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KÄHLERIAN NORMAL COMPLEX SURFACES

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Introduction. By Artin [2] a normal (compact) Moishezon surface X is projective if it has only rational singularities. The purpose of this note is to prove a kählerian analogue of this result. We call a normal isolated surface singularity (X, x) *nondegenerate* if for some (and hence for any) resolution $f: (\tilde{X}, A) \to (X, x)(A = f^{-1}(x))$ the natural homomorphism $R^1 f_* \boldsymbol{R} \to R^1 f_* \mathcal{O}_{\tilde{X}}$ is surjective. In particular a rational singularity is nondegenerate. Then our result is as follows.

THEOREM. *Let X be a normal compact complex surface in $\mathscr{C}$ with only nondegenerate isolated singularities. Then X is Kähler.*

Recall that in general a compact complex space X is said to be in $\mathscr{C}$ if it is a meromorphic image of a compact Kähler manifold [9]. In our case of a surface, it turns out, however, that this is equivalent to the condition that any nonsingular model of X is Kähler (Proposition 2). On the other hand, in general a complex space X is said to be *Kähler* if there exists an open covering $\{U_\mu\}$ of X and a system of C^∞ strictly plurisubharmonic functions $\{u_\mu\}$ with each u_μ defined on U_μ such that $u_\mu - u_\nu$ is pluriharmonic on $U_\mu \cap U_\nu$ (cf. [19], [9]). In this case $dd^c u_\mu = dd^c u_\nu$ on $U_\mu \cap U_\nu$ ($d^c = \sqrt{-1}(\bar{\partial} - \partial)$ where $d = \partial + \bar{\partial}$) and hence we get a globally defined real closed (1, 1)-form α on X which is called a *Kähler form* on X.

In Section 1 we shall prove a lemma on extension of a strictly plurisubharmonic function across an isolated singular point (Lemma 1) as well as draw some of its consequences (including Proposition 2 mentioned above). Using this lemma, we shall derive in Section 2 a necessary and sufficient condition for a normal compact complex surface to be Kähler in terms of its resolution (Proposition 4); indeed Theorem above is an immediate consequence of this proposition. Next in Section 3 we shall give some characterization of nondegenerate singularities. In particular, it turns out at once that nondegenerate singularities are nothing but pararational singularities of Brenton [6]. Finally in Section 4 we shall note that every small deformation of a normal compact Kähler surface with only nondegenerate singularities is again Kähler by merely checking the condition of Moishezon [19], in which the study of singular Kähler surfaces was initiated.

1. Extension of a strictly plurisubharmonic function. (1.1) Let Y be a complex space. Let $u: Y \to \boldsymbol{R} \cup \{-\infty\}$ be a map. Then u is said to be plurisubharmonic if for any point $y \in Y$ there exist a neighborhood $U \ni y$, an embedding $j: U \to G$ of U into a domain G in $\boldsymbol{C}^N$ and a plurisubharmonic function u_G on G such that $j^* u_G = u|_U$. On the other hand, u is said to be *weakly plurisubharmonic* if for any holomorphic map $h: D \to Y$ of the unit disc $D: = \{t \in \boldsymbol{C}; |t| < 1\}$ into Y, its pull-back $h^* u$ is either subharmonic or $= -\infty$. A plurisubharmonic function clearly is weakly plurisubharmonic. Conversely, it was recently shown by Fornaess-Narasimhan [7] that every weakly plurisubharmonic function is plurisubharmonic. Using this result we shall show the following lemma, which is indeed a main ingredient of our proof of Theorem.

LEMMA 1. *Let (X, x) be a normal isolated singularity with* $\dim(X, x) \geqq 2$. *Let u be a strictly plurisubharmonic function on $X - x$. Then for any sufficiently small neighborhood V of x there exists a strictly plurisubharmonic function v on X such that $u = v$ on $X - V$.*

PROOF. By Grauert-Remmert [11] u extends to a unique weakly plurisubharmonic function on X which we shall still denote by the same letter u. Then by the result of Fornaess-Narasimhan cited above there exists a neighborhood U of x, an embedding $j: U \to B$ of U into an open ball B in $\boldsymbol{C}^N = \boldsymbol{C}^N(z)$ with center $j(x)$ and plurisubharmonic function $\hat{u}$ on B such that $j^* \hat{u} = u$. We may assume that $j(x)$ is the origin of $\boldsymbol{C}^N$. Let $B_1 \Subset B$ be another concentric ball. Then it suffices to show the lemma for $V = B_1 \cap X$ where we identify X with $j(X)$. Take another concentric ball $B_2 \Subset B_1$. Let λ be a nonnegative C^∞ function on $\boldsymbol{C}^N$ with support contained in B_1 such that $\lambda(z)$ is a positive constant on B_2, λ depends only on $|z|$ and $\int_{\boldsymbol{C}^N} \lambda(z) dV(z) = 1$, where $dV(z)$ is the standard volume form on $\boldsymbol{C}^N$. Then as in Miyaoka [17] we set

$$\tilde{u}_\varepsilon(z) = \int_{\boldsymbol{C}^N} \hat{u}(z - \varepsilon\lambda(z)\zeta)\lambda(\zeta) dV(\zeta)$$

for any real number $\varepsilon > 0$. Then for any sufficiently small $\varepsilon > 0$, we see that $\tilde{u}_\varepsilon \equiv \hat{u}$ on $B - B_1$, $\tilde{u}_\varepsilon|_{U - B_2 \cap U}$ is strictly plurisubharmonic and $\tilde{u}_\varepsilon$ is plurisubharmonic on B_2 (cf. [17]). Then for any such ε we set

$$v_\delta(z) = \tilde{u}_\varepsilon(z) + \delta |z|^2 \lambda(z) , \quad \delta > 0 .$$

Then $v_\delta(z) \equiv \hat{u}(z)$ on $B - B_1$ and $v_\delta(z)$ is strictly plurisubharmonic on B_2.

Moreover it is easy to see that if δ is sufficiently small, $v_\delta(z)|_{U-B_2\cap U}$ also is strictly plurisubharmonic. For such a δ we can define a C^∞ strictly plurisubharmonic function v on X by setting $v = u$ on $X - V$ and $v = v_\delta|_V$ on V. q.e.d.

COROLLARY. *Let (X, x) be as above. Let α be a Kähler form on $X - x$. Suppose that α is written on $X - x$ in the form $\alpha = dd^c u$ for some C^∞ function u on $X - x$. Then for any sufficiently small neighborhood V of x there exists a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$.*

PROOF. Since α is a Kähler form, u is strictly plurisubharmonic. Hence we have only to set $\alpha' = dd^c v$ where v is as in Lemma 1. q.e.d.

(1.2) As an immediate application of the above corollary we shall show the following:

PROPOSITION 1. *Let X be an irreducible normal complex space with $\dim X \geqq 2$. Suppose that X has only isolated quotient singularities. Let U be a smooth Zariski open subset of X such that* $\dim(X - U) = 0$. *Let α be a Kähler form on U. Then for any sufficiently small neighborhood V of $X - U$ in X we can find a Kähler form $\tilde{\alpha}$ on X such that $\tilde{\alpha}|_{U-U\cap V} = \alpha|_{U-U\cap V}$. In particular, X is a Kähler space, if so is U.*

EXAMPLE. Let X be a Kähler manifold and G a finite group of biholomorphic automorphisms of X. Suppose that the set of those points which are fixed by some elements of G is isolated. Then X/G is a Kähler space.

