orbifold structure group $\Gamma_{[z]_{[\mathbf{a}]}}$ of any point $[z]_{[\mathbf{a}]} \in \breve{N}_{[\mathbf{a}],r}$ is the isotropy of the $K^{\mathbb{C}}_{\mathbf{a}}$-action at any point $z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1}$ with $z_r = 0$, and $z_k \neq 0$ for all $k \neq r$. It follows from (2.11) that such an isotropy subgroup is given by the elements $[(\eta, t)] \in K^{\mathbb{C}}_{\mathbf{a}}$ such that $\eta_k = t^{-a_k}$, for all $k \neq r$. Since $\eta_k \in \mathbb{Z}_{\hat{a}_k}$, this implies that $t \in \mathbb{Z}_{\hat{a}} \subset \mathbb{C}^*$, and so

$$\Gamma_{[z]_{[\mathbf{a}]}} \cong (\mathbb{Z}_{\hat{a}_r} \times \mathbb{Z}_{\hat{a}}) \,/\, \left((\zeta^{a_r}, \zeta^{-1}) \,,\ \zeta \in \mathbb{Z}_{\hat{a}} \right) \ .$$

The right-hand side is isomorphic to $\mathbb{Z}_{\hat{a}_r}$ via the map

$$[(\eta_r, \zeta)] \mapsto \eta_r \cdot \zeta^{a_r} \ ,\ \eta_r \in \mathbb{Z}_{\hat{a}_r} \ ,\ \zeta \in \mathbb{Z}_{\hat{a}} \ .$$

Q.E.D.

The natural candidate for labeled polytope corresponding to $\mathbb{CP}^n_{[\mathbf{a}]}$ is then the labeled simplex $P^n_{[\mathbf{a}]}$ in $(\mathbb{R}^n)^*$ defined by

$$P^n_{[\mathbf{a}]} = \bigcap_{r=1}^{n+1} \{x \in (\mathbb{R}^n)^* : \ell_r(x) \equiv \langle x, m_r \mu_r \rangle - \lambda_r \geq 0\} \ , \tag{2.14}$$

where $m_r = \hat{a}_r = -\lambda_r \ ,\ r = 1, \ldots, n+1$, and the μ_r's are as in (2.13).

Proposition 2.11. *The compact Kähler toric orbifold* $(M_{[\mathbf{a}]}, \omega_{[\mathbf{a}]}, J_{[\mathbf{a}]})$*, associated to the labeled polytope* $P^n_{[\mathbf{a}]}$ *via the construction of §2.2, is isomorphic as a complex toric orbifold to* $\mathbb{CP}^n_{[\mathbf{a}]}$.

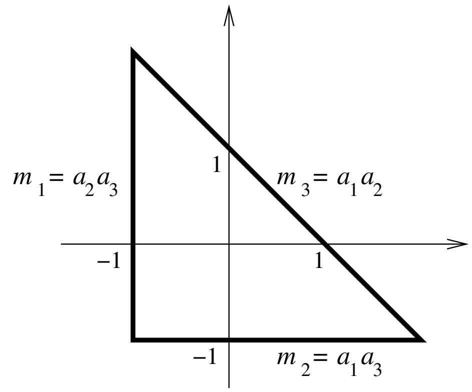

FIGURE 2. The labeled simplex $P^2_{[\mathbf{a}]}$ corresponding to $\mathbb{CP}^2_{[\mathbf{a}]}$.

Proof. With respect to the standard basis of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$, the linear map $\beta : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by (2.2) is given by the matrix

$$\begin{bmatrix} m_1 & 0 & \dots & 0 & -m_{n+1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & m_n & -m_{n+1} \end{bmatrix} = \begin{bmatrix} \hat{a}_1 & 0 & \dots & 0 & -\hat{a}_{n+1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & \hat{a}_n & -\hat{a}_{n+1} \end{bmatrix}$$

Using multiplicative notation, the kernel $K_{\mathbf{a}} \subset \mathbb{T}^{n+1}$ of the induced map $\beta : \mathbb{T}^{n+1} \to \mathbb{T}^n$ is then given by

$$K_{\mathbf{a}} = \left\{ (e^{i\theta_1}, \dots, e^{i\theta_{n+1}}) \in \mathbb{T}^{n+1} \ : \ e^{i\hat{a}_1\theta_1} = \cdots = e^{i\hat{a}_{n+1}\theta_{n+1}} \right\} .$$

$K_{\mathbf{a}}$ acts on $\mathbb{C}^{n+1}$ as a subgroup of $\mathbb{T}^{n+1}$, and from (2.6) and Remark 2.7 we have that $(M_{[\mathbf{a}]}, \omega_{[\mathbf{a}]}, J_{[\mathbf{a}]})$ is the Kähler reduction

$$M_{[\mathbf{a}]} = \phi^{-1}_{K_{\mathbf{a}}}(0)/K_{\mathbf{a}} \,, \tag{2.15}$$

where $\phi^{-1}_{K_{\mathbf{a}}}$ is the moment map defined by (2.5).

One easily checks that $K_{\mathbf{a}}$ is isomorphic to the Lie group

$$\left(\mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}} \times \mathbb{T}^1 \right) / \mathbb{Z}_{\hat{a}}$$

where

$$\begin{array}{rcl} \mathbb{Z}_{\hat{a}} & \hookrightarrow & \mathbb{Z}_{\hat{a}_1} \times \cdots \times \mathbb{Z}_{\hat{a}_{n+1}} \times \mathbb{T}^1 \\ \zeta & \mapsto & \left(\zeta^{a_1} \,, \ \dots \ , \zeta^{a_{n+1}} \,, \ \zeta^{-1} \right) \,, \end{array}$$

the isomorphism being given explicitly by

$$\left[(\eta, e^{i\theta}) \right] \mapsto (\eta_1 e^{ia_1\theta}, \dots, \eta_{n+1} e^{ia_{n+1}\theta}) \in K_{\mathbf{a}} \subset \mathbb{T}^{n+1} .$$

This means that the complex Lie group $K^{\mathbb{C}}_{\mathbf{a}}$ defined by (2.10) is the complexification of $K_{\mathbf{a}}$, and by (2.12) we know that

$$\mathbb{CP}^n_{[\mathbf{a}]} \cong (\mathbb{C}^{n+1} \setminus \{0\}) / K^{\mathbb{C}}_{\mathbf{a}} \ . \tag{2.16}$$

The statement of the proposition now follows from a general principle that gives an identification between the Kähler reduction (2.15) and the complex quotient (2.16). A good reference in our context is the appendix to [G2]. **Q.E.D.**

The construction of §2.2 applies of course to any labeled polytope, and hence to any labeled simplex

$$P^n_{\mathbf{m}} = \bigcap_{r=1}^{n+1} \{x \in (\mathbb{R}^n)^* : \ell_r(x) \equiv \langle x, m_r \mu_r \rangle - \lambda_r \geq 0\} \,, \tag{2.17}$$

with arbitrary $m_r = -\lambda_r \in \mathbb{N}\,,\ r = 1, \ldots, n+1$, and the μ_r's again as in (2.13).

