Tohoku Math. J.
63 (2011), 519–538

KAKEYA-NIKODYM AVERAGES AND L^p-NORMS OF EIGENFUNCTIONS

CHRISTOPHER D. SOGGE

(Received May 6, 2010)

Abstract. We provide a necessary and sufficient condition that L^p-norms, $2 < p < 6$, of eigenfunctions of the square root of minus the Laplacian on two-dimensional compact boundaryless Riemannian manifolds M are small compared to a natural power of the eigenvalue λ. The condition that ensures this is that their L^2-norms over $O(\lambda^{-1/2})$ neighborhoods of arbitrary unit geodesics are small when λ is large (which is not the case for the highest weight spherical harmonics on S^2 for instance). The proof exploits Gauss' lemma and the fact that the bilinear oscillatory integrals in Hörmander's proof of the Carleson-Sjölin theorem become better and better behaved away from the diagonal. Our results are related to a recent work of Bourgain who showed that L^2-averages over geodesics of eigenfunctions are small compared to a natural power of the eigenvalue λ provided that the $L^4(M)$ norms are similarly small. Our results imply that QUE cannot hold on a compact boundaryless Riemannian manifold (M, g) of dimension two if L^p-norms are saturated for a given $2 < p < 6$. We also show that eigenfunctions cannot have a maximal rate of L^2-mass concentrating along unit portions of geodesics that are not smoothly closed.

1. Introduction. The main purpose of this paper is to slightly sharpen a recent result of Bourgain [5] concerning two-dimensional compact boundaryless Riemannian manifolds. By doing so we shall be able to provide a natural necessary and sufficient condition concerning the growth rate of L^p-norms of eigenfunctions for $2 < p < 6$ and their L^2-concentration about geodesics.

There are different ways of measuring the concentration of eigenfunctions. One is by means of the size of their L^p-norms for various values of $p > 2$. If M is a compact boundaryless manifold with Riemannian metric $g = g_{jk}(x)$ and if Δ_g is the associated Laplace-Beltrami operator, then the eigenfunctions solve the equation $-\Delta_g e_{\lambda_j}(x) = \lambda_j^2 e_{\lambda_j}(x)$ for a sequence of eigenvalues $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \dots$. Thus, we are normalizing things so that λ_j are the eigenvalues of the first-order operator $\sqrt{-\Delta_g}$. We shall also usually assume that the e_{λ_j} have L^2-norm one, in which case $\{e_{\lambda_j}\}$ provides an orthonormal basis of $L^2(M, dx)$ where dx is the volume element coming from the metric. Earlier, in the two-dimensional case, we showed in [26] that if M is fixed then there is a uniform constant C so that for $2 \le p \le \infty$ and $j = 1, 2, 3, \dots$

$$\|e_{\lambda_j}\|_{L^p(M)} \le C\lambda_j^{\delta(p)} \|e_{\lambda_j}\|_{L^2(M)}\,, \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 35P99; Secondary 35L20, 42C99.

The author was supported in part by NSF Grant DMS-0555162.

with

$$\delta(p) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \le p \le 6\,, \\ \frac{1}{2} - \frac{2}{p}, & 6 \le p \le \infty\,. \end{cases}$$

These estimates are sharp for the round sphere S^2, and in this case they detect two types of concentration of eigenfunctions that occur there. Recall that on S^2 with the canonical metric the distinct eigenvalues are $\sqrt{k^2+k}$, $k = 0, 1, 2, \ldots$, which repeat with multiplicity $d_k = 2k+1$. If $\mathcal{H}_k$, the space of spherical harmonics of degree k, is the space of all eigenfunctions with eigenvalue $\sqrt{k^2+k}$, and if $H_k(x, y)$ is the kernel of the projection operator onto $\mathcal{H}_k$, then the k-th zonal function at $x_0 \in S^2$ is $Z_k(y) = (H_k(x_0, x_0))^{-1/2} H_k(x_0, y)$. Its L^2-norm is one but its mass is highly concentrated at $\pm x_0$ where it takes on the value $\sqrt{d_k/4\pi}$. Explicit calculations show that $\|Z_k\|_{L^p(S^2)} \approx k^{\delta(p)}$ for $p \ge 6$ (see e.g. [25]), which shows that in the case of $M = S^2$ with the round metric (1.1) cannot be improved for this range of exponents. Another extreme type of concentration is provided by the highest weight spherical harmonics which have mass concentrated on the equators of S^2, which are its geodesics. The ones concentrated on the equator $\gamma_0 = \{(x_1, x_2, 0)\,;\; x_1^2 + x_2^2 = 1\}$ are the functions Q_k, which are the restrictions of the $\boldsymbol{R}^3$ harmonic polynomials $k^{1/4}(x_1 + ix_2)^k$ to $S^2 = \{x\,;\, |x| = 1\}$. One can check that the Q_k have L^2-norms comparable to one and L^p-norms comparable to $k^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$ when $2 \le p \le 6$ (see e.g. [25]). Notice also that the Q_k have Gaussian type concentration about the equator γ_0. Specifically, if $\mathcal{T}_{k^{-1/2}}(\gamma_0)$ denotes all points on S^2 of distance smaller than $k^{-1/2}$ from γ_0 then one can check that

$$\liminf_{k\to\infty} \int_{\mathcal{T}_{k^{-1/2}}(\gamma_0)} |Q_k(x)|^2 dx > 0\,. \tag{1.2}$$

For future reference, obviously the Q_k also have the related property that

$$\int_{\gamma_0} |Q_k|^2 ds \approx k^{1/2}\,, \tag{1.3}$$

if ds is the measure on γ_0 induced by the volume element.

Thus, the sequence of highest weight spherical harmonics shows that the norms in (1.1) (for $2 < p < 6$), (1.2) and (1.3) are related. A goal of this paper is to show that this is true for general two-dimensional compact manifolds without boundary.

We remark that, although the estimates (1.1) are sharp for the round sphere, one expects that it should be the case that, for generic manifolds, and L^2-normalized eigenfunctions one has

$$\limsup_{j\to\infty} \lambda_j^{-\delta(p)} \|e_{\lambda_j}\|_{L^p(M)} = 0 \tag{1.4}$$

for every $2 < p \le \infty$. This was verified for exponents $p > 6$ by Zelditch and the author in [30] by showing that if there are no points x through which a positive measure of geodesics starting at x loop back through x then $\|e_\lambda\|_\infty = o(\lambda^{1/2})$. By interpolating with the estimate

(1.1) for $p = 6$, this yields (1.4) for all $p > 6$. Corresponding results were also obtained in [30] for higher dimensions. Recently, these results were strengthened by Toth, Zelditch and the author [29] to allow similar results for quasimodes under the weaker condition that at every point x the set of recurrent directions for the first return map for geodesic flow has measure zero in the cosphere bundle $S^*_x M$ over x.

Other than the partial results in Bourgain [5], there do not seem to be any results addressing when (1.4) holds for a given $2 < p < 6$ (although Zygmund [37] showed that on the torus L^2-normalized eigenfunctions have uniformly bounded L^4-norms). Furthermore, there do not seem to be results addressing the interesting endpoint case of $p = 6$, where one expects both types of concentration mentioned before to be relevant.

Recently authors have studied the L^2 norms of eigenfunctions over unit-length geodesics. Burq, Gérard and Tzvetkov [6] showed that if Π is the collection of all unit length geodesics then

$$\sup_{\gamma\in\Pi}\int_\gamma |e_{\lambda_j}|^2 ds \lesssim \lambda_j^{1/2}\|e_{\lambda_j}\|^2_{L^2(M)}, \quad j = 1, 2, 3, \dots, \tag{1.5}$$

which is sharp in view of (1.3). Related results for hyperbolic surfaces were obtained earlier by Reznikov [20], who opened up the present line of investigation. The proof of (1.5) boils down to bounds for certain Fourier integral operators with folding singularities (cf. Greenleaf and Seeger [12], Tataru [32]). In Section 3, we shall use ideas from [12], [32], and [10], [16], [29], [30] to show that if $\gamma \in \Pi$ and

$$\limsup_{j\to\infty} \lambda_j^{-1/2}\int_\gamma |e_{\lambda_j}|^2 ds > 0,$$

then the geodesic extension of γ must be a smoothly closed geodesic. Presumably it also has to be stable, but we cannot prove this. Further recent work on L^2-concentration along curves can be found in Toth [33].

