

KÄHLER–RICCI SOLITONS ON COMPACT COMPLEX MANIFOLDS WITH $C_1(M) > 0$

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Abstract. In this paper, we discuss the relation between the existence of Kähler–Ricci solitons and a certain functional associated to some complex Monge–Ampère equation on compact complex manifolds with positive first Chern class. In particular, we obtain a strong inequality of Moser–Trudinger type on a compact complex manifold admitting a Kähler–Ricci soliton.

0 Introduction

In this paper, we study the existence of Kähler–Ricci solitons by using properness of a certain functional. Our approach is similar to that of [T] (also see [TZ1]) for Kähler–Einstein metrics. A Kähler metric g on a compact complex manifold M with first Chern class $c_1(M) > 0$ is called a (homothetically shrinking) Kähler–Ricci soliton if there is a holomorphic vector field X on M such that the Kähler form ω_g of g satisfies

$$\text{Ric}(\omega_g) - \omega_g = L_X \omega_g,$$

where $\text{Ric}(\omega_g)$ denotes the Ricci form of ω_g and L_X is the Lie derivative operator along X . In particular, if $X = 0$, such a g is a Kähler–Einstein metric. So Kähler–Ricci solitons can be regarded as a generalization of Kähler–Einstein metrics of positive scalar curvature. Ricci solitons have been studied extensively in the recent years ([K], [H], [C], [M2], [T], [TZ2,3], [WZ], etc.). One motivation is that they are very closely related to the singular behavior of limit solutions of certain PDEs which arise from geometric analysis, such as Hamilton’s Ricci flow [H] and certain complex Monge–Ampère equations associated to Kähler–Einstein metrics [T]. Kähler–Ricci solitons are special Ricci solitons. It was proved recently in [TZ2] and [TZ3] that there exists at most one Kähler–Ricci soliton on any compact

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Kähler manifold with positive first Chern class, modulo holomorphic automorphisms. This extends the uniqueness theorem of Bando and Mabuchi for Kähler–Einstein metrics with positive scalar curvature [BM].

Let g be a Kähler metric on M with its Kähler form

$$\omega_g = \frac{\sqrt{-1}}{2\pi} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

representing $c_1(M)$. Then there is a smooth real-valued function h_g such that

$$\text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}h_g.$$

Suppose that X is a holomorphic vector field on M so that the integral curve K_X of the imaginary part $\text{Im}(X)$ of X consists of isometries of g . By the Hodge decomposition theorem, there exists a unique smooth real-valued function $\theta_X = \theta_X(\omega_g)$ on M such that

$$\begin{cases} i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_X, \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n = V. \end{cases}$$

Set

$$\mathcal{M}_X(\omega_g) = \left\{ \psi \in C^\infty(M) \mid \omega_\psi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi > 0, \text{Im}(X)(\psi) = 0 \right\}.$$

The following functional on $\mathcal{M}_X(\omega_g)$ was introduced in [TZ3],

$$\tilde{F}_{\omega_g}(\psi) = \tilde{J}_{\omega_g}(\psi) - \frac{1}{V} \int_M \psi e^{\theta_X} \omega_g^n - \log \left(\frac{1}{V} \int_M e^{h_g - \psi} \omega_g^n \right),$$

where

$$\begin{aligned} \tilde{J}_{\omega_g}(\psi) = \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 \int_0^1 t(st)^k (1-st)^{n-1-k} e^{\theta_X + stX(\psi)} dt ds \right) \\ \times \partial\psi \wedge \bar{\partial}\psi \wedge \omega_\psi^k \wedge \omega_g^{n-1-k}. \end{aligned}$$

Let $K_0(\supseteq K_X)$ be a maximal compact subgroup of the automorphisms group of M such that $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\eta \in K_0$ and any $\sigma \in K_X$. Then a Kähler–Ricci soliton with respect to X on M must be K_0 -invariant by the uniqueness theorem in [TZ2]. We introduce

DEFINITION 0.1. *The functional $\tilde{F}_{\omega_g}(\cdot)$ is said to be proper with respect to X if for any sequence $\{\psi_i\}$ of K_0 -invariant functions in $\mathcal{M}_X(\omega_g)$,*

$$\overline{\lim}_{i \rightarrow \infty} \tilde{F}_{\omega_g}(\psi_i) = +\infty,$$

whenever $\lim_{i \rightarrow \infty} I_{\omega_g}(\psi_i) = +\infty$. Here

$$I_{\omega_g}(\psi_i) = \frac{1}{V} \int_M \psi_i (\omega_g^n - \omega_{\psi_i}^n).$$

We note that the above definition is independent of the choice of ω_g since functional $\tilde{F}_{\omega_g}(\cdot)$ satisfies a co-cycle condition (cf. section 1). The following was essentially proved in [TZ2].

Theorem 0.1 [TZ2]. *Suppose that $\tilde{F}_{\omega_g}(\cdot)$ is proper with respect to a holomorphic vector field X on M . Then there is a Kähler–Ricci soliton with respect to X on M .*

This theorem says that the properness of $\tilde{F}_{\omega_g}(\cdot)$ is a sufficient condition for the existence of Kähler–Ricci solitons. Since we did not state this result as a theorem in [TZ2], we will include its proof here following [TZ2].

In [TZ3], we proved that $\tilde{F}_{\omega_g}(\cdot)$ is bounded from below if there is a Kähler–Ricci soliton on M . The main purpose of this paper is to show that $\tilde{F}_{\omega_g}(\cdot)$ is actually proper wherever M admits a Kähler–Ricci soliton. This shows that $\tilde{F}_{\omega_g}(\cdot)$ is also a necessary condition for the existence of Kähler–Ricci solitons (cf. section 5). Suppose that M admits a Kähler–Ricci soliton ω_{KS} with respect to some holomorphic vector field X . We define a weighted inner product on $C^\infty(M)$ by

$$(\phi, \psi) = \int_M \phi \psi e^{\theta_X(\omega_{KS})} \omega_{KS}^n,$$

and denote by

$$\Lambda_1(M, \omega_{KS}) = \{u \in C^\infty(M) \mid \Delta_{\omega_{KS}} u + X(u) = -u\}.$$

Theorem 0.2. *Let M be a compact complex manifold which admits a Kähler–Ricci soliton ω_{KS} with respect to some holomorphic vector field X and $G(\supseteq K_X)$ be a compact subgroup of K_0 with $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\sigma \in K_X$ and $\eta \in G$. Suppose that any G -invariant smooth function on M is perpendicular to the space $\Lambda_1(M, \omega_{KS})$ with respect to the weighted inner product defined above. Then there are two positive numbers c and C such that for any G -invariant ψ in $\mathcal{M}_X(\omega_g)$,*

$$\tilde{F}_{\omega_{KS}}(\psi) \geq c I_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - C. \tag{0.1}$$

In particular, $\tilde{F}_{\omega_{KS}}(\cdot)$ is proper under the same assumption.

The proof of Theorem 0.2 is inspired by [T] and [TZ1], where the authors proved a fully non-linear inequality of Moser–Trudinger type for Kähler–Einstein manifolds. In fact, inequality (0.1) in Theorem 0.2 is equivalent to the following inequality,

$$\int_M e^{-\psi} \omega_{KS}^n \leq C \exp \left\{ \tilde{J}_{\omega_{KS}}(\psi) - c \tilde{J}_{\omega_{KS}}(\psi)^{\frac{1}{4n+5}} - \frac{1}{V} \int_M \psi \omega_{KS}^n \right\}. \tag{0.2}$$

Inequality (0.2) generalizes the corresponding one in [T] and [TZ1]. Recall that on S^2 with the standard Riemannian metric such an inequality is just a kind of the stronger version of Moser–Trudinger inequality [A], [T].

The organization of this paper is as follows. In section 1, we give a proof of Theorem 0.1 following [TZ2]. In section 2, we give an estimate for a certain heat kernel on manifolds with modified positive Ricci curvature. In section 3, a C^0 -estimate for certain complex Monge–Ampère equations is obtained. In section 4, we prove a smoothing lemma by using Hamilton’s Ricci flow. Theorem 0.2 will be proved in section 5.