Definition 2.12. Given an arbitrary vector $\mathbf{m} = (m_1, \ldots, m_{n+1})$ of positive integer labels, we define the **labeled projective space** $(\mathbb{SP}^n_{\mathbf{m}}, \omega_{\mathbf{m}}, \tau_{\mathbf{m}})$ as the symplectic toric orbifold associated to the labeled simplex $P^n_{\mathbf{m}} \subset (\mathbb{R}^n)^*$ defined by (2.17).

Remark 2.13. It follows from Remark 2.7 that any labeled projective space $(\mathbb{SP}^n_{\mathbf{m}}, \omega_{\mathbf{m}}, \tau_{\mathbf{m}})$ comes equipped with a "canonical" compatible toric complex structure $J_{\mathbf{m}}$. Theorem 9.4 in [LT] implies that as a complex toric variety, not only with respect to $J_{\mathbf{m}}$ but also with respect to any toric complex structure J compatible with $\omega_{\mathbf{m}}$, $\mathbb{SP}^n_{\mathbf{m}}$ is equivariantly biholomorphic to $\mathbb{CP}^n$. The biholomorphism $[\pi_{\mathbf{a}}] : \mathbb{CP}^n_{[\mathbf{a}]} \to \mathbb{CP}^n$ defined by (2.9) is just a particular explicit instance of this more general fact.

Remark 2.14. In Definition 2.12 we have normalized all labeled simplices $P^n_{\mathbf{m}}$ by the conditions $m_r = -\lambda_r\,,\ r = 1, \ldots, n+1$, which amounts to the fact that the underlying simplex is always the same $P^n_{\mathbf{1}} \subset (\mathbb{R}^n)^*$. This also means that the cohomology class of $\omega_{\mathbf{m}}$ in $H^2(\mathbb{SP}^n_{\mathbf{m}})$ is apriori fixed. One can allow for an arbitrary positive scaling of this cohomology class by scaling the λ_r's in the same way.

Remark 2.15. Although labeled projective spaces might seem to be a more general class of toric orbifolds than orbifold projective spaces, that is not really the case. In fact one can easily check that, up to scaling, global coverings and/or finite quotients, the classes of labeled, orbifold and weighted projective spaces consist of the same Kähler toric orbifolds.

3. Toric Kähler metrics

Let (M_P, ω_P, τ_P) be the symplectic toric orbifold associated to a labeled polytope P. In this section we describe how all ω_P-compatible toric complex structures on M_P (in other words, all toric Kähler metrics) can be effectively parametrized by smooth functions on P, according to the statements of Theorems 1 and 2.

3.1. **The "canonical" toric Kähler metric.** Recall from the construction of §2.2 that (M_P, ω_P, τ_P) comes equipped with a "canonical" ω_P-compatible toric complex structure J_P, induced from the standard one on $\mathbb{C}^d$ through symplectic reduction. Following Guillemin [G1], we will now prove Theorem 1, which states that the potential g_P of J_P is given by

$$g_P(x) = \frac{1}{2} \sum_{r=1}^{d} \ell_r(x) \log \ell_r(x) \,, \tag{3.1}$$

where ℓ_r, $r = 1, \ldots, d$, are the affine functions on $(\mathbb{R}^n)^*$ defining the polytope P as in (2.1).

It is enough to show that the Kähler metric given in symplectic (x,θ)-coordinates by (1.4), with $G \equiv G_P = \mathrm{Hess}_x(g_P)$, corresponds to the Kähler metric $\langle\cdot,\cdot\rangle_P = \omega_P(\cdot, J_P\cdot)$ on M_P. Because both these metrics are invariant under the $\mathbb{T}^n$-action and $\breve{M}_P$ is open and dense in M_P, one just needs to find a suitable slice, orthogonal to the orbits of the $\mathbb{T}^n$-action on $\breve{M}_P$, and isometric to $\breve{P}$ via the moment map ϕ_P. Here the word "isometric" is with respect to the metric on the slice induced by $\langle\cdot,\cdot\rangle_P$, and the metric on $\breve{P}$ given by G_P.

Such a slice arises naturally as the fixed point set of an anti-holomorphic involution, induced from complex conjugation in $\mathbb{C}^d$:

$$\sigma : \mathbb{C}^d \to \mathbb{C}^d\,,\ \sigma(z) = \bar{z}\,,\ \mathrm{Fix}(\sigma) = \mathbb{R}^d \subset \mathbb{C}^d\ . \tag{3.2}$$

The construction of §2.2 is invariant (or equivariant) with respect to σ. In particular, the submanifold $Z = \phi_K^{-1}(0) \subset \mathbb{C}^d$, with ϕ_K defined by (2.5), is stable under σ. The K-action on Z commutes with σ, and so σ descends to give an involution on M_P.

Let $Z^\sigma \subset \mathbb{R}^d$ and M_P^σ denote the fixed point sets of σ on Z and M_P. Define

$$\breve{Z}^\sigma = Z^\sigma \cap \breve{\mathbb{R}}^d \ \text{ and }\ \breve{M}_P^\sigma = M_P^\sigma \cap \breve{M}_P\,,$$

where $\breve{\mathbb{R}}^d = \big\{(u_1,\dots,u_d) \in \mathbb{R}^d : u_r \neq 0 \text{ for all } r = 1,\dots,d\big\}$. The following can be easily checked from the construction in §2.2:

- the quotient map $\pi : Z \to M_P$ induces a covering map $\breve{\pi}^\sigma : \breve{Z}^\sigma \to \breve{M}_P^\sigma$, with group of deck transformations given by $\{\alpha \in K : \alpha^2 = 1\}$;
- $\breve{\pi}^\sigma$ is an isometry with respect to the metric on $\breve{Z}^\sigma$ induced by the Euclidean metric on $\mathbb{R}^d$, and the metric on $\breve{M}_P^\sigma$ induced by the metric $\langle\cdot,\cdot\rangle_P$ on M_P;
- the moment map $\phi_P : M_P \to P \subset (\mathbb{R}^n)^*$ induces a covering map $\breve{\phi}_P^\sigma : \breve{M}_P^\sigma \to \breve{P}$, with group of deck transformations given by $\{\theta \in \mathbb{T}^n : \theta^2 = 1\}$. Moreover, $\breve{M}_P^\sigma$ is $\langle\cdot,\cdot\rangle_P$-orthogonal to the orbits of the $\mathbb{T}^n$-action on $\breve{M}_P$.