In a recent paper [5], Bourgain proved an estimate that partially links the norms in (1.1) and (1.5), namely that for all $p \geq 2$

$$\sup_{\gamma\in\Pi}\int_\gamma |e_{\lambda_j}|^2 ds \lesssim \lambda_j^{1/p}\|e_{\lambda_j}\|^2_{L^p(M)}. \tag{1.6}$$

Of course for $p = 2$, this is just (1.5); however, an interesting feature of (1.6) is that the estimate for a given $2 < p \leq 6$ combined with (1.1) yields (1.5). Thus, if $e_{\lambda_{j_k}}$ is a sequence of eigenfunctions with (relatively) small $L^p(M)$ norms for a given $2 < p \leq 6$, it follows that its L^2-norms over unit geodesics must also be (relatively) small. Bourgain [5] also came close to establishing the equivalence of these two things by showing that given $\varepsilon > 0$ there is a constant C_ε so that for $j = 1, 2, \dots$

$$\|e_{\lambda_j}\|_{L^4(M)} \leq C_\varepsilon\Big(\lambda_j^{1/8+\varepsilon}\|e_{\lambda_j}\|_{L^2(M)}\Big)^{3/4}\Big[\lambda_j^{-1/2}\sup_{\gamma\in\Pi}\int_\gamma |e_{\lambda_j}|^2 ds\Big]^{1/8}. \tag{1.7}$$

Since $\delta(4) = 1/8$ in (1.1), if the preceding inequality held for $\varepsilon = 0$ one would obtain the linkage of the size of the norms in (1.5) for large energy with the size of the $L^4(M)$ norms.

Our main estimate in Theorem 1.1 is that a variant of (1.7) holds, which is strong enough to complete the linkage.

Bourgain's approach in proving (1.7) was to employ ideas going back to Córdoba [9] and Fefferman [11] that were used to give a proof of the Carleson-Sjölin theorem [7]. The key object that arose in Córdoba's work [9] was what he called the Kakeya maximal function in $\boldsymbol{R}^2$, namely,

$$\mathcal{M}f(x) = \sup_{x \in \mathcal{T}_{\lambda^{-1/2}}} |\mathcal{T}_{\lambda^{-1/2}}|^{-1} \int_{\mathcal{T}_{\lambda^{-1/2}}} |f(y)|dy\,, \quad f \in L^2(\boldsymbol{R}^2)\,, \tag{1.8}$$

with the supremum taken over all $\lambda^{-1/2}$-neighborhoods $\mathcal{T}_{\lambda^{-1/2}}$ of unit line segments containing x, and $|\mathcal{T}_{\lambda^{-1/2}}| \approx \lambda^{-1/2}$ denoting its area. The above maximal operator is now more commonly called the Nikodym maximal operator as this is the terminology in Bourgain's important papers [2]–[4] which established highly nontrivial progress towards establishing the higher dimensional version of the Carleson-Sjölin theorem for Euclidean spaces $\boldsymbol{R}^n$, $n \geq 3$.

One could also consider variable coefficient versions of the maximal operators in (1.8). In the present context if $\gamma \in \Pi$ is a unit geodesic, one could consider the $\lambda^{-1/2}$-tube about it given by

$$\mathcal{T}_{\lambda^{-1/2}}(\gamma) = \left\{ y \in M\,;\ \inf_{x \in \gamma} d_g(x,y) < \lambda^{-1/2} \right\},$$

with $d_g(x,y)$ being the geodesic distance between x and y. Then if $\mathrm{Vol}_g(\mathcal{T}_{\lambda^{-1/2}}(\gamma))$ denotes the measure of this tube, the analog of (1.8) would be

$$\mathcal{M}f(x) = \sup_{x \in \gamma \in \Pi} \frac{1}{\mathrm{Vol}_g(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} \int_{\mathcal{T}_{\lambda^{-1/2}}} |f(y)|dy\,.$$

These operators have been studied before because of their applications in harmonic analysis on manifolds. See e.g. [18], [28]. As was shown in [17], following the earlier paper [4], they are much better behaved in 2-dimensions compared to higher dimensions.

As (1.7) suggests, it is not the size of the L^2-norm of $\mathcal{M}f$ for $f \in L^2(M)$ that is relevant for estimating $L^4(M)$-norms of eigenfunctions but rather the sup-norm of this quantity with $f = |e_{\lambda_j}|^2$, which up to the normalizing factor in front of the integral is the quanitity

$$\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_{\lambda_j}(x)|^2 dx\,.$$

If the e_{λ_j} are L^2-normalized this is trivially bounded by one. In rough terms our results say that beating this trivial bound is equivalent to beating the bounds in (1.1) for a given $2 < p < 6$.

Let us now state our variant of (1.7):

THEOREM 1.1. *Fix a two-dimensional compact boundaryless Riemannian manifold (M,g). Then given $\varepsilon > 0$ there is a constant C_ε so that for eigenfunctions e_λ of $\sqrt{-\Delta_g}$ with* *eigenvalues* $\lambda \geq 1$ *we have*

$$\|e_\lambda\|_{L^4(M)}^4 \leq \varepsilon\lambda^{1/2}\|e_\lambda\|_{L^2(M)}^4 + C_\varepsilon\lambda^{1/2}\|e_\lambda\|_{L^2(M)}^2 \sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(x)|^2 dx + C\|e_\lambda\|_{L^2(M)}^4, \tag{1.9}$$

with C *being a fixed constant which is independent of* λ *and* ε.

We shall prove this not by adapting Córdoba's [9] proof of the Carleson-Sjölin theorem but rather that of Hörmander [15]. He obtained sharp oscillatory integral bounds in $\boldsymbol{R}^2$ that provided sharp Böchner-Riesz estimates for $L^4(\boldsymbol{R}^2)$ (i.e., the Carleson-Sjölin theorem), which turns out to be the endpoint case for this problem in 2-dimensions. Hörmander's approach was to turn this L^4-problem into an L^2-problem by squaring the oscillatory integrals and then estimating their L^2-norms. As his proof shows, the resulting bilinear operators that arise are better and better behaved away from the diagonal, and this fact is what allows us to take the constant in front of the first term in the right side of (1.9) to be arbitrarily small (at the expense of the 2nd term).

Stein [31] provided a generalization of Hörmander's oscillatory integral theorem to higher dimensions in a way that proved to be sharp because of a later construction of Bourgain [4]. Bourgain's example and related ones in [17] suggest that extending the results of this paper to higher dimensions (where the range of exponents would be $2 < p < 2(n+1)/(n-1)$) could be subtle. On the other hand, since the constructions tend to involve concentration about hypersurfaces as opposed to geodesics, their relevance is not plain.

We shall prove Theorem 1.1 by estimating an oscillatory integral operator, which up to a remainder term, reproduces eigenfunctions. The remainder term in this reproducing formula accounts for the last term in (1.9), which we could actually take to be $\leq C_N\lambda^{-N}\|e_\lambda\|_2^4$ for any N, but this is not important for our applications. Also, we remark that the proof of the theorem will show that the constant C_ε in (1.9) can be taken to be $O(\varepsilon^{-2})$ as $\varepsilon \to 0$.

Let us now state an immediate consequence of Theorem 1.1 which states that the size of L^4-norms of eigenfunctions is equivalent to size of L^2-mass near geodesics.

COROLLARY 1.2. *Let* $e_{\lambda_{j_k}}$ *be a sequence of eigenfunctions with eigenvalues* $\lambda_{j_1} \leq \lambda_{j_2} \leq \dots$ *and unit* $L^2(M)$*-norms. Then*

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma)} |e_{\lambda_{j_k}}(x)|^2 dx = 0 \tag{1.10}$$

if and only if

$$\limsup_{k\to\infty} \lambda_{j_k}^{-1/8}\|e_{\lambda_{j_k}}\|_{L^4(M)} = 0\,. \tag{1.11}$$

To prove this, we first notice that if we assume (1.10), then (1.11) must hold because of (1.9). Also, by Hölder's inequality

$$\left(\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda(x)|^2 dx\right)^{1/2} \le (\mathrm{Vol}_g(\mathcal{T}_{\lambda^{-1/2}}(\gamma)))^{1/4} \|e_\lambda\|_{L^4(M)} \lesssim \lambda^{-1/8} \|e_\lambda\|_{L^4(M)},$$

and so (1.11) trivially implies (1.10).