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1 An Analytic Criterion for Kähler–Ricci Solitons

Let (M, g) be an n -dimensional compact Kähler manifold with the first Chern class $c_1(M) > 0$. Denote by $\text{Aut}^\circ(M)$ the connected component of the automorphism group of M . Let K be a maximal compact subgroup of $\text{Aut}^\circ(M)$, then the Chevalley decomposition allows us to write $\text{Aut}^\circ(M)$ as a semi-direct decomposition [FM],

$$\text{Aut}^\circ(M) = \text{Aut}_r(M) \ltimes R_u,$$

where $\text{Aut}_r(M) \subset \text{Aut}^\circ(M)$ is a reductive algebraic subgroup and the complexification of K , and R_u is the unipotent radical of $\text{Aut}^\circ(M)$. Let $\eta_r(M)$ be the Lie subalgebra of $\text{Aut}_r(M)$.

Let $X \in \eta_r(M)$ such that the one-parameter subgroup K_X generated by $\text{Im}(X)$ is contained in K , where $\text{Im}(X)$ denotes the imaginary part of X . Choose a K_X -invariant Kähler metric g on M with the Kähler form ω_g . Then by the Hodge decomposition theorem, there is a unique smooth real-valued function $\theta_X = \theta_X(\omega_g)$ of M such that

$$\begin{cases} i_X \omega_g = \frac{\sqrt{-1}}{2\pi} \bar{\partial} \theta_X, \\ \int_M e^{\theta_X} \omega_g^n = \int_M \omega_g^n = V, \end{cases}$$

where $\omega_g^n = \omega_g \wedge \dots \wedge \omega_g$. The first relation above implies

$$L_X \omega_g = di_X(\omega_g) = \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \theta_X.$$

We consider the following complex Monge–Ampère equations with parameter $t \in [0, 1]$:

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp \{h - \theta_X - X(\varphi) - t\varphi\}, \\ (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0, \end{cases} \quad (1.1)$$

where $h = h_g$ is a smooth real-valued function on M defined by

$$\begin{cases} \text{Ric}(\omega_g) - \omega_g = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}h, \\ \int_M e^h \omega_g^n = \int_M \omega_g^n. \end{cases}$$

Then one can check that $\omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi$ is a Kähler–Ricci soliton with respect to X , i.e. ω_ϕ satisfies

$$\text{Ric}(\omega_\phi) - \omega_\phi = L_X \omega_\phi,$$

if and only if ϕ modulo a constant is a solution of equation (1.1) _{t} for $t = 1$.

In fact, (1.1) _{t} is equivalent to

$$\text{Ric}(\omega_{\phi_t}) - L_X(\omega_{\phi_t}) = t\omega_{\phi_t} + (1 - t)\omega_g, \tag{1.2}$$

where ϕ_t is a solution of (1.1) _{t} .

Since

$$\frac{d}{dt} \left(\int_M e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n \right) = \int_M (\Delta' \dot{\phi}_t + X(\dot{\phi}_t)) e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n = 0,$$

we have

$$\int_M e^{\theta_X + X(\phi_t)} \omega_{\phi_t}^n = \int_M e^{\theta_X} \omega_g^n = V. \tag{1.3}$$

Integrating (1.1) _{t} after multiplying by $e^{\theta_X + X(\phi_t)}$, it follows that

$$\int_M e^{h - t\phi_t} \omega_g^n = V.$$

Therefore, differentiating the above identity, we get

$$\int_M \dot{\phi}_t e^{h - t\phi_t} \omega_g^n = -\frac{1}{t} \int_M \phi_t e^{h - t\phi_t} \omega_g^n. \tag{1.4}$$

Set

$$\mathcal{M}(\omega_g) = \left\{ \phi \in C^\infty(M) \mid \omega_\phi = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\phi > 0 \right\}$$

and

$$\mathcal{M}_X(\omega_g) = \{ \phi \in \mathcal{M}(\omega_g) \mid \text{Im}(X)(\phi) = 0 \}.$$

We define the following two functionals on $\mathcal{M}_X(\omega_g)$ (cf. [TZ2]):

$$\tilde{I}_{\omega_g}(\phi) = \frac{1}{V} \int_M \phi (e^{\theta_X} \omega_g^n - e^{\theta_X + X(\phi)} \omega_\phi^n)$$

and

$$\tilde{J}_{\omega_g}(\phi) = \frac{1}{V} \int_0^1 \int_M \dot{\phi}_s (e^{\theta_X} \omega_g^n - e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n) \wedge ds, \tag{1.5}$$

where $\phi_s (0 \leq s \leq 1)$ is a path from 0 to ϕ in $\mathcal{M}_X(\omega_g)$. Since $\tilde{J}_{\omega_g}(\phi)$ is independent of the choice of a path, by choosing $\phi_s = s\phi$, one can show that $\tilde{J}_{\omega_g}(\phi)$ defined here coincides with one in the introduction. It is known that there are two uniform positive constants c_1 and c_2 such that

$$c_1 I_{\omega_g}(\phi) \leq \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \leq c_2 I_{\omega_g}(\phi), \tag{1.6}$$

where $I_{\omega_g}(\phi) = \frac{1}{V} \int_M \phi(\omega_g^n - \omega_\phi^n)$.

Let

$$\begin{aligned} \hat{F}_{\omega_g}(\phi) &= \tilde{J}_{\omega_g}(\phi) - \frac{1}{V} \int_M \phi e^{\theta_X} \omega_g^n \\ &= -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_s e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \wedge ds. \end{aligned}$$

By simple computations, one can show that for any two functions ϕ and ψ in $\mathcal{M}_X(\omega_g)$, the following co-cycle condition is satisfied,

$$\hat{F}_{\omega_g}(\psi) = \hat{F}_{\omega_g}(\phi) + \hat{F}_{\omega_\phi}(\psi - \phi),$$

where $\hat{F}_{\omega_\phi}(\psi - \phi) = -\frac{1}{V} \int_0^1 \int_M \dot{\phi}_s e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \wedge ds$ and ϕ_s is a path from 0 to $(\psi - \phi)$ in $\mathcal{M}_X(\omega_g)$.

PROPOSITION 1.1. *Let ϕ_s be a solution of (1.1)_s for all $s \leq t$. Then*

$$\hat{F}_{\omega_g}(\phi_t) = -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds.$$

Proof. Differentiating (1.1)_s on s , we have

$$\Delta' \dot{\phi}_s + X(\dot{\phi}_s) = -(s\dot{\phi}_s + \phi_s). \quad (1.7)$$

Then by using (1.7) and (1.4), we get

$$\begin{aligned} \frac{d}{ds} (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) &= -\frac{1}{V} \int_M \phi_s \frac{d}{ds} (e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n) \\ &= -\frac{1}{V} \int_M \phi_s (\Delta' \dot{\phi}_s + X(\dot{\phi}_s)) e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \\ &= \frac{1}{V} \int_M \phi_s (s\dot{\phi}_s + \phi_s) e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \\ &= \frac{1}{V} \int_M \phi_s (s\dot{\phi}_s + \phi_s) e^{h - s\phi_s} \omega_g^n \\ &= \frac{1}{V} \frac{d}{ds} \left(\int_M (-\phi_s) e^{h - s\phi_s} \omega_g^n \right) + \frac{1}{V} \int_M \dot{\phi}_s e^{h - s\phi_s} \omega_g^n \\ &= \frac{1}{sV} \frac{d}{ds} \left(\int_M s(-\phi_s) e^{h - s\phi_s} \omega_g^n \right) \\ &= \frac{1}{sV} \frac{d}{ds} \left(\int_M s(-\phi_s) e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \right). \end{aligned}$$

It follows that

$$\begin{aligned} \frac{d}{ds} (s(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s))) - (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) \\ = \frac{1}{V} \frac{d}{ds} \left(\int_M s(-\phi_s) e^{\theta_X + X(\phi_s)} \omega_{\phi_s}^n \right), \quad (1.8) \end{aligned}$$

and consequently,

$$\begin{aligned} \hat{F}_{\omega_g}(\phi_t) &= -\frac{1}{V} \int_M \phi e^{\theta_X + X(\phi_t)} \omega_t^n - (\tilde{I}_{\omega_g}(\phi_t) - \tilde{J}_{\omega_g}(\phi_t)) \\ &= -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds. \end{aligned} \quad \square$$

