Published for SISSA by Springer

Received: *February 18, 2015*
Accepted: *April 7, 2015*
Published: *April 30, 2015*

Kähler potential and ambiguities in 4d $\mathcal{N} = 2$ SCFTs

Jaume Gomis[a,b] **and Nafiz Ishtiaque**[a]

[a] *Perimeter Institute for Theoretical Physics, Waterloo, Ontario, N2L 2Y5, Canada*

[b] *Department of Physics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada*

E-mail: jgomis@perimeterinstitute.ca, nishtiaque@perimeterinstitute.ca

Abstract: The partition function of four-dimensional $\mathcal{N} = 2$ superconformal field theories on S^4 computes the exact Kähler potential on the space of exactly marginal couplings [1]. We present a new elementary proof of this result using supersymmetry Ward identities. The partition function is a section rather than a function, and is subject to ambiguities coming from Kähler transformations acting on the Kähler potential. This ambiguity is realized by a local supergravity counterterm in the underlying SCFT. We provide an explicit construction of the Kähler ambiguity counterterm in the four dimensional $\mathcal{N} = 2$ off-shell supergravity theory that admits S^4 as a supersymmetric background.

Keywords: Supersymmetric gauge theory, Extended Supersymmetry, Supergravity Models

ArXiv ePrint: 1409.5325

doi:10.1007/JHEP04(2015)169

Contents

1 Introduction

Recent years have witnessed remarkable progress in obtaining the exact partition function of supersymmetric field theories in various background geometries. When the geometry is $S^1 \times \mathcal{M}_{d-1}$ the partition function admits a standard Hilbert space interpretation as a supertrace over the states of the theory on $\mathcal{M}_{d-1}$. In other geometries, such as on a sphere S^d, the physical interpretation of the partition function must be sought.

In [1] it has been shown that the partition function of 4d $\mathcal{N} = 2$ superconformal field theories (SCFTs) on S^4 computes the exact Kähler potential K on the space of exactly marginal couplings, also referred to as the conformal manifold. This result was proven both by using supersymmetric localization [2] and by conformal dimension regularization on S^4, and extends the proof in [3] that the S^2 partition function of 2d $\mathcal{N} = (2, 2)$ SCFTs computes the exact Kähler potential on the conformal manifold, as conjectured by [4] based on the exact formulae in [5, 6] (see also [3, 7]). In detail, [1] demonstrated that

$$Z_{S^4} = e^{K/12} \,. \tag{1.1}$$

These identifications provide a physical and geometrical interpretation of the sphere partition function of 4d $\mathcal{N} = 2$ and 2d $\mathcal{N} = (2, 2)$ SCFTs. These results also provide a computational pathway for obtaining the exact metric in the conformal manifold, which are interesting new observables in these theories, acted on by dualities (see e.g. recent work [8, 9]).

Here we present an elementary proof of the formula (1.1) using supersymmetry Ward identities. This new proof does not require localization nor that the 4d $\mathcal{N} = 2$ SCFT admits a Lagrangian description. By virtue of the relation (1.1) identifying the S^4 partition function with the Kähler potential K on the conformal manifold, it follows that the partition function is subject to the Kähler ambiguity transformations

$$K(\tau, \bar{\tau}) \to K(\tau, \bar{\tau}) + \mathcal{F}(\tau) + \bar{\mathcal{F}}(\bar{\tau}) \,, \tag{1.2}$$

JHEP04(2015)169

where $\mathcal{F}$ is an arbitrary holomorphic function and τ are holomorphic coordinates on the conformal manifold. This ambiguity implies that the partition function is a section over the space of exactly marginal couplings.

We also give the microscopic realization of the Kähler ambiguity (1.2) by constructing the local supergravity counterterm in 4d $\mathcal{N}=2$ off-shell supergravity that when evaluated on the supersymmetric S^4 background yields (1.2). This is the 4d counterpart of the Kähler ambiguity counterterm for 2d $\mathcal{N}=(2,2)$ SCFTs constructed in [1].

The plan is as follows. In section 2 we use supersymmetry Ward identities to show that the S^4 partition function of 4d $\mathcal{N}=2$ SCFTs computes the Kähler potential in the conformal manifold. In section 3 we identify the off-shell 4d $\mathcal{N}=2$ Poincaré supergravity theory in which the S^4 is a supersymmetric background. In section 4 we construct the supergravity invariant in the relevant Poincaré supergravity theory that once evaluated on S^4 provides a first principles realization of the Kähler transformation (1.2).

2 Kähler potential from S^4 partition function

An exactly marginal operator in a four dimensional $\mathcal{N}=2$ SCFT is a scalar operator of dimension four which is a superconformal descendant of a scalar chiral primary operator of $\mathrm{U}(1)_R$ charge $w=2$. An $\mathcal{N}=2$ SCFT can be deformed while preserving superconformal invariance by[1]

$$\frac{1}{\pi^2}\int d^4x \sum_I \left(\tau_I O_I + \bar{\tau}_{\bar{I}}\bar{O}_{\bar{I}}\right)\,. \tag{2.1}$$

The exactly marginal couplings τ_I are holomorphic coordinates in the space of exactly marginal deformations, known as the conformal manifold. The canonical metric in the conformal manifold $g_{I\bar{J}}$ is the Zamolodchikov metric

$$\langle O_I(x)\bar{O}_{\bar{J}}(0)\rangle = \frac{g_{I\bar{J}}}{x^8}\,, \tag{2.2}$$

which in four dimensional $\mathcal{N}=2$ SCFTs is Kähler, that is

$$g_{I\bar{J}} = \frac{\partial}{\partial\tau_I}\frac{\partial}{\partial\bar{\tau}_{\bar{J}}}K(\tau,\bar{\tau}) \equiv \partial_I\partial_{\bar{J}}K(\tau,\bar{\tau})\,. \tag{2.3}$$

An $\mathcal{N}=2$ SCFT can be canonically placed on S^4 by the stereographic projection. The $\mathcal{N}=2$ superconformal transformations on S^4 are parametrized by chiral conformal Killing spinors ϵ^i and ϵ_i of opposite chirality transforming as doublets of the $\mathrm{SU}(2)_R$ R-symmetry, which obey[2]

$$\nabla_m\epsilon^i = \gamma_m\eta^i \qquad \nabla_m\epsilon_i = \gamma_m\eta_i\,, \tag{2.4}$$

so that $\eta^i = \frac{1}{4}\not{\nabla}\epsilon^i$ and $\eta_i = \frac{1}{4}\not{\nabla}\epsilon_i$.

An exactly marginal operator in an $\mathcal{N}=2$ SCFT can be represented as the top component of a four dimensional $\mathcal{N}=2$ *chiral multiplet* of R-charge $w=2$, whose bottom component realizes the parent chiral primary operator. The holomorphic coordinates on

[1]We use the same conventions as in [1].
[2]Γ^a denotes tangent space gamma matrices while $\gamma^m = e^m_a\Gamma^a$ denotes curved space ones.