Now Proposition 1 is clearly a consequence of the following local version of it.

LEMMA 2. *Let (X, x) be an isolated quotient singurality with* $\dim(X, x) \geqq 2$. *Let α be a Kähler form on $X - x$. Then for any sufficiently small neighborhood V of x there eixsts a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$.*

PROOF. Let $n = \dim(X, x)$. We can find a neighborhood D of the origin o of C^N and a finite subgroup $G \subseteqq GL(n, C)$ leaving D invariant such that (1) $X \cong D/G$ for a suitable representative X of (X, x) and (2) the natural map $\pi: D \to X$ is unramified over $X' := X - x$ (cf. Prill [20]). Let $\pi': D' \to X'$ be the induced map, where $D' = D - \{o\}$. Let $\tilde{\alpha} = \pi'^*\alpha$. Then $\tilde{\alpha}$ is a G-invariant Kähler form on D'. Since $n \geqq 2$, there exists a C^∞ strictly plurisubharmonic function $\tilde{u}$ on D' such that $\tilde{\alpha} = dd^c\tilde{u}$ by Shiffman [22]. Replacing $\tilde{u}$ by $(1/|G|)\sum_{g\in G} g^*\tilde{u}$ we may assume that $\tilde{u}$

also is G-invariant. Then $\tilde{u}$ is a pull-back of a C^∞ function u on X' and $\alpha = dd^c u$ on X'. The lemma then follows from Corollary to Lemma 1. q.e.d.

(1.3) As an application of Proposition 1 we shall show that any compact smooth analytic surface X in $\mathscr{C}$ is Kähler. The following special case is essential:

LEMMA 3. *Let X be a compact smooth analytic surface which is bimeromorphic to a K3 surface of algebraic dimension zero. Then X is Kähler if $X \in \mathscr{C}$.*

PROOF. We may assume that X is minimal; otherwise X is obtained by a blowing up of the minimal one, say X_0, and hence is Kähler if X_0 is (cf. [9]). Then X contains only finite number of irreducible curves C_i, $i = 1, \cdots, m$, and C_i are nonsingular rational curves with self-intersection number $C_i \cdot C_i = -2$ (cf. [13]). Moreover the intersection matrix $(C_i \cdot C_j)$ is negative definite. Hence there is a contraction $f: X \to X'$ of $C := \bigcup_{i=1}^m C_i$ to a finite number of rational double points $p_1, \cdots, p_s$ of X'. In particular, X' has only isolated quotient singularities. We show that X' is a Kähler space. Since f is projective, this would show that X is Kähler (cf. [9]). Now by our assumption, there exists a compact Kähler manifold Y and a surjective holomorphic map $h: Y \to X$. Let $h' = fh$. Since X' contains no curve, h' must be smooth over a a Zariski open subset $U \subseteq X'$ whose complement consists of a finite number of points. Let α be a Kähler form on Y. By integration along the fibers, $h'_* \alpha^{r+1}$ defines a Kähler form α' on U, where $r = \dim h$. The result then follows from Proposition 1. (Since a rational singularity is nondegenerate, we can also use Proposition 3 below instead of Proposition 1.) q.e.d.

REMARK 1. The proof actually shows that for any tubular neighborhood of C in X there exists a Kähler form $\hat{\alpha}$ on X such that $\hat{\alpha} = h_* \alpha^{r+1}$ on its complement in the notation above.

PROPOSITION 2. *Let X be a compact complex manifold of dimension 2. Suppose that $X \in \mathscr{C}$. Then X is Kähler.*

PROOF. Since $b_1(X)$ is even, we have only to consider the case where X is a $K3$ surface of algebraic dimension zero (cf. Miyaoka [18]). In this case X is Kähler by Lemma 3. q.e.d.

2. Extension of a Kähler form. (2.0) First we shall fix some notations, terminologies and conventions.

(a) Let (X, x) be a normal isolated singularity. Let $f: (\tilde{X}, A) \to (X, x)$ be a resolution of (X, x) where $A = f^{-1}(x)$. We usually identify $\tilde{X}$ with any tubular neighborhood of A in $\tilde{X}$; in this note a *tubular neighborhood* shall mean a relatively compact open neighborhood U with C^∞ strongly pseudoconvex boundary such that A is a strong deformation retract of U. (Note that X is then necessarily Stein.) In particular, the restriction maps $H^i(\tilde{X}, \boldsymbol{R}) \to H^i(A, \boldsymbol{R})$ are isomorphic for all $i \geqq 0$, and $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong (R^i f_* \mathcal{O}_{\tilde{X}})_x$ for all $i > 0$. Thus (when $\dim(X, x) = 2$), (X, x) is nondegenerate if and only if the natural map $e: H^2(\tilde{X}, \boldsymbol{R}) \to H^2(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is surjective.

Take a local embedding $(X, x) \to (\boldsymbol{C}^N, o)$. Then we can always take U as above in such a way that $H^i(U - A, \boldsymbol{R}) \cong H^i(U - f^{-1}(\bar{B}_\varepsilon), \boldsymbol{R}) \cong H^i(K_\varepsilon, \boldsymbol{R})$ for all sufficiently small $\varepsilon > 0$, where $\bar{B}_\varepsilon$ is the closed ball of radius ε with center o and K_ε is the boundary of $\bar{B}_\varepsilon$. In fact, we may set $U = f^{-1}(B_{\varepsilon_0})$ for some $\varepsilon_0 > 0$, where B_{ε_0} is the interior of $\bar{B}_{\varepsilon_0}$.

(b) Let Y be a reduced complex space. We denote by $\mathscr{P}_Y$ the sheaf of germs of pluriharmonic functions on Y. Then we have the short exact sequence of abelian groups on Y;

$$(1) \qquad 0 \to \boldsymbol{R} \xrightarrow{\iota} \mathcal{O}_Y \xrightarrow{\mu_Y} \mathscr{P}_Y \to 0 ,$$

where ι is the natural inclusion and $\mu(g) = -$(imaginary part of g). (When Y is smooth, this is well-known and the general case follows from the smooth case readily.) From this we get the long exact sequence of cohomology

$$(2) \qquad \to H^1(Y, \boldsymbol{R}) \to H^1(Y, \mathcal{O}_Y) \to H^1(Y, \mathscr{P}_Y) \xrightarrow{\delta} H^2(Y, \boldsymbol{R}) .$$

On the other hand, let $\mu^*: \mathcal{O}_Y^* \to \mathscr{P}_Y$ be defined by $\mu^*(g) = -(1/2\pi)\log|g|$. Let $\hat{c}: H^1(Y, \mathcal{O}_Y^*) \to H^1(Y, \mathscr{P}_Y)$ be the resulting homomorphism. Then for any line boundle L on Y we call $\hat{c}(L)$ the *refined Chern class* of L. One checks readily that $\delta\hat{c}(L) = c(L)$, where $c(L)$ is the real Chern class of L.