REMARK 1.1. It was proved in Lemma 3.2 in [TZ2] that

$$\frac{d}{ds}(\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) \geq 0.$$

Recall that the functional $\tilde{F}_{\omega_g}(\cdot)$ is defined as

$$\begin{aligned} \tilde{F}_{\omega_g}(\psi) &= \hat{F}_{\omega_g}(\psi) - \log\left(\frac{1}{V} \int_M e^{h-\psi} \omega_g^n\right) \\ &= \tilde{J}_{\omega_g}(\psi) - \frac{1}{V} \int_M \psi e^{\theta_X} \omega_g^n - \log\left(\frac{1}{V} \int_M e^{h-\psi} \omega_g^n\right), \end{aligned}$$

and $\tilde{F}_{\omega_g}(\cdot)$ is called proper with respect to X if for any sequence $\{\psi_i\}$ of K_0 -invariant functions in $\mathcal{M}_X(\omega_g)$,

$$\overline{\lim}_{i \rightarrow \infty} \tilde{F}_{\omega_g}(\psi_i) = +\infty,$$

whenever $\lim_{i \rightarrow \infty} I_{\omega_g}(\psi_i) = +\infty$, where $K_0(\supseteq K_X)$ is a maximal compact subgroup of automorphisms group $\text{Aut}(M)$ of M such that $\sigma \cdot \eta = \eta \cdot \sigma$ for any $\sigma \in K_X$ and $\eta \in K_0$.

Proof of Theorem 0.1. We use arguments from [TZ2]. Assuming the functional \tilde{F}_{ω_g} is proper, we shall prove the existence of a Kähler-Ricci soliton with respect to X on M . This is equivalent to proving that there is a solution of (1.1)_t for $t = 1$. It suffices to prove that $I_{\omega_g}(\phi_t)$ is uniformly bounded for any solution of (1.1)_t for $0 \leq t \leq 1$. This is because C^3 -norm of ϕ_t can be uniformly bounded by $I_{\omega_g}(\phi_t)$ and the set of parameter t for which there exists a smooth solution of (1.1)_t is non-empty and open (cf. [TZ2]). By the implicit function theorem, one can show the solution of (1.1)_t varies smoothly with $t < 1$. Without the loss of generality, we may assume that the Kähler form ω_g is K_0 -invariant. Then all solutions ϕ_t are all K_0 -invariant.

By Proposition 1.1, we have

$$\begin{aligned} \tilde{F}_{\omega_g}(\phi_t) &= -\frac{1}{t} \int_0^t (\tilde{I}_{\omega_g}(\phi_s) - \tilde{J}_{\omega_g}(\phi_s)) ds - \log\left(\frac{1}{V} \int_M e^{h-\phi_t} \omega_g^n\right) \\ &\leq -\log\left(\frac{1}{V} \int_M e^{h-\phi_t} \omega_g^n\right). \end{aligned} \quad (1.9)$$

On the other hand, by using (1.1)_t and concavity of the logarithmic function, one can deduce

$$\begin{aligned} -\log\left(\frac{1}{V}\int_M e^{h-\phi_t}\omega_g^n\right) &= \frac{1-t}{V}\int_M \phi_t e^{\theta_X+X(\phi_t)}\omega_{\phi_t}^n \\ &\leq \frac{1-t}{V}\int_M \phi_t e^{h-t\phi_t}\omega_g^n \leq C. \end{aligned} \quad (1.10)$$

Combining (1.9) and (1.10), we get

$$\tilde{F}_{\omega_g}(\phi_t) \leq C.$$

Therefore, the assumption of properness on $\tilde{F}_{\omega_g}(\cdot)$ implies

$$I_{\omega_g}(\phi_t) \leq C',$$

and consequently, there is a Kähler–Ricci soliton on M with respect to X . \square

2 A Heat Kernel Estimate

In this section, we give an estimate on the heat kernel of a linear elliptic operator P associated to a Kähler form ω and a holomorphic vector field X on M , where $P = P_\omega = \Delta + X(\cdot)$ is defined on the space $\mathcal{N}_X = \{u \in C^\infty(M) \mid \text{Im}(X(u)) = 0\}$. As a consequence, we derive a lower bound of the Green function of P . The method here follows that of T. Mabuchi in [M1] with modifications which in turn were inspired by Li–Yau [LY]. Note that P is a self-adjoint elliptic operator on \mathcal{N}_X with respect to the inner product,

$$(\phi, \psi) = \int_M \phi \psi e^{\theta_X} \omega^n.$$

LEMMA 2.1. *Let ω be a Kähler form on M and X a holomorphic vector field on M with*

$$\text{Ric}(\omega) - L_X \omega \geq 0,$$

and

$$\Delta \theta_X \leq k,$$

for some positive number k , where $\theta_X = \theta_X(\omega)$ is defined as in section 1 for Kähler form ω . Let $v(x, t)$ be a positive smooth solution on $M \times (0, \infty)$ of equation $(P - \frac{\partial}{\partial t})v = 0$. Suppose that

$$\limsup_{t \rightarrow 0} \sup_{M \times \{t\}} t(v^{-2} \langle \bar{\partial} v, \bar{\partial} v \rangle - v^{-1} v_t) \leq 2n,$$

where $v_t = \partial v / \partial t$. Then there is a positive number C depending only on

$$m_1 = -\max_M \theta_X \text{ and } m_2 = -\min_M \theta_X$$

such that

$$v(x, t_1) \leq v(y, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{n}{C}} \exp \left\{ (t_2 - t_1)^{-1} r(x, y)^2 / 2 + C^{-1} k(t_2 - t_1) \right\}, \quad \forall t_1 < t_2, \tag{2.1}$$

where $r(x, y)$ denotes the distance between x and y associated to metric ω .

Proof. Let $f = \ln v$ and $\hat{F} = t(\langle \bar{\partial}f, \bar{\partial}f \rangle - f_t)$. Then

$$\begin{aligned} P\hat{F} + tPf_t &= t(\langle \text{tr}_\omega(\bar{\nabla}\nabla)\bar{\partial}f, \bar{\partial}f \rangle + \langle \text{tr}_\omega(\bar{\nabla}\nabla)\partial f, \partial f \rangle) \\ &\quad + X(\langle \bar{\partial}f, \bar{\partial}f \rangle) + \|\nabla\bar{\nabla}f\|^2 + \|\nabla\nabla f\|^2. \end{aligned} \tag{2.2}$$

For simplicity, we choose a local holomorphic coordinate system (z_1, \dots, z_n) near each point p such that $g_{i\bar{j}}(p) = \delta_{ij}$ and $f_{i\bar{j}}(p) = \delta_{ij} f_{i\bar{i}}(p)$. By a direct computation, one sees that

$$\begin{aligned} f_{\bar{i}}f_{ij\bar{j}} &= f_{\bar{i}}(f_{j\bar{j}i} + f_l R_{i\bar{l}}), \\ f_i f_{i\bar{j}\bar{j}} &= f_i f_{j\bar{j}\bar{i}}, \end{aligned}$$

and

$$X_{\bar{i}}(f_j f_{\bar{j}})_i + f_{\bar{i}} f_j X_{\bar{j}i} = f_i (X_{\bar{j}} f_j)_{\bar{i}} + f_{\bar{i}} (X_{\bar{j}} f_j)_i.$$

Inserting the above identities into (2.2), we obtain

$$\begin{aligned} P\hat{F} &\geq t(\langle \partial f, \partial(Pf) \rangle + \langle \partial(Pf), \partial f \rangle) - tPf_t + \frac{t}{n}(\Delta f)^2 \\ &= 2t\langle \partial(Pf), \partial f \rangle - tPf_t + \frac{t}{n}(\Delta f)^2. \end{aligned} \tag{2.3}$$

Since

$$Pf - f_t = -\langle \bar{\partial}f, \bar{\partial}f \rangle,$$

we have

$$\hat{F} = -tPf,$$

and

$$\frac{\partial}{\partial t}\hat{F} - \frac{1}{t}\hat{F} = -tPf_t.$$

Hence by (2.3), we get

$$\left(P - \frac{\partial}{\partial t}\right)\hat{F} \geq -2\langle \partial\hat{F}, \partial f \rangle - \frac{1}{t}\hat{F} + \frac{t}{n}\left(\frac{1}{t}\hat{F} + X(f)\right)^2. \tag{2.4}$$