JHEP04(2015)169

the conformal manifold can be promoted to supersymmetric background chiral superfields with vanishing R-charge $w = 0$. The $\mathcal{N} = 2$ superconformal transformations of a chiral multiplet with R-charge w on S^4 are given by [10] (we use [11]):[3]

$$
\begin{aligned}
\delta A &= \frac{1}{2}\bar{\epsilon}^i \Psi_i \\
\delta \Psi_i &= \nabla\!\!\!/(A\epsilon_i) + \frac{1}{2}B_{ij}\epsilon^j + \frac{1}{4}\Gamma^{ab}F^-_{ab}\varepsilon_{ij}\epsilon^j + (2w-4)A\eta_i \\
\delta B_{ij} &= \bar{\epsilon}_{(i}\nabla\!\!\!/\Psi_{j)} - \bar{\epsilon}^k\Lambda_{(i}\varepsilon_{j)k} + 2(1-w)\bar{\eta}_{(i}\Psi_{j)} \\
\delta F^-_{ab} &= \frac{1}{4}\varepsilon^{ij}\bar{\epsilon}_i\nabla\!\!\!/\Gamma_{ab}\Psi_j + \frac{1}{4}\bar{\epsilon}^i\Gamma_{ab}\Lambda_i - \frac{1}{2}(1+w)\varepsilon^{ij}\bar{\eta}_i\Gamma_{ab}\Psi_j \\
\delta \Lambda_i &= -\frac{1}{4}\Gamma^{ab}\nabla\!\!\!/(F^-_{ab}\epsilon_i) - \frac{1}{2}\nabla\!\!\!/B_{ij}\varepsilon^{jk}\epsilon_k + \frac{1}{2}C\varepsilon_{ij}\epsilon^j - (1+w)B_{ij}\varepsilon^{jk}\eta_k + \frac{1}{2}(3-w)\Gamma^{ab}F^-_{ab}\eta_i \\
\delta C &= -\nabla_m(\varepsilon^{ij}\bar{\epsilon}_i\gamma^m\Lambda_j) + (2w-4)\varepsilon^{ij}\bar{\eta}_i\Lambda_j\,,
\end{aligned}
\tag{2.5}
$$

where in Euclidean signature F^-_{ab} is a self-dual rank-two tensor. Indeed, for $w = 2$, the integrated top component is superconformal invariant and we have the identification

$$C_I = O_I \qquad \text{for } w = 2\,. \tag{2.6}$$

For $w = 0$, an arbitrary covariantly constant background value for the bottom component of the chiral multiplet[4] is superconformal invariant, and serves as the spurion field for the holomorphic coordinates on the conformal manifold

$$A_I = \tau_I \qquad \text{for } w = 0\,. \tag{2.7}$$

We denote by $\mathcal{A}_I$ the chiral multiplets to which the coordinates in the conformal manifold have been promoted.

Consider now the SCFT partition function on S^4 as a function of the exactly marginal couplings $Z_{S^4}(\tau, \bar{\tau})$. The second derivative

$$\partial_I\partial_{\bar{J}}\log Z_{S^4} = \frac{1}{\pi^4}\left\langle \int_{S^4} d^4x\sqrt{g}\,C_I(x)\ \int_{S^4} d^4y\sqrt{g}\,\bar{C}_{\bar{J}}(y)\right\rangle \tag{2.8}$$

is the *integrated* connected two-point function of exactly marginal operators. This correlator is ultraviolet divergent, divergences arising when the operators collide. These ultraviolet divergences can be regularized by introducing a massive deformation. Regulating divergences in a supersymmetric manner leads us to consider the OSp(2|4) massive subalgebra of the $\mathcal{N} = 2$ superconformal algebra on S^4, which is the supersymmetry algebra of an arbitrary massive four dimensional $\mathcal{N} = 2$ theory on S^4.

The OSp(2|4) massive subalgebra on S^4 is generated by supercharges that anticommute to the SO(5) isometries of S^4 and an $\mathrm{SO}(2)_R \subset \mathrm{SU}(2)_R$ R-symmetry. Conformal generators and $\mathrm{U}(1)_R$ are projected out. The OSp(2|4) transformations are generated by Killing spinors which obey

$$\nabla_m\chi^j = \frac{i}{2r}\gamma_m\chi^j\,, \tag{2.9}$$

[3]Throughout a barred spinor is $\bar{\lambda} = \lambda^T\mathcal{C}$, where $\mathcal{C}$ is the charge conjugation matrix.
[4]All other components in multiplet must vanish.

where

$$\chi^j = \epsilon^j + \tau_1^{jk}\epsilon_k \tag{2.10}$$

so that[5]

$$\epsilon^i = \chi_L^i \qquad \epsilon_i = \tau_{1ij}\chi_R^j \tag{2.11}$$

and $\tau_p^{jk} = (i\sigma_3, -1, -i\sigma_1) = (\tau_{p\,jk})^*$, where σ_p are the Pauli matrices. In stereographic coordinates, where $ds^2 = \frac{1}{\left(1+\frac{x^2}{4r^2}\right)^2} dx_m dx^m$, we have

$$\chi^j = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(1 + \frac{i}{2r}x_m\Gamma^m\right)\chi_0^j \,. \tag{2.12}$$

The constant spinors χ_0^j parametrize the transformations of the eight supercharges in OSp(2|4). If these parameters are chiral

$$P_L\chi_0^j = 0\,, \tag{2.13}$$

the corresponding spinors generate an OSp(2|2) subalgebra OSp(2|4). The chiral components of these spinors χ_L^j and χ_R^j

$$\chi_L^j = P_L\chi^j = \frac{i/2r}{\sqrt{1+\frac{x^2}{4r^2}}} x_m\Gamma^m\chi_{0R}^j \qquad \chi_R^j = P_R\chi^j = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\chi_{0R}^j\,, \tag{2.14}$$

vanish at the North and the South poles of the sphere respectively. If the parameters are further constrained by

$$\chi_0^i = \tau_1^{ij}\varepsilon_{jk}\Gamma_1\Gamma_2\chi_0^k\,, \tag{2.15}$$

the corresponding spinors generate a further SU(1|1) subalgebra

$$Q^2 = J + R \tag{2.16}$$

of OSp(2|2) $\subset$ OSp(2|4), where $J = J_{12} + J_{34}$ is a self-dual rotation on S^4 and R is the $\mathrm{SO}(2)_R \subset \mathrm{SU}(2)_R$ R-symmetry.

