Kalman filter for vision tracking

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Abstract

The Kalman filter has been used successfully in different prediction applications or state determination of a system. One important field in computer vision is the object tracking. Different movement conditions and occlusions can hinder the vision tracking of an object. In this report we present the use of the Kalman filter in the vision tracking. We consider the capacity of the Kalman filter to allow small occlusions and also the use of the extended Kalman filter (EKF) to model complex movements of objects.

1 Introduction

The celebrated *Kalman filter*, rooted in the state-space formulation or linear dynamical systems, provides a recursive solution to the linear optimal filtering problem. It applies to stationary as well as nonstationary environments. The solution is recursive in that each updated estimate of the state is computed from the previous estimate and the new input data, so only the previous estimate requires storage. In addition to eliminating the need for storing the entire past observed data, the Kalman filter is computationally more efficient than computing the estimate directly from the entire past observed data at each step of the filtering process.

In this section, we present an introductory treatment of Kalman filters to pave the way for their application in vision tracking.

Consider a *linear, discrete-time dynamical system* described by the block diagram shown in Figure 1. The concept of *state* is fundamental to this description. The *state vector* or simply state, denoted by $\mathbf{x}_k$, is defined as the minimal set of data that is sufficient to uniquely describe the unforced dynamical behavior of the system; the subscript k denotes discrete time. In other words, the state is the least amount of data on the past behavior of the system that is needed to predict its future behavior. Typically, the state $\mathbf{x}_k$ is unknown. To estimate it, we use a set of observed data, denoted by the vector $\mathbf{y}_k$.

In mathematical terms, the block diagram of Figure 1 embodies the following pair of equations:

1. Process equation

$$\mathbf{x}_{k+1} = \mathbf{F}_{k+1,k}\mathbf{x}_k + \mathbf{w}_k \tag{1}$$

where $\mathbf{F}_{k+1,k}$ is the *transition matrix* taking the state $\mathbf{x}_k$ from time k to time $k+1$. The process noise $\mathbf{w}_k$ is assumed to be additive, white, and Gaussian, with zero mean and with covariance matrix defined by

$$E\left[\mathbf{w}_n\mathbf{w}_k^T\right] = \begin{cases} \mathbf{Q}_k \text{ for } n{=}k \\ 0 \text{ for } n \neq k \end{cases} \tag{2}$$

where the superscript T denotes matrix transposition. The dimension of the state space is denoted by M.

2. Measurement equation

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k \tag{3}$$

where $\mathbf{y}_k$ is the observable at time k and $\mathbf{H}_k$ is the *measurement matrix.* The measurement noise $\mathbf{v}_k$ is assumed to be addilive, white, and Gaussian, with zero mean and with covariance matrix defined by

$$E\left[\mathbf{v}_n\mathbf{v}_k^T\right] = \begin{cases} \mathbf{R}_k \text{ for } n{=}k \\ 0 \text{ for } n \neq k \end{cases} \tag{4}$$

Moreover, the measurement noise $\mathbf{v}_k$is uncorrelated with the process noise $\mathbf{w}_k$. The dimension of the measurement space is denoted by N.

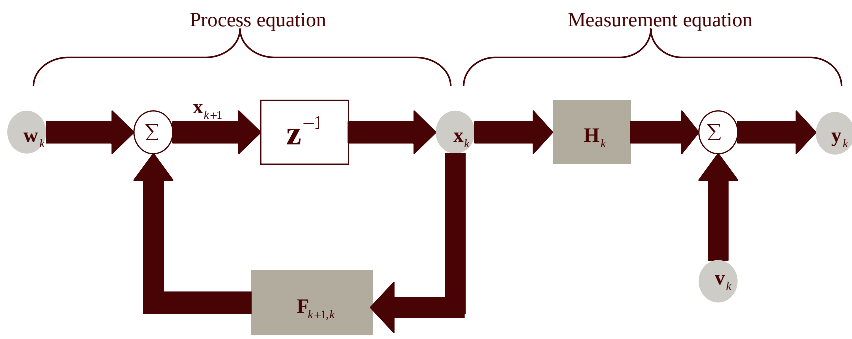

Figure 1: Signal-flow graph representation of a linear, distrete-time dynamical system.

The Kalman filtering problem, namely, the problem of jointly solving the process and measurement equations for the unknown state in an optimum manner may now be formally stated as follows:

Use the entire observed data, consisting of the vectors $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_k$, to find for each $k \geq 1$ the minimum mean-square error estimate of the state $\mathbf{x}_k$.

The problem is called filtering if $i = k$, prediction if $i > k$ and smoothing if $1 \leq i < k$.

2 Optimum estimates

Before proceeding to derive the Kalman filter, we find it useful to review some concepts basic to optimum estimation. To simplify matters, this review is presented in the context of scalar random variables; generalization of the theory to vector random variables is a straightforward matter. Suppose we are given the observable

$$y_k = x_k + v_k \tag{5}$$

where x_k is an unknown signal and v_k is an additive noise component. Let x_k denote the a posteriori estimate of the signal x_k, given the observations $y_1, y_2, ..., y_k$. In general, the estimate $\hat{x}_k$ is different from the unknown signal x_k. To derive this estimate in an optimum manner, we need a cost (loss) function for incorrect estimates. The cost function should satisfy two requirements:

The cost function is nonnegative.