(c) Let Y be as in (b). Let $\alpha = \{dd^c u_\mu\}$ be a Kähler form on Y with respect to an open covering $\mathfrak{U} = \{U_\mu\}$ as in the introduction. Then α defines a class $\bar{\alpha} \in H^1(Y, \mathscr{P}_Y)$ which is the image of the class $\{u_\mu - u_\nu\} \in H^1(\mathfrak{U}, \mathscr{P}_Y)$ under the natural homomorphism $H^1(\mathfrak{U}, \mathscr{P}_Y) \to H^1(Y, \mathscr{P}_Y)$. We call $\bar{\alpha}$ the *Kähler class* defined by α.

When Y is nonsingular, we have the natural isomorphism $H^1(Y, \mathscr{P}_Y) \cong Z^{1,1}/dd^c E$, where $Z^{1,1}$ is the real vector space of real d-closed $C^\infty(1, 1)$-forms on Y and E is that of real C^∞ functions on Y (cf. [22]). Under this isomorphism the Kähler class $\bar{\alpha}$ above corresponds precisely to the Kähler form α considered modulo $dd^c E$.

(2.1) Let (X, x) be a normal isolated surface singularity. Let $f: (\tilde{X}, A) \to (X, x)$ be a resolution. For a Kähler form α on $\tilde{X}$ we shall consider the following condition (E):

(E) Regard α as a Kähler form on $X - x$ by restriction to $\tilde{X} - A$ followed by the isomorphism $\tilde{X} - A \cong X - x$. Then for any sufficiently small neighborhood V of x in X there exists a Kähler form α' on X such that $\alpha' = \alpha$ on $X - V$ (i.e., the conclusion of Corollary to Lemma 1 holds true for α).

On the other hand, let $A_1, \cdots, A_m$ be the irreducible components of A. Then we shall denote by $H(A)$ the (real) linear subspace of $H^1(\tilde{X}, \mathscr{P}_{\tilde{X}})$ generated by the refined Chern classes $\hat{c}([A_i])$ of $[A_i]$, where $[A_i]$ is the line bundle defined by A_i. Further, recall the exact sequence (2) for $Y = \tilde{X}$;

$$\to H^1(\tilde{X}, \boldsymbol{R}) \xrightarrow{e} H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) \xrightarrow{d} H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^2(\tilde{X}, \boldsymbol{R}) . \tag{3}$$

PROPOSITION 3. *Let (X, x) be a normal isolated surface singularity. Let $f: (\tilde{X}, A) \to (X, x)$ be a resolution. Then*:

(1) *we have the natural direct sum decomposition* $H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) = H(A) \oplus \operatorname{Im} d$, *where* Im *denotes the image, and*

(2) *for any Kähler form α on $\tilde{X}$, the following conditions are equivalent*;

(a) *α satisfies the condition* (E) *above, and*

(b) *the Kähler class $\bar{\alpha} \in H^1(\tilde{X}, \mathscr{P}_{\tilde{X}})$ defined by α belongs to $H(A)$.*

PROOF. (1) We consider the following commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & H^1_A(\tilde{X}, \mathscr{O}_{\tilde{X}}) & \longrightarrow & H^1_A(\tilde{X}, \mathscr{P}_{\tilde{X}}) & \xrightarrow{h} & H^2_A(\tilde{X}, \boldsymbol{R}) & \xrightarrow{b} & H^2_A(\tilde{X}, \mathscr{O}_{\tilde{X}}) \\
 & & \downarrow & & \downarrow c & & \downarrow a & & \downarrow \\
H^1(\tilde{X}, \boldsymbol{R}) & \xrightarrow[e]{} & H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) & \xrightarrow[d]{} & H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) & \xrightarrow[\delta]{} & H^2(\tilde{X}, \boldsymbol{R}) & \longrightarrow & H^2(\tilde{X}, \mathscr{O}_{\tilde{X}}) \\
 & & & & \downarrow r & & & & \\
 & & & & H^1(\tilde{X} - A, \mathscr{P}_{\tilde{X}}) & & & &
\end{array}$$

where the horizontal sequences come from (1) for $\tilde{X}$ and the vertical lines are part of the local cohomology exact sequences. We first show that a is isomorphic. First, we have the natural isomorphisms $H^2_A(\tilde{X}, \boldsymbol{R}) \cong \bigoplus_{i=1}^m H^2_{A_i}(\tilde{X}, \boldsymbol{R}) \cong \bigoplus_{i=1}^m H^0(A_i, \boldsymbol{R}) \cong \boldsymbol{R}^{\oplus m}$ and $H^2(\tilde{X}, \boldsymbol{R}) \cong H^2(A, \boldsymbol{R}) \cong \bigoplus_{i=1}^m H^2(A_i, \boldsymbol{R}) \cong \boldsymbol{R}^{\oplus m}$, where $A_1, \cdots, A_m$ are the irreducible components of A. Then a is given by; $a(y_1, \cdots, y_m) = (\cdots, \sum_{i=1}^m y_i(A_i \cdot A_j), \cdots)$ where $(A_i \cdot A_j)$ is the intersection number of A_i and A_j. Indeed, the image of

$(0, \cdots, 1, \cdots, 0)$(1 in the i-th place) in $H^2(\tilde{X}, \boldsymbol{R})$ by a is nothing but the real Chern class $c([A_i])$ of $[A_i]$. Since the intersection matrix is negative definite [10], a is isomorphic as was desired. Next we show that b is the zero map. First note that the (topological) dual map b' of b is given by $b'\colon H^0(A, \Omega^2_{\tilde{X}}|_A) \to H^2(A, \boldsymbol{R})$ (cf. [3]). But since the restriction of any holomorphic 2-form α on $\tilde{X}$ to A vanishes (so that the integral of α over each A_i is zero), b' is the zero map. Hence so is b. Thus h is isomorphic, since $H^1_A(\tilde{X}, \mathscr{O}_{\tilde{X}}) = 0$ by Grauert-Riemenschneider [12]. Then δc also is isomorphic and we have the direct sum decomposition $H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) = \operatorname{Im} c \oplus \operatorname{Im} d$. It suffices to show that $\operatorname{Im} c = H(A)$. Consider the following commutative diagram

$$\begin{array}{ccccc} H^1_A(\tilde{X}, \mathscr{O}^*_{\tilde{X}}) & \xrightarrow{\hat{c}_A} & H^1_A(\tilde{X}, \mathscr{P}_{\tilde{X}}) & \xrightarrow{h} & H^2_A(\tilde{X}, \boldsymbol{R}) \\ \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle a} \\ H^1(\tilde{X}, \mathscr{O}^*_{\tilde{X}}) & \xrightarrow{\hat{c}} & H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) & \xrightarrow{\delta} & H^2(\tilde{X}, \boldsymbol{R}) \end{array},$$

where $\hat{c}_A$ is defined in the same way as $\hat{c}$. (The sequences are in general not exact.) Obviously the line bundles $[A_i]$ are in the image of k and hence $H(A) \subseteq \operatorname{Im} c$. On the other hand, the above description of the map a shows that $\delta(H(A)) = H^2(\tilde{X}, \boldsymbol{R})$. Thus $H(A) = \operatorname{Im} c$ as was desired.