Our strategy is to first prove that the *integrated* top component of the chiral multiplet in (2.8) can be written as an SU(1|1) $\subset$ OSp(2|4) supersymmetry transformation δ everywhere except at the North pole of S^4, where the corresponding Killing spinor vanishes. The proof is completed by showing that the correlator of the integrated top component C with an arbitrary operator $\mathcal{O}$ invariant under the SU(1|1) supersymmetry transformation δ reduces to the correlator of the bottom component A at the North pole with $\mathcal{O}$. In detail

$$\left\langle \int_{S^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle = 32\pi^2r^2\left\langle A(N)\,\mathcal{O}\right\rangle\,. \tag{2.17}$$

[5] P_L and P_R are the spinor chirality projectors: $P_L^2 = P_L$, $P_R^2 = P_R$ and $P_L + P_R = 1$. The Killing spinors obey $P_L\epsilon^i = \epsilon^i$ and $P_R\epsilon_i = \epsilon_i$.

The supersymmetry transformation of the fermions in a chiral multiplet with R-charge $w = 2$ can be written as (2.5)[6]

$$\delta\Psi_i = \tau_{1ij}\nabla\!\!\!/(A\chi_R^j) + \frac{1}{2}\vec{B}\cdot\vec{\tau}_{ij}\chi_L^j + \frac{1}{4}\Gamma^{ab}F^-_{ab}\varepsilon_{ij}\chi_L^j \tag{2.18a}$$

$$\begin{aligned}\delta\Lambda_i =& -\frac{1}{4}\Gamma^{ab}\nabla\!\!\!/ F^-_{ab}\tau_{1ij}\chi_R^j - \frac{i}{4r}\Gamma^{ab}F^-_{ab}\tau_{1ij}\chi_L^j + \frac{1}{2}C\varepsilon_{ij}\chi_L^j \\ &- \frac{1}{2}\nabla\!\!\!/\vec{B}\cdot\vec{\tau}_{ij}\tau_1^{jk}\varepsilon_{kl}\chi_R^l - \frac{3i}{2r}\vec{B}\cdot\vec{\tau}_{ij}\tau_1^{jk}\varepsilon_{kl}\chi_L^l\end{aligned} \tag{2.18b}$$

Using the SU(1|1) supersymmetry transformation δ obtained by imposing the constraints (2.13) and (2.15) on the Killing spinors, we get after multiplying (2.18a) by $\tau_{2ij}\tau_1^{jk}{\chi_L^i}^\dagger$ and (2.18b) by $\tau_{2ij}\varepsilon^{jk}{\chi_L^i}^\dagger$ that

$$B_1 = -\,\delta\left(\frac{{\chi_L^i}^\dagger\Psi_k}{\|\chi_L\|^2}\right)\tau_{2ij}\tau_1^{jk} + \frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\nabla\!\!\!/(A\chi_R^j)\tau_{2ij} - \frac{1}{4}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\chi_L^j}{\|\chi_L\|^2}\tau_{3ij}F^-_{ab} \tag{2.19a}$$

$$\begin{aligned}C =& -\,\delta\left(\frac{{\chi_L^i}^\dagger\Lambda_k}{\|\chi_L\|^2}\right)\tau_{2ij}\varepsilon^{jk} - \frac{1}{2}\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\tau_{2ij}\nabla_m B_1 + \frac{3i}{r}B_1 \\ &+ \frac{1}{4}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\gamma^m\chi_R^j}{\|\chi_L\|^2}\tau_{3ij}\nabla_m F^-_{ab} + \frac{i}{4r}\frac{{\chi_L^i}^\dagger\Gamma^{ab}\chi_L^j}{\|\chi_L\|^2}\tau_{3ij}F^-_{ab}\end{aligned} \tag{2.19b}$$

where we have used that for the SU(1|1) Killing spinors $\|\chi_L\|^2 := \|\chi_L^1\|^2 = \|\chi_L^2\|^2$, where $\|\lambda\|^2 = \lambda^\dagger\lambda$. The terms proportional to F^-_{ab} and $\nabla_\mu F^-_{ab}$ in (2.18a) (2.18b) also vanish. Their coefficients are anti-self-dual in the tangent space indices since

$${\chi_L^i}^\dagger\Gamma^{ab}\gamma^{(r)}\chi_{L/R}^j = {\chi_L^i}^\dagger\Gamma_*\Gamma^{ab}\gamma^{(r)}\chi_{L/R}^j = -\frac{1}{2}\varepsilon^{ab}{}_{cd}{\chi_L^i}^\dagger\Gamma^{cd}\gamma^{(r)}\chi_{L/R}^j\,, \tag{2.20}$$

where $\gamma^{(r)}$ is the product of r distinct gamma matrices. Since F^-_{ab} is self-dual in Euclidean signature, all the terms involving F^-_{ab} vanish. We can eliminate B_1 from (2.19b) by using (2.19a), which yields

$$\begin{aligned}C =& -\frac{1}{2}\frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\nabla\!\!\!/\left(\left[\frac{{\chi_L^k}^\dagger}{\|\chi_L\|^2}\nabla\!\!\!/(A\chi_R^l)\right]\chi_R^j\right)\tau_{2ij}\tau_{2kl} + \frac{i}{r}\frac{{\chi_L^i}^\dagger}{\|\chi_L\|^2}\nabla\!\!\!/(A\chi_R^j)\tau_{2ij} \\ &+ \delta\left(\Xi(\Lambda_i,\Psi_i,\chi^i)\right)\,,\end{aligned} \tag{2.21}$$

where, for brevity, we have defined

$$\Xi(\Lambda_i,\Psi_i,\chi^i) := -\frac{{\chi_L^i}^\dagger\Lambda_k}{\|\chi_L\|^2}\tau_{2ij}\varepsilon^{jk} + \frac{1}{2}\frac{{\chi_L^i}^\dagger\gamma^m\chi_R^j}{\|\chi_L\|^2}\nabla_m\left(\frac{{\chi_L^k}^\dagger\Psi_t}{\|\chi_L\|^2}\right)\tau_{2ij}\tau_{2kl}\tau_1^{lt} - \frac{3i}{r}\frac{{\chi_L^i}^\dagger\Psi_k}{\|\chi_L\|^2}\tau_{2ij}\tau_1^{jk}\,. \tag{2.22}$$

We now show that the sum of the terms in (2.21) involving A are a total derivative.

For any OSp(2|2) supersymmetry parameter χ^j and any scalar quantity X we have that[7]

$$\frac{{\chi_L^j}^\dagger}{\|\chi_L^j\|^2}\nabla\!\!\!/\left(X\chi_R^j\right) = \nabla_m\left(\frac{{\chi_L^j}^\dagger\gamma^m\chi_R^j}{\|\chi_L^j\|^2}X\right) + \frac{4irX}{x^2}\,. \tag{2.23}$$

[6] $\vec{B} = (B_1, B_2, B_3)$ such that $B_{ij} = \vec{B}\cdot\vec{\tau}_{ij} = \sum_p B_p\tau_{pij}$.

[7] By using that $\nabla_m\chi_L^\dagger = -\frac{i}{2r}\chi_R^\dagger\gamma_m$.