The cost function is a nondecreasing function of the *estimation error* x_k defined by

$$\tilde{x}_k = x_k - \hat{x}_k \tag{6}$$

These two requirements are satisfied by the *mean-square error* defined by

$$J_k = E[(x_k - \hat{x}_k)^2]$$

$$J_k = E[(\tilde{x}_k)^2] \tag{7}$$

where E is the expectation operator. The dependence of the cost function J_k on time k emphasizes the nonstationary nature of the recursive estimation process.

To derive an optimal value for the estimate $\hat{x}_k$ we may invoke two theorems taken from stochastic process theory [1, 2]:

Theorem 1. Conditional mean estimator If the stochastic processes $\{x_k\}$ and $\{y_k\}$ are jointly Gaussian, then the optimum estimate $\hat{x}_k$ that minimizes the mean-square error $J_k$is the conditional mean estimator:

$$\hat{x}_k = E[x_k \,|y_1 \;, y_2, ..., y_k] \tag{8}$$

Theorem 2. Principle of orthogonality Let the stochastic processes $\{x_k\}$ and $\{y_k\}$ be of zero means; that is,

$$E[x_k] = E[y_k] = 0 \text{ for all } k \tag{9}$$

Then:

(i) the stochastic process $\{x_k\}$ and $\{y_k\}$ are jointly Gaussian; or

(ii) if the optimal estimate $\hat{x}_k$ is restricted to be a linear function of the observables and the cost function is the mean-square error,

(iii) then the optimum estimate $\hat{x}_k$ given the observables $y_1, y_2, ..., y_k$ is the orthogonal projection of x_k on the space spanned by these observables.

3 Kalman filter

Suppose that a measurement on a linear dynamical system, described by Eqs. (1) and (3), has been made at time k. The requirement is to use the information contained in the new measurement $\mathbf{y}_k$ to update the estimate of the unknown state $\mathbf{x}_k$. Let $\hat{\mathbf{x}}_k^-$ denote a priori estimate of the state, which is already available at time k. With a linear estimator as the objective, we may express the a posteriori estimate $\hat{\mathbf{x}}_k$ as a linear combination of the a priori estimate and the new measurement, as shown by

$$\hat{\mathbf{x}}_k = \mathbf{G}_k^{(1)}\hat{\mathbf{x}}_k^- + \mathbf{G}_k\mathbf{y}_k \tag{10}$$

where the multiplying matrix factors $\mathbf{G}_k^{(1)}$ and $\mathbf{G}_k$ are to be determined. The *state-error* vector is defined by

$$\tilde{\mathbf{x}}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k \tag{11}$$

Applying the principle of orthogonality to the situation at hand, we may thus write

$$E\left[\tilde{\mathbf{x}}_k\mathbf{y}_i^T\right] = 0 \text{ for } i = 1, 2, ..., k-1 \tag{12}$$

Using Eqs. (3), (10), and (11) in (12), we get

$$E[(\mathbf{x}_k - \mathbf{G}_k^{(1)}\hat{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{H}_k\mathbf{x}_k - \mathbf{G}_k\mathbf{v}_k)\mathbf{y}_i^T] = 0 \text{ for } i = 1, 2..., k. \tag{13}$$

Since the process noise $\mathbf{w}_k$ and measurement noise $\mathbf{v}_k$ are uncorrelated, it follows that

$$E[\mathbf{v}_k\mathbf{y}_i^T] = 0 \tag{14}$$

Using this relation and adding the element $\mathbf{G}_k^{(1)}\mathbf{x}_k - \mathbf{G}_k^{(1)}\mathbf{x}_k$, we may rewrite Eq. (13) as

$$E[(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k - \mathbf{G}_k^{(1)})\mathbf{x}_k\mathbf{y}_i^T + \mathbf{G}_k^{(1)}(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)\mathbf{y}_i^T] = 0 \tag{15}$$

where $\mathbf{I}$ is the identity matrix. From the principle of orthogonality, we now note that

$$E[(\mathbf{x}_k - \mathbf{x}_k^-)\mathbf{y}_i^T] = 0 \tag{16}$$

Accordingly, Eq. (15) simplifies to

$$(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k - \mathbf{G}_k^{(1)})E[\mathbf{x}_k\mathbf{y}_i^T] = 0 \text{ for } i = 1, 2, ..., k-1 \tag{17}$$

For arbitrary values of the state $\mathbf{x}_k$ and observable $\mathbf{y}_i$, Eq. (1.17) can only be satisfied if the scaling factors $\mathbf{G}_k^{(1)}$ and $\mathbf{G}_k$ are related as follows:

$$\mathbf{I} - \mathbf{G}_k\mathbf{H}_k - \mathbf{G}_k^{(1)} = 0 \tag{18}$$

or, equivalently, $\mathbf{G}_k^{(1)}$ is defined in terms of $\mathbf{G}_k$ as

$$\mathbf{G}_k^{(1)} = \mathbf{I} - \mathbf{G}_k\mathbf{H}_k \tag{19}$$

Substituting Eq. (19) into (10), we may express the a posteriori estimate of the state at time k as

$$\mathbf{x}_k = \hat{\mathbf{x}}_k^- + \mathbf{G}_k(\mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-) \tag{20}$$

in light of which, the matrix $\mathbf{G}_k$ is called the *Kalman gain.*

There now remains the problem of deriving an explicit formula for $\mathbf{G}_k$. Since, from the principle of orthogonality, we have