Hence, any connected component of $\breve{M}_P^\sigma$ can be taken to be the slice we were looking for. It is isometric via $\breve{\pi}^\sigma$ to any connected component of $\breve{Z}^\sigma \subset \mathbb{R}^d$.

Let $\breve{Z}_+^\sigma = \breve{Z}^\sigma \cap \mathbb{R}_+^d$, where $\mathbb{R}_+^d = \{(u_1,\dots,u_d) \in \mathbb{R}^d : u_r > 0 \text{ for all } r = 1,\dots,d\}$. From (2.5) we have that $\breve{Z}_+^\sigma$ is the subset of $\mathbb{R}_+^d$ defined by the quadratic equation

$$\sum_{r=1}^d \left(\frac{u_r^2}{2} + \lambda_r\right) \iota^*(e_r^*) = 0\ .$$

Consider the change of coordinates in $\mathbb{R}_+^d$ given by

$$s_r = \frac{u_r^2}{2}\,,\ r = 1,\dots,d\,.$$

$\breve{Z}_+^\sigma$ is now defined by the linear equation

$$\sum_{r=1}^d (s_r + \lambda_r)\,\iota^*(e_r^*) = 0\,,$$

and the Euclidean metric $\sum_r (du_r)^2$ becomes

$$\frac{1}{2}\sum_{r=1}^d \frac{(ds_r)^2}{s_r}\ . \tag{3.3}$$

The commutative diagram (2.7) can be written here as

$$\begin{tikzcd} \check{Z}^\sigma_+ \arrow[r, "\check{\phi}^\sigma_{\mathbb{T}^d}"] \arrow[d, "\check{\pi}^\sigma"'] & (\mathbb{R}^d)^* \\ \check{M}^\sigma_P \arrow[r, "\check{\phi}^\sigma_P"] & \check{P} \subset (\mathbb{R}^n)^* \arrow[u, "\beta^*"'] \end{tikzcd} \tag{3.4}$$

and we want to determine the form of the metric (3.3) on $\check{P}$. The map β^*, being dual to the surjective linear map defined by (2.2), is an injective linear map given by

$$\beta^*(x) = \sum_{r=1}^d \langle x, m_r \mu_r \rangle \, e_r^* \ . \tag{3.5}$$

The map $\check{\phi}^\sigma_{\mathbb{T}^d}$, being the restriction of $\phi_{\mathbb{T}^d}$ defined by (2.4), is given in the s-coordinates by

$$\check{\phi}^\sigma_{\mathbb{T}^d}(s) = \sum_{r=1}^d (s_r + \lambda_r) \, e_r^* \ . \tag{3.6}$$

From (3.5) and (3.6) we conclude that

$$(s_r + \lambda_r) = \langle x, m_r\mu_r \rangle \Rightarrow s_r = \langle x, m_r\mu_r \rangle - \lambda_r \equiv \ell_r(x) \, , \text{ for all } r = 1, \ldots, d \, .$$

Hence the metric (3.3) can be written in the x-coordinates of the polytope P as

$$\frac{1}{2} \sum_{r=1}^d \frac{(ds_r)^2}{s_r} = \frac{1}{2} \sum_{r=1}^d \frac{(d\ell_r)^2}{\ell_r} = \sum_{i,j=1}^n \frac{\partial^2 g_P(x)}{\partial x_i \partial x_j} \, dx_i dx_j \, ,$$

where g_P is given by (3.1) and the last equality is a trivial exercise.

This completes the proof of Theorem 1.

3.2. **General toric Kähler metrics.** We will now prove Theorem 2, which states that on a symplectic toric orbifold (M_P, ω_P, τ_P), associated to a labeled polytope P, compatible toric complex structures J are in one to one correspondence with potentials $g \in C^\infty(\check{P})$ of the form

$$g = g_P + h \, , \tag{3.7}$$

where g_P is given by (3.1), h is smooth on the whole P, and the matrix $G = \mathrm{Hess}(g)$ is positive definite on $\check{P}$ and has determinant of the form

$$\mathrm{Det}(G) = \left[\delta \prod_{r=1}^d \ell_r \right]^{-1} \, , \tag{3.8}$$

with δ being a smooth and strictly positive function on the whole P.

The proof of this theorem for symplectic toric orbifolds given in the Appendix of [A2], generalizes with very minor modifications to our orbifold context.

We first prove that any potential $g \in C^\infty(\check{P})$ of the form (3.7) and satisfying (3.8), defines through (1.2) a compatible toric complex structure J on (M_P, ω_P, τ_P). It is clear that J is well defined on $\check{M}_P \cong \check{P} \times \mathbb{T}^n$. To see that it extends to the whole M_P one has to check that the singular behaviour of J near the boundary of P is the same as the singular behaviour of J_P, which we know extends to the whole M_P.

This singular behaviour is best described in terms of the Hessians $G_P = \mathrm{Hess}(g_P)$ and $G = \mathrm{Hess}(g)$. Explicit calculations show that although G_P is singular on the boundary of the polytope P, G_P^{-1} is smooth on the whole P and its determinant has the form

$$\mathrm{Det}(G_P^{-1}) = \delta_P \prod_{r=1}^{d} \ell_r \,,$$

where δ_P is a smooth and strictly positive function on the whole P. This formula captures the relevant singular behaviour and has the following geometric interpretation. Given $x \in P$, let $\mathcal{F}(x)$ be the set of facets of P that contain x, i.e.

$$\mathcal{F}(x) = \{r \in \{1, \dots, d\} : \ell_r(x) = 0\} \ .$$

The kernel of $G_P^{-1}(x)$ is precisely the linear span of the normals $\mu_r \in \mathbb{R}^n$ for $r \in \mathcal{F}(x)$. Due to Remark 2.6, this kernel is also the Lie algebra of the isotropy group $\Gamma_p \subset \mathbb{T}^n$ of any $p \in M_P$ such that $\phi_P(p) = x$. Conditions (3.7) and (3.8) guarantee that G^{-1} has these same degeneracy properties, and that is enough for the corresponding J to extend to a compatible toric complex structure well defined on the whole M_P.

We now prove that any compatible toric complex structure J on (M_P, ω_P, τ_P) corresponds, in suitable symplectic coordinates on $\breve{M}_P \cong \breve{P} \times \mathbb{T}^n$, to a potential $g \in C^\infty(\breve{P})$ of the form (3.7). Because J is apriori defined on the whole M_P, the matrix $G = \mathrm{Hess}(g)$ will automatically satisfy (3.8). The idea of the proof is to translate to symplectic coordinates some well known facts from Kähler geometry.