(2) (b) $\Rightarrow$ (a). Suppose that $\bar{\alpha} \in H(A) = \operatorname{Im} c$. Then $r(\bar{\alpha}) = 0$ and hence $\alpha = dd^c u$ for some C^∞ function u on $\tilde{X} - A$. Then α satisfies the condition (E) by Corollary to Lemma 1. (a) $\Rightarrow$ (b). Suppose that α satisfies (E). For any V and α' as in (E), write $\alpha' = dd^c v$ on X for some C^∞ function v. Then with respect to some local embedding $(X, x) \to (\boldsymbol{C}^n, 0)$, there exists a small open ball B with center 0 in $\boldsymbol{C}^n$ such that $\alpha = dd^c v$ on $X - X \cap B$. Write $B' = X \cap B$. We show that the restriction map $j\colon H^1(X - x, \mathscr{P}_X) \to H^1(X - B', \mathscr{P}_X)$ is injective. In view of the commutative diagram of exact sequences

$$\begin{array}{ccccccc} H^1(X - x, \boldsymbol{R}) & \longrightarrow & H^1(X - x, \mathscr{O}_X) & \longrightarrow & H^1(X - x, \mathscr{P}_X) & \longrightarrow & H^2(X - x, \boldsymbol{R}) \\ \wr\downarrow{\scriptstyle s} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \wr\downarrow{\scriptstyle s} \\ H^1(X - B', \boldsymbol{R}) & \longrightarrow & H^1(X - B', \mathscr{O}_X) & \longrightarrow & H^1(X - B', \mathscr{P}_X) & \longrightarrow & H^2(X - B', \boldsymbol{R}) \end{array}$$

coming from the sequence (1) on $X - x$, we see that it suffices to show that i is injective (cf. (2.0) (a)). But this is shown in Andreotti and Grauert [1, Théorème 15]. Now by the injectivity of j we can write $\alpha = dd^c v'$ for some C^∞ function v' on $X - x$. This implies that $r(\bar{\alpha}) = 0$ in $H^1(\tilde{X} - A, \mathscr{P}_{\tilde{X}}) = H^1(X - x, \mathscr{P}_X)$, or $\bar{\alpha} \in \operatorname{Im} c = H(A)$. q.e.d.

REMARK 2. Let (X, x) be a normal isolated singularity of dimension $\geqq$

3. Let $f: (\tilde{X}, A) \to (X, x)$ be a resolution such that $\dim A = 1$. Then the same proof as above shows that no Kähler form on X satisfies the condition (E), unless (X, x) is smooth.

(2.2) Next we globalize Proposition 3.

PROPOSITION 4. *Let X be a normal complex surface with isolated singular points $x_1, x_2, \cdots, x_s$. Let $f: \tilde{X} \to X$ be a resolution. Let $A_\nu = f^{-1}(x_\nu)$ and U_ν a tubular neighborhood of A_ν in $\tilde{X}$. Then the following conditions are equivalent*:

(1) *X is a Kähler space.*

(2) *$\tilde{X}$ is Kähler and there exists a Kähler form α on $\tilde{X}$ such that for each ν the Kähler class $\bar{\alpha}_\nu \in H^1(U_\nu, \mathscr{P}_{\tilde{X}})$ defined by $\alpha|_{U_\nu}$ belongs to $H(A_\nu)$.*

Note that Theorem in the introduction follows immediately from Propositions 2, 3 and 4 together with (3), in view of the definition of nondegeneracy.

For the implication $(1) \Rightarrow (2)$ we need lemmas.

LEMMA 4. *Let L be a line bundle on a complex manifold X. Let $b_1, \cdots, b_m$ be C^∞ sections of L on X which have no common zeroes. Then the refined Chern class $\hat{c}(L)$ of L is represented by the real closed* (1, 1)-*form* (*cf.* 2.0) (c))

$$\gamma = (-1/4\pi) dd^c \log\left(\sum_{i=1}^m |b_i|^2\right).$$

Here if $b_j(x) \neq 0$ at $x \in X$, then by definition $dd^c \log(\sum_{i=1}^m |b_i|^2) = dd^c \log((\sum_{i=1}^m |b_i|^2)/|b_j|^2)$ at x, which is independent of the choice of such a j. The proof of the lemma is straightforward and is omitted.

LEMMA 5. *Let $\tilde{X}$ be a smooth complex surface and $A \subsetneqq \tilde{X}$ an exceptional divisor. Let $U \Supset U' \supsetneqq A$ be tubular neighborhoods of A. Let $A_1, \cdots, A_m$ be the irreducible components of A. Then there exists a line bundle F on $\tilde{X}$ of the form $F = \bigotimes_{i=1}^m [A_i]^{n_i}$ whose refined Chern class $\hat{c}(F)$ is represented by a real closed $C^\infty(1, 1)$-form β which is positive definite* (*i.e., Kähler*) *on U' and vanishes identically on $\tilde{X} - U$.*

PROOF. (cf. [21, §4, 3]). By [14, Lemma 4.10] there exist integers n_i such that $(\sum_{i=1}^m n_i A_i \cdot A_j) < 0$ for any j. Set $L = \bigotimes_{i=1}^m [A_i]^{-n_i}$. Then L is positive, and hence is ample, on U' (cf. [8, Lemmas 4 and 3]). So for a sufficiently large $b > 0$ we can find holomorphic sections $\psi_1, \cdots, \psi_s$ of $L^{\otimes b}$ on U' which embeds U' into the projective space $\boldsymbol{P}^{s-1}$. Let σ be the canonical meromorphic section of $L = \bigotimes [A_i]^{-n_i}$, which has thus no zeroes and poles outside A. (Actually $n_i > 0$ for all i, and σ is holomorphic.) Let ρ be a C^∞ funciton on $\tilde{X}$ with support contained in U such that $\rho \equiv 1$ on U'. Then $\rho\psi_1, \cdots, \rho\psi_s, (1-\rho)\sigma^b$ are C^∞ sections of $L^{\otimes b}$ which have no common zeroes on $\tilde{X}$. Hence by Lemma 4

$$\gamma := -(1/4\pi b)dd^c \log\left(\sum_{i=1}^{s} |\rho\psi_i|^2 + |(1-\rho)\sigma^b|^2\right)$$

represents $\hat{c}(L)$. Since $\gamma = -(1/4\pi b)dd^c \log(\sum_{i=1}^s |\psi_i|^2)$ on U' it is negative definite there and clearly $\gamma \equiv 0$ on $\tilde{X} - U$. Then it suffices to set $F = L^*$ (the dual of L) and $\beta = -\gamma$. q.e.d.

PROOF OF PROPOSITION 4. We may assume that $U_\mu \cap U_\nu = \emptyset$ for $\mu \neq \nu$.

