

Kalman Filtering Over A Packet Dropping Network: A Probabilistic Approach

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Abstract—We consider the problem of state estimation of a discrete time process over a packet dropping network. Previous pioneering work on Kalman filtering with intermittent observations is concerned with the asymptotic behavior of $\mathbb{E}[P_k]$, i.e., the expected value of the error covariance, for a given packet arrival rate. We consider a different performance metric, $\Pr[P_k \leq M]$, i.e., the probability that P_k is bounded by a given M , and we derive lower and upper bounds on $\Pr[P_k \leq M]$. We are also able to recover the results in the literature when using $\Pr[P_k \leq M]$ as a metric for scalar systems. Examples are provided to illustrate the theory developed in the paper.

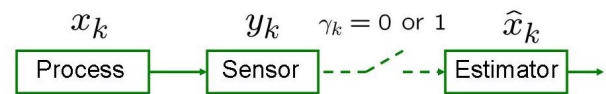
Index Terms—Networked estimation, Kalman filtering, Packet-dropping network.

I. INTRODUCTION

In the past decade, networked control systems (NCS) have gained much attention from both the control community and the network and communication community [1]. When compared with classical feedback control systems, networked control systems have several advantages. For example, they can reduce the system wiring, make the system easy to operate and maintain and later diagnose in case of malfunctioning, and increase system agility.

Although NCS have advantages, inserting a network in between the plant and the controller can introduce many problems as well. For example, in communication networks, data packets that carry the information can be dropped or delayed due to the network traffic conditions. When closing the control loop over such communication networks, the overall system might have poor performance or even become unstable. Thus the effect that those issues have on the closed loop system performance must be fully analyzed before networked control systems become commonplace.

Recently, many researchers have investigated these issues and some significant results were obtained and many are in progress. The problem of state estimation and stabilization of a linear time invariant (LTI) system over a digital communication channel which has a finite bandwidth capacity was introduced by Wong and Brockett [2], [3]. In [4], Sinopoli et al. discussed how packet loss can affect state estimation. They showed there exists a certain threshold of the packet arrival rate below which, $\mathbb{E}[P_k]$, the expected value of the error covariance matrix, becomes unbounded as time goes to infinity. The authors extended their result from estimation to closed loop



State Estimation Over a Network

Fig. 1. System Block Diagram

control in [5] where stability region of packet arrival rates are provided. A scheme based on multi-description coding for packet dropping networks, but limited to the estimation, is considered in [6]. The readers are referred to [7] and references therein for some recent results in the area of networked control systems.

The problem of state estimation of a dynamical system where measurements are sent across a packet dropping network is also the focus of this work. Despite the great progress of the previous researchers, the problems they have studied have certain limitations. For example, in [4], the authors assumed that packets are dropped independently, which is not true when burst packets are dropped or in queuing networks where adjacent packets are not dropped independently. They also use $\mathbb{E}[P_k]$ as the measure of performance, which can conceal the fact that events with arbitrarily low probability can drive expected value diverge, and it might be better to ignore such events that occur with arbitrarily low probability.

The goal of the present work is to give a more complete characterization of the estimator performance by instead considering a probabilistic description of the error covariance, i.e., $\Pr[P_k \leq M]$. In [8] the present authors first introduced this notion for the same problem setting but under the additional assumption that the measurement matrix, C , is invertible. In [9], the present authors extended the result to the case when C is not invertible. However, extra assumptions are made, e.g., A is assumed to be purely unstable. The main contribution of this paper can be summarized as follows. 1) Unlike almost all previous work where the *a priori* error covariance is studied, we consider the *a posteriori* error covariance in this paper. 2) We remove the constraint in [9] that requires A to be unstable and work with arbitrary A . 3) We are able to recover the result in [4] for scalar systems.

The rest of the paper is organized as follows. In Section II,

the mathematical model of the system that we consider is given. In Section III, some frequently used terms are defined, a quick summary of Kalman filter updating equations is provided and some results on $\mathbb{E}[P_k]$ from [4] is reviewed. In Section IV we derive lower and upper bounds for $\Pr[P_k \leq M]$. In Section V we provide an example to demonstrate the theory developed. The paper concludes with a summary of our results and a discussion of the work that lies ahead.