It follows from Theorem 9.4 in [LT] that there is an equivariant biholomorphism

$$\varphi_J : (M_P, J_P, \tau_P) \to (M_P, J, \tau_P) \,,$$

with φ_J acting as the identity in cohomology. This means that (M_P, ω_P, J) is equivariantly Kähler isomorphic to (M_P, ω_J, J_P), where $\omega_J = (\varphi_J)^*(\omega_P)$ and $[\omega_J] = [\omega_P] \in H^2(M_P)$. It follows from [Ba] that the $\partial\overline{\partial}$-lemma is valid on Kähler orbifolds, and hence there exists a $\mathbb{T}^n$-invariant smooth function $f_J \in C^\infty(M_P)$ such that

$$\omega_J = \omega_P + 2i\partial\overline{\partial} f_J \,,$$

where the ∂- and $\overline{\partial}$-operators are defined with respect to the complex structure J_P.

In the symplectic (x, θ)-coordinates on $\breve{M}_P \cong \breve{P} \times \mathbb{T}^n$, obtained via the "canonical" moment map $\phi_P : M_P \to P \subset (\mathbb{R}^n)^*$, we then have a function $f_J \equiv f_J(x)$, smooth on the whole polytope P, and such that

$$\omega_J = dx \wedge d\theta + 2i\partial\overline{\partial} f_J \ .$$

The rest of the proof consists of the following three steps:

(i) write down on P the change of coordinates $\tilde{\varphi}_J : P \to P$ that corresponds to the equivariant biholomorphism $\varphi_J : M_P \to M_P$ and transforms the symplectic (x, θ)-coordinates for ω_P into symplectic $(\tilde{x} = \tilde{\varphi}_J(x), \theta)$-coordinates for ω_J. These $(\tilde{x}, \theta)$-coordinates are the suitable symplectic coordinates we were looking for;

(ii) find the potential $g = g(\tilde{x})$ for the transformed $J = (\tilde{\varphi}_J)_*(J_P)$ in these $(\tilde{x}, \theta)$-coordinates;

(iii) check that the function $h : \breve{P} \to \mathbb{R}$ given by $h(\tilde{x}) = g(\tilde{x}) - g_P(\tilde{x})$, with $g_P(\tilde{x}) = \frac{1}{2}\sum_r \ell_r(\tilde{x}) \log \ell_r(\tilde{x})$, is actually defined and smooth on the whole

P. Here, as always, $\ell_r(\tilde{x}) \equiv \langle \tilde{x}, m_r \mu_r \rangle - \lambda_r$, $r = 1, \ldots, d$, are the defining functions of the polytope P.

All these steps can be done in a completely explicit way. We refer the reader to the Appendix in [A2] for details. The change of coordinates in step (i) is given in vector form by

$$\tilde{x} = \tilde{\varphi}_J(x) = x + G_P^{-1} \cdot \frac{\partial f_J}{\partial x} \,,$$

where $\partial f_J / \partial x = (\partial f_J/\partial x_1, \ldots, \partial f_J/\partial x_n)^t \equiv$ column vector. $\tilde{\varphi}_J$ is a diffeomorphism of the whole P and, due to the degeneracy behaviour of the matrix G_P^{-1} on the boundary of P, preserves each of its faces (i.e. each vertex, edge, ... ,facet and interior $\breve{P}$). In step (ii) one finds that

$$g(\tilde{x}) = \left\langle \tilde{x} - \tilde{\varphi}_J^{-1}(\tilde{x}), (\frac{\partial g_P}{\partial x} \circ \tilde{\varphi}_J^{-1})(\tilde{x}) \right\rangle + (g_P \circ \tilde{\varphi}_J^{-1})(\tilde{x}) - (f_J \circ \tilde{\varphi}_J^{-1})(\tilde{x}) \,.$$

Since $(f_J \circ \tilde{\varphi}_J^{-1}) \in C^\infty(P)$ and $\tilde{\varphi}_J$ is a smooth diffeomorphism of the whole P, step (iii) reduces to checking that

$$\left\langle \tilde{\varphi}_J(x) - x, \frac{\partial g_P}{\partial x}(x) \right\rangle + g_P(x) - g_P(\tilde{\varphi}_J(x)) \in C^\infty(P) \,.$$

Simple explicit computations show that this is true provided

$$\frac{\ell_r(x)}{\ell_r(\tilde{\varphi}_J(x))} \in C^\infty(P) \text{ for all } \ r = 1, \ldots, d \,,$$

and this follows from the fact that $\tilde{\varphi}_J$ preserves the combinatorial structure of P.

The proof of Theorem 2 is completed.

4. Extremal metrics

In this section, after some preliminaries on extremal Kähler metrics, we use the framework of Sections 2 and 3 to prove Theorem 3, i.e. we give a simple description of a toric extremal Kähler metric on any labeled projective space. Due to Proposition 2.11 and Remark 2.9, this gives rise in particular to a toric extremal Kähler metric on any weighted projective space, as stated in Corollary 1. Theorem 4 is proved in the last subsection.

4.1. **Preliminaries on extremal metrics.** In [C1] and [C2], Calabi introduced the notion of **extremal** Kähler metrics. These are defined, for a fixed closed complex manifold (M, J_0), as critical points of the square of the L^2-norm of the scalar curvature, considered as a functional on the space of all symplectic Kähler forms ω in a fixed Kähler class $\Omega \in H^2(M, \mathbb{R})$. The extremal Euler-Lagrange equation is equivalent to the gradient of the scalar curvature being an holomorphic vector field (see [C1]), and so these metrics generalize constant scalar curvature Kähler metrics. Calabi illustrated this in [C1] by constructing families of extremal Kähler metrics of non-constant scalar curvature. Moreover, Calabi showed in [C2] that extremal Kähler metrics are always invariant under a maximal compact subgroup of the group of holomorphic transformations of (M, J_0). Hence, on a complex toric manifold or orbifold, extremal Kähler metrics are automatically toric Kähler metrics, and one should be able to write them down using the framework of Section 3. This was carried out in [A1] for Calabi's simplest family, having $\mathbb{CP}\# \overline{\mathbb{CP}}^2$ as underlying toric manifold.

We now recall from [A1] some relevant differential-geometric formulas in symplectic (x,θ)-coordinates. A Kähler metric of the form (1.4) has scalar curvature S given by[1]

$$S = -\sum_{j,k} \frac{\partial}{\partial x_j}\left(g^{jk}\,\frac{\partial \log \operatorname{Det}(G)}{\partial x_k}\right), \tag{4.1}$$

which after some algebraic manipulations becomes the more compact

$$S = -\sum_{j,k} \frac{\partial^2 g^{jk}}{\partial x_j \partial x_k}, \tag{4.2}$$

where the $g^{jk},\ 1 \le j,k \le n$, are the entries of the inverse of the matrix $G = \operatorname{Hess}_x(g)$, $g \equiv$ potential. The Euler-Lagrange equation defining an extremal Kähler metric can be shown to be equivalent to

$$\frac{\partial S}{\partial x_j} \equiv \text{ constant},\ j = 1,\dots,n, \tag{4.3}$$

i.e. *the metric is extremal if and only if its scalar curvature S is an affine function of x.* One can express (4.3) in more invariant terms, giving a symplectic analogue of the complex extremal condition saying that the gradient of the scalar curvature is an holomorphic vector field.