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\mathbf{y}_i^T] = 0 \tag{21}$$

it follows that

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\hat{\mathbf{y}}_i^T] = 0 \tag{22}$$

where $\hat{\mathbf{y}}_k^T$ is an estimate of $\mathbf{y}_k$ given the previous measurement $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_{k-1}$. Define the innovations process

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k \tag{23}$$

The innovation process represents a measure of the "new" information contained in $\mathbf{y}_k$; it may also be expressed as

$$\tilde{\mathbf{y}}_k = \mathbf{y}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-$$

$$= \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k - \mathbf{H}_k\hat{\mathbf{x}}_k^-$$

$$= \mathbf{v}_k + \mathbf{H}_k\tilde{\mathbf{x}}_k^- \tag{24}$$

Hence, subtracting Eq. (22) from (21) and then using the definition of Eq. (23), we may write

$$E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)\tilde{\mathbf{y}}_k^T] = 0 \tag{25}$$

Using Eqs. (3) and (20), we may express the state-error vector $\mathbf{x}_k - \hat{\mathbf{x}}_x$ as

$$\mathbf{x}_k - \hat{\mathbf{x}}_k = \tilde{\mathbf{x}}_k^- - \mathbf{G}_k(\mathbf{H}_k\tilde{\mathbf{x}}_k^- + \mathbf{v}_k)$$

$$= (\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k \tag{26}$$

Hence, substimting Eqs. (24) and (26) into (25), we get

$$E[\{(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\tilde{\mathbf{x}}_k^- - \mathbf{G}_k\mathbf{v}_k\}(\mathbf{H}_k\tilde{\mathbf{x}}_k^- + \mathbf{v}_k)] = 0 \tag{27}$$

Since the measurement noise $\mathbf{v}_k$ is independent of the state $\mathbf{x}_k$ and therefore the error $\tilde{\mathbf{x}}_k^-$ the expectation of Eq. (27) reduces to

$$(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)E[\tilde{\mathbf{x}}_k^-\tilde{\mathbf{x}}_k^{-T}]\mathbf{H}_k^T - \mathbf{G}_kE[\mathbf{v}_k\mathbf{v}_k^T] = 0 \tag{28}$$

Define the *a priori covariance matrix*

$$\mathbf{P}_k^- = E[(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)(\mathbf{x}_k - \hat{\mathbf{x}}_k^-)^T]$$

$$= E[\tilde{\mathbf{x}}_k^-\tilde{\mathbf{x}}_k^{-T}] \tag{29}$$

Then, invoking the covariance definitions of Eqs. (4) and (29), we may rewrite Eq. (28) as

$$(\mathbf{I} - \mathbf{G}_k\mathbf{H}_k)\mathbf{P}_k^-\mathbf{H}_k^T - \mathbf{G}_k\mathbf{R}_k = 0 \tag{30}$$

Solving this equation for $\mathbf{G}_k$, we get the desired formula

$$\mathbf{G}_k = \mathbf{P}_k^-\mathbf{H}_k^T[\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^T + \mathbf{R}_k]^{-1} \tag{31}$$

where the symbol $[\bullet]^{-1}$ denotes the inverse of the matrix inside the square brackets. Equation (22) is the desired formula for computing the Kalman gain $\mathbf{G}_k$, which is defined in terms of the a priori covariance matrix $\mathbf{P}_k^-$. To complete the recursive estimation procedure, we consider the *error covariance propagation*, which describes the effects of time on the covariance matrices of estimation errors. This propagation involves two stages of computation:

1. The a priori covariance matrix $\mathbf{P}_k^-$ at time k is defined by Eq. (1.21). Given $\mathbf{P}_k^-$, compute the a posteriori covariance matrix $\mathbf{P}_k$, which, at time k, is defined by

$$\mathbf{P}_k = E[\tilde{\mathbf{x}}_k \tilde{\mathbf{x}}_k{}^T]$$

$$= E[(\mathbf{x}_k - \hat{\mathbf{x}}_k)(\mathbf{x}_k - \hat{\mathbf{x}}_k)^T] \tag{32}$$

2. Given the "old" a posteriori covariance matrix, $\mathbf{P}_{k-1}$, compute the "updated" a priori covariance matrix $\mathbf{P}_k^-$.

To proceed with stage 1, we substitute Eq. (26) into (32) and note that the noise process $\mathbf{v}_k$ is independent of the a priori estimation error $\tilde{\mathbf{x}}_k^-$. We thus obtain

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) E[\tilde{\mathbf{x}}_k^- \tilde{\mathbf{x}}_k^{-T}](\mathbf{I} - \mathbf{G}_k \mathbf{H}_k)^T + \mathbf{G}_k E[\mathbf{v}_k \mathbf{v}_k^T] \mathbf{G}_k^T$$

$$= (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) \mathbf{P}_k^- (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k)^T + \mathbf{G}_k \mathbf{R}_k \mathbf{G}_k^T \tag{33}$$