If we use Bourgain's estimate (1.6) and (1.1) we can say a bit more.

COROLLARY 1.3. *Let* $\{e_{\lambda_{j_k}}\}_{k=1}^\infty$ *be as above and suppose that* $2 < p < 6$. *Then the following are equivalent*

$$\limsup_{k\to\infty} \lambda_{j_k}^{-1/2} \sup_{\gamma\in\Pi} \int_\gamma |e_{\lambda_{j_k}}(s)|^2 ds = 0 \tag{1.12}$$

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma)} |e_{\lambda_{j_k}}(x)|^2 dx = 0 \tag{1.13}$$

$$\limsup_{k\to\infty} \lambda_{j_k}^{-\delta(p)} \|e_{\lambda_{j_k}}\|_{L^p(M)} = 0\,. \tag{1.14}$$

To prove this result, we first note that, by the M. Riesz interpolation theorem and (1.1) for $p = 2$ and $p = 6$, (1.14) holds for a given $2 < p < 6$ if and only if it holds for $p = 4$, which we just showed is equivalent to (1.13). Clearly (1.12) implies (1.13). Finally, since Bourgain's estimate (1.6) shows that (1.14) implies (1.12), the proof of Corollary 1.3 is complete.

Let us conclude this section by describing one more application. Recall that a sequence of L^2-normalized eigenfunctions $\{e_{\lambda_{j_k}}\}_{k=1}^\infty$ satisfies the quantum unique ergodicity property (QUE) if the associated Wigner measures $|e_{\lambda_{j_k}}|^2 dx$ tend to the Liouville measure on S^*M. If this is the case, then one certainly cannot have

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \int_{\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma)} |e_{\lambda_{j_k}}(x)|^2 dx > 0\,,$$

since the tubes are shrinking.

In the case where M has negative sectional curvature Schnirelman's [22] theorem says there is a density one subsequence $\{e_{\lambda_{j_k}}\}_{k=1}^\infty$ of all the $\{e_{\lambda_j}\}$ satisfying QUE. Rudnick and Sarnak [21] conjectured that in the negatively curved case there should be no exceptional subsequences violating QUE, i.e., in this case QUE should hold for the full sequence $\{e_{\lambda_j}\}$ of L^2-normalized eigenfunctions. On the other hand, by Corollary 1.3, we have the following.

COROLLARY 1.4. *Let* M *be a two-dimensional compact boundaryless Riemannian manifold. Then QUE cannot hold for* M *if for a given* $2 < p < 6$ *there is saturation of* L^p *norms, i.e.,*

$$\limsup_{j\to\infty} \lambda_j^{-\delta(p)} \|e_{\lambda_j}\|_{L^p(M)} > 0\,,$$

with e_{λ_j} *being the* L^2*-normalized eigenfunctions.*

See e.g. [36] for connections between QUE and the Lindelöf hypothesis, and see [8] for recent developments regarding the QUE conjecture.

2. Proof of Theorem 1: Gauss' lemma and the Carleson-Sjölin condition.

As in [5] and [6] we shall prove our estimate by using certain convenient operators that reproduce eigenfunctions. Specifically, we shall use a slight variant of a result from [27], Chapter 5 that was presented in [6].

LEMMA 2.1. *Let $\delta > 0$ be smaller than half of the injectivity radius of (M, g). Then there is a function $\chi \in \mathcal{S}(\boldsymbol{R})$ with $\chi(0) = 1$ so that if $d_g(x, y)$ is the geodesic distance between $x, y \in M$*

$$\chi_\lambda f(x) = \chi\big(\sqrt{-\Delta_g} - \lambda\big) f(x) = \lambda^{1/2} \int_M e^{i\lambda d_g(x,y)} \alpha(x, y, \lambda) f(y) dy + R_\lambda f(x)\,, \tag{2.1}$$

where

$$\|R_\lambda f\|_{L^\infty(M)} \le C_N \lambda^{-N} \|f\|_{L^1(M)} \quad \textit{for all } N = 1, 2, \dots,$$

and $\alpha \in C^\infty$ has the property that

$$|\partial^\alpha_{x,y} \alpha(x, y, \lambda)| \le C_\alpha \quad \textit{for all } \alpha\,,$$

and, moreover,

$$\alpha(x, y, \lambda) = 0 \quad \textit{if } d_g(x, y) \notin (\delta/2, \delta)\,. \tag{2.2}$$

Since $\chi_\lambda e_\lambda = e_\lambda$ and since the 4th power of the L^4-norm of $R_\lambda e_\lambda$ is dominated by the last term in (1.9), we conclude that in order to prove Theorem 1.1 it is enough to show that, given $\varepsilon > 0$ there is a constant C_ε so that when $\lambda \ge 1$

$$\begin{aligned} \int_M \left| \lambda^{1/2} \int_M e^{i\lambda d_g(x,y)} \alpha(x, y, \lambda) f(y) dy \right|^2 |f(x)|^2 dx \\ \le \varepsilon \lambda^{1/4} \|f\|^2_{L^2(M)} \|f\|^2_{L^4(M)} \\ + C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(M)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |f(x)|^2 dx\,, \end{aligned} \tag{2.3}$$

for, if $f = e_\lambda$, the first term in the right is bounded by a fixed constant times $\varepsilon \lambda^{1/2} \|e_\lambda\|^4_{L^2(M)}$, because of (1.1).

After applying a partition of unity (and abusing notation a bit), we may assume that in addition to (2.2), $\alpha(x, y, \lambda)$ vanishes unless x is in a small neighborhood of some $x_0 \in M$ and y is in a small neighborhood of some $y_0 \in M$ with $\delta/2 < d_g(x_0, y_0) < \delta$. We may assume both of these neighborhoods are contained in the geodesic ball $B(x_0, 10\delta) = \{y \in M\,;\ d_g(x_0, y) < 10\delta\}$. As mentioned before, we are also at liberty to take $\delta > 0$ to be small.

To simplify the calculations to follow, it is convenient to choose a natural coordinate system. Specifically, we shall choose Fermi normal coordinates about the geodesic γ_0 which passes through x_0 and is perpendicular to the geodesic connecting x_0 and y_0. These coordinates will be well defined on $B(x_0, 10\delta)$ if δ is small. Furthermore, we may assume that the

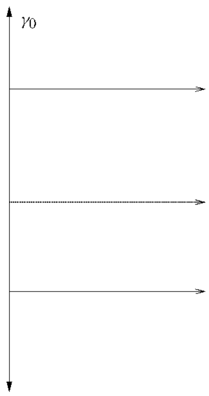

FIGURE 1. Fermi normal coordinates about γ_0.

image of $\gamma_0 \cap B(x_0, 10\delta)$ in the resulting coordinates is a line segment which is parallel to the 2nd coordinate axis and that all horizontal line segments $s \to \{(s, t_0)\}$ are geodesic with the property that $d_g((s_1, t_0), (s_2, t_0)) = |s_1 - s_2|$. See Figure 1 below.

If we use these coordinates and apply Schwarz's inequality, we conclude that, in order to prove (2.3), it suffices to show that given $\varepsilon > 0$ we can find $C_\varepsilon < \infty$ so that when $\lambda \geq 1$

$$\int \left(\int \left| \lambda^{1/2} \int e^{i\lambda d_g(x,(s,t))} \alpha(x, (s,t), \lambda) f(s,t) dt \right|^2 |f(x)|^2 dx \right) ds \\ \leq \varepsilon \lambda^{1/4} \|f\|^2_{L^2(M)} \|f\|^2_{L^4(M)} + C_\varepsilon \lambda^{1/2} \|f\|^2_{L^2(M)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |f(x)|^2 dx \,.$$

This, in turn would follow if we could show that given $\varepsilon > 0$

$$\int \left| \lambda^{1/2} \int e^{i\lambda d_g(x,(s,t))} \alpha(x, (s,t), \lambda) h(t) dt \right|^2 |f(x)|^2 dx \\ \leq \varepsilon \lambda^{1/4} \|h\|^2_{L^2(dt)} \|f\|^2_{L^4(M)} + C_\varepsilon \lambda^{1/2} \|h\|^2_{L^2(dt)} \sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |f(x)|^2 dx \,, \tag{2.4}$$

with C_ε depending on $\varepsilon > 0$ but not on s or on $\lambda \geq 1$.