$(2) \Rightarrow (1)$. Let α_ν be the restriction of α to U_ν. By further restriction, regard α_ν as a Kähler form on $f(U_\nu) - x_\nu \cong U_\nu - A_\nu$. Then by Proposition 3 after modification of α_ν within a relatively compact open subset of $f(U_\nu)$, α_ν extends to a Kähler form α'_ν on $f(U_\nu)$. Define a C^∞ form α' on X by: $\alpha' = \alpha'_\nu$ on $f(U_\nu)$ and $\alpha' = \alpha$ on $X - \bigcup_\nu f(U_\nu) = \tilde{X} - \bigcup_\nu U_\nu$. Then α' is a Kähler form on X.

$(1) \Rightarrow (2)$. Let α' be a Kähler form on X. Take a tubular neighborhood U'_ν of A_ν with $U'_\nu \Subset U_\nu$. Let $A_{\nu,i}$, $i = 1, \cdots, m_\nu$, be the irreducible components of A_ν. Then by Lemma 5 for each ν there exists a line bundle F_ν on $\tilde{X}$ of the form $\bigotimes_{i=1}^{m_\nu} [A_{\nu,i}]^{k_i^\nu}$ whose refined Chern class is represented by a real closed (1, 1)-form β_ν which is positive definite on U'_ν and vanishes identically on $\tilde{X} - U_\nu$. Since $f^*\alpha'$ is semipositive on $\tilde{X}$ and positive on $\tilde{X} - \bigcup_\nu A_\nu$, $\alpha := M f^*\alpha' + \sum_{\nu=1}^s \beta_\nu$ is a Kähler form on $\tilde{X}$ if we take a real number M sufficiently large. Moreover $\bar{\alpha}_\nu = \bar{\beta}_\nu$ on each U_ν, where $\bar{\beta}_\nu$ is the class of β_ν in $H^1(U_\nu, \mathscr{P}_{\tilde{X}})$. Hence $\bar{\alpha}_\nu \in H(A_\nu)$. q.e.d.

REMARK 3. Let $\tilde{X}$ be a Kähler manifold (of arbitrary dimension) and A an exceptional (compact) connected submanifold of codimension 1 in $\tilde{X}$ in the sense of Grauert [10]. Let $f: \tilde{X} \to X$ be the contraction of A to a normal point x of a complex space X. Then a modification of the proof of Theorem shows that if $h^{1,1}(A) = 1$ and $H^1(A, \mathscr{N}^{*\nu}) = 0$ for all $\nu > 0$, X also is Kähler where $\mathscr{N}^*$ is the conormal sheaf of A in $\tilde{X}$. This is a kählerian analogue of Grauert's criterion for projective contraction ([10, Satz 8]).

(2.3) The following proposition shows that the condition of Theorem, i.e., the nondegeneracy of the singularity, is a consequence of a certain global condition on the variety X.

PROPOSITION 5. *Let X be a normal compact complex surface in $\mathscr{C}$. Then X has only nondegenerate singularities if the natural homomorphism*

$f^*: H^2(X, O_X) \to H^2(\tilde{X}, O_{\tilde{X}})$ is injective for some (and hence for any) resolution $f: \tilde{X} \to X$ of X. In particular, X is Kähler. Moreover, under the same condition, $\tilde{X}$ is projective if X is projective.

PROOF. By Theorem, the second assertion follows from the first. The first assertion follows from the commutative diagram

$$\begin{array}{ccccccc} H^1(\tilde{X}, \boldsymbol{R}) & \longrightarrow & H^0(X, R^1 f_* \boldsymbol{R}) & \longrightarrow & H^2(X, \boldsymbol{R}) & \longrightarrow & H^2(\tilde{X}, \boldsymbol{R}) \\ \downarrow a & & \downarrow & & \downarrow & & \downarrow \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^0(X, R^1 f_* \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^2(X, \mathcal{O}_X) & \longrightarrow & H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \end{array}$$

of exact sequences coming from the Leray spectral sequences for f, if we note the surjectivity of a ($X \in \mathscr{C}$). To show the last assertion, recall from the proof of Theorem that for any Kähler form β on $\tilde{X}$ we can find a Kähler form α on X such that $\beta = f^*\alpha$ on a complement of a tubular neighborhood of $A := f^{-1}(\mathrm{Sing}\, X)$. Let $\bar{\beta} \in H^2(\tilde{X}, \boldsymbol{R})$ (resp. $\bar{\alpha} \in H^2(X, \boldsymbol{R})$) be the class defined by β (resp. α). Then by the above remark, $\bar{\beta} - f^*\bar{\alpha}$ is contained in the image of the natural map $H^2_A(\tilde{X}, \boldsymbol{R}) \to H^2(\tilde{X}, \boldsymbol{R})$, which is generated by the real Chern classes $c(A_i) := c([A_i])$ of $[A_i]$, where A_i are the irreducible components of A. Namely $\bar{\beta} - f^*\bar{\alpha} = \sum_i r_i c(A_i)$ for some real numbers r_i. Suppose now that $\tilde{X}$ is projective and $\bar{\beta}$ is a real Chern class of an ample line bundle L on $\tilde{X}$. Then r_i are all rational numbers; indeed $0 = f^*\bar{\alpha}\cdot c(A_j) = \bar{\beta}\cdot c(A_j) - \sum r_i(A_i \cdot A_j)$ for any j, $\bar{\beta}\cdot c(A_j)$ is an integer and $\{(A_i \cdot A_j)\}$ is a negative definite integral matrix. Thus $mf^*\bar{\alpha} = f^*(m\bar{\alpha})$ is a real Chern class of a line bundle for some $m > 0$. Then from our assumption and from the commutative diagram of exact sequences

$$\begin{array}{ccccc} H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \boldsymbol{Z}) & \longrightarrow & H^2(X, \mathcal{O}_X) \\ \downarrow f^* & & \downarrow f^* & & \downarrow \\ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}^*) & \longrightarrow & H^2(\tilde{X}, \boldsymbol{Z}) & \longrightarrow & H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) \end{array}$$

it follows that $m\bar{\alpha}$ itself is a real Chern class of a line bundle, say L'. Then by Grauert [10, Satz 3] L' is ample and hence X is projective.

REMARK 4. The last assertion is originally due Brenton (cf. [5, Prop. 10]).

3. Criteria for nondegeneracy. We use the convention and notation of (2.0) (a).

(3.1) We first note the following:

LEMMA 6. *Let (X, x) be a normal isolated singularity. Let $f: (\tilde{X}, A) \to$*

(X, x) be a resolution. Then the natural map $H^1(\tilde{X}, \boldsymbol{R}) \to H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is injective.

PROOF. We consider the commutative diagram of exact sequences (2) for $\tilde{X}$ and A;

$$\begin{array}{ccccccc} H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^0(\tilde{X}, \mathscr{P}_{\tilde{X}}) & \xrightarrow{b} & H^1(\tilde{X}, \boldsymbol{R}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \\ \downarrow & & \downarrow & & \downarrow r & & \downarrow \\ H^0(A, \mathcal{O}_A) & \xrightarrow{d} & H^0(A, \mathscr{P}_A) & \xrightarrow{\bar{b}} & H^1(A, \boldsymbol{R}) & \longrightarrow & H^1(A, \mathcal{O}_A) , \end{array}$$

where the vertical arrows are the restriction maps. It suffices to show that b is the zero map. Since r is isomorphic, this follows, if $\bar{b}$ is the zero map, or d is surjective. Since A is compact and connected, by the maximum principle d is isomorphic to the map $d'\colon \boldsymbol{C} \to \boldsymbol{R}$ defined by $d'(s) = -$(imaginary part of s). Thus d is surjective as was desired. q.e.d.