Expanding terms in Eq. (33) and then using Eq. (31), we may reformulate the dependence of the a posteriori covariance matrix $\mathbf{P}_k$ on the a priori covariance matrix $\mathbf{P}_k^-$ in the simplified form

$$\mathbf{P}_k = (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) \mathbf{P}_k^- - (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) \mathbf{P}_k^- \mathbf{H}_k^T \mathbf{G}_k^T + \mathbf{G}_k \mathbf{R}_k \mathbf{G}_k^T$$

$$= (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) \mathbf{P}_k^- - \mathbf{G}_k \mathbf{R}_k \mathbf{G}_k^T + \mathbf{G}_k \mathbf{R}_k \mathbf{G}_k^T$$

$$= (\mathbf{I} - \mathbf{G}_k \mathbf{H}_k) \mathbf{P}_k^- \tag{34}$$

For the second stage of error covariance propagation, we first recognize that the a priori estimate of the state is defined in terms of the "old" a posteriori estimate as follows:

$$\tilde{\mathbf{x}}_k^- = \mathbf{F}_{k,k-1} \hat{\mathbf{x}}_{k-1} \tag{35}$$

We may therefore use Eqs. (1) and (35) to express the a priori estimation error in yet another form:

$$\tilde{\mathbf{x}}_k^- = \mathbf{x}_k - \hat{\mathbf{x}}_k^-$$

$$= (\mathbf{F}_{k,k-1}\mathbf{x}_{k-1} + \mathbf{w}_{k-1}) - (\mathbf{F}_{k,k-1}\hat{\mathbf{x}}_{k-1})$$

$$= \mathbf{F}_{k,k-1}(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1}) + \mathbf{w}_{k-1}$$

$$= \mathbf{F}_{k,k-1}\tilde{\mathbf{x}}_{k-1} + \mathbf{w}_{k-1} \tag{36}$$

Accordingly, using Eq. (36) in (29) and noting that the process noise $\mathbf{w}_k$ is independent of $\hat{\mathbf{x}}_{k-1}$ we get

$$\mathbf{P}_k^- = \mathbf{F}_{k,k-1}E[\tilde{\mathbf{x}}_{k-1}\tilde{\mathbf{x}}_{k-1}^T]\mathbf{F}_{k,k-1}^T + E[\mathbf{w}_{k-1}\mathbf{w}_{k-1}^T]$$

$$= \mathbf{F}_{k,k-1}\mathbf{P}_{k-1}\mathbf{F}_{k,k-1}^T + \mathbf{Q}_{k-1} \tag{37}$$

which defines the dependence of the a priori covariance matrix $\mathbf{P}_k^-$ on the "old" a posteriori covariance matrix $\mathbf{P}_{k-1}$.

With Eqs. (35), (37), (31), (20), and (34) at hand, we may now summarize the recursive estimation of state as shown in figure 2. This figure also includes the initialization. In the absence of any observed data at time $k = 0$, we may choose the initial estimate of the state as

$$\mathbf{x}_0 = E[\mathbf{x}_0] \tag{38}$$

and the initial value of the a posteriori covariance matrix as

$$\mathbf{P}_0 = E[(\mathbf{x}_0 - E[\mathbf{x}_0])((\mathbf{x}_0 - E[\mathbf{x}_0])^T] \tag{39}$$

This choice for the initial conditions not only is intuitively satisfying but also has the adventage of yielding an *unbiased* estimate of the state $\mathbf{x}_k$.

The Kalman filter uses Gaussian probability density in the propagation process, the diffusion is purely linear and the density function evolves as a gaussian pulse that translates, spreads and reinforced, remaining gaussian throughout.

The random component of the dynamical model $\mathbf{w}_k$ leads to spreading (increasing uncertainty) while the deterministic component $\mathbf{F}_{k+1,k}\mathbf{x}_k$ causes the density function to drift bodily. The effect of an external observation y is to superimpose a reactive effect on the diffusion in which the density tends to peak in the vicinity of observations. The figure 3 shows the propagation form of the density function using the Kalman filter.