To simplify the notation, we shall establish this estimate for a particular value of s, which, after relabeling, we may assume to be $s = 0$. Since the proof of (2.4) for this case relies only on Gauss' lemma and the related Carleson-Sjölin condition, it also yields the uniformity in s, assuming, as we may, that α has small support.

To prove this inequality, let us choose a function $\eta \in C_0^\infty(\boldsymbol{R})$ satisfying $\eta(t) = 0$, $|t| > 1$, and $\sum_{j=-\infty}^{\infty} \eta(t - j) \equiv 1$. Given $\lambda \geq 1$ fixed, we shall then set

$$\eta_j(t) = \eta_{\lambda,j}(t) = \eta(\lambda^{1/2} t - j) \,.$$

Then, given $N = 1, 2, \dots$, we have that

$$\begin{aligned}\Big|\lambda^{1/2}\int e^{i\lambda d_g(x,(0,t))}\alpha(x,(0,t),\lambda)h(t)dt\Big|^2 \\ \le N\sum_j\Big|\lambda^{1/2}\int e^{i\lambda d_g(x,(0,t))}\eta_j(t)\alpha(x,(0,t),\lambda)h(t)dt\Big|^2 \\ +\Big|\lambda\iint e^{i\lambda(d_g(x,(0,t))+d_g(x,(0,t')))}a_N(x,t,t')h(t)h(t')dtdt'\Big|,\end{aligned} \tag{2.5}$$

where

$$a_N(x,t,t') = \sum_{|j-k|>N}\eta_j(t)\alpha(x,(0,t),\lambda)\eta_k(t')\alpha(x,(0,t'),\lambda)$$

vanishes when $|t-t'| \le (N-1)\lambda^{-1/2}$. The first term in the right side of the preceding inequality comes from applying Young's inequality to handle the double-sum over indices with $|j-k| \le N$. Because of (2.5), we conclude that (2.4) would follow if we could show that there is a constant independent of $\lambda \ge 1$ and $N = 2, 3, 4\dots$ so that

$$\begin{aligned}\Big\|\lambda\iint e^{i\lambda[d_g(x,(0,t))-d_g(x,(0,t')]}a_N(x,t,t')h(t)h(t')dtdt'\Big\|_{L^2(dx)} \\ \le C\lambda^{1/4}N^{-1/2}\|h\|^2_{L^2(dt)},\end{aligned} \tag{2.6}$$

and also that there is a constant C independent of $j \in \boldsymbol{Z}$ and $\lambda \ge 1$ so that

$$\begin{aligned}\int\Big|\lambda^{1/2}\int e^{i\lambda d_g(x,(0,t))}\eta_j(t)\alpha(x,(0,t),\lambda)h(t)dt\Big|^2|f(x)|^2dx \\ \le C\lambda^{1/2}\|h\|^2_{L^2(dt)}\sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)}|f(x)|^2dx.\end{aligned} \tag{2.7}$$

Indeed, by using the finite overlapping of the supports of the η_j, if we set $\varepsilon = CN^{-1/2}$, then we see that these two inequalities and (2.5) imply (2.4) with $C_\varepsilon \approx \varepsilon^{-2}$. Since the proof of (2.7) only uses Gauss' lemma and the fact that coordinates have been chosen so that $s \to (s, t_0)$ are unit speed geodesics for fixed t_0, we shall just verify (2.7) for $j = 0$, as the argument for this case will yield the other cases as well.

The next step is to see that these two inequalities are consequences of the following two propositions.

PROPOSITION 2.2. *Let $a(x,t,t')$, $x \in \boldsymbol{R}^2$, $t, t' \in \boldsymbol{R}$ satisfy $|\partial_x^\alpha a| \le C_\alpha$ for all multi-indices α and $a(x,t,t') = 0$ if $|x| > \delta$ or $|t-t'| > \delta$ where $\delta > 0$ is small. Suppose also that $\phi \in C^\infty(\boldsymbol{R}^2 \times \boldsymbol{R})$ is real and satisfies the Carleson-Sjölin condition on the support of a, i.e.,*

$$\det\begin{pmatrix}\phi''_{x_1t} & \phi''_{x_2t}\\ \phi'''_{x_1tt} & \phi'''_{x_2tt}\end{pmatrix} \ne 0. \tag{2.8}$$

Then if the $\delta > 0$ *above is sufficiently small, there is a uniform constant* C *so that when* $\lambda, N \geq 1$

$$\left\| \iint_{|t-t'|\geq N\lambda^{-1/2}} e^{i\lambda[\phi(x,t)+\phi(x,t')]} a(x,t,t')F(t,t')dtdt' \right\|^2_{L^2(\boldsymbol{R}^2)} \leq C\lambda^{-3/2}N^{-1}\|F\|^2_{L^2(\boldsymbol{R}^2)}. \tag{2.9}$$

To state the next Proposition, we need to introduce one more coordinate system, which finally explains where the L^2 norms over small tubular neighborhoods of geodesics comes into play. Since we are proving (2.7) with $j = 0$ and since η_0 is supported in the small interval $[-\lambda^{-1/2}, \lambda^{-1/2}]$, it is natural to take geodesic normal coordinates about $(0, 0)$. If we recall that the 1st coordinate axis is a unit-speed geodesic in our original Fermi normal coordinates, we shall naturally choose the geodesic normal coordinates $x \to \kappa(x)$ that preserve this axis (and its orientation). Such a system is unique up to reflection about this axis, and we shall just fix one of these two choices.

PROPOSITION 2.3. *Let* $\psi(x,t) = d_g(x,(0,t))$, *and suppose that* $\rho \in C_0^\infty(\boldsymbol{R} \times \boldsymbol{R}^2)$ *satisfies*

$$|\partial_t^m \rho(t;x)| \leq C_m(\lambda^{1/2})^m, \quad \textit{and} \quad \rho(t;x) = 0, \ |t| \geq \lambda^{-1/2}. \tag{2.10}$$

Suppose also that ρ *vanishes when* x *is outside of a small neighborhood* $\mathcal{N}$ *of a fixed point* $(-s_0, 0)$ (*in the Fermi normal coordinates*) *with* $s_0 > 0$. *If* $x \to \kappa(x) = (\kappa_1(x), \kappa_2(x))$ *are the coordinates described above*, *assume that points* $x_j \in \mathcal{N}$ *are chosen so that*

$$\left| \frac{\kappa_2(x_j)}{|\kappa(x_j)|} - \frac{\kappa_2(x_k)}{|\kappa(x_k)|} \right| \geq c\lambda^{-1/2}|j-k|, \quad \textit{if} \quad |j-k| \geq 10, \tag{2.11}$$

with $c > 0$ *fixed. It then follows that, if* $\mathcal{N}$ *is sufficiently small*, *then there is a uniform constant* C, *which is independent of the* $\{x_j\}$ *chosen as above*, *so that*

$$\lambda^{1/2} \int \left| \sum_j e^{i\lambda\psi(x_j,t)} \rho(t;x_j) a_j \right|^2 dt \leq C \sum |a_j|^2. \tag{2.12}$$

Proposition 2.2 would imply (2.6) if $\phi(x,t) = d_g(x,(0,t))$ satisfies the Carleson-Sjölin condition. The fact that this is the case is well known. See e.g., Section 5.1 in [27]. It follows from our choice of coordinates and the fact that if $x_0 \in M$ is fixed then the set of points $\{\nabla_x d_g(x,y)\,;\ x = x_0, d_g(x_0,y) \in (\delta/2,\delta)\}$ is the cosphere at x_0, $S^*_{x_0}M = \{\xi\,;\ \sum g^{jk}(x_0)\xi_j\xi_k = 1\}$, where $g^{jk}(x)$ is the cometric (inverse to $g_{jk}(x)$). If we choose geodesic normal coordinates $\kappa(y)$ vanishing at x_0 then the gradient becomes $\kappa(y)$. This turns out to be equivalent to the usual formulation of Gauss' lemma, saying that this exponential map $y \to \kappa(y)$ is a local radial isometry. More specifically, it says that small geodesic spheres centered at x_0 get sent to spheres centered at the origin and small geodesic rays through x_0 intersect these geodesic spheres orthogonally and get sent to rays through the origin, which is what allows Proposition 2.3 to be true.