Let $m_1 = -\sup_{x \in M} \theta_X(x)$ and $m_2 = -\inf_{x \in M} \theta_X(x)$. As in [M1], we define a monotone-increasing function η on $[m_1, m_2]$ by

$$\eta(s) = \exp \int_m^s \frac{1}{b_0 e^{-y} - 1} dy,$$

where $b_0 = e^{m_2}(1+n)$. Then η is a solution of the ODE on $[m_1, m_2]$,

$$\frac{\eta''}{\eta} - \frac{\eta'}{\eta} = 2\left(\frac{\eta'}{\eta}\right)^2.$$

Moreover, one can check that the number C defined by

$$C = \min_{s \in [m_1, m_2]} \eta(s)^{-1} (1 - (1 - \eta(s)^{-1} \eta(s)' n)^2)$$

is positive, and

$$0 < \frac{\eta'}{\eta} \leq \frac{1}{n}.$$

Let $F = \eta(-\theta_X)\hat{F}$. Then by (2.4), one can show that

$$\begin{aligned} (P - \frac{\partial}{\partial t})F &\geq -2\langle \partial F, \partial f \rangle - \frac{1}{t}F \\ &\quad + \left(\frac{\eta''}{\eta} - \frac{\eta'}{\eta} - 2\left(\frac{\eta'}{\eta}\right)^2 \right) \|X\|^2 F \\ &\quad + \eta^{-1}\eta'(-2X(F) - F\Delta\theta_X) \\ &\quad + n^{-1}t^{-1}F^2\eta^{-1}(1 - (1 - \eta^{-1}\eta'n)^2) \\ &\quad + \frac{t}{n}\eta \left[-\frac{1}{t}\eta^{-1}F(1 - n\eta^{-1}\eta')^2 - X(f) \right]^2. \end{aligned}$$

By the assumption in the lemma, we have

$$(P - \frac{\partial}{\partial t})F \geq -2\langle \partial F, \partial f \rangle - 2\eta^{-1}\eta'X(F) - \left(\frac{1}{t} + \frac{k}{n}\right)F + \frac{C}{nt}F^2. \tag{2.5}$$

Applying the maximal principle to the function $F(x, t)$ on $M \times (0, T]$, we get from (2.5),

$$F(x, t) \leq \tilde{C}^{-1}(2n + kt),$$

for any $(x, t) \in M \times (0, T]$, and consequently,

$$v(x, t)^{-2}\langle \bar{\partial}v(x, t), \bar{\partial}v(x, t) \rangle - v(x, t)^{-1}v_t(x, t) \leq \tilde{C}^{-1}\left(\frac{2n}{t} + k\right),$$

for any $(x, t) \in M \times (0, \infty)$. Now by integrating the above estimate as in [LY], we can immediately obtain (2.1). \square

Let $H = H(x, y, t)$ be a fundamental solution on $M \times M \times [0, \infty)$ of equation

$$(P - \frac{\partial}{\partial t})v(x, y, t) = 0, \tag{2.6}$$

i.e. H is a smooth solution of (2.6) satisfying

$$\begin{cases} H(x, y, t) = H(y, x, t), \\ H(x, y, t) = \int_M H(x, z, t - s)H(z, y, s)e^{\theta_x}\omega^n, \\ \lim_{t \rightarrow 0} H(x, y, t) = \delta_x(y). \end{cases}$$

By using the asymptotic behavior of H , for each fixed $x \in M$, one sees that

$$\lim_{t \rightarrow 0} \sup_{M \times \{t\}} t(H^{-2}\langle \bar{\partial}H(x, \cdot, t), \bar{\partial}H(x, \cdot, t) \rangle - H^{-1}H_t) \leq 2n.$$

Moreover, following an argument by Li and Yau (cf. [LY, Lemma 3.2]), we can deduce

LEMMA 2.2. *Let Z_1 and Z_2 be two measurable subset of M . Let T, δ, τ be three positive numbers with $\tau < (1 + 2\delta)T$. For each $x, y \in M$ and*

$0 < t \leq \tau$, denote

$$F_{x,T}(y, t) = \int_{Z_1} H(y, \cdot, t)H(x, \cdot, T)e^{\theta x} \omega^n.$$

Then

$$\int_{Z_2} F_{x,T}(\cdot, t)^2 e^{\theta x} \omega^n \leq \exp \left\{ \frac{-r(x, Z_1)^2}{2(1+2\delta)T} + \frac{R(x, Z_2)^2}{2(1+2\delta)T - 2t} \right\} F_{x,T}(x, T),$$

where $r(x, Z_1) = \inf_{z \in Z_1} r(x, z)$ and $R(x, Z_2) = \sup_{z \in Z_2} r(x, z)$.

PROPOSITION 2.1. Let $H(x, y, t)$ be a fundamental solution on $M \times M \times [0, \infty)$ of equation (2.6). Suppose

$$\text{Ric}(\omega) - L_X \omega \geq 0,$$

and

$$\Delta \theta_X \leq k,$$

for some positive number k . Then for any $\delta > 0$ we have

$$\begin{aligned} & \widetilde{\text{vol}}(B_x(\sqrt{t}))^{1/2} \widetilde{\text{vol}}(B_y(\sqrt{t}))^{1/2} H(x, y, t) \\ & \leq (1+\delta)^{4n/C} \exp \left\{ -\frac{\{r(x, y) - \sqrt{t}\}_+^2}{4t(1+3\delta+2\delta^2)} + t\delta(2+\delta)C^{-1}k + \frac{3}{4\delta(1+\delta)} + \frac{1}{2\delta} \right\}, \end{aligned} \tag{2.7}$$

where $\widetilde{\text{vol}}(B_y(\sqrt{t})) = \int_{B_y(\sqrt{t})} e^{\theta x} \omega^n$, and

$$\{r(x, y) - \sqrt{t}\}_+ = \max \{0, r(x, y) - \sqrt{t}\}.$$

Proof. First applying Lemma 2.1 to the function $F_{x,T}(y, t)$ with $(t_1, t_2) = (T, \tau = (1 + \delta)T)$, we have

$$F_{x,T}(x, T) \leq F_{x,T}(y, \tau)(1 + \delta)^{\frac{2n}{C}} \exp \left\{ \frac{T^{-1}\delta^{-1}r(x, y)^2}{2} + C^{-1}kT\delta \right\}. \tag{2.8}$$

Let $Z_1 = B_y(\sqrt{t})$ and $Z_2 = B_x(\sqrt{t})$. Then integrating the square of the above inequality over all $y \in Z_2$ and using Lemma 2.2, it follows that

$$\begin{aligned} & \widetilde{\text{vol}}(B_x(\sqrt{t})) F_{x,T}(x, T)^2 \\ & \leq (1 + \delta)^{4n/C} \exp \left\{ \frac{-r(x, Z_1)^2}{2(1+2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta} \right\} F_{x,T}(x, T), \end{aligned}$$

and consequently,

$$\begin{aligned} & \widetilde{\text{vol}}(B_x(\sqrt{t})) \int_{Z_1} H(x, \cdot, T)^2 e^{\theta x} \omega^n \\ & = \widetilde{\text{vol}}(B_x(\sqrt{t})) F_{x,T}(x, T) \\ & \leq (1 + \delta)^{4n/C} \exp \left\{ \frac{-r(x, Z_1)^2}{2(1+2\delta)T} + 2C^{-1}kT\delta + \frac{3t}{2T\delta} \right\}. \end{aligned} \tag{2.9}$$

On the other hand, applying Lemma 2.1 to the function $H(x, z, t)$ in z with $(t_1, t_2) = (t, T = (1 + \delta)t)$, we have for any $x, y, z \in M$,

$$H(x, y, t)^2 \leq (1 + \delta)^{4n/C} H(x, z, T)^2 \exp \{T^{-1} \delta^{-1} r(y, z)^2 + 2\tilde{C}^{-1} kT\delta\}.$$

Integrating this inequality over all $z \in Z_1$, and using (2.9), we can get (2.7). □