II. PROBLEM SETUP

We consider the networked control systems as seen in Fig. 1. The process dynamics and sensor measurement equation are given as follows:

$$x_k = Ax_{k-1} + w_{k-1}, \quad (1)$$

$$y_k = Cx_k + v_k. \quad (2)$$

In the above equations, $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $w_{k-1} \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero mean white Gaussian random vectors with $\mathbb{E}[w_k w_j'] = \delta_{kj} Q$, $Q \geq 0$, $\mathbb{E}[v_k v_j'] = \delta_{kj} R$, $R > 0$, $\mathbb{E}[w_k v_j'] = 0 \forall j, k$, where $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ otherwise. We assume that the pair (A, C) is observable and (A, \sqrt{Q}) is controllable.

After taking a measurement at time k , the sensor sends y_k to a remote estimator for generating the state estimate. We assume that the measurement data packets from the sensor are to be sent across a packet dropping network, with negligible quantization effects, to the estimator. Let γ_k be the random variable indicating whether a packet is dropped at time k or not, i.e., $\gamma_k = 0$ if a packet is dropped and $\gamma_k = 1$ otherwise. In addition, we assume the sensor has the ability to store some previous measurements in a buffer when needed. Therefore each packet sent through the network could contain a finite number of the previous measurements.

Let us define the following state quantities at the remote state estimator:

$$\hat{x}_k \triangleq \mathbb{E}[x_k | \text{all data packets up to } k],$$

$$P_k \triangleq \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)'].$$

As mentioned in Section I, we are interested in finding a closed form solution to $\Pr[P_k \leq M]$ given M . Before we present our main results in Section IV, we go over some preliminaries first.

III. PRELIMINARIES

A. Definitions

It is assumed that (A, C, Q, R) are the same as they appear in Section II; $\lambda_i(A)$ is the i th eigenvalue of the matrix A ; $X \in \mathbb{S}_+^n$ where \mathbb{S}_+^n is the set of n by n positive semi-definite matrices; $f_i : \mathbb{S}_+^n \rightarrow \mathbb{S}_+^n$, $i = 1, 2$; Y_i is a random variable where the underlying sample spaces will be clear from its context.

$$\rho(A) \triangleq \max_i |\lambda_i(A)|,$$

$$h(X) \triangleq AXA' + Q,$$

$$g(X) \triangleq h(X) - AXC'[CXC' + R]^{-1}CXA',$$

$$\tilde{g}(X) \triangleq X - XC'[CXC' + R]^{-1}CX.$$

B. Kalman Filtering Preliminaries

If the network between the sensor and the estimator is perfect, i.e., no packet is dropped, then it is well known that the optimal linear estimator for the system described by Eqn (1) and (2) is a standard Kalman filter, denoted as **KF**. We write (\hat{x}_k, P_k) in compact form as

$$(\hat{x}_k, P_k) = \mathbf{KF}(\hat{x}_{k-1}, P_{k-1}, y_k)$$

which represents the follow set of equations:

$$\begin{cases} \hat{x}_k^- &= A\hat{x}_{k-1}, \\ P_k^- &= AP_{k-1}A' + Q, \\ K_k &= P_k^- H_k' [H_k P_k^- H_k' + R_k]^{-1}, \\ \hat{x}_k &= A\hat{x}_{k-1} + K_k(y_k - H_k A\hat{x}_{k-1}), \\ P_k &= (I - K_k H_k) P_k^-. \end{cases}$$