Let us next see that Proposition 2.3 implies (2.7) for $j = 0$. If we take $\rho(t;x) = \eta_0(t)\alpha(x,(0,t),\lambda)$, then ρ satisfies (2.10). Also, if we let

$$S_j = \{y\,;\ \theta(y) \in (\lambda^{-1/2}j, \lambda^{-1/2}(j+1)]\}\,,$$

where $\theta(y) \in [0,2\pi)$ is defined so that $y = |y|(\cos\theta(y), \sin\theta(y))$, then, if $y = \kappa(x)$ are the geodesic normal coordinates about $(0,0)$ in the Proposition 2.3, then the left side of (2.7) is dominated by

$$\sum_j \left\| \lambda^{1/2} \int e^{i\lambda\psi(x,t)} \rho(t;x) h(t)\,dt \right\|^2_{L^\infty(\kappa^{-1}(S_j))} \|f\|^2_{L^2(\kappa^{-1}(S_j)\cap K)}$$
$$\leq \sup_k \|f\|^2_{L^2(\kappa^{-1}(S_k)\cap K)} \sum_j \left\| \lambda^{1/2} \int e^{i\lambda\psi(x,t)} \rho(t;x) h(t)\,dt \right\|^2_{L^\infty(\kappa^{-1}(S_j))}\,,$$

where K is the x-support of ρ. Since the first factor on the right is dominated by the last factor in the right-hand side of (2.7) (the sup can just be taken over $(0,0) \in \gamma \in \varPi$ here), we conclude that we would obtain this inequality if we could show that there is a uniform constant so that for all choices of $x_j \in \kappa^{-1}(S_j)$

$$\lambda^{1/2} \sum_j \left| \int e^{i\lambda\psi(x_j,t)} \rho(t;x_j) h(t)\,dt \right|^2 \leq C\|h\|^2_{L^2(dt)}\,. \tag{2.13}$$

This inequality is an estimate for an operator from $L^2(dt) \to \ell^2$. The dual operator is the one in Proposition 2.3. Therefore since, by duality, (2.13) follows from (2.12) we get (2.7). To verify this assertion, we use the fact that if ρ has small support then the terms in (2.13) with $\rho(t;x_j) \neq 0$ will fulfill the hypotheses in Proposition 2.3.

To finish the proof of Theorem 1.1 we must prove the two propositions. Let us start with the first one since it is pretty standard. It is based on the well known fact that the bilinear oscillatory integrals arising in Hörmander's [15] proof of the Carleson-Sjölin [7] theorem become better and better behaved away from the diagonal.

PROOF OF PROPOSITION 2.2. Let $\varPhi(x;t,t') = \phi(x,t) + \phi(x,t')$ be the phase function in (2.9). Then $\varPhi$ is a symmetric function in the (t,t') variables. So if we make the change of variables

$$u = (t - t', t + t')\,,$$

then since $|du/d(t,t')| = 2$, we see that (2.8) implies that the Hessian determinant of $\varPhi$ satisfies

$$\left|\det\left(\frac{\partial^2\varPhi}{\partial x \partial u}\right)\right| \geq c|u_1|\,,$$

for some $c > 0$ on the support of a, if the latter is small. Since $\varPhi(x;u)$ is an even function of the diagonal variable u_1, it must be a C^∞ function of u_1^2. So if we make the final change of variables

$$v = \left(\frac{1}{2}u_1^2, u_2\right),$$

then since $|dv/du| = |u_1|$, it follows that

$$\left|\det\left(\frac{\partial^2 \Phi}{\partial x \partial v}\right)\right| \geq c,$$

for some $c > 0$. This in turn implies that if v and $\tilde{v}$ are close then

$$|\nabla_x[\Phi(x,v) - \Phi(x,\tilde{v})]| \geq c'|v - \tilde{v}|,$$

for some $c' > 0$, and since $x, v \to \Phi$ is smooth, we also have that

$$|\partial_x^\alpha[\Phi(x,v) - \Phi(x,\tilde{v})]| \leq C_\alpha|v - \tilde{v}|,$$

for all multi-indices α. Therefore, if we let

$$K_\lambda(v,\tilde{v}) = \int_{\boldsymbol{R}^2} a(x,t,t')\overline{a(x,\tilde{t},\tilde{t}')}e^{i\lambda[\Phi(x,v)-\Phi(x,\tilde{v})]}dx,$$

then by integrating by parts, we find that if the number $\delta > 0$ in the statement of the Proposition is small then for $j = 1, 2, 3, \ldots$

$$\begin{aligned} |K_\lambda(v,\tilde{v})| &\leq C_j(1 + \lambda|v - \tilde{v}|)^{-2j} \\ &\leq C_j(1 + \lambda|(t+t') - (\tilde{t}+\tilde{t}')|)^{-j}(1 + \lambda|(t-t')^2 - (\tilde{t}-\tilde{t}')^2|)^{-j}. \end{aligned} \tag{2.14}$$

Note that the left side of (2.9) equals

$$\int \cdots \int_{|t-t'|,|\tilde{t}-\tilde{t}'| \geq N\lambda^{-1/2}} K_\lambda(t,t';\tilde{t},\tilde{t}')F(t,t')\overline{F(\tilde{t},\tilde{t}')}dtdt'd\tilde{t}d\tilde{t}'.$$

We next claim that there is a uniform constant C so that for $\lambda, N \geq 1$

$$\sup_{\tilde{t},\tilde{t}'} \int_{|t-t'| \geq N\lambda^{-1/2}} |K_\lambda| dtdt', \quad \sup_{t,t'} \int_{|\tilde{t}-\tilde{t}'| \geq N\lambda^{-1/2}} |K_\lambda| d\tilde{t}d\tilde{t}' \leq C\lambda^{-2}(\lambda^{1/2}/N). \tag{2.15}$$

This follows from (2.14) and the fact that if $\tau = s^2$ then $2sds = d\tau$ and so, given $\tau_0 \in \boldsymbol{R}$, we have

$$\begin{aligned} \int_{s \geq N\lambda^{-1/2}} (1 + \lambda|s^2 - \tau_0|)^{-2} ds &= \frac{1}{2}\int_{\sqrt{\tau} \geq N\lambda^{-1/2}} (1 + \lambda|\tau - \tau_0|)^{-2} \frac{d\tau}{\sqrt{\tau}} \\ &\leq (\lambda^{1/2}/N) \int_{-\infty}^{+\infty} (1 + \lambda|\tau|)^{-2} d\tau \leq C\lambda^{-1}(\lambda^{1/2}/N). \end{aligned}$$

Since (2.15) and Young's inequality yield (2.9), the proof is complete. □

To finish our task we need to prove the other Proposition, which is a straightforward application of Gauss' lemma.

PROOF OF PROPOSITION 2.3. The support assumptions on the amplitude will allow us to linearize the function $t \to \psi$ in the proof, which is a tremendous help. Specifically,

$$\psi(x,t) = \psi(x,0) + t(\partial_t \psi(x,0)) + r(x,t)\,,$$

where

$$|\partial_t^m r(x,t)| \le C_m |t|^{2-m}\,, \quad 0 \le m \le 2\,, \quad \text{and} \quad |\partial_t^m r| \le C_m\,, \quad m \ge 2\,. \tag{2.16}$$

Our choice of coordinates implies that

$$\partial_t \psi(x,0) = \langle \nu, \kappa(x)/|\kappa(x)|\rangle\,,$$

where the inner-product is the euclidean one and $\nu \in \boldsymbol{R}^2$ is chosen so that $\langle \nu, \nabla\rangle$ is the pushforward of $\partial/\partial x_2$ at $(0,0)$ under the map $x \to \kappa(x)$—i.e., tangent vector to the curve $t \to \kappa((0,t))$. Since the pushforward of $\partial/\partial x_1$ is itself under this map, it follows that the second coordinate of ν is nonzero. (See Figure 2 below.) Therefore, if $\mathcal{N} \ni (s_0, 0)$ is small enough, then our assumption (2.11) implies that

$$|\partial_t \psi(x_j,0) - \partial_t \psi(x_k,0)| \ge c'\lambda^{-1/2}|j-k|\,, \text{ if } |j-k| \ge 10\,, \text{ and } x_j, x_k \in \mathcal{N}\,, \tag{2.17}$$

for some constant $c' > 0$.