Proposition 4.1. *Let (M_P, ω_P, τ_P) be a compact symplectic toric orbifold with moment map $\phi_P : M_P \to P \subset (\mathbb{R}^n)^*$. A toric compatible complex structure J gives rise to an extremal Kähler metric $\langle\cdot,\cdot\rangle = \omega_P(\cdot, J\cdot)$ if and only if its scalar curvature S is a constant plus a linear combination of the components of the moment map ϕ_P.*

In other words, the metric is extremal if and only if there exists $\xi \in \mathbb{R}^n \equiv$ Lie algebra of $\mathbb{T}^n$, such that

$$dS = d\langle \xi, \phi_P\rangle\ .$$

4.2. **Extremal orbifold metrics on S^2.** Here we prove Theorem 3 when $n = 1$. This very simple case is already interesting and motivates well the formula for the potential g in the general case.

Consider the one dimensional labeled polytope defined by

$$\ell_1(x) = m_1(1+x) \ \text{ and } \ \ell_2(x) = m_2(1-x)\,,\ \text{with } m_1, m_2 \in \mathbb{N}\,.$$

The corresponding labeled projective space $\mathbb{SP}^2_{\mathbf{m}}$ is homeomorphic to the 2-sphere S^2, and the orbifold structure at each pole can be geometrically interpreted as a conical singularity with angle $2\pi/m_r\,,\ r = 1,2$. We look for an extremal metric generated by a potential $g \in C^\infty(-1,1)$ of the form

$$g(x) = \frac{1}{2}\left(m_1(1+x)\log(m_1(1+x)) + m_2(1-x)\log(m_2(1-x)) + h(x)\right),$$

with $h \in C^\infty[-1,1]$. Formula (4.2) for the scalar curvature becomes

$$S(x) = -\left(\frac{1}{g''(x)}\right)'' = \left(-(1-x^2)\overline{h}(x)\right)''$$

[1] The normalization for the value of the scalar curvature we are using here is the same as in [Be]. It differs from the one used in [A1, A2] by a factor of 1/2.

where

$$\overline{h}(x) = \frac{2}{m_1(1-x) + m_2(1+x) + (1-x^2)h''(x)} \in C^\infty[-1,1]\,.$$

Equation (4.3) says that the metric is extremal if and only if S is a first degree polynomial, hence if and only if $\overline{h}$ is a first degree polynomial. Since $\overline{h}(-1) = 1/m_1$ and $\overline{h}(1) = 1/m_2$ we must have

$$\overline{h}(x) = \frac{1}{2m_2}(1+x) + \frac{1}{2m_1}(1-x) = \frac{\ell_1(x)+\ell_2(x)}{2m_1m_2}\,.$$

Solving for $h''(x)$ and integrating one gets

$$h(x) = -(m_1(1+x) + m_2(1-x))\log(m_1(1+x)+m_2(1-x))\,,$$

i.e.

$$h = -\ell_\Sigma \log \ell_\Sigma \quad \text{with} \quad \ell_\Sigma = \ell_1 + \ell_2\,.$$

Note that, because ℓ_Σ is strictly positive on $[-1,1]$, h is defined and smooth on $[-1,1]$. Moreover,

$$G^{-1} = \frac{1}{g''} = \frac{\ell_1\,\ell_2\,\ell_\Sigma}{2\,m_1^2\,m_2^2}$$

is strictly positive on $(-1,1)$ and has the degeneracy behaviour at the boundary points -1 and 1 required by (3.8).

Hence the potential

$$g = \frac{1}{2}\left(\ell_1\log\ell_1 + \ell_2\log\ell_2 - \ell_\Sigma\log\ell_\Sigma\right) \tag{4.4}$$

defines a toric extremal Kähler metric on $\mathbb{SP}^2_{\mathbf{m}}$. Its scalar curvature is given by

$$S(x) = \frac{(m_1+m_2) + 3x(m_1-m_2)}{m_1m_2}\,.$$

As a function on $\mathbb{SP}^2_{\mathbf{m}}$ it can be written as

$$S = \left(\frac{1}{m_1}+\frac{1}{m_2}\right) + 3\left(\frac{1}{m_2}-\frac{1}{m_1}\right)\phi_{\mathbf{m}}\,,$$

where $\phi_{\mathbf{m}} : \mathbb{SP}^2_{\mathbf{m}} \to [-1,1] \subset \mathbb{R}^*$ is the moment map. Hence

$$dS = d\langle \xi_{\mathbf{m}}, \phi_{\mathbf{m}}\rangle \quad \text{for} \quad \xi_{\mathbf{m}} = 3\left(\frac{1}{m_2}-\frac{1}{m_1}\right) \in \mathbb{R} \cong \text{Lie algebra of } \mathbb{T}^1\,.$$

4.3. **Extremal metrics on $\mathbb{SP}^n_{\mathbf{m}}$.** We now consider a general labeled simplex $P^n_{\mathbf{m}} \subset (\mathbb{R}^n)^*$ defined by

$$\ell_r(x) = m_r(1+x)\,,\ r = 1,\ldots,n\,,\ \ell_{n+1}(x) = m_{n+1}(1-\psi)\,,\ \psi = \sum_{j=1}^n x_j\,,$$

with $m_r \in \mathbb{N}$, for all $r = 1,\ldots,n+1$. The corresponding labeled projective space $\mathbb{SP}^n_{\mathbf{m}}$ is homeomorphic to $\mathbb{CP}^n$ (see Remark 2.13). Under this homeomorphism the pre-image of the r-th facet

$$F_r = \{x \in P^n_{\mathbf{m}} : \ell_r(x) = 0\}$$

by the moment map $\phi_{\mathbf{m}} : \mathbb{SP}^n_{\mathbf{m}} \to P^n_{\mathbf{m}}$ corresponds to

$$N_r = \{[z_1,\ldots,z_{n+1}] \in \mathbb{CP}^n : z_r = 0\} \cong \mathbb{CP}^{n-1}\,.$$

The orbifold structure of $\mathbb{SP}^n_{\mathbf{m}}$ can be geometrically interpreted on $\mathbb{CP}^n$ as a conical singularity with angle $2\pi/m_r$ around each $N_r \cong \mathbb{CP}^{n-1}$, for $r = 1,\ldots,n+1$.