Theorem 2.1. *Let $\phi \in \mathcal{M}_X(\omega_g)$. Suppose that*

$$\text{Ric}(\omega_\phi) - L_X \omega_\phi \geq \lambda \omega_\phi, \tag{2.10}$$

and

$$\Delta \theta_X(\omega_\phi) \leq k,$$

for some positive numbers λ and k . Then there is a uniform constant C depending only on λ and k such that

$$\sup_M (-\phi) \leq \frac{1}{V} \int_M (-\phi) e^{\theta_X(\omega_\phi)} \omega_\phi^n + C. \tag{2.11}$$

Proof. Let $\mu_i (\mu_0 = 0), i = 0, 1, \dots$, be the increasing sequence of eigenvalues of operator $-P = -(\Delta_{\omega_\phi} + X(\cdot))$ associated to metric ω_ϕ . Then by using the standard Bochner technique, one can obtain $\mu_1 \geq \lambda$ (cf. [TZ2]). Let $G(x, y)$ be the Green function with $\int_M G(x, \cdot) e^{\theta_X(\omega_\phi)} \omega_\phi^n = 0$ associated to the operator P . Then

$$G(x, y) = \int_0^\infty (H(x, y, t) - \frac{1}{V}) dt.$$

Since

$$H_0(x, y, t) = H(x, y, t) - \frac{1}{V} = \sum_{i=1}^\infty e^{-\mu_i t} f_i(x) f_i(y),$$

we have

$$H_0(x, x, t + t_0) \leq e^{-\mu_1 t} H_0(x, x, t_0), \tag{2.12}$$

for any $t_0, t > 0$, where $f_i(x)$ denote the eigenfunctions of μ_i .

In [M2], it was proved under the condition (2.10) that there is a uniform constant C_1 such that

$$\text{Diam}(M, \omega_\phi) \leq \frac{C_1}{\sqrt{\lambda}}.$$

Choose $t_0 = \frac{1}{4} \text{Diam}(M, \omega_\phi)^2$. Then by Proposition 2.1 we have

$$H_0(x, x, t_0) \leq C_2, \tag{2.13}$$

for some uniform constant C_2 depending only on λ and k .

By using (2.12) and (2.13), we get

$$\begin{aligned} G(x, y) &\geq - \int_0^{t_0} \frac{1}{V} dt - \int_{t_0}^\infty e^{-\mu_1(t-t_0)} (H_0(x, x, t_0) H_0(y, y, t_0))^{1/2} dt \\ &\geq -C_3. \end{aligned}$$

Note that $\Delta_{\omega_\phi}(-\phi) \geq -n$ and

$$\sup_{\psi \in \mathcal{M}_X(\omega_g)} \|X(\psi)\|_{C^0(M)} \leq c \tag{2.14}$$

for some uniform constant $c = c(\omega_g, X)$ (cf. Lemma 5.1 in [Z]). Therefore applying the Green formula to function $-\phi$, we prove

$$\begin{aligned} \sup_M(-\phi) &= \frac{1}{V} \int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n - \inf_M \int_M P(-\phi(\cdot))G(x, \cdot)e^{\theta_X(\omega_\phi)}\omega_\phi^n \\ &= \frac{1}{V} \int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n - \inf_M \int_M (\Delta_{\omega_\phi}(-\phi) - X(\phi))G(x, \cdot)e^{\theta_X(\omega_\phi)}\omega_\phi^n \\ &\leq \frac{1}{V} \int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n + C_3V(n + \|X(\phi)\|_{C^0(M)}) \\ &\leq \frac{1}{V} \int_M (-\phi)e^{\theta_X(\omega_\phi)}\omega_\phi^n + C. \end{aligned} \quad \square$$

3 A C^0 -Estimate for Solutions of (1.1)_t

LEMMA 3.1. *There are two positive numbers c_1 and $c_2 < 1$ such that for any $\phi \in \mathcal{M}_X(\omega_g)$,*

$$c_1\tilde{I}_{\omega_g}(\phi) \leq \tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \leq c_2\tilde{I}_{\omega_g}(\phi). \tag{3.1}$$

Proof. Let $\omega_{s\phi} = \omega + s\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\phi$. Then one can compute

$$\begin{aligned} \tilde{I}_{\omega_g}(\phi) &= \frac{1}{V} \int_M \phi \int_0^1 \frac{d}{ds} (e^{\theta_X+sX(\phi)}\omega_{s\phi}^n) \wedge ds \\ &= \frac{n\sqrt{-1}}{2\pi V} \int_0^1 ds \int_M \partial\phi \wedge \bar{\partial}\phi e^{\theta_X+sX(\phi)} \wedge \omega_{s\phi}^{n-1} \\ &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 s^k(1-s)^{n-1-k} e^{\theta_X+sX(\phi)} ds \right) \\ &\quad \times \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \\ &\leq \frac{n\sqrt{-1}}{2\pi V} C_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}. \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \tilde{J}_{\omega_g}(\phi) &= \frac{n\sqrt{-1}}{2\pi V} \int_0^1 dt \int_0^1 ds \int_M t\partial\phi \wedge \bar{\partial}\phi e^{\theta_X+stX(\phi)} \wedge \omega_{st\phi}^{n-1} \end{aligned}$$

$$\begin{aligned}
 &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 \int_0^1 t(st)^k (1-st)^{n-1-k} e^{\theta_X + stX(\phi)} dt \wedge ds \right) \\
 &\quad \times \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \tag{3.3} \\
 &\geq \frac{n\sqrt{-1}}{2\pi V} C'_1 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
 \end{aligned}$$

Combining (3.2) and (3.3), we get

$$\tilde{J}_\omega(\phi) \geq \frac{C'_1}{C_1} \tilde{I}_\omega(\phi),$$

and consequently, prove the second inequality of (3.1).

On the other hand, we have (cf. [TZ2]),

$$\begin{aligned}
 &\tilde{I}_{\omega_g}(\phi) - \tilde{J}_{\omega_g}(\phi) \\
 &= \frac{n\sqrt{-1}}{2\pi V} \sum_{k=0}^{n-1} C_{n-1}^k \int_M \left(\int_0^1 s^{k+1} (1-s)^{n-1-k} e^{\theta_X + sX(\phi)} ds \right) \\
 &\quad \times \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k} \tag{3.4} \\
 &\geq \frac{n\sqrt{-1}}{2\pi V} C_2 \sum_{k=0}^{n-1} C_{n-1}^k \partial\phi \wedge \bar{\partial}\phi \wedge \omega_\phi^k \wedge \omega_g^{n-1-k}.
 \end{aligned}$$

Hence, combining (3.2) and (3.4), we also prove the first inequality of (3.1). \square

PROPOSITION 3.1. *Let $\phi = \phi_t (t \geq t_0 > 0)$ be a solution of equation (1.1)_t. Then there are two uniform constants C_1 and C_2 depending only on X and t_0 such that*

$$\text{osc}_M \phi \leq C_1 \int_M \phi(\omega_g^n - \omega_\phi^n) + C_2. \tag{3.5}$$

Proof. Let $\theta'_X = \theta_X(\omega_\phi)$. First we note

$$\Delta\theta'_X = -\theta'_X - X(h_{\omega_\phi}) + c,$$

for some constant c . Clearly $c \leq \text{osc}_M |\theta'_X|$, since θ'_X changes the sign. By using the fact (cf. the relation (1.2)),

$$h_{\omega_\phi} = \theta'_X - (1-t)\phi + \text{const.},$$

we have

$$\begin{aligned}
 \Delta\theta'_X &= -\theta'_X - \|X\|_{\omega_\phi} + (1-t)X(\phi) + c \\
 &\leq 2 \text{osc}_M |\theta_X| + 3|X(\phi)| \leq C'_1,
 \end{aligned}$$

for some uniform constant C'_1 . Applying Theorem 2.1, we see that there is some uniform constant C'_2 depending only on X and t_0 such that

$$\sup_M (-\phi) \leq \frac{1}{V} \int_M (-\phi) e^{\theta'_X} \omega_\phi^n + C'_2. \tag{3.6}$$