It is easy now to finish the proof of (2.12). If we let

$$\rho(x_j, x_k; t) = \rho(t; x_j)\overline{\rho(t; x_k)} e^{i\lambda(\psi(x_j,0)+r(x_j,t))} e^{-i\lambda(\psi(x_k,0)+r(x_k,t))}\,,$$

it follows from (2.10) and (2.16) that

$$|\partial_t^m \rho(x_j, x_k; t)| \le C_m \lambda^{m/2}\,,$$

and

$$\rho(x_j, x_k; t) = 0\,, \quad \text{if } |t| \ge \lambda^{-1/2}\,, \quad x_j \notin \mathcal{N}\,, \quad \text{or } x_k \notin \mathcal{N}\,.$$

We can use this since the left side of (2.12) equals

$$\lambda^{1/2} \sum_{j,k} a_j \overline{a_k} \Big(\int e^{it\lambda(\partial_t\psi(x_j,0) - \partial_t\psi(x_k,0))} \rho(x_j, x_k; t)\,dt \Big)\,,$$

which, after integrating by parts $N = 1, 2, 3\ldots$ times, we conclude is dominated by a fixed constant C_N times

$$\sum_{j,k} |a_j a_k| (1 + |j-k|)^{-N}\,.$$

Since, by Young's inequality, this is dominated by the right side of (2.12) when $N = 2$, the proof is complete. □

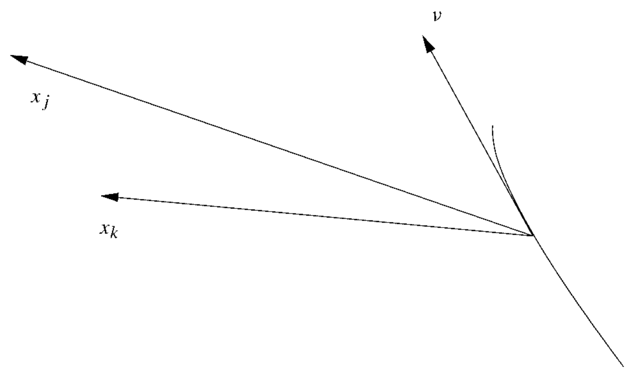

FIGURE 2. Image of $\{(0, t)\}$ in geodesic normal coordinates about $(0, 0)$.

3. Local restrictions of eigenfunctions to non-smoothly closed geodesics. We have shown above that if $\{e_{\lambda_{j_k}}\}_{k=1}^{\infty}$ is a sequence of L^2-normalized eigenfunctions satisfying

$$\limsup_{k\to\infty} \sup_{\gamma\in\Pi} \lambda_{j_k}^{-1/2} \int_\gamma |e_{\lambda_{j_k}}|^2 ds = 0\,, \tag{3.1}$$

then $\lambda_{j_k}^{-\delta(p)} \|e_{\lambda_{j_k}}\|_{L^p(M)} = 0$, $2 < p < 6$. While it seems difficult to determine when this holds, one can show the following.

PROPOSITION 3.1. *Suppose that $\gamma \in \Pi$ is not contained in a smoothly closed geodesic. Then if $\{e_{\lambda_j}\}$ is the full sequence of L^2-normalized eigenfunctions, we have*

$$\limsup_{j\to\infty} \lambda_j^{-1/2} \int_\gamma |e_{\lambda_j}|^2 ds = 0\,. \tag{3.2}$$

In proving this proposition we may assume, after possible multiplying the metric by a constant, that the injectivity radius is more than 10. This will allow us to write down Fourier integral operators representing the solution of the wave equation up to times $|t| \le 10$. More important, though, is that we shall use an observation of Tataru [32] that the map from Cauchy data to the solution of the wave equation restricted to $\gamma \times \boldsymbol{R}$ is a Fourier integral operator with a one-sided fold. Using this fact and the standard method of long-time averages (see e.g. [10], [16], [30], [29]), we shall be able to prove Proposition 3.1.

To set up our proof, let us choose Fermi normal coordinates about γ so that, in these coordinates, γ becomes $\{(s, 0)\,;\ 0 \le s \le 1\}$. Note that in these coordinates the metric takes the form $g_{11}(x)dx_1^2 + dx_2^2$. As a consequence if $p(x, \xi) = \sqrt{\sum g^{jk}(x)\xi_j\xi_k}$ is the principal symbol of $P = \sqrt{-\Delta_g}$ then $p((s, 0), \xi) = \sqrt{g_{11}((s, 0))\xi_1^2 + \xi_2^2}$ is an even function of ξ_2.

To proceed, let us fix a real-valued function $\chi \in \mathcal{S}(\mathcal{R})$ with $\chi(0) = 1$ and $\hat{\chi}(t) = 0$, $|t| > 1/2$. Then if e_λ is an eigenfunction with eigenvalue λ it follows that $\chi(N(P - \lambda))e_\lambda = e_\lambda$. Thus, in order to prove (3.2), it would suffice to prove that given λ, $N \ge 1$

$$\big\|\chi(N(P - \lambda))f\big\|_{L^2(\gamma)} \le CN^{-1/2}\lambda^{1/4}\|f\|_{L^2(M)} + C_N\|f\|_{L^2(M)}\,. \tag{3.3}$$

Note that

$$\chi(N(P-\lambda))f(x) = N^{-1}\int \hat{\chi}(t/N)e^{-it\lambda}\big(e^{itP}f\big)(x)dt\,, \tag{3.4}$$

and because of the support properties of the $\hat{\chi}$ the integrand vanishes when $|t| \geq N/2$.

The operator

$$f \to \big(e^{itP}f\big)(x)$$

is a Fourier operator with canonical relation

$$\{(x,t,\xi,\tau;\,y,\eta)\,;\ \Phi_t(x,\xi) = (y,\eta),\ \pm\tau = p(x,\xi)\}\,,$$

with $\Phi_t : T^*M \to T^*M$ being geodesic flow on the cotangent bundle and $p(x,\xi)$, as above, being the principal symbol of $\sqrt{-\Delta_g}$. Given that we want to restrict the operator in (3.4) to $\gamma = (s,0)$, $0 \leq s \leq 1$, we really need to also focus on the the Fourier integral operator

$$f \to \big(e^{itP}f\big)(s,0)\,.$$

Given the above, its canonical relation is

$$\mathcal{C} = \big\{\Pi_{\gamma\times\boldsymbol{R}}(x,t,\xi,\tau;\,y,\eta) \in T^*(\gamma\times\boldsymbol{R})\times T^*M\,;\ \Phi_t(x,\xi) = (y,\eta),\ \pm\tau = p(x_1,0,\xi)\big\}\,,$$

with $\Pi_{\gamma\times\boldsymbol{R}}$ being the projection map from $T^*(M\times\boldsymbol{R})$ to $T^*(\gamma\times\boldsymbol{R})$. Note that the projection from the latter canonical relation to $T^*(\gamma\times\boldsymbol{R})$ is the map

$$(s,t,\xi) \to (s,t,\xi_1,p((s,0),\xi))\,,$$

which has a fold singularity when $\xi_2 = 0$ but has surjective differential away from this set (given the aforementioned properties of p).