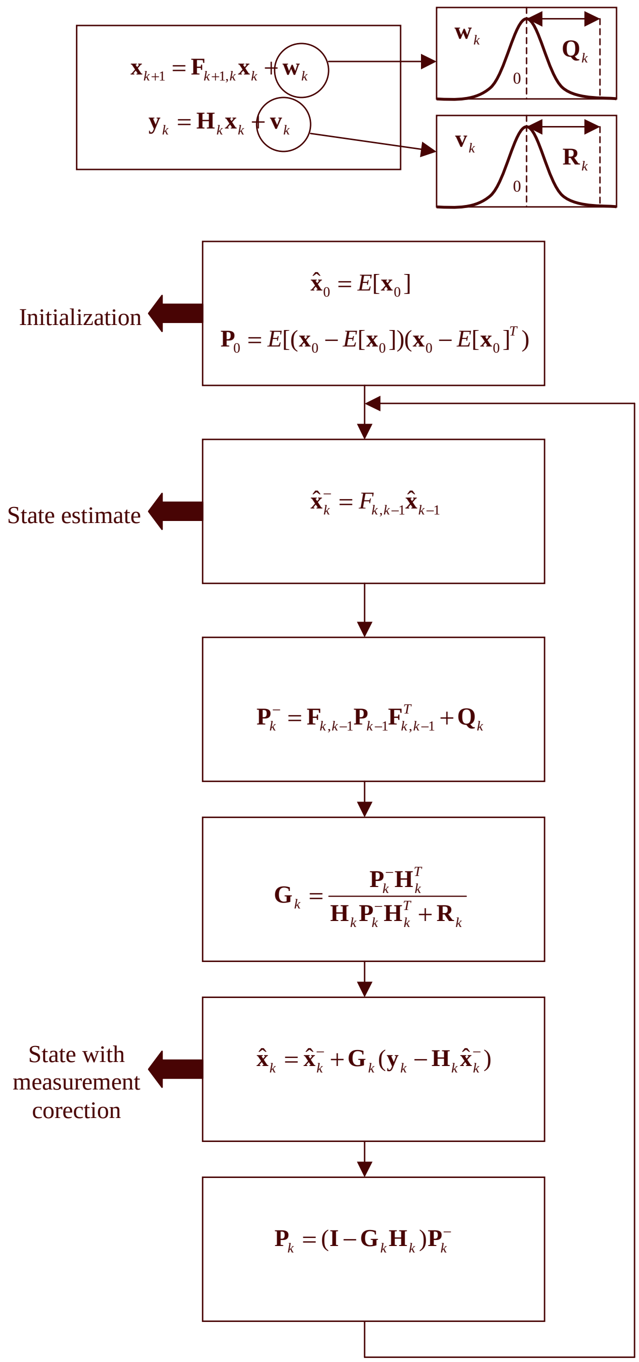

Figure 2: Sumary of the Kalman filter

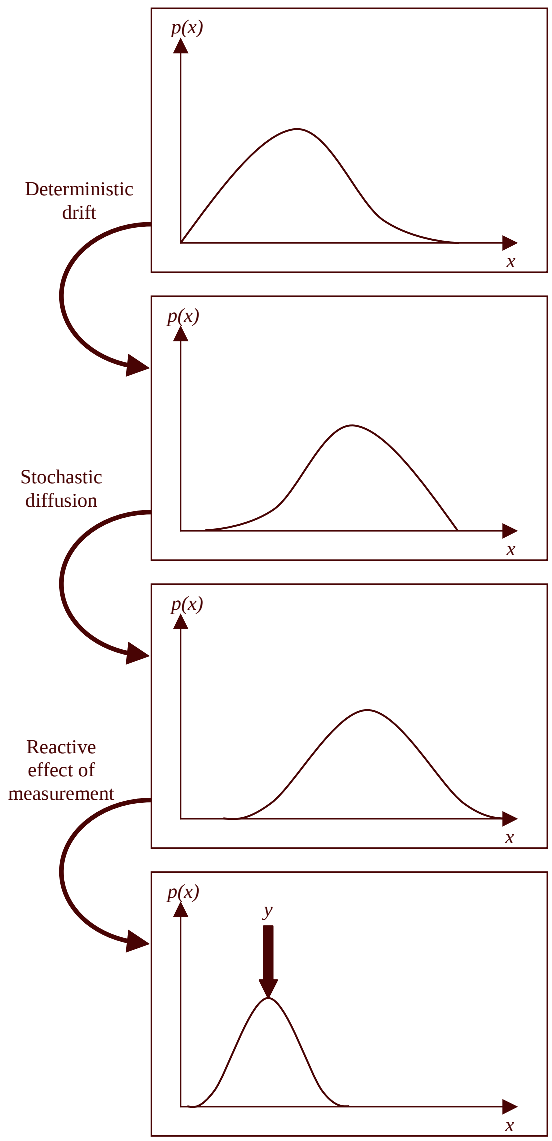

Figure 3: Kalman filter as density propagation

4 Extended Kalman filter

The Kalman filtering problem considered up to this point in the discussion has addressed the estimation of a state vector in a linear model of a dynamical system. If, however, the model is *nonlinear*, we may extend the use of Kalman filtering through a linearization procedure. The resulting filter is referred to as *the extended Kalman filter* (**EKF**) [3-5]. Such an extension is feasible by virtue of the fact that the Kalman filter is described in terms of difference equations in the case of discrete-time systems.

To set the stage for a development of the extended Kalman filter, consider a nonlinear dynamical system described by the state-space model

$$\mathbf{x}_{k+1} = \mathbf{f}(k, \mathbf{x}_k) + \mathbf{w}_k \tag{40}$$

$$\mathbf{y}_k = \mathbf{h}(k, \mathbf{x}_k) + \mathbf{v}_k \tag{41}$$

where, as before, $\mathbf{w}_k$ and $\mathbf{v}_k$ are independent zero-mean white Gaussian noise processes with covariance matrices $\mathbf{R}_k$ and $\mathbf{Q}_k$ respectively. Here, however, the functional $\mathbf{f}(k, \mathbf{x}_k)$ denotes a nonlinear transition matrix function that is possibly time-variant. Likewise, the functional $\mathbf{h}(k, \mathbf{x}_k)$ denotes a nonlinear measurement matrix that may be time-variant, too.

The basic idea of the extended Kalman filter is to linearize the state-space model of Eqs. (52) and (53) at each time instant around the most recent state estimate, which is taken to be either $\hat{\mathbf{x}}_k$ or $\hat{\mathbf{x}}_k^-$ epending on which particular functional is being considered. Once a linear model is obtained, the standard Kalman filter equations are applied.

More explicitly, the approximation proceeds in two stages.

Stage 1. The following two matrices are constructed:

$$\mathbf{F}_{k+1,k} = \left.\frac{\partial \mathbf{f}(k, \mathbf{x}_k)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k} \tag{42}$$

$$\mathbf{H}_k = \left.\frac{\partial \mathbf{h}(k, \mathbf{x}_k)}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}_k} \tag{43}$$

That is, the ijth entry of $\mathbf{F}_{k+1,k}$ is equal to the partial derivative of the ith component of $\mathbf{F}(k, \mathbf{x})$ with respect to the yth component of $\mathbf{x}$. Likewise, the ijth entry of $\mathbf{H}_k$ is equal to the partial derivative of the ith component of $\mathbf{H}(k, \mathbf{x})$ with respect to the jth component of $\mathbf{x}$. In the former case, the derivatives are evaluated at $\hat{\mathbf{x}}_k$ while in the latter case, the derivatives are evaluated at $\hat{\mathbf{x}}_k^-$. The entries of the matrices $\mathbf{F}_{k+1,k}$ and $\mathbf{H}_k$ are all known (i.e., computable), by having $\hat{\mathbf{x}}_k$ and $\hat{\mathbf{x}}_k^-$ available at time k.