Motivated by the form of the potential (4.4) for the toric extremal Kähler metric on $\mathbb{SP}^2_{\mathbf{m}}$, we consider now the potential $g \in C^\infty(\breve{P}^n_{\mathbf{m}})$ given by

$$g = \frac{1}{2}\left(\sum_{r=1}^{n+1} \ell_r \log \ell_r - \ell_\Sigma \log \ell_\Sigma\right) \quad \text{with} \quad \ell_\Sigma = \sum_{r=1}^{n+1} \ell_r \ . \tag{4.5}$$

Note that since ℓ_Σ is strictly positive on $P^n_{\mathbf{m}}$, this potential g is of the general form (3.7).

The entries of the matrix $G = \mathrm{Hess}(g)$ are easily computed from (4.5):

$$g_{jk} = \frac{\partial^2 g}{\partial x_j \partial x_k} = \frac{1}{2}\left(\delta_{jk}\frac{m_j^2}{\ell_j} + \frac{m_{n+1}^2}{\ell_{n+1}} - \frac{(m_j - m_{n+1})(m_k - m_{n+1})}{\ell_\Sigma}\right) , \tag{4.6}$$

where δ_{jk} is equal to 1 if $j = k$ and equal to 0 otherwise. The proof of the following lemma is left as an exercise to the reader.

Lemma 4.2. *The matrix $G = \mathrm{Hess}(g) = (g_{jk})_{j.k=1}^{n,n}$ is positive definite on $\breve{P}^n_{\mathbf{m}}$ with determinant given by*

$$\mathrm{Det}(G) = \left[\left(\prod_{r=1}^{n+1} \ell_r\right) \frac{2^n \ell_\Sigma}{(n+1)^2 \prod_{r=1}^{n+1} m_r^2}\right]^{-1} . \tag{4.7}$$

The entries of the matrix $G^{-1} = (g^{jk})_{j,k=1}^{n,n}$ are given by

$$g^{jk} = 2\left(\delta_{jk}\frac{\ell_j}{m_j^2} - \frac{m_j + m_k}{n+1}\,\frac{\ell_j\,\ell_k}{m_j^2\, m_k^2} + \frac{1}{(n+1)^2}\,\frac{\ell_j\,\ell_k}{m_j\, m_k}\left(\sum_{r=1}^{n+1}\frac{\ell_r}{m_r^2}\right)\right) . \tag{4.8}$$

It follows that the potential g defined by (4.5) satisfies the conditions of Theorem 2, and hence defines a toric Kähler metric on $\mathbb{SP}^n_{\mathbf{m}}$. Moreover, since each g^{jk} is a third degree polynomial, it is clear from (4.2) that the scalar curvature S is a first degree polynomial. By (4.3) this means that the metric defined by g is indeed extremal, thus finishing the proof of Theorem 3.

More explicitly, we have that the scalar curvature is given by

$$S(x) = \frac{2n}{n+1}\left(\sum_{r=1}^{n+1}\frac{1}{m_r}\right) + \frac{2(n+2)}{n+1}\sum_{j=1}^{n}\left(\frac{1}{m_{n+1}} - \frac{1}{m_j}\right) x_j \ .$$

As a function on $\mathbb{SP}^n_{\mathbf{m}}$ it can be written as

$$S = \frac{2n}{n+1}\left(\sum_{r=1}^{n+1}\frac{1}{m_r}\right) + \langle \xi_{\mathbf{m}}, \phi_{\mathbf{m}}\rangle ,$$

where $\phi_{\mathbf{m}}$ is the moment map and

$$\xi_{\mathbf{m}} = \frac{2(n+2)}{n+1}\left(\frac{1}{m_{n+1}} - \frac{1}{m_1}, \ldots , \frac{1}{m_{n+1}} - \frac{1}{m_n}\right) \in \mathbb{R}^n \cong \text{Lie algebra of } \mathbb{T}^n \ .$$

4.4. **Conical extremal metrics on $\mathbb{SP}^n_{\mathbf{1}}$.** The purpose of this subsection is to prove Theorem 4, i.e. we will describe natural "conical" compactifications of extremal Kähler metrics defined by potentials g of the form (4.5), for any positive real vector of labels $\mathbf{m} \in \mathbb{R}^{n+1}_+$.

The symplectic toric orbifold where this compactification takes place is obtained by forgetting the labels. Hence we consider the standard smooth symplectic toric manifold $(\mathbb{SP}^n_{\mathbf{1}}, \omega_{\mathbf{1}}, \tau_{\mathbf{1}})$ associated to the simplex $P^n_{\mathbf{1}} \subset (\mathbb{R}^n)^*$, and denote by $\phi_{\mathbf{1}}$:

$\mathbb{SP}^n_{\mathbf{1}} \to P^n_{\mathbf{1}}$ the corresponding moment map. Note that $(\mathbb{SP}^n_{\mathbf{1}}, \omega_1, \tau_1)$ is equivariantly symplectomorphic to $\mathbb{CP}^n$ with a suitably normalized Fubini-Study symplectic form and standard torus action.

For any $m \in \mathbb{R}^{n+1}_+$, the potential $g \in C^\infty(\breve{P}^n_{\mathbf{1}})$ given by (4.5) defines an extremal Kähler metric $\langle\cdot,\cdot\rangle_{\mathbf{m}}$ on $\breve{\mathbb{SP}}^n_{\mathbf{1}} = \phi_{\mathbf{1}}^{-1}(\breve{P}^n_{\mathbf{1}}) \cong \breve{P}^n_{\mathbf{1}} \times \mathbb{T}^n$ given by (1.4). Consider the pre-image $N_r \equiv \phi_{\mathbf{1}}^{-1}(F_r)$ of each facet $F_r \subset P^n_{\mathbf{1}}$, $r = 1, \ldots , n+1$. Each N_r is a real codimension 2 symplectic toric submanifold of $\mathbb{SP}^n_{\mathbf{1}}$, symplectomorphic to a suitably normalized $\mathbb{SP}^{n-1}_{\mathbf{1}} \cong \mathbb{CP}^{n-1}$. The restriction $\phi_{\mathbf{1}}|_{N_r} : N_r \to F_r$ is a corresponding moment map. We want to show that $\langle\cdot,\cdot\rangle_{\mathbf{m}}$ extends to an extremal metric on the whole $\mathbb{SP}^n_{\mathbf{1}}$ with conical singularities of angles $2\pi/m_r$ around each N_r.