Using this, the top component C of a chiral multiplet with $w = 2$ can be written locally as the sum of an SU(1|1) supersymmetry transformation δ and total derivatives

$$C = \delta\left(\Xi(\Lambda_i, \Psi_i, \chi^i)\right) - \frac{1}{2}\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}\nabla_n\left[\frac{\chi_L^{k\,\dagger}\gamma^n\chi_R^l}{\|\chi_L\|^2}A\right]\right)\tau_{2ij}\tau_{2kl}$$
$$+ 8ir\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j A}{\|\chi_L\|^2 x^2}\right)\tau_{2ij} + \frac{i}{r}\nabla_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}A\right)\tau_{2ij}\,. \tag{2.24}$$

This formula fails at the North pole, where $\|\chi_L^1\| = \|\chi_L^2\| = 0$ and Ξ diverges. Therefore the integrated top component is non-trivial in correlation functions, as it is not supersymmetry-exact globally, but the entire contribution localizes to the North pole, just as in the analysis of 2d $\mathcal{N} = (2,2)$ SCFTs in [1].[8]

Let us consider the integrated correlator with an operator $\mathcal{O}$ obeying $\delta\mathcal{O} = 0$

$$\left\langle \int_{S^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle = \lim_{R\to 0}\left[\left\langle\int_{S^4\backslash B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle + \left\langle\int_{B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle\right]. \tag{2.25}$$

We have divided S^4 into two-regions: a four-dimensional ball B_R^4 of radius R around the North pole and its complement $S^4\backslash B_R^4$. In the $R \to 0$ limit the ball contribution vanishes[9] and we are left with

$$\lim_{R\to 0}\left\langle\int_{S^4\backslash B_R^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle\,. \tag{2.26}$$

Using (2.24), which is valid in $S^4\backslash B_R^4$, and $\delta\Phi = 0$, we can replace C by the last three terms in (2.24), which inside (2.26) can be written as an integral over the three-sphere S_R^3 of radius R at the boundary of $S^4\backslash B_R^4$. For any OSp(2|2) Killing spinor χ^j (2.14), we have that in the $R \to 0$ limit

$$\chi_L^i(R) \sim O(R),\ \chi_R^i(R) \sim O(1) \Rightarrow \frac{\chi_L^{i\,\dagger}\gamma^\mu\chi_R^i}{\|\chi_L\|^2} \sim O\left(\frac{1}{R}\right)\,. \tag{2.27}$$

Therefore, a simple scaling argument shows that the last term in (2.24) cannot compensate for the R^3 measure factor coming from S_R^3 and gives a vanishing contribution in the $R \to 0$ limit. Therefore, we have shown that in the presence of δ-closed operators

$$\int_{S^4} d^4x\sqrt{g}\,C(x) = \lim_{R\to 0}\int_{S^4\backslash B_R^4} d^4x\sqrt{g}\,C(x)$$
$$= -\frac{1}{2}\lim_{R\to 0}\int_{S^4\backslash B_R^4} \mathrm{d}^4x\,\partial_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}\partial_n\left[\frac{\chi_L^{k\,\dagger}\gamma^n\chi_R^l}{\|\chi_L\|^2}A(x)\sqrt{g}\right]\right)\tau_{2ij}\tau_{2kl}$$
$$+ 8ir\lim_{R\to 0}\int_{S^4\backslash B_R^4} \mathrm{d}^4x\,\partial_m\left(\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2 x^2}A(x)\sqrt{g}\right)\tau_{2ij}\,. \tag{2.28}$$

[8] We note that had we assumed that the partition function can be regulated while preserving full $\mathcal{N} = 2$ superconformal invariance, we would have concluded that the partition function is independent of the moduli, as the top component C is globally superconformal-exact.

[9] The R^4 measure factor suppresses the ball contribution in the $R \to 0$ limit.

JHEP04(2015)169

In the limit $R \to 0$ we can replace the bottom component $A(x)$ by its value at the North pole $A(N)$, as higher order terms in the expansion in R vanish in the limit, and using Stoke's theorem

$$\int_{S^4} d^4x\sqrt{g}\,C(x) = \lim_{R\to 0}\int_{S^3_R} V\cdot\hat{\eta}\,, \tag{2.29}$$

where

$$V^m := -\frac{1}{2}\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2}\partial_n\left(\frac{\chi_L^{k\,\dagger}\gamma^n\chi_R^l}{\|\chi_L\|^2}\sqrt{g}\right)A(N)\tau_{2ij}\tau_{2kl} + 8ir\frac{\chi_L^{i\,\dagger}\gamma^m\chi_R^j}{\|\chi_L\|^2x^2}A(N)\sqrt{g}\tau_{2ij}\,, \tag{2.30}$$

and $\hat{\eta}$ is the unit vector towards the North pole of S^4 along the radial direction.[10] Going to spherical coordinates, where R is the radial coordinate, we find that

$$V\cdot\hat{\eta} = \frac{512A(N)r^6(R^2-2r^2)}{R^3(R^2+4r^2)^3} + \frac{2048A(N)r^8}{R^3(R^2+4r^2)^3} = \frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}\,. \tag{2.31}$$

The integration in (2.29) is over S^3_R, therefore

$$\int_{S^3_R} V\cdot\hat{\eta} = \frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}2\pi^2R^3\,, \tag{2.32}$$

and

$$\lim_{R\to 0}\frac{512A(N)r^6(R^2+2r^2)}{R^3(R^2+4r^2)^3}2\pi^2R^3 = 32A(N)\pi^2r^2\,. \tag{2.33}$$

This yields the desired formula

$$\left\langle\int_{S^4} d^4x\sqrt{g}\,C(x)\,\mathcal{O}\right\rangle = 32\pi^2r^2\left\langle A(N)\,\mathcal{O}\right\rangle\,. \tag{2.34}$$

The integrated top component C of a chiral multiplet is equivalent to inserting the bottom component A at the North pole. A very similar analysis yields

$$\left\langle\int_{S^4} d^4x\sqrt{g}\,\bar{C}(x)\,\mathcal{O}\right\rangle = 32\pi^2r^2\left\langle \bar{A}(S)\,\mathcal{O}\right\rangle\,. \tag{2.35}$$

The integrated top component $\bar{C}$ of an anti-chiral multiplet is equivalent to inserting the bottom component $\bar{A}$ at the South pole.

We can now use (2.34) and (2.35) to express the derivative of the partition function in (2.8) as an *unintegrated* two-point function

$$\partial_I\partial_{\bar{J}}\log Z_{S^4} = \frac{1}{\pi^4}\left\langle\int_{S^4} d^4x\sqrt{g}\,C_I(x)\int_{S^4} d^4y\sqrt{g}\,\bar{C}_{\bar{J}}(y)\right\rangle = \left(32r^2\right)^2\left\langle A_I(N)\bar{A}_{\bar{J}}(S)\right\rangle\,. \tag{2.36}$$

It follows from the first equation in (2.5) that the correlator $\left\langle A_I(N)\bar{A}_{\bar{J}}(S)\right\rangle$ is δ invariant, since SU(1|1) supersymmetry parameters ϵ^j and ϵ_j vanish at the North pole and South pole respectively, and therefore $\delta A_I(N) = \delta\bar{A}_{\bar{J}}(S) = 0$.