Stage 2. Once the matrices $\mathbf{F}_{k+1,k}$ and $\mathbf{H}_k$ are evaluated, they are then employed in a *first-order Taylor approximation* of the nonlinear functions $\mathbf{F}(k, \mathbf{x})$ and $\mathbf{H}(k, \mathbf{x})$ around $\hat{\mathbf{x}}_k$ and $\hat{\mathbf{x}}_k^-$, respectively. Specifically, $\mathbf{F}(k, \mathbf{x})$ and $\mathbf{H}(k, \mathbf{x})$ are approximated as follows

$$\mathbf{F}(k, \mathbf{x}_k) \approx \mathbf{F}(\mathbf{x}, \hat{\mathbf{x}}_k) + \mathbf{F}_{k+1,k}(\mathbf{x}, \hat{\mathbf{x}}_k) \tag{44}$$

$$\mathbf{H}(k, \mathbf{x}_k) \approx \mathbf{H}(\mathbf{x}, \hat{\mathbf{x}}_k^-) + \mathbf{H}_{k+1,k}(\mathbf{x}, \hat{\mathbf{x}}_k^-) \tag{45}$$

With the above approximate expressions at hand, we may now proceed to approximate the nonlinear state equations (40) and (41) as shown by, respectively,

$$\mathbf{x}_{k+1} \approx \mathbf{F}_{k+1,k}\mathbf{x}_k + \mathbf{w}_k + \mathbf{d}_k \tag{46}$$

$$\bar{\mathbf{y}}_k \approx \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k \tag{47}$$

where we have introduced two new quantities:

$$\bar{\mathbf{y}}_k = \mathbf{y}_k - \{\mathbf{h}(\mathbf{x}, \hat{\mathbf{x}}_k^-) - \mathbf{H}_k\hat{\mathbf{x}}_k^-\} \tag{48}$$

$$\mathbf{d}_k = \mathbf{f}(\mathbf{x}, \hat{\mathbf{x}}_k) - \mathbf{F}_{k+1,k}\hat{\mathbf{x}}_k \tag{49}$$

The entries in the term $\bar{\mathbf{y}}_k$ are all known at time k, and, therefore, $\bar{\mathbf{y}}_k$ can be regarded as an observation vector at time n. Likewise, the entries in the term $\mathbf{d}_k$ are all known at time k.

Given the linearized slate-space model of Eqs. (48) and (49), we may then proceed and apply the Kalman filler theory of section 3 to derive the extended Kalman filter. Figure 4 summarizes the recursions involved in computing the extended Kalman filter.

5 Vision Tracking with the Kalman filter

The main application of the Kalman filter in robot vision is the following object, also called *tracking*. To carry out this, it is necessary to calculate the object position and speed in each instant. As input is considered a sequence of images captured by a camera containing the object. Then using a image processing method the object is segmented and later calculated their position in the image. Therefore we will take as system state $\mathbf{x}_k$the position x and y of the object in the instant k. Considering the above-mentioned we can use the Kalman filter to make more efficient the localization method of the object, that is to say instead