The potential g, although only smooth on the interior $\breve{P}^n_{\mathbf{1}}$, is a continuous function on the whole polytope $P^n_{\mathbf{1}}$. Denote by $g_r \in C^\infty(\breve{F}_r) \cap C^0(F_r)$ the restriction of g to F_r (here $\breve{F}_r$ denotes the relative interior of F_r). Using the explicit form of the matrix $G = \mathrm{Hess}(g)$ given by (4.6), one can easily check that the extremal metric $\langle\cdot,\cdot\rangle_{\mathbf{m}}$, defined on $\breve{\mathbb{SP}}^n_{\mathbf{1}}$, extends to a well defined smooth extremal metric on $\breve{N}_r \equiv \phi_{\mathbf{1}}^{-1}(\breve{F}_r)$ whose potential is exactly given by g_r. Note that the hyperplane in $(\mathbb{R}^n)^*$ that contains F_r has an induced affine structure, and so it makes sense to consider $G_r = \mathrm{Hess}(g_r)$.

Because of the equivariant version of Darboux's theorem, we can understand what happens in the normal directions to each point $p \in \breve{N}_r$ by analysing a neighborhood of zero in $\mathbb{R}^2$. In (r,θ)-polar coordinates the standard symplectic form is $rdr \wedge d\theta$, and the moment map for the standard circle action is given by $x = r^2/2$. The moment polytope is $[0, +\infty)$ defined by the single affine function $\ell(x) = x$. The standard smooth Kähler metric is defined by the potential $g_1 = \frac{1}{2} x \log x$, hence given by

$$\langle\cdot,\cdot\rangle_1 = G_1''\, dx^2 + \frac{1}{g_1''}\, d\theta^2 = \frac{1}{2x}\, dx^2 + 2x\, d\theta^2 \,,$$

while the "orbifold" one is defined for any $m \in \mathbb{R}_+$ by the potential $g_m = \frac{1}{2} mx \log(mx)$, and hence given by

$$\langle\cdot,\cdot\rangle_m = g_m''\, dx^2 + \frac{1}{g_m''}\, d\theta^2 = \frac{m}{2x}\, dx^2 + \frac{2x}{m}\, d\theta^2 \,.$$

In (r,θ)-polar coordinates we get

$$\langle\cdot,\cdot\rangle_1 = dr^2 + r^2 d\theta^2 \equiv \text{ standard smooth flat metric}\,,$$

while

$$\langle\cdot,\cdot\rangle_m = m\left(dr^2 + \left(\frac{r}{m}\right)^2 d\theta^2\right)$$

which is the polar form of a metric with a conical singularity of angle $2\pi/m$ around the origin.

Hence we have an extension of each extremal Kähler metric $\langle\cdot,\cdot\rangle_{\mathbf{m}}$, $\mathbf{m} \in \mathbb{R}^{n+1}_+$, from $\breve{\mathbb{SP}}^n_{\mathbf{1}}$ to $\breve{\mathbb{SP}}^n_{\mathbf{1}} \cup \left(\cup_{r=1}^{n+1} \breve{N}_r\right)$, having normal conical singularities around each $\breve{N}_r$. The same argument can be used to show that the metric on each $\breve{N}_r$ extends to the moment map pre-images of the relative interior of each facet of F_r (an $(n-2)$-dimensional simplex and face of $P^n_{\mathbf{1}}$). One can continue this process until the metric is extended to the whole $\mathbb{SP}^n_{\mathbf{1}}$. For example, at the last step one extends the metric to the fixed points of the $\mathbb{T}^n$-action, corresponding to the vertices of $P^n_{\mathbf{1}}$. There the metric looks like the product of n cones of dimension two and angles $2\pi/m_{r_i}$, where $m_{r_1},\ldots,m_{r_n}$ are the positive real labels of the n facets of $P^n_{\mathbf{1}}$ that meet at the relevant vertex.

5. A family of self-dual Einstein metrics

Recall from the introduction that the extremal Kähler metrics given by Theorem 4 are actually Bochner-Kähler (see [Br]). In dimension four ($n=2$) "Bochner-Kähler" is the same as "self-dual Kähler". It follows from the work of Derdzinski [Der] and Apostolov-Gauduchon [AG] that, whenever its scalar curvature S is nonzero, a self-dual Kähler metric is conformally Einstein, with conformal factor given by S^{-2}. In this section we explore this relation for a particular one-parameter family of metrics arising from Theorem 4.

Consider the smooth symplectic toric manifold $(\mathbb{SP}^2_{\mathbf{1}} \cong \mathbb{CP}^2, \omega_{\mathbf{1}}, \tau_{\mathbf{1}})$ associated to the simplex $P^2_{\mathbf{1}} \subset (\mathbb{R}^2)^*$. For any $\mathbf{m} = (1,1,m)$, $m \in \mathbb{R}_+$, let $\langle\cdot,\cdot\rangle_m$ be the extremal Kähler metric defined by the potential

$$g_m(x) = \frac{1}{2}\left(\sum_{r=1}^{3} \ell_r(x)\log\ell_r(x) - \ell_\Sigma(x)\log\ell_\Sigma(x)\right), \tag{5.1}$$

where $\ell_1(x) = 1+x_1$, $\ell_2(x) = 1+x_2$, $\ell_3(x) = m(1-\psi)$ and $\ell_\Sigma(x) = 2+m-(m-1)\psi$. Here and in the rest of this section $\psi = x_1 + x_2$. Note that in (5.1) the two terms with ℓ_1 and ℓ_2 correspond to the standard flat metric on $\mathbb{R}^4$, while the terms with ℓ_3 and ℓ_Σ only depend on the "radial" coordinate ψ. This means that the metric $\langle\cdot,\cdot\rangle_m$ defined by the potential g_m is $U(2)$-invariant (see [A3] for a general discussion of this type of metrics).

The scalar curvature S_m of $\langle\cdot,\cdot\rangle_m$ is given by

$$S_m(x) = \frac{4}{3m}(2m+1+2(1-m)\psi), \tag{5.2}$$

which is strictly positive on $\mathbb{SP}^2_{\mathbf{1}}$ if $m > 1/2$. Hence, for any $1/2 < m < +\infty$, the metric $\langle\cdot,\cdot\rangle^*_m \equiv S_m^{-2}\langle\cdot,\cdot\rangle_m$ is a self-dual Einstein metric on $\mathbb{SP}^2_{\mathbf{1}} \cong \mathbb{CP}^2$ with a normal conical singularity of angle $2\pi/m$ around a $\mathbb{SP}^1_{\mathbf{1}} \cong \mathbb{CP}^1$.

In this simple case it is not hard to check explicitly that $\langle\cdot,\cdot\rangle^*_m$ is Einstein and compute its scalar curvature. A result of Derdzinski [Der] (see also [Be]) states that for any 4-dimensional extremal Kähler metric $\langle\cdot,\cdot\rangle$ with non constant scalar curvature S, the metric $\langle\cdot,\cdot\rangle^* \equiv S^{-2}\langle\cdot,\cdot\rangle$ is Einstein if and only if

$$S^3 - 6S\Delta S - 12|dS|^2 = \text{ constant}. \tag{5.3}$$

Moreover, a standard formula for the scalar curvatures of conformally related metrics (see e.g. [Be]) states that the scalar curvature S^* of $\langle\cdot,\cdot\rangle^*$ is given by

$$S^* = S^3(6\Delta(S^{-1}) + 1). \tag{5.4}$$

In both these formulas Δ is the Laplacian with respect to the metric $\langle\cdot,\cdot\rangle$.