[10] The unit radial vector in cartesian coordinates is given by $\hat{\eta}^a = -\frac{x^a}{\sqrt{x^2}}$.

Using the supersymmetry Ward identity $\langle A_I(N)\bar{A}_{\bar{J}}(S)\rangle = \frac{r^4}{48}\langle C_I(N)\bar{C}_{\bar{J}}(S)\rangle$ [1], that $\langle C_I(N)\bar{C}_{\bar{J}}(S)\rangle = \frac{1}{(2r)^8}g_{I\bar{J}}$ defines the Zamolodchikov metric $g_{I\bar{J}}$ and that the metric is Kähler (2.3) we arrive at

$$\partial_I\partial_{\bar{J}}\log Z_{S^4} = \frac{1}{12}g_{I\bar{J}} = \frac{1}{12}\partial_I\partial_{\bar{J}}K\,. \tag{2.37}$$

Therefore, the four sphere partition function of a four dimensional $\mathcal{N}=2$ SCFT computes the Kähler potential in the conformal manifold (1.1), and is subject to Kähler transformation ambiguities (1.2), which do not affect the Zamolodchikov metric.

3 Off-shell $\mathcal{N}=2$ Poincaré supergravity for S^4

The partition function of a field theory in a curved geometry can be ambiguous. These ambiguities are encoded in finite counterterms for the background fields that capture the background geometry and the parameters of the theory. When the partition function of a supersymmetric theory can be regulated in a diffeomorphism invariant and supersymmetric manner, the counterterms are *supergravity* invariants constructed out of the supergravity multiplet encoding the background geometry and the supersymmetry multiplets to which the other parameters of the theory can be promoted, since all parameters in a supersymmetric field theory can be promoted to background supermultiplets [12].

Constructing these supergravity invariants requires identifying first the supergravity theory in which the curved geometry over which the partition function is computed is a supersymmetric background. This can be analyzed in the framework of off-shell supergravity [13]. In this section we identify the four dimensional $\mathcal{N}=2$ off-shell Poincaré supergravity theory and the background fields in that supergravity multiplet that give rise to the OSp(2|4)-invariant four-sphere background geometry.

A conceptual way of constructing off-shell Poincaré supergravity theories is to start with off-shell conformal supergravity and partially gauge fix the conformal symmetries down to Poincaré by adding compensating supermultiplets. Different choices of compensating multiplets give rise to different off-shell Poincaré supergravity theories, with different sets of auxiliary fields.[11] The Poincaré supersymmetry transformations of the gauge fixed theory are constructed by combining the Poincaré supersymmetry transformations in conformal supergravity with field dependent superconformal transformations that are needed to preserve the gauge choice.[12]

Our starting point is four dimensional $\mathcal{N}=2$ conformal supergravity [15] (we refer to [14] for more details). Off-shell $\mathcal{N}=2$ superconformal transformations are realized on the Weyl multiplet, whose independent fields are

$$\begin{aligned} \text{bosonic:}\;\; & e_m^a, b_m, V_m{}^j{}_i, A_m^R, T_{ab}^-, D \\ \text{fermionic:}\;\; & \psi_m^i, \chi^i\,. \end{aligned} \tag{3.1}$$

[11]For instance, old and new minimal four dimensional $\mathcal{N}=1$ Poincaré supergravity arises from $\mathcal{N}=1$ conformal supergravity by using a compensating chiral and tensor multiplet respectively.

[12]We refer to the [14] for more background material and references, in particular for 4d $\mathcal{N}=2$ supergravity.

JHEP04(2015)169

The fields $e_m^a, b_m, V_m{}^j{}_i, A_m^R, \psi_m^i$ are the gauge fields for translations, dilatations, $\mathrm{SU}(2)_R$, $\mathrm{U}(1)_R$ and Poincaré supersymmetry generators in the $\mathcal{N}=2$ superconformal algebra. The Weyl multiplet is completed by the bosonic auxiliary fields T_{ab}^- and D, and the fermionic auxiliary field χ^i. In Euclidean signature T_{ab}^- is a self-dual rank-two tensor. The embedding of the OSp(2|4)-invariant S^4 in conformal supergravity appeared in [16, 17].

Four dimensional $\mathcal{N}=2$ Poincaré supergravity [18] contains a graviphoton gauge field A_m. This field is furnished in the conformal approach by coupling an abelian vector multiplet to the Weyl multiplet [19, 20]. An $\mathcal{N}=2$ vector multiplet, also known as a restricted chiral multiplet, is an $\mathcal{N}=2$ chiral multiplet (2.5) with $w=1$ subject to constraints, and consists of

$$\begin{aligned} \text{bosonic:} &\ \ X, A_m, Y_{ij} \\ \text{fermionic:} &\ \ \Omega_i \end{aligned} \tag{3.2}$$

a complex scalar X, a gauge field A_m, a triplet of real auxiliary fields $Y_{ij}=Y_{ji}$ and gauginos Ω_i. The vielbein e_m^a and gravitino ψ_m^i of the Weyl multiplet and the gauge field A_m in the vector multiplet complete the on-shell content of four dimensional $\mathcal{N}=2$ Poincaré supergravity multiplet.

The first step in constructing a Poincaré supergravity theory is to gauge fix special conformal transformations. This can be accomplished by setting

$$b_m = 0\,. \tag{3.3}$$

In order to preserve this gauge, supersymmetry transformations must be accompanied by a compensating special conformal transformation, which acts nontrivially on b_m. Fortunately, all elementary fields in conformal supergravity and all fields in $\mathcal{N}=2$ matter multiplets transform trivially under special conformal transformations, and therefore the supersymmetry transformations of these fields are not modified by the gauge choice (3.3).

Dilatations and $\mathrm{U}(1)_R$ are gauge fixed by setting [15]

$$X = \mu\,, \tag{3.4}$$

where μ is an arbitrary mass scale, while [15]

$$\Omega_i = 0 \tag{3.5}$$

fixes the special conformal supersymmetry transformations. Under supersymmetry [14]

$$\delta X = \frac{1}{2}\bar{\epsilon}^i\Omega_i \tag{3.6}$$

$$\delta\Omega_i = \not{D}X\epsilon_i + \frac{1}{4}\Gamma^{ab}\mathcal{F}_{ab}\varepsilon_{ij}\epsilon^j + \frac{Y_{ij}}{2}\epsilon^j + 2X\eta_i\,, \tag{3.7}$$

where $\delta \equiv \delta_\epsilon + \delta_\eta$, and (ϵ^i, ϵ_i) and (η^i, η_i) parametrize the Poincaré and conformal supersymmetry transformations. $\mathcal{F}_{ab}$ is the superconformal covariant field strength (see equation (20.77) in [14]) and

$$\mathcal{D}_\mu X = (\partial_\mu - b_\mu - iA_\mu^R)X - \frac{1}{2}\bar{\psi}_\mu^i\Omega_i \tag{3.8}$$