Let (X, x) be a normal isolated surface singularity and $f\colon (\tilde{X}, A) \to (X, x)$ a resolution. Then we set $b_1(X, x) = b_1(\tilde{X})$ and call it the *first Betti number* of (X, x). This is independent of the chosen resolution f. Let $p_g(X, x) := \dim(R^1 f_* \mathcal{O}_{\tilde{X}})_x = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}})$ be the geometric genus of (X, x). Then the above lemma shows the inequality $b_1(X, x) \leqq 2p_g(X, x)$. From the definitions we obtain:

PROPOSITION 6. *A normal isolated surface singularity (X, x) is nondegenerate if and only if $b_1(X, x) = 2p_g(X, x)$.*

(3.2) According to Brenton [6], a normal isolated surface singularity (X, x) is called *pararational*, if, for some (and hence for any) resolution $f\colon (\tilde{X}, A) \to (X, x)$ of (X, x) such that A is of normal crossings in $\tilde{X}$, we have (1) (the dual graph of) A contains no cycles and (2) $R^1 f_* \mathscr{I} = 0$ where $\mathscr{I}$ is the ideal sheaf of A with the reduced structure.

PROPOSITION 7. *Let (X, x) be a normal isolated surface singulality. Then (X, x) is nondegenerate if and only if (X, x) is pararational.*

PROOF. Let $A_1, \cdots, A_m$ be the irreducible components of A and $p_1, \cdots, p_n$ the singular points of A. Since A is a curve with normal crossings, we have $b_1(A) = n - m + 1 + \sum_{i=1}^m 2g(A_i)$ and $g(A) := \dim H^1(A, \mathcal{O}_A) = n - m + 1 + \sum_{i=1}^m g(A_i)$, where b_1 is the first Betti number and $g(A_i)$ is the genus of A_i. Hence we see that the following conditions are equivalent: (a) $\dim_{\boldsymbol{R}} H^1(A, \boldsymbol{R}) \geqq 2 \dim H^1(A, \mathcal{O}_A)$, (b) $\dim_{\boldsymbol{R}} H^1(A, \boldsymbol{R}) = 2 \dim H^1(A, \mathcal{O}_A)$ and (c) $n - m + 1 = 0$. Note further that (c) is equivalent to the condition (1) of (3.2) above. Now we consider the following commutative diagram

$$\begin{array}{ccccccccc} & & & & H^1(\tilde{X}, \boldsymbol{R}) & \xrightarrow{\sim} & H^1(A, \boldsymbol{R}) & & \\ & & & & \downarrow e & & \downarrow & & \\ 0 & \longrightarrow & H^1(\tilde{X}, \mathscr{I}) & \longrightarrow & H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) & \longrightarrow & H^1(A, \mathscr{O}_A) & \longrightarrow & 0\,, \end{array}$$

where the bottom sequence is exact, since $H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}) \to H^0(A, \mathscr{O}_A) \cong \boldsymbol{C}$ is surjective and $H^2(\tilde{X}, \mathscr{I}) = 0$. Then in view of the above remark together with the injectivity of e (Lemma 6), it follows that (1) and (2) imply the surjectivity of e. Conversely if e is surjective, then the above equivalent conditions are satisfied, and $\dim H^1(\tilde{X}, \mathscr{I}) + \dim H^1(A, \mathscr{O}_A) = \dim H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) \leqq (1/2) \dim_{\boldsymbol{R}} H^1(\tilde{X}, \boldsymbol{R}) = \dim H^1(A, \mathscr{O}_A)$ by (b). Hence $H^1(\tilde{X}, \mathscr{I}) = 0$ as was desired. q.e.d.

EXAMPLE. Let A be a compact Riemann surface and L a negative line bundle on A. Let (X, x) be the normal isolated singularity obtained by contracting the zero section of L. Then (X, x) is nondegenerate if and only if $H^1(A, L^{*\otimes m}) = 0$ for any $m > 0$, where L^* denotes the dual of L. In particular simple-elliptic singularity of K. Saito are all nondegenerate.

(3.3) Modifying Laufer's argument [15], we shall give a criterion which makes no use of a resolution.

PROPOSITION 8. *Let (X, x) be a normal isolated surface singularity. Then (X, x) is nondegenerate if and only if the following condition is satisfied; any holomorphic 2-form ω on $X - x$ is L^2-integrable whenever the class $[\omega + \bar{\omega}] \in H^2(X - x, \boldsymbol{R})$ defined by $\omega + \bar{\omega}$ vanishes. Here $\bar{\omega}$ denotes the complex conjugate of ω.*

First we recall some results needed for the proof from the duality theory.

(a) Let $U = X - x$. Then $H^1(U, \mathscr{O}_U)$ has the natural QFS structure (QFS = quotient of Frechet-Schwartz) which is separated in our case where X is Stein (cf. [3, p. 82, Theorem 6.1]). On the other hand, by a theorem of Malgrange [16] $H^2(U, \Omega_U^2) = 0$ because $\dim U = 2$. Hence by Serre duality, (1) the natural QDFS structure (QDFS = quotient of dual of FS) on $H_c^1(U, \Omega_U^2)$ also is separated and (2) $H^1(U, \mathscr{O}_U)$ and $H_c^1(U, \Omega_U^2)$ are in topological duality, where c denotes the compact supports (cf. [3]). Let $\omega_X := j_* \Omega_U^2$ be the Grothendieck dualizing sheaf on X where $j: U \to X$ is the inclusion (cf. [12]). Then depth $\omega_X = 2$ so that $H_c^0(X, \omega_X) = H_c^1(X, \omega_X) = 0$ (cf. [3, p. 41, Corollary 3.10]). Therefore from the exact sequence

$$H_c^0(X, \omega_X) \to \omega_{X,x} \to H_c^1(U, \Omega_U^2) \to H_c^1(X, \omega_X)$$

we have the natural (topological) isomorphism $\delta: \omega_{X,x} \cong H^1_c(U, \Omega^2_U)$.

We now proceed to describe the induced duality between $H^1(U, \mathcal{O}_U)$ and $\omega_{X,x}$, or the pairing $\langle\ ,\ \rangle$ giving this duality. Let $\alpha \in H^1(U, \mathcal{O}_U)$ and $\beta \in \omega_{X,x}$ be arbitrary. Let $\tilde{\alpha}$ (resp. $\tilde{\beta}$) be a $\bar{\partial}$-closed C^∞ (0, 1)-form on U (resp. a holomorphic 2-form defined on $V - x$ for some open neighborhood V of x) representing α (resp. β). Let ρ be a C^∞ function with compact support on X such that $\rho \equiv 1$ on a neighborhood V' of x with $V' \Subset V$. Then $\bar{\partial}(\rho\tilde{\beta})$ has compact support on U and hence $\int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta})$ is finite. Further, the integral is independent of the choice of $\tilde{\alpha}$, $\tilde{\beta}$ and ρ as above depending only on α and β. Then from the definition of δ it follows that $\langle\alpha, \beta\rangle = \int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta})$.