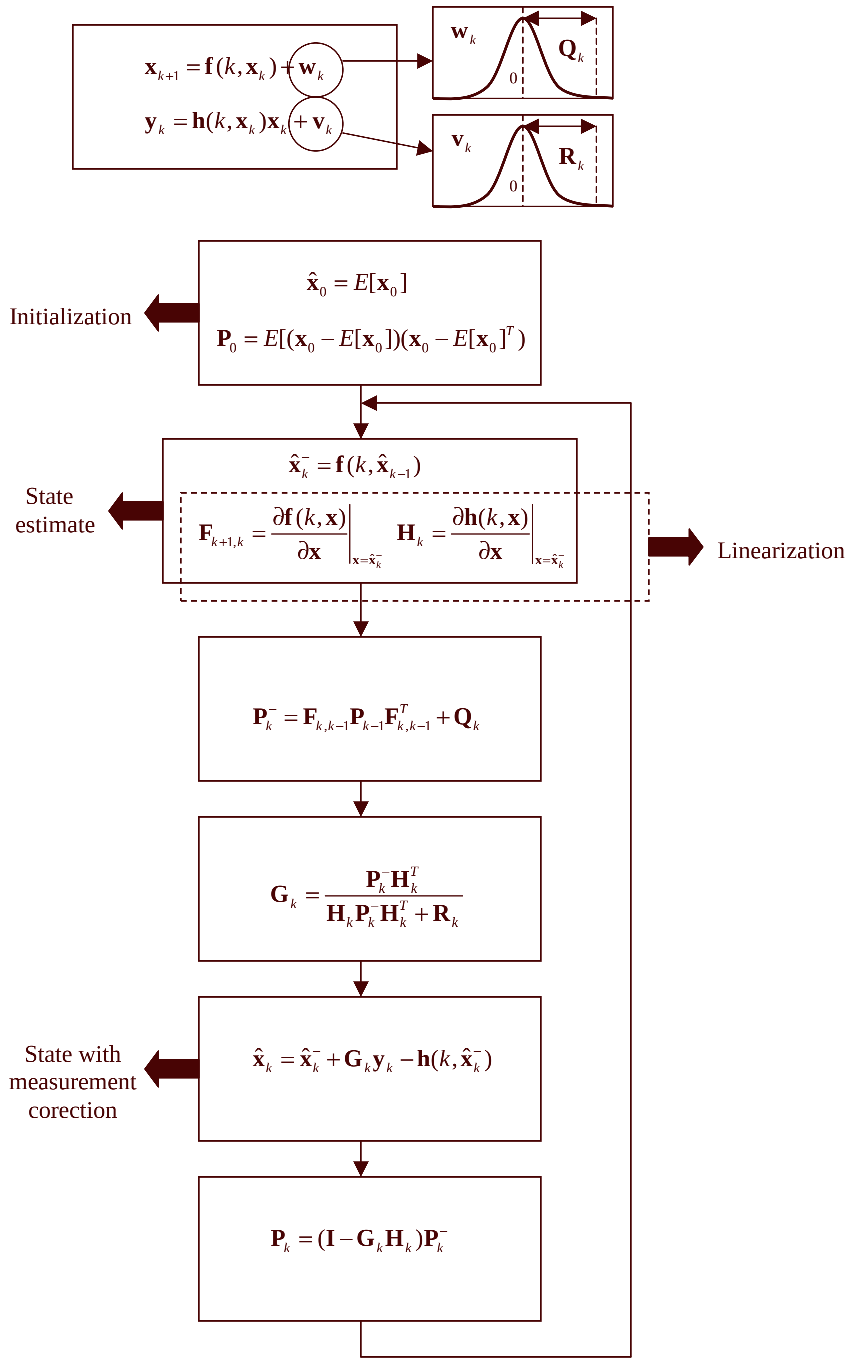

Figure 4: Summary of the Extended Kalman Filter

of looking for to the object in the whole image plane we define a search window centered in the predicted value $\hat{\mathbf{x}}_k^-$ of the filter.

The steps to use the Kalman filter for vision tracking are:

1. Initialization (k=0). In this step it is looked for the object in the whole image due we do not know previously the object position. We obtain this way $\mathbf{x}_0$. Also we can considerer initially a big error tolerance ($\mathbf{P}_0 = 1$).

2. Prediction (k>0). In this stage using the Kalman filter we predict the relative position of the object, such position $\hat{\mathbf{x}}_k^-$ is considered as search center to find the object.

3. Correction (k>0). In this part we locate the object (which is in the neighborhood point predicted in the previous stage $\hat{\mathbf{x}}_k^-$) and we use its real position (measurement) to carry out the state correction using the Kalman filter finding this way $\hat{\mathbf{x}}_k$.

The steps 2 and 3 are carried out while the object tracking runs. To exemplify the results of the use of the Kalman filter in vision tracking, we choose the tracking of a soccer ball and consider the following cases:

a) In this test we carry out the ball tracking considering a lineal uniform movement, which could be described by the following system equations

$$\mathbf{x}_{k+1} = \mathbf{F}_{k+1,k}\mathbf{x}_k + \mathbf{w}_k$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \Delta x_k \\ \Delta y_k \end{bmatrix} + \mathbf{w}_k$$

$$\mathbf{y}_k = \mathbf{H}_k\mathbf{x}_k + \mathbf{v}_k$$

$$\begin{bmatrix} xm_k \\ ym_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \Delta x_k \\ \Delta y_k \end{bmatrix} + \mathbf{v}_k$$

In the figure 5 the prediction of the object position is shown for each instant as well as the real trajectory.

b) One advantage of the Kalman filter for the vision tracking is that can be used to tolerate small occlusions. The form to carrying out it, is to consider the two work phases of the filter, prediction and correction. That is to say, if the object

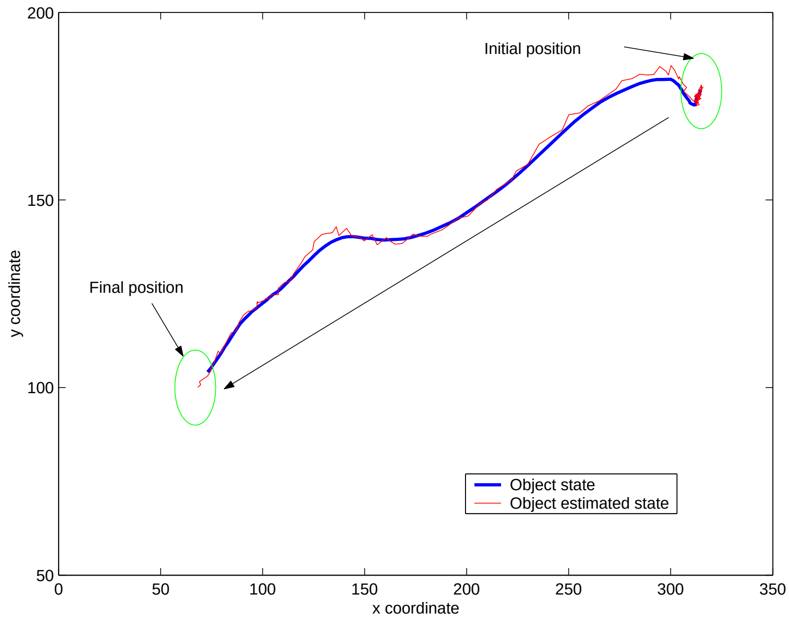

Figure 5: Position prediction with the Kalman filter

localization is not in the neighborhood of the predicted state by the filter (in the instant k), we can consider that the object is hidden by some other object, consequently we will not use the measurement correction and will only take as object position the filter prediction. The figure 6 shows the filter performance during the object occlusion. The system was modeled with the same equations used in the previous case.