Because of this, given the explicit formula in Fermi coordinates, if we choose $\psi \in C_0^\infty(M)$ equal to one on γ and $\alpha \in C_0^\infty(\boldsymbol{R})$ satisfying $\alpha = 1$ on $[-1/2,1/2]$ but $\alpha(\tau) = 0$, $|\tau| \geq 1$, then

$$b_\varepsilon(x,\xi) = \psi(x)\alpha(\xi_2/\varepsilon|\xi|)$$

equals one on a conic neighborhood of the set that projects onto the set where the left projection of $\mathcal{C}$ has a folding singularity. This means that

$$B_\varepsilon(x,\xi) = \psi(x)(1-\alpha(\xi_2/\varepsilon|\xi|))$$

has symbol vanishing in a conic neighborhood of this set and consequently the map

$$f \to \big(B_\varepsilon\circ e^{itP}f\big)((s,0))\,,\quad 0\leq s\leq 1$$

is a nondegenerate Fourier integral operator of order zero. Therefore, Hörmander's theorem [14] about the L^2 boundedness of Fourier integral operators yields

$$\int_{-N}^{N}\int_0^1 \Big|\big(B_\varepsilon\circ e^{itP}f\big)(s,0)\Big|^2\,dsdt \leq C_{N,B_\varepsilon}\|f\|^2_{L^2(M)}\,.$$

Therefore, an application of Schwarz's inequality yields

$$\|\chi_\lambda^{N,B_\varepsilon}f\|_{L^2(\gamma)} \leq C'_{N,B_\varepsilon}\|f\|_{L^2(M)}\,,$$

if

$$\chi_\lambda^{N,B_\varepsilon} f = B_\varepsilon \circ \chi(N(P-\lambda))f = N^{-1}\int \hat{\chi}(t/N)e^{-it\lambda}\big(B_\varepsilon \circ e^{itP}\big) f\,dt\,.$$

Therefore if we similarly define $\chi_\lambda^{N,b_\varepsilon} f = b_\varepsilon \circ \chi(N(P-\lambda))f$, then $\chi_\lambda^{N,B_\varepsilon} f + \chi_\lambda^{N,b_\varepsilon} f = \psi\chi(N(P-\lambda))f$ and since $\psi = 1$ on γ, the proof of (3.3) would be complete if we could show that if $\varepsilon > 0$ is small enough (depending on N) then for $\lambda \geq 1$ we have for a constant C independent of ε, N and $\lambda \geq 1$

$$\|\chi_\lambda^{N,b_\varepsilon} f\|_{L^2(\gamma)} \leq CN^{-1/2}\lambda^{1/4}\|f\|_{L^2(M)} + C_{N,b_\varepsilon}\|f\|_{L^2(M)}\,. \tag{3.5}$$

In addition to taking $\varepsilon > 0$ to be small, we shall also take the support of ψ about γ to be small.

It is in proving (3.5) of course where we shall use our assumption that γ is not part of a smoothly closed geodesic. A consequence of this is that, given fixed N, if ε and the support of ψ are small enough then

$$b_\varepsilon(y,\eta) = 0 \quad \text{whenever } (y,\eta) = \Phi_t(x,\xi)\,, \quad (x,\xi) \in \operatorname{supp} b_\varepsilon\,, \quad 2 \leq |t| \leq N\,. \tag{3.6}$$

In what follows, we shall assume that ε and ψ have been chosen so that this is the case. The point here is that if $\gamma(s)$, $s \in \boldsymbol{R}$, is the geodesic starting at $(0,0)$ and containing $\{\gamma(s) = (s,0)\,;\, 0 \leq s \leq 1\}$, points on the curve $\gamma(s)$, $|s| \leq N+1$ might intersect γ, but the intersection must be transverse as $s \to \gamma(s)$ is not a smoothly closed geodesic. Then if ε is chosen to be a small multiple of the smallest angle of intersection and if ψ has small enough support about γ, then we get (3.6). Using the canonical relation for e^{itP}, we can deduce from this that

$$b_\varepsilon \circ e^{itP} \circ b_\varepsilon^* \quad \text{is a smoothing operator when} 2 \leq |t| \leq N+1\,, \tag{3.7}$$

i.e., for such times this operator's kernel is smooth.

Let T be the operator $\chi_\lambda^{N,b_\varepsilon} f|_\gamma$, i.e., the truncated approximate spectral projection operator restricted to γ. Our goal is to show (3.5) which says that

$$\|T\|_{L^2(M)\to L^2(\gamma)} \leq CN^{-1/2}\lambda^{1/4} + C_{N,b_\varepsilon}\,.$$

This is equivalent to saying that the dual operator $T^* : L^2(\gamma) \to L^2(M)$ with the same norm, and since

$$\|T^*g\|^2_{L^2(M)} = \int_M T^*g\overline{T^*g}dx = \int_\gamma TT^*g\;\overline{g}ds \leq \|TT^*g\|_{L^2(\gamma)}\|g\|_{L^2(\gamma)}\,,$$

we would be done if we could show that

$$\|TT^*g\|_{L^2(\gamma)} \leq \big(CN^{-1}\lambda^{1/2} + C_{N,b_\varepsilon}\big)\|g\|_{L^2(\gamma)}\,. \tag{3.8}$$

But the kernel of TT^* is $K(\gamma(s),\gamma(s'))$, where $K(x,y)$, $x, y \in M$ is the kernel of the operator $b_\varepsilon \circ \rho(N(P-\lambda)) \circ b_\varepsilon^*$ with $\rho(\tau) = (\chi(\tau))^2$ being the square of χ. Its Fourier transform, $\hat{\rho}$, is the convolution of $\hat{\chi}$ with itself, and thus $\hat{\rho}(t) = 0$, $|t| \geq 1$. Consequently, we can write

$$b_\varepsilon \circ \rho(N(P - \lambda)) \circ b_\varepsilon^* = N^{-1} \int \hat{\rho}(t/N) e^{-it\lambda} \big(b_\varepsilon \circ e^{itP} \circ b_\varepsilon^*\big) dt\,. \tag{3.9}$$

Thus, if $\alpha \in C_0^\infty(\boldsymbol{R})$ is as above, then by (3.6) and (3.7), the difference of the kernel of the operator in (3.9) and the kernel of the operator given by

$$N^{-1} \int \alpha(t/10) \hat{\rho}(t/N) e^{-it\lambda} \big(b_\varepsilon \circ e^{itP} \circ b_\varepsilon^*\big) dt \tag{3.10}$$

is $O(\lambda^{-J})$ for any J. Thus, if we restrict the kernel of the difference to $\gamma \times \gamma$, it contributes a portion of TT^* that maps $L^2(\gamma) \to L^2(\gamma)$ with norm $\leq C_{N,b_\varepsilon}$.

To finish, we need to estimate the remaining piece, which has the kernel of the operator in (3.10) restricted to $\gamma \times \gamma$. Since we are assuming that the injectivity radius of M is 10 or more one can use the Hadamard parametrix for the wave equation and standard stationary phase arguments (similar to ones in [27], Chapter 5, or the proof of Lemma 4.1 in [6]) to see that the kernel $K(x, y)$ of the operator in (3.10) satisfies

$$|K(x, y)| \leq CN^{-1} \lambda^{1/2} (d_g(x, y))^{-1/2} + C_{b_\varepsilon}\,.$$

The first term comes from the main term in the stationary phase expansion for the kernel and the other one is the resulting remainder term in the one-term expansion. Since this kernel restricted to $\gamma \times \gamma$ gives rise to an integral operator satisfying the estimates in (3.8), the proof is complete. □

4. Further questions. While as we explained before the condition that for the L^2-normalized eigenfunctions

$$\limsup_{j\to\infty} \sup_{\gamma\in\Pi} \lambda_j^{-1/2} \int_\gamma |e_{\lambda_j}|^2 ds = 0$$

is a natural one to quantify non-concentration, it would be interesting to formulate a geometric condition involving the long-time dynamics of the geodesic flow that would imply it and its equivalent version that $\lambda_j^{-\delta(p)} \|e_{\lambda_j}\|_p \to 0$, $2 < p < 6$. Presumably if $\gamma \in \Pi$ and

$$\limsup_{j\to\infty} \lambda_j^{-1/2} \int_\gamma |e_{\lambda_j}|^2 ds > 0\,, \tag{4.1}$$

then γ would have to be part of a stable smoothly closed geodesic, and not just a closed geodesic as we showed above. Toth and Zeldtich made a similar conjecture to this in [34], saying that, in n-dimensions, if γ is a closed stable geodesic then one should be able to find a sequence of eigenfunctions on which sup-norms are blowing up like $\lambda^{(n-1)/2}$. In [1], [19], it was shown that there is a sequence of quasimodes blowing up at this rate.