For any toric Kähler metric defined by a potential $g \in C^\infty(\check{P})$, the Laplacian Δ of a function $f \in C^\infty(P)$ (i.e. a smooth $\mathbb{T}^n$-invariant function on M_P) is given by

$$\Delta f = -(\mathrm{Det}G)\sum_{j,k=1}^{n} g^{jk}\frac{\partial}{\partial x_j}\left(\frac{1}{\mathrm{Det}G}\frac{\partial f}{\partial x_k}\right),$$

where $G = \mathrm{Hess}(g) = (g_{jk})$ and g^{jk} are the entries of G^{-1}. This formula, together with the simple form of g_m and S_m, makes the calculations involved in (5.3) and (5.4) easy enough.

For example, one computes that the scalar curvature S_m^* of $\langle\cdot,\cdot\rangle_m^*$ is given by

$$S_m^* = \left(\frac{4}{m}\right)^3 (2m-1)\,.$$

One sees that the self-dual Einstein metric $\langle\cdot,\cdot\rangle_m^*$ has positive scalar curvature when $1/2 < m < +\infty$, but is actually Ricci-flat when $m = 1/2$ or $m = +\infty$ provided we can make sense of it.

When $m = 1/2$ the extremal scalar curvature $S_{1/2}$ is given by

$$S_{1/2}(x) = \frac{8}{3}(2+\psi)\,,$$

and hence vanishes at the unique point of $\mathbb{SP}^2_{\mathbf{1}} \cong \mathbb{CP}^2$ corresponding to the vertex $(-1,-1) \in P^2_{\mathbf{1}}$. The complement of this point in $\mathbb{CP}^2$ is just the normal bundle of the "opposite" $\mathbb{CP}^1$ (corresponding to the facet $F_3 \subset P^2_{\mathbf{1}}$), i.e. a line bundle with first Chern class $c_1 = 1$. The label $m = 1/2$ means that the normal conical singularity can be resolved by passing to a $\mathbb{Z}_2$-quotient, i.e. to the line bundle with $c_1 = 2$ given by $T\,\mathbb{CP}^1$. This means that the self-dual Ricci-flat Einstein metric $\langle\cdot,\cdot\rangle_{1/2}^* = S_{1/2}^{-2}\langle\cdot,\cdot\rangle_{1/2}$ is smooth and complete when considered on $T\,\mathbb{CP}^1$. Being $U(2)$-invariant, it must coincide with the well-known Eguchi-Hanson metric [EH].

When $m \to \infty$ the matrix $G_m = \mathrm{Hess}(g_m)$ converges to the matrix

$$G_\infty(x) = \frac{1}{2}\begin{bmatrix} \frac{1}{1+x_1} + \frac{4-\psi}{(1-\psi)^2} & \frac{4-\psi}{(1-\psi)^2} \\ \frac{4-\psi}{(1-\psi)^2} & \frac{1}{1+x_2} + \frac{4-\psi}{(1-\psi)^2} \end{bmatrix}.$$

One easily checks that $G_\infty = \mathrm{Hess}(g_\infty)$ where

$$g_\infty(x) = \frac{1}{2}\left(\sum_{r=1}^{3} \ell_r(x)\log \ell_r(x) - 3\log(1-\psi)\right).$$

The metric $\langle\cdot,\cdot\rangle_\infty$ defined by this potential does not extend to the whole $\mathbb{SP}^2_{\mathbf{1}} \cong \mathbb{CP}^2$. However it is a well-defined smooth complete extremal Kähler metric of finite volume on $B = \mathbb{SP}^2_{\mathbf{1}} \setminus \mathbb{SP}^1_{\mathbf{1}}$, where the sphere $\mathbb{SP}^1_{\mathbf{1}} \cong \mathbb{CP}^1$ corresponds to the facet $F_3 \subset P^2_{\mathbf{1}}$. In the normal directions to this sphere at infinity, the extremal metric $\langle\cdot,\cdot\rangle_\infty$ looks like a complete hyperbolic cusp (this can be seen by considering for example $m_1 = 1$ and $m_2 \to +\infty$ for the orbifold metrics on S^2 discussed in §4.2). Note that B is symplectomorphic to an open ball in $\mathbb{R}^4$ and, with respect to the complex structure J_∞ defined by g_∞, biholomorphic to $\mathbb{C}^2$.

The scalar curvature of $\langle\cdot,\cdot\rangle_\infty$ is given by

$$S_\infty(x) = \frac{8}{3}(1-\psi)\,,$$

which vanishes exactly at the sphere at infinity. Hence, the metric $\langle\cdot,\cdot\rangle_\infty^* \equiv S_\infty^{-2}\langle\cdot,\cdot\rangle_\infty$ is a smooth complete self-dual Ricci-flat Einstein metric on B, obviously with infinite volume. Being $U(2)$-invariant and B being diffeomorphic to $\mathbb{R}^4$, it must coincide with the well-known Taub-NUT metric [EGH].

As promised in the introduction, we get in this way a one parameter family of $U(2)$-invariant self-dual Einstein metrics $\langle\cdot,\cdot\rangle_m^*$, $1/2 \le m \le +\infty$, having positive scalar curvature when $1/2 < m < +\infty$ and connecting the Ricci-flat Eguchi-Hanson metric on $T\,\mathbb{CP}^1$ ($m = 1/2$) with the Ricci-flat Taub-NUT metric on $\mathbb{R}^4 \cong \mathbb{C}^2$ ($m = +\infty$). Note that one of the metrics in between is the Kähler-Einstein Fubini-Study metric on $\mathbb{CP}^2$ ($m = 1$).

In [GL] Galicki and Lawson use quaternionic reduction to produce self-dual Einstein metrics on certain weighted projective spaces. These include $\mathbb{CP}^2_{(p+q,p+q,2p)}$, which up to covering/quotient correspond in the above family to $m = (p+q)/2p$. Galicki-Lawson assume that $p, q \in \mathbb{N}$, $q \leq p$ and $(p, q) = 1$. They point out that when $q/p \to 1$ their metrics converge to Fubini-Study on $\mathbb{CP}^2$, while when $q/p \to 0$ they converge to Eguchi-Hanson on $T\,\mathbb{CP}^1$. This is consistent with the $m = 1$ and $m = 1/2$ cases in our family. In fact, it follows from the classification results of [AG] that the Galicki-Lawson metrics, whenever defined, are the same as the ones constructed here for the corresponding value of the parameter m.

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