On the other hand, by using the Green formula, we have

$$\sup_M \phi \leq \frac{1}{V} \int_M \phi e^{\theta_X} \omega_g^n + C'_3 \tag{3.7}$$

for some uniform constant C'_3 (cf. Lemma 5.3 in [TZ3]). Hence, combining (3.6) and (3.7), we get

$$\text{osc}_M \phi \leq \frac{1}{V} \int_M \phi (e^{\theta_X} \omega_g^n - e^{\theta'_X} \omega_\phi^n) + C'_2 + C'_3.$$

By using (1.6) and (3.1), we prove (3.5). □

4 A Smoothing Lemma

In this section, following an approach in [T], we will prove a smoothing lemma by using Hamilton’s Ricci flow. This lemma will be used in the proof of Theorem 0.2. Let ω be any Kähler form in $c_1(M) > 0$ such that

$$\begin{cases} \text{Ric}(\omega) - L_X \omega \geq (1 - \epsilon)\omega, \\ |X(h_\omega - \theta_X(\omega))| \leq \epsilon c_1, \end{cases} \tag{4.1}$$

for some constant c_1 and $0 < \epsilon < 1$. We consider the heat equation

$$\begin{cases} \frac{\partial u}{\partial t} = \log \left(\frac{(\omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u)^n}{\omega^n} \right) + u - h_\omega + \theta_X(\omega), \\ u|_{t=0} = 0. \end{cases} \tag{4.2}$$

Note that eq. (4.2) is the scalar version of the modified Kähler–Ricci flow

$$\frac{\partial}{\partial t} \omega_t = -\text{Ric}(\omega_t) + \omega_t + L_X \omega_t.$$

Here $\omega_t = \omega + \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} u_t$, and $u_t = u(x, \cdot)$. Denote by $h_t = h_{\omega_t}$ and $\theta_t = \theta_X(\omega_t)$, then it follows from the above equation and maximum principle that

$$h_t - \theta_t = -\frac{\partial u}{\partial t} + \tilde{c}_t$$

where \tilde{c}_t depends on t only. Also, $u_0 = 0$ and hence $\tilde{c}_0 = 0$.

We list a few basic estimates for the solution $u(x, t)$. Differentiating (4.2), we get

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right) = (\Delta + X) \left(\frac{\partial u}{\partial t} \right) + \left(\frac{\partial u}{\partial t} \right). \tag{4.3}$$

Applying the maximum principle, we have

LEMMA 4.1. *Let u_t be a solution of (4.2). Then*

$$\|u_t\|_{C^0} \leq e^t \|h_\omega - \theta_X(\omega)\|_{C^0},$$

and

$$\left\| \frac{\partial u}{\partial t} \right\|_{C^0} \leq e^t \|h_\omega - \theta_X(\omega)\|_{C^0}.$$

LEMMA 4.2.

$$(\Delta + X)(h_t - \theta_t) \geq -(c_1 + n)\epsilon e^t.$$

Proof. From eq. (4.3), we have

$$\begin{aligned} \frac{\partial}{\partial t} \left((\Delta + X) \left(\frac{\partial u}{\partial t} \right) \right) &= (\Delta + X)^2 \left(\frac{\partial u}{\partial t} \right) + (\Delta + X) \left(\frac{\partial u}{\partial t} \right) - \left| \nabla \bar{\nabla} \left(\frac{\partial u}{\partial t} \right) \right|^2 \\ &\leq (\Delta + X)^2 \left(\frac{\partial u}{\partial t} \right) + (\Delta + X) \left(\frac{\partial u}{\partial t} \right). \end{aligned}$$

It follows from the maximum principle that

$$(\Delta + X)(h_t - \theta_t) \geq e^t \inf_M (\Delta + X)(h_\omega - \theta_X(\omega)).$$

On the other hand, (4.1) implies that, at $t = 0$,

$$(\Delta + X)(h_\omega - \theta_X(\omega)) \geq -(c_1 + n)\epsilon.$$

Then the lemma follows directly. \square

LEMMA 4.3.

$$\left\| \frac{\partial u}{\partial t} \right\|_{C^0}^2 + t \left\| \nabla \left(\frac{\partial u}{\partial t} \right) \right\|_{C^0}^2 \leq e^{2t} \|h_\omega - \theta_X(\omega)\|_{C^0}^2.$$

Proof. By direct computations, we have

$$\frac{\partial}{\partial t} \left(\frac{\partial u}{\partial t} \right)^2 = (\Delta + X) \left(\frac{\partial u}{\partial t} \right)^2 - \left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 + 2 \left(\frac{\partial u}{\partial t} \right)^2,$$

and

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 \right) \\ &= (\Delta + X) \left(\left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 \right) - \left| \nabla \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 - \left| \nabla \bar{\nabla} \left(\frac{\partial u}{\partial t} \right) \right|^2 + \left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\left(\frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 \right) \\ &\leq (\Delta + X) \left(\left(\frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 \right) + 2 \left(\left(\frac{\partial u}{\partial t} \right)^2 + t \left| \nabla \left(\frac{\partial u}{\partial t} \right) \right|^2 \right). \end{aligned}$$

Lemma 4.3 follows from the maximum principle again. \square

Set

$$v = h_1 - \theta_1 - \frac{1}{V} \int_M (h_1 - \theta_1) e^{\theta_1 \omega_1^n}.$$

LEMMA 4.4.

$$\|v\|_{L^2}^2 \leq \frac{2(c_1 + n)e^2 V}{\lambda_1} \epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}.$$

Proof. By Lemma 4.2, we have

$$(\Delta + X)v + (c_1 + n)\epsilon e \geq 0.$$

It follows that

$$\begin{aligned} \int_M |(\Delta + X)v + (c_1 + n)\epsilon e| e^{\theta_1 \omega_1^n} &= \int_M ((\Delta + X)v + (c_1 + n)\epsilon e) e^{\theta_1 \omega_1^n} \\ &= (c_1 + n)\epsilon e V. \end{aligned}$$

Hence, by applying the Poincaré inequality and Lemma 4.1, we have

$$\begin{aligned} \frac{\lambda_1}{V} \int_M |v|^2 e^{\theta_1} \omega_1^n &\leq \frac{1}{V} \int_M |\nabla v|^2 e^{\theta_1} \omega_1^n \\ &= \frac{1}{V} \int_M (-v)(\Delta + X)v e^{\theta_1} \omega_1^n \\ &= \frac{1}{V} \int_M (-v)[(\Delta + X)v + c_1 e\epsilon] e^{\theta_1} \omega_1^n \\ &\leq \frac{1}{V} \|v\|_{C^0} \int_M ((\Delta + X)v + (c_1 + n)e\epsilon) e^{\theta_1} \omega_1^n \\ &\leq 2e^2(c_1 + n)\epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}. \end{aligned}$$

This shows the lemma is true. □

LEMMA 4.5. *We have*

$$\|v\|_{C^0} \leq C(n, c_1 a, \lambda_1) (1 + \|h_\omega - \theta_X(\omega)\|_{C^0}) \epsilon^{1/2(n+1)}, \tag{4.4}$$

provided that the following condition holds: there exists a constant $a > 0$ such that for any $x_0 \in M$ and $0 < r < 1$,

$$(B_r(x_0)) \geq ar^{2n} \tag{4.5}$$

with respect to the metric ω_1 .