JHEP04(2015)169

is the superconformal covariant derivative acting on the scalar field X. In order to preserve the gauge choice (3.4)(3.5), we must accompany the Poincaré supersymmetry transformations δ_ϵ with a field dependent compensating conformal supersymmetry transformation δ_η with parameter[13]

$$\eta_i = \frac{i}{2}\not{A}^R\epsilon_i - \frac{1}{2\mu}\left(\frac{1}{4}\Gamma^{ab}\mathcal{F}_{ab}\varepsilon_{ij} + \frac{Y_{ij}}{2}\right)\epsilon^j\,. \tag{3.9}$$

Different Poincaré supergravity theories depend on the choice of a second multiplet which gauge fixes the remaining $\mathrm{SU}(2)_R$ symmetry. Three choices for this compensating multiplet have been considered in the literature (see [21]): a non-linear multiplet, a hypermultiplet and a tensor multiplet. We now demonstrate that the OSp(2|4)-invariant S^4 is a supersymmetric background of the $\mathcal{N}=2$ Poincaré supergravity theory constructed with a tensor multiplet (and not with the non-linear or hypermultiplet).

Consider the off-shell $\mathcal{N}=2$ Poincaré supergravity multiplet constructed by coupling a vector multiplet and a tensor multiplet to the Weyl multiplet. An $\mathcal{N}=2$ tensor multiplet [21]

$$\begin{aligned}\text{bosonic}:&\ \ L_{ij}, G, E_{mn}\\ \text{fermionic}:&\ \ \phi^i\end{aligned} \tag{3.10}$$

consists of a triplet of real scalars $L_{ij}=L_{ji}$, a tensor gauge field E_{mn}, a complex scalar G and a doublet of spinors ϕ^i. The $\mathrm{SU}(2)_R$ symmetry can be gauge fixed by setting

$$L_{ij} = \tau_{1ij}\varphi\,, \tag{3.11}$$

which breaks $\mathrm{SU}(2)_R$ down to $\mathrm{SO}(2)_R$. The supersymmetry transformation [11]

$$\delta L_{ij} = \bar{\epsilon}_{(i}\phi_{j)} + \varepsilon_{ik}\varepsilon_{jl}\bar{\epsilon}^{(k}\phi^{l)} \tag{3.12}$$

implies that to preserve (3.11), we must accompany the Poincaré supersymmetry transformation δ_ϵ with a compensating $\mathrm{SU}(2)_R$ transformation $\delta_{\mathrm{SU}(2)_R}(\Lambda^k{}_j)$ with parameter[14]

$$\Lambda^k{}_j = -\tau_1^{km}\frac{(\bar{\epsilon}_m\phi_j - \varepsilon_{im}\varepsilon_{jl}\bar{\epsilon}^i\phi^l)}{\varphi}\,. \tag{3.13}$$

In summary, this off-shell Poincaré supergravity multiplet constructed by gauge fixing a Weyl, vector and tensor multiplet completes the on-shell multiplet e^a_m, ψ^i_m, A_m with bosonic auxiliary fields and fermionic auxiliary fields χ^i, ϕ^i. The Poincaré supersymmetry transformations in this $\mathcal{N}=2$ Poincaré supergravity theory are given by the following combination of superconformal transformations

$$\delta_\epsilon + \delta_\eta + \delta_{\mathrm{SU}(2)_R}(\Lambda^k{}_j) \tag{3.14}$$

with η in (3.9) and $\Lambda^k{}_j$ in (3.13).

[13] Since (3.5) preserves δX, no other compensating transformation in required.
[14] The parameter is determined only up to an $\mathrm{SO}(2)_R$ transformation.

JHEP04(2015)169

In this $\mathcal{N}=2$ Poincaré supergravity theory the supersymmetric backgrounds where the background values of all fermions vanish are solutions to the following equations

$$(\delta_\epsilon+\delta_\eta)\,\psi^i_m=0 \qquad (\delta_\epsilon+\delta_\eta)\,\chi^i=0 \qquad (\delta_\epsilon+\delta_\eta)\,\phi^i=0 \tag{3.15}$$

with η in (3.9), since $\Lambda^k{}_j=0$ vanish on bosonic backgrounds. The explicit form of these transformations are [19–21] (we use [11])

$$\begin{aligned}
\delta\psi^i_m &= \left(\partial_m+\frac{1}{2}b_m+\frac{1}{4}\Gamma^{ab}\omega_{mab}-\frac{1}{2}iA^R_m\right)\epsilon^i+V_m{}^i{}_j\epsilon^j-\frac{1}{16}\Gamma^{ab}T^-_{ab}\varepsilon^{ij}\gamma_m\epsilon_j-\gamma_m\eta^i\\
\delta\chi^i &= \frac{1}{2}D\epsilon^i+\frac{1}{6}\Gamma^{ab}\left[-\frac{1}{4}\not{\mathcal{D}}T^-_{ab}\varepsilon^{ij}\epsilon_j-\widehat{R}_{ab}(U_j{}^i)\epsilon^j+i\widehat{R}_{ab}(T)\epsilon^i+\frac{1}{2}T^-_{ab}\varepsilon^{ij}\eta_j\right]\\
\delta\phi^i &= \frac{1}{2}\not{\mathcal{D}}L^{ij}\epsilon_j+\frac{1}{2}\varepsilon^{ij}\not{E}\epsilon_j-\frac{1}{2}G\epsilon^i+2L^{ij}\eta_j\,,
\end{aligned} \tag{3.16}$$

with η in (3.9). $\mathcal{D}$ is superconformal covariant derivative and $\widehat{R}_{ab}(T)$ and $\widehat{R}_{ab}(U_j{}^i)$ are covariant curvatures for $\mathrm{U}(1)_R$ and $\mathrm{SU}(2)_R$.

The $\mathrm{OSp}(2|4)$- supersymmetric S^4 background is described by the following Killing spinor equations (2.9)

$$\nabla_m\epsilon^i=\frac{i}{2r}\gamma_m\tau_1^{ij}\epsilon_j \qquad \nabla_m\epsilon_i=\frac{i}{2r}\gamma_m\tau_{1ij}\epsilon^j\,. \tag{3.17}$$

From (3.16) we find that S^4 is a supersymmetric background of this supergravity theory with the following non-vanishing background fields turned on

$$e^a_m=e^a_m|_{S^4} \qquad Y^{ij}=-\frac{2i\mu}{r}\tau_1^{ij} \qquad Y_{ij}=-\frac{2i\mu}{r}\tau_{1ij} \qquad \text{other}=0\,. \tag{3.18}$$

With these background fields turned on $\delta\psi^i_m$ realizes the S^4 Killing spinor equations (3.17), while $\delta\xi^i$ and $\delta\phi^i$ vanish identically.[15] The algebra of supergravity transformations when evaluated on the background (3.18) realizes the $\mathrm{OSp}(2|4)$ symmetry of S^4.