c) Most of the complex dynamic trajectories (changes of acceleration) cannot be modeled by lineal systems, which results in that we have to use for the modeling nonlinear equations, therefore in these cases we will use the extended Kalman filter. The figure 7 shows the acting of the extended Kalman filter for the vision tracking of a complex trajectory versus the poor performance of the normal Kalman filter. For the extended Kalman filter the dynamic system was modeled using the unconstrained Brownian motion equations

$$\mathbf{x}_{k+1} = \mathbf{f}(k, \mathbf{x}_k) + \mathbf{w}_k$$

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ \Delta x_{k+1} \\ \Delta y_{k+1} \end{bmatrix} = \begin{bmatrix} \exp\left(-\frac{1}{4}(x_k + 1.5\Delta x_k)\right) \\ \exp\left(-\frac{1}{4}(y_k + 1.5\Delta y_k)\right) \\ \exp\left(-\frac{1}{4}\Delta x_k\right) \\ \exp\left(-\frac{1}{4}\Delta y_k\right) \end{bmatrix} + \mathbf{w}_k$$

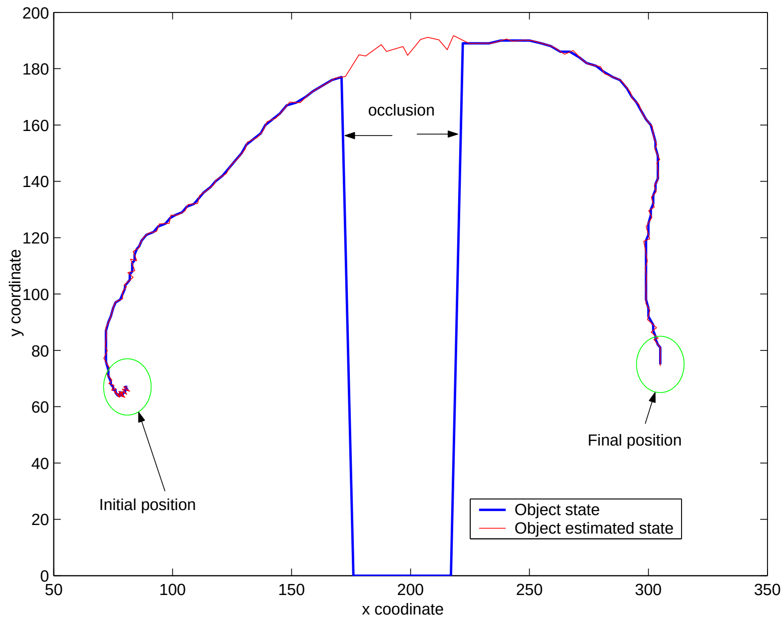

Figure 6: Kalman filter during the occlusion

$$\mathbf{y}_k = \mathbf{h}(k, \mathbf{x}_k) + \mathbf{v}_k$$

$$\begin{bmatrix} xm_k \\ ym_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ \Delta x_k \\ \Delta y_k \end{bmatrix} + \mathbf{v}_k$$

while for the normal Kalman filter, the system was modeled using the equations of the case a).

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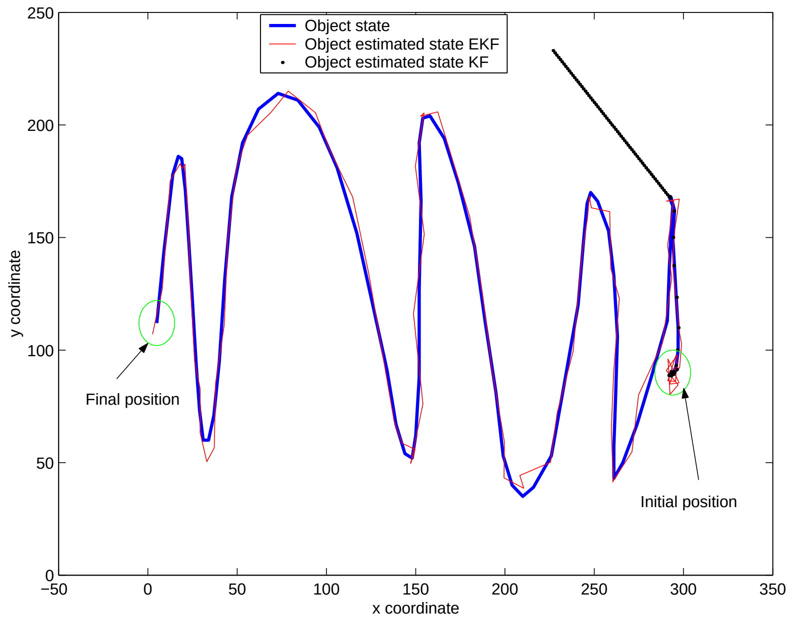

Figure 7: Tracking of a complex movement using an EKF.

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