Proof. Pick $r = \epsilon^{1/2(n+1)}$ and cover M by geodesic balls of radius r . For any $x \in M$, we have $x \in B_r(x_0)$ for some $x_0 \in M$. Now

$$\begin{aligned} \inf_{B_r(x_0)} |v|^2 \epsilon^{\frac{n}{n+1}} &\leq \frac{1}{a} \int_{B_r(x_0)} |v|^2 e^{\theta_1} \omega_1^n \\ &\leq \frac{2(c_1 + n)e^2 V^2}{a\lambda_1} \epsilon \|h_\omega - \theta_X(\omega)\|_{C^0}. \end{aligned}$$

Hence

$$\inf_{B_r(x_0)} |v| \leq C(n, c_1, a, \lambda_1) \epsilon^{1/2(n+1)} \|h_\omega - \theta_X(\omega)\|_{C^0}^{1/2}.$$

Assuming $\inf_{B_r(x_0)} |v| = v(x'_0)$, then

$$\begin{aligned} |v(x)| &\leq |v(x) - v(x'_0)| + v(x'_0) \leq r \sup_{B_r(x_0)} |\nabla v| + v(x'_0) \\ &\leq e \|h_\omega - \theta_X(\omega)\|_{C^0} \epsilon^{1/2(n+1)} + C \epsilon^{1/2(n+1)} \|h_\omega - \theta_X(\omega)\|_{C^0}^{1/2} \\ &\leq C'(1 + \|h_\omega - \theta_X(\omega)\|_{C^0}) \epsilon^{1/2(n+1)}. \end{aligned}$$

This finishes the proof of Lemma 4.5. □

PROPOSITION 4.1 (Smoothing lemma). *Let $\omega \in c_1(M) > 0$ be any Kähler metric satisfying (4.1). Then there is another Kähler form $\omega' = \omega + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u$ such that*

- (1) $\|u\|_{C^0} \leq e\|h_\omega - \theta_X(\omega)\|$
- (2) $\|h' - \theta'\|_{C^{1/2}} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{\frac{1}{4(n+1)}}$, where $C(n, c_1, a, \lambda_1)$ is a constant depending only on the dimension n , the Poincare constant λ_1 , constants c_1 and a appeared in (4.5).

Proof. We shall prove that ω_1 satisfies the above two conditions of the proposition under the assumption (4.5). By Lemma 4.1, it suffices to check the second condition only.

Since

$$\frac{1}{V} \int_M e^{h' - \theta'} e^{\theta_1} \omega_1^n = 1,$$

by (4.4) in Lemma 4.5, we have

$$\|h' - \theta'\|_{C^0} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{1/2(n+1)}. \tag{4.6}$$

For any two points x, y in M , if the distance $d(x, y) \leq \epsilon^{1/2(n+1)}$, then Lemma 4.3 implies that $|\nabla(h' - \theta')| \leq e\|h_\omega - \theta_X(\omega)\|_{C^0}$ and hence

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \leq e\|h_\omega - \theta_X(\omega)\|_{C^0}\epsilon^{1/4(n+1)}.$$

On the other hand, if $d(x, y) \geq \epsilon^{1/2(n+1)}$ then (4.6) implies that

$$\frac{|(h' - \theta')(x) - (h' - \theta')(y)|}{\sqrt{d(x, y)}} \leq C(n, c_1, a, \lambda_1)(1 + \|h_\omega - \theta_X(\omega)\|_{C^0})\epsilon^{1/4(n+1)}.$$

This completes the proof of Proposition 4.1. □

5 Properness of $\tilde{F}_{\omega_{KS}}(\psi)$

We are now ready to prove Theorem 0.2 stated in the introduction. The method here is analogous to one in [T] for Kähler–Einstein manifolds with positive scalar curvature.

Proof of Theorem 0.2. Let ω_{KS} be the Kähler form of Kähler–Ricci soliton g_{KS} and $\omega_g = \omega_\psi = \omega_{KS} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi$. We consider the complex Monge–Ampère equations with parameter $t \in [0, 1]$:

$$\begin{cases} \det(g_{i\bar{j}} + \varphi_{i\bar{j}}) = \det(g_{i\bar{j}}) \exp\{h_g - \theta_X - X(\varphi) - t\varphi\}, \\ (g_{i\bar{j}} + \varphi_{i\bar{j}}) > 0. \end{cases} \tag{5.1}$$

Clearly, $-\psi$ modulo a constant is a solution of (5.1) $_t$ for $t = 1$. Since ψ is G -invariant, by the implicit function theorem, there are G -invariant solutions of (5.1) $_t$ for t sufficiently close to 1. In fact, in [TZ2], it was proved that there are G -invariant solutions of (5.1) $_t$ for any $t \in [0, 1]$. This is because

$\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)$ is nondecreasing in t (cf. Remark 1.1 in section 1), and consequently the C^3 -norm of φ_t can be uniformly bounded (cf. [TZ2], [Y]).

Put $\omega_t = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_t$. Then $\omega_1 = \omega_{KS}$. Moreover, by (1.2), we have

$$\begin{cases} h_{\omega_t} - \theta_X(\omega_t) = -(1-t)\varphi_t + c_t, \\ \text{Ric}(\omega_t) - L_X(\omega_t) = t\omega_t + (1-t)\omega \geq t\omega_t, \end{cases}$$

where c_t is determined by

$$\int_M e^{-(1-t)\varphi_t + c_t} e^{\theta_X(\omega_t)} \omega_t^n = V.$$

In particular,

$$|c_t| \leq (1-t)\|\varphi_t\|_{C^0(M)}, \quad \|h_{\omega_t} - \theta_X(\omega_t)\|_{C^0(M)} \leq 2(1-t)\|\varphi_t\|_{C^0(M)},$$

and

$$\begin{aligned} |X(h_{\omega_t} - \theta_X(\omega_t))| &= |(1-t)X(\varphi_t)| \\ &\leq (1-t)(|X(\varphi_t - \varphi_1)| + |X(\psi)|) \leq (1-t)c_1(\omega_{KS}, X). \end{aligned}$$

Hence by applying Proposition 4.1 to each ω_t , we obtain a modified Kähler form $\omega'_t = \omega_t + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}u_t$ satisfying

$$\|u_t\|_{C^0(M)} \leq 2e(1-t)\|\varphi_t\|_{C^0(M)},$$

$$\|h_{\omega'_t}\|_{C^{1/2}} \leq C(n, c_1, a, \lambda_1)(1 + \|(1-t)\varphi_t\|_{C^0(M)})(1-t)^{1/4(n+1)}.$$

As before, there are $\tilde{\psi}_t$ such that $\omega_{KS} = \omega'_t + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\tilde{\psi}_t$ and

$$\omega_{KS}^n = (\omega'_t)^n e^{h_{\omega'_t} - \theta_X(\omega'_t) - X(\tilde{\psi}_t) - \tilde{\psi}_t}.$$

It follows from the maximum principle that

$$\varphi_t = \varphi_1 - \tilde{\psi}_t + \mu_t, \tag{5.2}$$

where μ_t are constants with

$$|\mu_t| \leq 2(e+1)(1-t)\|\varphi_t\|_{C^0(M)} + c_1(\omega_{KS}, X). \tag{5.3}$$

Hence, φ_t is uniformly equivalent to φ_1 as long as $\tilde{\psi}_t$ is uniformly bounded. Consider the operator $\Phi_t : C^{2,1/2}(M) \rightarrow C^{0,1/2}$ by

$$\Phi_t(\tilde{\psi}) = \log \left(\frac{(\omega_{KS} - \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\tilde{\psi})^n}{\omega_{KS}^n} \right) + h_{\omega'_t} - \theta_X(\omega'_t) - X(\tilde{\psi}) - \tilde{\psi}.$$

Its linearization at $\tilde{\psi} = 0$ is $(-\Delta - 1 - X(\cdot))$, so it is invertible in the space of G -invariant functions by the assumption of Theorem 0.2. Then by the implicit function theorem, there is a $\delta > 0$, such that if the Hölder norm $\|h_{\omega'_t}\|_{C^{1/2}(\omega_{KS})}$ with respect to ω_{KS} is less than δ , then there is a unique $\tilde{\psi}_t$ such that $\Phi_t(\tilde{\psi}_t) = 0$ and $\|\tilde{\psi}_t\|_{C^{2,1/2}} \leq C(\delta)$.