It would also be interesting to formulate a condition that would ensure that $\|e_\lambda\|_{L^6(M)} = o(\lambda^{\delta(6)}) = o(\lambda^{1/6})$, for L^2-normalized eigenfunctions. Presumably, such a condition would have to involve both ones like those in the present paper and conditions of the type in [29], [30]. Since L^6 is an endpoint for (1.1) one expects that one would need a condition that both guarantees that L^p bounds for $2 < p < 6$ and $p > 6$ be small. Formally, the proof of Theorem 1.1 suggests that L^4-norms over geodesics might be relevant for the problem of determining when the $L^6(M)$ norms of eigenfunctions are small. This is interesting because the L^4-norm is the unique L^p-norm taken over geodesics that captures both the concentration of the highest weight spherical harmonics on geodesics and the concentration of zonal functions at points. Indeed, the highest weight spherical harmonics saturate these norms for $2 \le p \le 4$, while the zonal functions saturate them for $p \ge 4$ (see [6]).

Also, it would be interesting to see whether the results here generalize to the case of two-dimensional compact manifolds with boundary. Recently, Smith and the author [24] were able to obtain sharp eigenfunction estimates in this case. In this case, the critical estimate was an L^8 one. So the results here suggest that size estimates for the Kakeya-Nikodym maximal operator associated with broken unit geodesics and applied to squares of eigenfunctions could be relevant for improving the bounds in [24], which are known to be sharp in the case of the disk (see [13]). An observation of Grieser [13] involving the Rayleigh whispering gallery modes suggests that in order to obtain a variant of Corollary 1.2 for compact domains one would have to consider L^2-norms over $\lambda_j^{-2/3}$-neighborhoods of broken geodesics. Smith and the author [23] also showed that for compact manifolds with geodesically concave boundary one has better estimates than one does for compact domains in $\boldsymbol{R}^n$. For example, when $n = 2$ (1.1) holds. Based on this and the better behavior of the geodesic flow, it seems reasonable that the analog of Corollary 1.2 might hold (with the same scales) in this setting.

Finally, as mentioned before it would be interesting to see to what extent the results for the boundaryless case extend to higher dimensions. The arguments given here and in [5], though, rely very heavily on special features of the two-dimensional case.

Acknowledgments. It is a real pleasure to thank J. Bourgain for sharing an early version of his paper [5] and to also thank W. Minicozzi for helpful conversations and for going over a key step in the proof. The author would also like to express his gratitude to J. Toth and S. Zeldtich for helpful discussions and suggestions.

REFERENCES

[1] V. M. BABIČ AND V. F. LAZUTKIN, The eigenfunctions which are concentrated near a closed geodesic, (Russian) Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov (LOMI) 9 (1967), 15–25.

[2] J. BOURGAIN, Besicovitch type maximal operators and applications to Fourier analysis, Geom. Funct. Anal. 1 (1991), 147–187.

[3] J. BOURGAIN, Some new estimates on oscillatory integrals, Essays on Fourier analysis in honor of Elias M. Stein (Princeton, NJ, 1991), 83–12, Princeton Math. Ser. 42, Princeton Univ. Press, Princeton, NJ, 1995.

[4] J. BOURGAIN, L^p-estimates for oscillatory integrals in several variables, Geom. Funct. Anal. 1 (1991), 321–374.

[5] J. BOURGAIN, Geodesic restrictions and L^p-estimates for eigenfunctions of Riemannian surfaces, Linear and complex analysis, 27–35, Amer. Math. Soc. Tranl. Ser. 2, 226, Amer. Math. Soc., Providence, RI, 2009.

[6] N. BURQ, P. GÉRARD AND N. TZVETKOV, Restriction of the Laplace-Beltrami eigenfunctions to submanifolds, Duke Math. J. 138 (2007), 445–486.

[7] L. CARLESON AND P. SJÖLIN, Oscillatory integrals and a multiplier problem for the disc, Sudia Math. 44

(1972), 287–299.

[8] Y. COLIN DE VERDIÈRE, Semi-classical measures and entropy [after Nalini Anantharaman and Stéphane Nonnenmacher], (English summary) Séminaire Bourbaki. Vol. 2006/2007, Astrisque No. 317 (2008), Exp. No. 978, ix, 393–414.

[9] A. CÓRDOBA, A note on Bochner-Riesz operators, Duke Math. J. 46 (1979), 505–511.

[10] J. J. DUISTERMAAT AND V. W. GUILLEMIN, The spectrum of positive elliptic operators and periodic bicharacteristics, Invent. Math. 29 (1975), 39–79.

[11] C. FEFFERMAN, A note on spherical summation operators, Israel J. Math. 15 (1973), 44–52.

[12] A. GREENLEAF AND A. SEEGER, Fourier integrals with fold singularities, J. Reine Angew. Math. 455 (1994), 35–56.

[13] D. GRIESER, L^p bounds for eigenfunctions and spectral projections of the Laplacian near concave boundaries, Ph. D. Thesis, University of California, Los Angeles, 1992.

[14] L. HÖRMANDER, Fourier integral operators. I, Acta Math. 127 (1971), 79–183.

[15] L. HÖRMANDER, Oscillatory integrals and multipliers on FL^p, Ark. Math. II (1973), 1–11.

[16] V. IVRII, The second term of the spectral asymptotics for a Laplace-Beltrami operator on manifolds with boundary, (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), 25–34.

[17] W. P. MINICOZZI AND C. D. SOGGE, Negative results for Nikodym maximal functions and related oscillatory integrals in curved space, Math. Res. Lett. 4 (1997), 221–237.

[18] G. MOCKENHAUPT, A. SEEGER AND C. D. SOGGE, Local smoothing of Fourier integral operators and Carleson-Sjölin estimates, J. Amer. Math. Soc. 6 (1993), 65–130.

[19] J. V. RALSTON, On the construction of quasimodes associated with stable periodic orbits, Comm. Math. Phys. 51 (1976), 219–242.

[20] A. REZNIKOV, Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory, arXiv:math.AP/0403437.

[21] Z. RUDNICK AND P. SARNAK, The behaviour of eigenstates of arithmetic hyperbolic manifolds, Comm. Math. Phys 161 (1994), 195–213.

[22] A. SCHNIRELMAN, Ergodic properties of eigenfunctions, Usp. Math. Nauk. 29 (1974), 181–182.

[23] H. SMITH AND C. D. SOGGE, On the critical semilinear wave equation outside convex obstacles, J. Amer. Math. Soc. 8 (1995), 879–916.

[24] H. SMITH AND C. D. SOGGE, On the L^p norm of spectral clusters for compact manifolds with boundary, Acta Math. 198 (2007), 107–153.

[25] C. D. SOGGE, Oscillatory integrals and spherical harmonics, Duke Math. J. 53 (1986), 43–65.

[26] C. D. SOGGE, Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds, J. Funct. Anal. 77 (1988), 123–138.

[27] C. D. SOGGE, Fourier integrals in classical analysis, Cambridge Tracts in Math., Cambridge Univ. Press, Cambridge, 1993.

[28] C. D. SOGGE, Concerning Nikodym-type sets in 3-dimensional curved spaces, J. Amer. Math. Soc. 12 (1999), 1–31.

[29] C. D. SOGGE, J. TOTH AND S. ZELDITCH, About the blowup of quasimodes on Riemannian manifolds, to appear, J. Geom. Anal.

[30] C. D. SOGGE AND S. ZELDITCH, Riemannian manifolds with maximal eigenfunction growth, Duke Math. J. 114 (2002), 387–437.

[31] E. M. STEIN, Oscillatory integrals in Fourier analysis, Beijing lectures in harmonic analysis (Beijing, 1984), 307–355, Ann. of Math. Stud.,112, Princeton Univ. Press, Princeton, NJ, 1986.

[32] D. TATARU, On the regularity of boundary traces for the wave equation, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 26 (1998), 185–206.

[33] J. TOTH, L^2-restriction bounds for eigenfunctions along curves in the quantum completely integrable case, Comm. Math. Phys. 288 (2009), 379–401.

[34] J. TOTH AND S. ZELDITCH, L^p norms of eigenfunctions in the completely integrable case, Ann. Henri Poincaré 4 (2003), 343–368.

[35] S. ZELDITCH, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (1987), 919–941.

[36] S. ZELDITCH, Mean Lindelöf hypothesis and equidistribution of cusp forms and Eisenstein series, J. Funct. Anal. 97 (1991), 1–49.

[37] A. ZYGMUND, On Fourier coefficients and transforms of two variables, Studia Math. 50 (1974), 189–201.

DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD 21218
U.S.A.

E-mail address: sogge@jhu.edu