Let B_ε be the ball of radius ε with center x with respect to a local embedding of X into $\boldsymbol{C}^N$. Let S_ε be the boundary sphere B_ε. Let $U_\varepsilon = X - B_\varepsilon \cap X$ and $K_\varepsilon = X \cap S_\varepsilon$. Take ε so small that $B_\varepsilon \cap X \subseteq V'$. Then using Stokes' theorem we obtain

$$\int_U \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta}) = \int_{U_\varepsilon} \tilde{\alpha} \wedge \bar{\partial}(\rho\tilde{\beta}) = \int_{U_\varepsilon} d(\tilde{\alpha} \wedge \rho\tilde{\beta}) - \int_{U_\varepsilon} d\tilde{\alpha} \wedge \rho\tilde{\beta} = \int_{K_\varepsilon} \tilde{\alpha} \wedge \tilde{\beta}\ .$$

Since these are independent of ε as above, we write $K = K_\varepsilon$ symbolically. We have thus $\langle\alpha, \beta\rangle = \int_K \tilde{\alpha} \wedge \tilde{\beta}$.

(b) Let $f: (\tilde{X}, A) \to (X, x)$ be resolution. Consider the coboundary map $\zeta: H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) \to H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}})$ in the local cohomology exact sequence. Note that ζ is surjective since $H^2(\tilde{X}, O_{\tilde{X}}) = 0$. Both terms have the natural QFS structures for which ζ is continuous (cf. [3, pp. 82, 287]). Since $H^1(A, \Omega^2_{\tilde{X}})$ is finite dimensional, $H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}})$ is separated and $H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}})$ and $H^0(A, \Omega^2_{\tilde{X}})$ are in natural topological duality. On the other hand, since $H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}}) \cong H^1(X - x, \mathcal{O}_X)$, the transpose ζ' of ζ gives rise to a homomorphism $H^0(A, \Omega^2_{\tilde{X}}) \to \omega_{X,x}$ by (a). Further, with a little more efforts this homomorphism is identified with the natural homomorphism $(f_*\Omega^2_{\tilde{X}})_x \to \omega_{X,x}$ obtained by the restriction to $\tilde{X} - A \cong X - x$, which we shall still denote by ζ'.

(c) Let $K = K_\varepsilon$ be as in (a). Then for $\boldsymbol{K} = \boldsymbol{C}$ or $\boldsymbol{R}$ we have $H^i(\tilde{X} - A, \boldsymbol{K}) \cong H^i(X - x, \boldsymbol{K}) \cong H^i(K, \boldsymbol{K})$ for all i (cf. (2.0) (a)) and they are finite dimensional over $\boldsymbol{K}$. So they have the natural FS structure with its (topological) dual given by $H^i(\tilde{X} - A, \boldsymbol{K})' \cong H^i(K, \boldsymbol{K})' \cong H^{3-i}(K, \boldsymbol{K}) \cong H^{3-i}(\tilde{X} - A, \boldsymbol{K})$ by the Poincaré duality.

The natural map $\gamma_C: H^1(\tilde{X} - A, \boldsymbol{C}) \to H^1(\tilde{X} - A, \mathcal{O}_{\tilde{X}})$ is given by $\gamma_C(\alpha) =$ the class of $\tilde{\alpha}_1$, where $\tilde{\alpha}_1$ is the (0, 1)-component of a representing d-closed 1-form $\tilde{\alpha}$ of α. Then from the description of the duality in (a) it follows that the dual map $\gamma_C'\colon \omega_{X,x} \to H^2(\tilde{X}-A, \boldsymbol{C})$ is given by $\gamma_C'(\beta)=\tilde{\beta}$ modulo d-closed forms via the natural isomorphism $H^2(\tilde{X}-A, \boldsymbol{C}) \cong H^2(V-x, \boldsymbol{C})$, where $\tilde{\beta}$ and V are as in (a) with V suitably restricted.

PROOF OF PROPOSITION 8. Let $f\colon (\tilde{X}, A) \to (X, x)$ be a resolution. Consider the commutative diagram of local cohomology exact sequences

$$\begin{array}{ccccccc} & & H^1(\tilde{X}, \boldsymbol{R}) & \xrightarrow{b} & H^1(\tilde{X}-A, \boldsymbol{R}) & \longrightarrow & H^2_A(\tilde{X}, \boldsymbol{R}) \xrightarrow{a} H^2(\tilde{X}, \boldsymbol{R}) \\ & & \downarrow e & & \downarrow \gamma & & \downarrow \\ H^1_A(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \xrightarrow{\varepsilon} & H^1(\tilde{X}-A, \mathcal{O}_{\tilde{X}}) & \xrightarrow{\zeta} & H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}}), \end{array}$$

where the vertical arrows are the natural maps. First by Grauert-Riemenschneider [12] $H^1_A(\tilde{X}, \mathcal{O}_{\tilde{X}})=0$ so that ε is injective. Since a is injective (cf. the proof of Proposition 3), b is surjective. This implies that (X, x) is nondegenerate, i.e., e is surjective if and only if $\operatorname{Im}\gamma \supseteqq \operatorname{Im}\varepsilon = \operatorname{Ker}\zeta$, where Im and Ker denote the image and the kernel respectively; in other words, the complex

$$H^1(\tilde{X}-A, \boldsymbol{R}) \xrightarrow{\gamma} H^1(\tilde{X}-A, \mathcal{O}_{\tilde{X}}) \xrightarrow{\zeta} H^2_A(\tilde{X}, \mathcal{O}_{\tilde{X}})$$

is exact. In view of the remarks preceding the proof, the topological dual of this sequence is given by

$$H^2(\tilde{X}-A, \boldsymbol{R}) \xleftarrow{\gamma'} \omega_{X,x} \xleftarrow{\zeta'} (f_*\Omega^2_{\tilde{X}})_x$$

and this sequence is exact if and only if the original sequence is exact (cf. [3, p. 248 (C)]). Further, $\operatorname{Im}\zeta'$ is precisely the space of L^2-integrable holomorphic 2-forms on $X-x$ (cf. [12], [15]). On the other hand, with K as in (c) above, by the description of γ_C' above we see that γ' is given by the map $\gamma'(\beta)=$ the class of $(1/2)(\tilde{\beta}+\bar{\tilde{\beta}})$ where $\tilde{\beta}$ is as in (a) and $\bar{\tilde{\beta}}$ denotes its complex conjugate. The proposition thus follows.

4. Small deformations and nondegeneracy. We show that every small deformation of a normal compact Kähler surface with only nondegenerate isolated singularities is again Kähler.

PROPOSITION 9. *Let X be a normal compact complex surface in $\mathscr{C}$. Suppose that X has only nondegenerate singularities so that X is Kähler (cf. Theorem). Then every (flat) small deformation X_t of X is Kähler.*

This is a consequence of the following theorem of Moishezon [19] in view of Lemma 7 below.