We observe that

$$\lambda_{1,\omega'} \geq 2^{-n-1} \lambda_{1,\omega_{KS}}, \quad a \geq \frac{1}{2^{2n}} a_0,$$

whenever $\frac{1}{2}\omega_{KS} \leq \omega' \leq 2\omega_{KS}$, where a is a constant appeared in (4.5) and a_0 is a constant such that

$$a_0 r^{2n} \leq \omega_{KS}(B_r(x)), \quad \forall x \in M.$$

Now we choose t_0 such that $(1 - t_0) \leq (\delta/4C_0)^{4(n+1)}$ and

$$(1 - t_0) \|\varphi_{t_0}\|_{C^0(M)} (1 - t_0)^{1/4(n+1)} = \frac{\delta}{4C_0}, \tag{5.4}$$

where $C_0 = C_0(n, c_1, a_0, \lambda_{1,\omega_{KS}})$. Then by Proposition 4.1 and the above argument, one can prove that for any $t \geq t_0$, we have

$$\|u_t\|_{C^0(M)} \leq 2e(1 - t) \|\varphi_t\|_{C^0}, \quad \|\tilde{\psi}_t\|_{C^0(M)} \leq \frac{1}{4}.$$

Therefore, by using (5.2) and (5.3), we get

$$\|\varphi_1 - \varphi_t\|_{C^0(M)} \leq 6e(1 - t) \|\varphi_t\|_{C^0(M)} + c_2, \tag{5.5}$$

and

$$\frac{1}{2} \|\varphi_1\|_{C^0} - c_2 \leq \|\varphi_t\|_{C^0(M)} \leq 2\|\varphi_1\|_{C^0(M)} + c_3, \tag{5.6}$$

for some uniform constants c_2 and c_3 , as long as $1 - t \leq \min\{1/12e, 1 - t_0\}$.

Since $\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)$ is nondecreasing, by (3.1), we have

$$\begin{aligned} \tilde{F}_{\omega_{KS}}(\psi) &= -\tilde{F}_{\omega_g}(\varphi_1) \\ &= \int_0^1 (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) dt \\ &\geq (1 - t) (\tilde{I}_{\omega_g}(\varphi_t) - \tilde{J}_{\omega_g}(\varphi_t)) \\ &\geq \varepsilon(1 - t) \tilde{J}_{\omega_g}(\varphi_t), \end{aligned}$$

where $\varepsilon > 0$ is a uniform constant. On the other hand, by using the co-cycle condition of $\hat{F}_{\omega_g}(\cdot)$, we have

$$\begin{aligned} \tilde{J}_{\omega_g}(\varphi_t) &= \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V} \int_M (\varphi_1 - \varphi_t) e^{\theta x} \omega_g^n + \hat{F}_{\varphi_1}(\varphi_t - \varphi_1) \\ &\geq \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V} \int_M (\varphi_1 - \varphi_t) (e^{\theta x} \omega_g^n - e^{\theta x + X(\varphi_1)} \omega_{\varphi_1}^n) \\ &\geq \tilde{J}_{\omega_g}(\varphi_1) - \text{osc}_M(\varphi_1 - \varphi_t). \end{aligned}$$

It follows by (1.6) and (3.1),

$$\tilde{J}_{\omega_g}(\varphi_t) \geq \varepsilon' I_{\omega_g}(\varphi_1) - \text{osc}_M(\varphi_1 - \varphi_t).$$

Thus by (5.5) and (5.6), we get

$$\begin{aligned} \tilde{F}_{\omega_{KS}}(\psi) &\geq \varepsilon \varepsilon' (1 - t) I_{\omega_g}(\varphi_1) - (1 - t) \text{osc}_M(\varphi_t - \varphi_1) \\ &\geq \varepsilon \varepsilon' (1 - t) I_{\omega_g}(\varphi_1) - 12e(1 - t)^2 \|\varphi_1\|_{C^0(M)} - C_1 \end{aligned} \tag{5.7}$$

$$= \varepsilon\varepsilon'(1-t)I_{\omega_{KS}}(\psi) - 12e(1-t)^2 \text{osc}_M \psi - C_1, \quad \text{for any } t \geq t_0.$$

Therefore, in case

$$\text{osc}_M \psi \leq \tilde{C}(1 + I_{\omega_{KE}}(\psi))$$

for some uniform constant \tilde{C} , then by choosing $t=t_0$ in (5.7) and using (5.4), we see that there are two positive numbers C and C' such that

$$\tilde{F}_{\omega_{KS}}(\psi) \geq CI_{\omega_{KS}}(\psi)^{1/4n+5} - C', \tag{5.8}$$

and consequently this would prove the theorem.

In the general case, we shall use a trick in [TZ1]. First by Proposition 3.1, we have for any $t \geq 1/2$,

$$\text{osc}_M(\varphi_t - \varphi_1) \leq C_2(1 + I_{\omega_{KS}}(\varphi_t - \varphi_1)).$$

Set $\psi' = \varphi_t - \varphi_1$. Then applying inequality (5.8) to function ψ' , we get

$$\begin{aligned} \tilde{F}_{\omega_g}(\varphi_t) - \tilde{F}_{\omega_g}(\varphi_1) &= \tilde{F}_{\omega_{KS}}(\psi') \\ &\geq C_3I_{\omega_{KS}}(\psi')^{1/4n+5} - C_4. \end{aligned} \tag{5.9}$$

On the other hand, by integrating (1.8) from t to 1, we have

$$\begin{aligned} \hat{F}_{\omega_g}(\varphi_1) - \hat{F}_{\omega_g}(\varphi_t) &\geq \tilde{J}_{\omega_g}(\varphi_1) - \frac{1}{V} \int_M \varphi_1 e^{\theta x} \omega_g^n - t \left(\tilde{J}_{\omega_g}(\varphi_t) - \frac{1}{V} \int_M \varphi_t e^{\theta x} \omega_g^n \right) \\ &\geq -(1-t)(\tilde{I}_{\omega_g}(\varphi_1) - \tilde{J}_{\omega_g}(\varphi_1)) \\ &\geq -C_5(1-t)I_{\omega_g}(\varphi_1) = -C_5(1-t)I_{\omega_{KS}}(\psi). \end{aligned} \tag{5.10}$$

By using the concavity of the logarithmic function and (3.6), we also have

$$\begin{aligned} -\log \left(\frac{1}{V} \int_M e^{h-\varphi_t} \omega_g^n \right) &\leq \frac{1-t}{V} \int_M \varphi_t e^{\theta x(\omega_t)} \omega_{\varphi_t}^n \\ &\leq -\frac{1-t}{V} \sup_M(-\varphi_t) + C_6 \leq C_6. \end{aligned} \tag{5.11}$$

Hence combining (5.10) and (5.11), we get

$$\tilde{F}_{\omega_g}(\varphi_t) - \tilde{F}_{\omega_g}(\varphi_1) \leq C_5(1-t)I_{\omega_{KS}}(\psi) + C_6. \tag{5.12}$$

From (5.9) and (5.12), we deduce

$$(1-t)I_{\omega_{KS}}(\psi) \geq c_3 \text{osc}_M(\varphi_t - \varphi_1)^{1/4n+5} - c_4.$$

Then as in (5.7), we prove (cf. [TZ1]),

$$\begin{aligned} \tilde{F}_{\omega_{KS}}(\psi) &\geq \varepsilon\varepsilon'(1-t)I_{\omega_{KS}}(\psi) - (1-t) \text{osc}_M(\varphi_t - \varphi_1) \\ &\geq \varepsilon\varepsilon'(1-t)I_{\omega_{KS}}(\psi) - (1-t)(c_3^{-1})^{4n+5}((1-t)I_{\omega_{KS}}(\psi) + c_4)^{4n+5} \\ &\geq cI_{\omega_{KS}}(\psi)^{1/4n+5} - C, \end{aligned} \tag{5.13}$$

for some small positive number c and large number C . Thus Theorem 0.2 is proved. \square

REMARK 5.1. By (1.6) and (3.1), we see that the inequality (5.13) is equivalent to the following non-linear inequality of Moser–Trudinger type,

$$\int_M e^{-\psi} \omega_{KS}^n \leq C \exp \left\{ \tilde{J}_{\omega_{KS}}(\psi) - c \tilde{J}_{\omega_{KS}}(\psi)^{1/4n+5} - \frac{1}{V} \int_M \psi \omega_{KS}^n \right\}$$

for some positive numbers c and C .

REMARK 5.2. In view of results in [T] and [TZ1], one should be able to generalize Theorem 0.2 by proving the following:

$$\tilde{F}_{\omega_{KS}}(\psi) \geq c I_{\omega_{KS}}(\psi)^{1/4n+5} - C$$

holds for any $\psi \in \Lambda_1(M, \omega_{KS})^\perp$. In the proof of Theorem 0.2, we used a technical assumption on subgroup $G(\subseteq K_0)$ in order to apply the implicit function theorem. We also notice from Theorem 0.2 that (5.13) holds for any almost plurisubharmonic function on a Kähler–Einstein manifold without any nontrivial holomorphic vector field.

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