4 The Kähler ambiguity supergravity counterterm

In this section we construct the $\mathcal{N}=2$ Poincaré supergravity invariant constructed out of the supergravity multiplet and the $w=0$ chiral multiplets $\mathcal{A}_I$ (see below (2.7)) which when evaluated on the $\mathrm{OSp}(2|4)$-supersymmetric background (4.6) realizes the Kähler ambiguity (1.2).

Our approach is to construct a superconformal invariant constructed out of the Weyl multiplet, the compensating vector multiplet Φ, the compensating tensor multiplet and the chiral multiplets $\mathcal{A}_I$, the supermultiplets to which the coordinates in the conformal manifold τ_I have been promoted. This invariant, when evaluated on the Poincaré gauge fixing choice

[15] A similar analysis for the Poincaré supergravity theories constructed with a compensating non-linear multiplet and hypermultiplet demonstrates that the background fields that yield the S^4 Killing spinor equations are incompatible with the vanishing of the supersymmetry variations of the fermions in these multiplets. Therefore, S^4 is not a supersymmetric background of these supergravity theories.

described in the previous section yields an invariant in the associated $\mathcal{N} = 2$ Poincaré supergravity theory. We first recall some facts about the construction of superconformal invariants.

Consider an abstract chiral multiplet (2.5) with $w = 2$, which we denote by $\hat{\mathcal{A}}$, coupled to the Weyl multiplet (3.1). The following superconformal invariant can be constructed from such a chiral multiplet [15]

$$I[\hat{\mathcal{A}}] = \int d^4x\sqrt{g}\left[\hat{C}(x) - \frac{1}{4}\hat{A}\left(T^+_{ab}\right)^2 + \text{fermions}\right], \tag{4.1}$$

where $\hat{C}$ and $\hat{A}$ denote the top and bottom components of the multiplet $\hat{\mathcal{A}}$. The coupling of the chiral multiplet to the Weyl multiplet is responsible for the appearance of the terms after $\hat{C}$ in (4.1). The product of two chiral multiplets with R-charge w_1 and w_2 yields another chiral multiplet of R-charge $w_1 + w_2$. Therefore, superconformal invariants can be constructed from products of chiral multiplets with total R-charge $w = 2$.

Consider now the compensating vector multiplet that appears in the construction of $\mathcal{N} = 2$ Poincaré supergravity, which we denote by Φ. It is important to note that an $\mathcal{N} = 2$ vector multiplet is a chiral multiplet with $w = 1$ subject to reducibility constraints [22], which express the last two components of the chiral multiplet in terms of the previous ones. It is also known as a restricted chiral multiplet. The components of a chiral multiplet (2.5) are given in terms of the fields in the abelian vector multiplet (3.2) by

$$\begin{aligned}
A|_\Phi &= X \\
\Psi_i|_\Phi &= \Omega_i \\
B_{ij}|_\Phi &= Y_{ij} \\
F^-_{ab}|_\Phi &= \mathcal{F}^-_{ab} \\
\Lambda_i|_\Phi &= -\,\varepsilon_{ij}\not{D}\Omega^j \\
C|_\Phi &= -\,2D_aD^a\bar{X} - \frac{1}{2}\mathcal{F}^+_{ab}T^{ab+} - 3\bar{\chi}_i\Omega^i
\end{aligned} \tag{4.2}$$

where $\mathcal{F}_{ab}$ is the superconformal covariant field strength. Expressing a vector multiplet as a $w = 1$ chiral multiplet provides a way of constructing a superconformal invariant out of Φ using (4.1).

We now write down the supergravity counterterm responsible for the Kähler ambiguity (1.2). It is the superconformal invariant (4.1) constructed from the $w = 2$ composite chiral multiplet

$$\Phi^2\mathcal{F}(\mathcal{A}_I)\,, \tag{4.3}$$

where $\mathcal{F}$ is an arbitrary holomorphic function of the $w = 0$ chiral multiplets $\mathcal{A}_I$ describing the coordinates in the conformal manifold.[16] The associated $\mathcal{N} = 2$ Poincaré supergravity

[16] Another natural guess for the Kähler counterterm can be constructed from the $w = 2$ chiral multiplet $W^{ab}W_{ab}\mathcal{F}(\mathcal{A}_I)$, where W_{ab} is a chiral multiplet that encodes the covariant Weyl multiplet (3.1). However, upon evaluating these terms on the background (4.6) they all vanish, as these terms involve the Weyl tensor, which vanishes on S^4. Supergravity couplings involving W^2 have been considered in the literature [23]. For other higher derivative invariants in $\mathcal{N} = 2$ supergravity see e.g. [24–28].

JHEP04(2015)169

invariant is

$$I[\Phi^2\mathcal{F}(\mathcal{A}_I)] \tag{4.4}$$

with

$$A|_\Phi = X = \mu\,. \tag{4.5}$$

We now want to evaluate the $\mathcal{N}=2$ Poincaré supergravity invariant (4.4) on the OSp(2|4)-invariant background field configuration

$$\begin{aligned} &\text{Weyl:} && e_m^a = e_m^a|_{S^4} \\ &\text{vector:} && A|_\Phi = X = \mu\,, \qquad B_{ij}|_\Phi = -\frac{2i\mu}{r}\tau_{1ij}\,, \qquad C|_\Phi = \frac{4\mu}{r^2} \\ &\text{chiral:} && A|_{\mathcal{F}(\mathcal{A}_I)} = \mathcal{F}(\tau_I)\,, \end{aligned} \tag{4.6}$$

where we have used (3.18)(4.2) and (4.5).

The product of two chiral multiplets with bosonic components (A, B_{ij}, F^-_{ab}, C) and (a, b_{ij}, f^-_{ab}, c) yields a new chiral multiplet with bosonic components (setting all fermions to zero, as they vanish on the OSp(2|4)-invariant background (4.6))

$$\left(Aa,\, Ab_{ij}+aB_{ij},\, Af^-_{ab}+aF^-_{ab},\, Ac+aC-\frac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}b_{kl}+F^-_{ab}f^{-ab}\right)\,. \tag{4.7}$$

Therefore, on the supersymmetric background (4.6)

$$\begin{aligned} A|_{\Phi^2} &= \mu^2 \\ B_{ij}|_{\Phi^2} &= -\frac{4i\mu^2}{r}\tau_{1ij} \\ C|_{\Phi^2} &= \frac{12\mu^2}{r^2} \end{aligned} \tag{4.8}$$

and finally

$$C|_{\Phi^2\mathcal{F}(\mathcal{A}_I)} = \frac{12\mu^2}{r^2}\mathcal{F}(\tau_I)\,. \tag{4.9}$$