THEOREM (Moishezon). *Let X be a normal compact Kähler space.*

If the natural map $H^2(X, \boldsymbol{R}) \to H^2(X, \mathcal{O}_X)$ is surjective, then every small deformation X_t of X is again Kähler.

LEMMA 7. *Let X be as in Proposition 9. Then the natural map $H^2(X, \boldsymbol{R}) \to H^2(X, \mathcal{O}_X)$ is surjective.*

PROOF. Let $f: \tilde{X} \to X$ be a resolution. We consider the following commutative diagram of exact sequences derived from the Leray spectral sequence for f

$$\begin{array}{ccccccc} & & & & H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) & & \\ & & & & \downarrow e & & \\ H^0(X, R^1f_*\boldsymbol{R}) & \longrightarrow & H^2(X, \boldsymbol{R}) & \longrightarrow & H^2(\tilde{X}, \boldsymbol{R}) & \xrightarrow{d} & H^0(X, R^2f_*\boldsymbol{R}) \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow \\ H^0(X, R^1f_*\mathcal{O}_{\tilde{X}}) & \longrightarrow & H^2(X, \mathcal{O}_X) & \longrightarrow & H^2(\tilde{X}, \mathcal{O}_{\tilde{X}}) & \longrightarrow & 0\,. \end{array}$$

Our assumption is that a is surjective and we have to show that b is surjective. Since $\tilde{X} \in \mathscr{C}$, c is surjective. Hence by diagram chasing it suffices to show that $de: H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^0(X, R^2f_*\boldsymbol{R})$ is surjective. Let $x_1, \cdots, x_m$ be the singular points of X. Let $A_\nu = f^{-1}(x_\nu)$ and $A = \bigcup_{\nu=1}^m A_\nu$. Then we show even that the composite map $H_A^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^0(X, R^2f_*\boldsymbol{R})$ is surjective. First, we have the natural isomorphisms $H_A^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \cong \bigoplus_{\nu=1}^m H_{A_\nu}^1(\tilde{X}, \mathscr{P}_{\tilde{X}})$ and $H^0(X, R^2f_*\boldsymbol{R}) \cong \bigoplus_{\nu=1}^m H^2(A_\nu, \boldsymbol{R})$ and thus it suffices to show that the composite map $H_{A_\nu}^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^1(\tilde{X}, \mathscr{P}_{\tilde{X}}) \to H^2(A_\nu, \boldsymbol{R})$ is surjective for any ν. This is indeed proved in the course of the proof of Proposition 3. q.e.d.

EXAMPLE. Let C be any nonsingular cubic curve in the complex projective plane $\boldsymbol{P}^2$. Take ten points $q_1, \cdots, q_{10}$ on C arbitrarily and blow up $\boldsymbol{P}^2$ with center $q_1, \cdots, q_{10}$. Let $\tilde{X}$ be the resulting surface and E the proper transform of C in $\tilde{X}$. Then the self-intersection number $E\cdot E = -1$ so that E can be blown down to a simple-elliptic singularity x of a normal Moishezon surface X. Then by our Theorem together with example in (3.2) X is Kähler, though X is not projective for 'general' choice of q_i.

This example of kählerian non-projective Moishezon surface is originally due to Moishezon [19], where he has proved that the above $X = X(q_1, \cdots, q_{10})$ is Kähler if it is sufficiently near to a projective one by using the above theorem. (Note that his proof amounts to showing that (X, x) is nondegenerate in our sense.) Many other examples of kählerian non-projective normal Moishezon surfaces can now be obtained by using our criterion. For instance, let $h: Y \to C$ be a projective nonsingular elliptic surface with $q(Y) = g(C)$, where $q(Y)$ and $g(C)$ are the irregularity and the genus of Y and C, respectively. Let $c \in C$ be a general point and $p \in Y_c$ the general point of Y_c. Let X be the normal surface obtained by blowing up p and then contracting the proper transform of Y_c to a unique singular point $x \in X$. Then X satisfies the desired condition.

REFERENCES

[1] A. ANDREOTTI AND H. GRAUERT, Théorème de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90 (1962), 193–259.

[2] M. ARTIN, Some numerical criteria for contractibility of curves on algebraic surfaces, Amer. J. Math. 84 (1962), 485–496.

[3] C. BANICA AND O. STANSILA, Algebraic methods in the global theory of complex spaces, John Wiley and Sons, London-New York-Sydney-Tronto and Editura Academiei, Bucuresti, 1976.

[4] J. BINGENER, On deformations of Kähler maps, preprint.

[5] L. BRENTON, Some algebraicity criteria for singular surfaces, Inventiones Math. 41 (1977), 129–147.

[6] L. BRENTON, On singular complex surfaces with vanishing geometric genus and pararational singularities, Composito Math. 43 (1981), 297–315.

[7] J. E. FORNAESS AND R. NARASIMHAN, The Levi problem on complex spaces with singularities, Math. Ann. 248 (1980), 47–72.

[8] A. FUJIKI, On the blowing down of analytic spaces, Publ. RIMS, Kyoto Univ. 10 (1975), 473–507.

[9] A. FUJIKI, Closedness of the Douady spaces of compact Kähler spaces, Publ. RIMS, Kyoto Univ. 14 (1978), 1–52.

[10] H. GRAUERT, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. 146 (1962), 331–368.

[11] H. GRAUERT AND R. REMMERT, Plurisubharmonische Funktionen in Komplexen Räumen, Math. Z. 65 (1956), 175–194.

[12] H. GRAUERT AND O. RIEMENSCHNEIDER, Verschwindungassätze für analytische Kohomologie Gruppen auf Komplexen Räumen, Inventiones Math. 11 (1970), 263–292.

[13] K. KODAIRA, On compact analytic surfaces I, Ann. of Math. 71 (1960), 111–152.

[14] H. LAUFER, Normal two-dimensional singularities, Ann. of Math. Studies 71, 1971.

[15] H. LAUFER, On rational singularities, Amer. J. Math. 94 (1972), 597–608.

[16] B. MALGRANGE, Faisceaux sur les variété analytique-reélles, Bull. Soc. Math. France 85 (1957), 231–237.

[17] Y. MIYAOKA, Extension theorems of Kähler metrics, Proc. Japan Acad. 50 (1974), 407–410.

[18] Y. MIYAOKA, Kähler metrics on elliptic surfaces, Proc. Japan Acad. 50 (1974), 533–536.

[19] B. MOISHEZON, Singular kählerian spaces, Proc. Int. Conf. on manifolds and related topics in topology, Univ. of Tokyo Press, 1974, 343–351.

[20] D. PRILL, Local classification of quotients of complex manifolds by discontinuous groups, Duke Math. J. 34 (1967), 375–386.

[21] O. RIEMENSCHENEIDER, Characterizing Moishezon spaces by almost positive coherent analytic sheaves, Math. Z. 123 (1971), 263–284.

[22] B. SHIFFMAN, Extension of positive line bundles and meromorphic maps, Inventiones Math. 15 (1972), 332-347.

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