Using that μ is a fiducial scale introduced in gauge fixing to Poincaré supergravity, so that we can write $\mu = a/r$, we have find that the invariant (4.4) evaluates to

$$I[\Phi^2\mathcal{F}(\mathcal{A}_I)] = \int d^4x\sqrt{g}\,C|_{\Phi^2\mathcal{F}(\mathcal{A}_I)} = 32\pi^2a^2\mathcal{F}(\tau_I)\,. \tag{4.10}$$

Therefore, the marginal supergravity counterterm

$$\frac{1}{384\pi^2a^2}\left(I[\Phi^2\mathcal{F}(\mathcal{A}_I)] + I[\bar{\Phi}^2\bar{\mathcal{F}}(\bar{\mathcal{A}}_{\bar{I}})]\right) \tag{4.11}$$

is responsible for the Kähler ambiguity (1.2) in the four sphere partition function of four dimensional $\mathcal{N}=2$ SCFTs

$$Z_{S^4} \simeq Z_{S^4}e^{\frac{1}{12}\left(\mathcal{F}(\tau)+\bar{\mathcal{F}}(\bar{\tau})\right)}\,. \tag{4.12}$$

This provides a microscopic realization of Kähler ambiguities in these SCFTs.

JHEP04(2015)169

Acknowledgments

We would like to thank B. de Wit, E. Gerchkovitz, Z. Komargodski and A. Van Proeyen for useful discussions. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation. J.G. also acknowledges further support from an NSERC Discovery Grant and from an ERA grant by the Province of Ontario.

References

[1] E. Gerchkovitz, J. Gomis and Z. Komargodski, *Sphere partition functions and the Zamolodchikov metric*, JHEP **11** (2014) 001 [arXiv:1405.7271] [INSPIRE].

[2] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[3] J. Gomis and S. Lee, *Exact Kähler potential from gauge theory and mirror symmetry*, JHEP **04** (2013) 019 [arXiv:1210.6022] [INSPIRE].

[4] H. Jockers, V. Kumar, J.M. Lapan, D.R. Morrison and M. Romo, *Two-Sphere Partition Functions and Gromov-Witten Invariants*, Commun. Math. Phys. **325** (2014) 1139 [arXiv:1208.6244] [INSPIRE].

[5] F. Benini and S. Cremonesi, *Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices*, Commun. Math. Phys. **334** (2015) 1483 [arXiv:1206.2356] [INSPIRE].

[6] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact results in $D = 2$ supersymmetric gauge theories*, JHEP **05** (2013) 093 [arXiv:1206.2606] [INSPIRE].

[7] N. Doroud and J. Gomis, *Gauge theory dynamics and Kähler potential for Calabi-Yau complex moduli*, JHEP **1312** (2013) 99 [arXiv:1309.2305] [INSPIRE].

[8] M. Baggio, V. Niarchos and K. Papadodimas, *tt^* equations, localization and exact chiral rings in 4d $\mathcal{N} = 2$ SCFTs*, JHEP **02** (2015) 122 [arXiv:1409.4212] [INSPIRE].

[9] M. Baggio, V. Niarchos and K. Papadodimas, *Exact correlation functions in* SU(2) $\mathcal{N} = 2$ *superconformal QCD*, Phys. Rev. Lett. **113** (2014) 251601 [arXiv:1409.4217] [INSPIRE].

[10] P. Breitenlohner and M.F. Sohnius, *An Almost Simple Off-shell Version of* SU(2) *Poincaré Supergravity*, Nucl. Phys. **B 178** (1981) 151 [INSPIRE].

[11] A.V. Proeyen, *$\mathcal{N} = 2$ supergravity in $d = 4,\ 5,\ 6$ and its matter couplings*, http://itf.fys.kuleuven.be/ toine/LectParis.pdf.

[12] N. Seiberg, *Naturalness versus supersymmetric nonrenormalization theorems*, Phys. Lett. **B 318** (1993) 469 [hep-ph/9309335] [INSPIRE].

[13] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, JHEP **06** (2011) 114 [arXiv:1105.0689] [INSPIRE].

[14] D.Z. Freedman and A.V. Proeyen, *Supergravity*, Cambridge University Press, New York, U.S.A. (2012).

[15] B. de Wit, J.W. van Holten and A. Van Proeyen, *Structure of $N = 2$ Supergravity*, *Nucl. Phys.* **B 184** (1981) 77 [INSPIRE].

[16] N. Hama and K. Hosomichi, *Seiberg-Witten theories on ellipsoids*, *JHEP* **09** (2012) 033 [arXiv:1206.6359] [INSPIRE].

[17] C. Klare and A. Zaffaroni, *Extended supersymmetry on curved spaces*, *JHEP* **10** (2013) 218 [arXiv:1308.1102] [INSPIRE].

[18] S. Ferrara and P. van Nieuwenhuizen, *Consistent supergravity with complex spin 3/2 gauge fields*, *Phys. Rev. Lett.* **37** (1976) 1669 [INSPIRE].

[19] B. de Wit, J.W. van Holten and A. Van Proeyen, *Transformation Rules of $N = 2$ Supergravity Multiplets*, *Nucl. Phys.* **B 167** (1980) 186 [INSPIRE].

[20] B. de Wit and J.W. van Holten, *Multiplets of Linearized* SO(2) *Supergravity*, *Nucl. Phys.* **B 155** (1979) 530 [INSPIRE].

[21] B. de Wit, R. Philippe and A. Van Proeyen, *The Improved Tensor Multiplet in $N = 2$ Supergravity*, *Nucl. Phys.* **B 219** (1983) 143 [INSPIRE].

[22] M. de Roo, J.W. van Holten, B. de Wit and A. Van Proeyen, *Chiral Superfields in $N = 2$ Supergravity*, *Nucl. Phys.* **B 173** (1980) 175 [INSPIRE].

[23] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082] [INSPIRE].

[24] D. Butter and S.M. Kuzenko, *New higher-derivative couplings in 4D $N = 2$ supergravity*, *JHEP* **03** (2011) 047 [arXiv:1012.5153] [INSPIRE].

[25] B. de Wit, S. Katmadas and M. van Zalk, *New supersymmetric higher-derivative couplings: full $N = 2$ superspace does not count!*, *JHEP* **01** (2011) 007 [arXiv:1010.2150] [INSPIRE].

[26] D. Butter, B. de Wit, S.M. Kuzenko and I. Lodato, *New higher-derivative invariants in $N = 2$ supergravity and the Gauss-Bonnet term*, *JHEP* **12** (2013) 062 [arXiv:1307.6546] [INSPIRE].

[27] S.M. Kuzenko, *Super-Weyl anomalies in $N = 2$ supergravity and (non)local effective actions*, *JHEP* **10** (2013) 151 [arXiv:1307.7586] [INSPIRE].

[28] D. Butter, B. de Wit and I. Lodato, *Non-renormalization theorems and $N = 2$ supersymmetric backgrounds*, *JHEP* **03** (2014) 131 [arXiv:1401.6591] [INSPIRE].

JHEP04(2015)169