P_k^- and P_k are easy shown to satisfy

$$P_k^- = g(P_{k-1}^-), \quad P_k = \tilde{g} \circ h(P_{k-1}).$$

Let P^* be the unique positive semi-definite solution¹ to $g(X) = X$, i.e., $P^* = g(P^*)$. Define \bar{P} as $\bar{P} \triangleq \tilde{g}(P^*)$. Then we have

$$\tilde{g} \circ h(\bar{P}) = \tilde{g} \circ h \circ \tilde{g}(P^*) = \tilde{g} \circ g(P^*) = \tilde{g}(P^*) = \bar{P},$$

where we use the fact that $h \circ \tilde{g} = g$. In other words,

$$P^* = \lim_{k \rightarrow \infty} P_k^-, \quad \bar{P} = \lim_{k \rightarrow \infty} P_k.$$

C. Kalman Filtering with Intermittent Observations

Upon receiving the measurement data from the sensor, it was shown in [4] that the optimal linear filter has the same equations as a standard Kalman filter except that

$$\hat{x}_k = \hat{x}_k^- + \gamma_k K_k (y_k - C\hat{x}_k^-) \quad (3)$$

$$P_k = P_k^- - \gamma_k K_k C P_k^-. \quad (4)$$

Due to the randomness of data packet drops, P_k is a random variable as well. When γ_k 's are independent and identically distributed Bernoulli random variables with mean γ , Sinopoli et al. in [4] showed that there exists a critical value γ_c such that if $\gamma > \gamma_c$, $\mathbb{E}[P_k]$ converges as $k \rightarrow \infty$ and diverges otherwise. When C^{-1} exists, γ_c is given in exact form as

$$\gamma_c = 1 - \frac{1}{\rho(A)^2}. \quad (5)$$

Using $\mathbb{E}[P_k]$ as a metric, however, may conceal the fact that events with arbitrarily small probability can make the expected value diverge, and it might be better to ignore such events when evaluate the performance of the estimator. For example, consider the unstable scalar system with $a = 2$, $q = 1$, $P_0 = 1$ in Eqn (1). Let the packet arrival rate γ satisfy

$$\gamma = 0.74 < \gamma_c = 1 - \frac{1}{a^2} = 0.75.$$

¹Since (A, C) is assumed to be observable and (A, \sqrt{Q}) controllable, from standard Kalman filtering analysis, P^* exists.

Then from [4] we conclude that $\lim_{k \rightarrow \infty} \mathbb{E}[P_k] = \infty$. This is easily verifiable by considering the event σ that no packets are received in k time steps. Then

$$\mathbb{E}[P_k] \geq \mathbb{E}[P_k | \sigma] \Pr[\sigma] > (0.26^k) 4^k P_0 = 1.04^k P_0 = 1.04^k.$$

By letting k go to infinity, we see that $\mathbb{E}[P_k]$ diverges. Thus σ alone can make $\mathbb{E}[P_k]$ diverge, and the probability that σ occurs approaches zero when k goes to infinity. This partially motivates us to consider $\Pr[P_k \leq M]$ as a metric to evaluate the performance of the estimator subject to packet drops.

IV. MAIN RESULTS

In this section, we assume C is full rank. Without loss of generality, we assume C^{-1} exists. We extend the results to the general case in Appendix B.

A. Lower and Upper Bounds of $\Pr[P_k \leq M]$

Similar to [4], the optimal state estimate \hat{x}_k and its error covariance matrix P_k are given by

$$(\hat{x}_k, P_k) = \begin{cases} (A\hat{x}_{k-1}, h(P_{k-1})), & \text{if } \gamma_k = 0 \\ \mathbf{KF}(\hat{x}_{k-1}, P_{k-1}, y_k), & \text{if } \gamma_k = 1 \end{cases}$$

As a result,

$$P_k = \begin{cases} h(P_{k-1}), & \text{if } \gamma_k = 0 \\ \tilde{g} \circ h(P_{k-1}), & \text{if } \gamma_k = 1 \end{cases}$$

Define $\bar{M} \triangleq C^{-1}RC^{-1}$. Then we have the following result that shows the relationship between P_k and \bar{M} .

Lemma 4.1: For any $k \geq 1$, if $\gamma_k = 1$, then $P_k \leq \bar{M}$.

Proof: As $\gamma_k = 1$, we have $P_k = \tilde{g} \circ h(P_{k-1}) \leq \bar{M}$, where the inequality is due to Lemma A.2 in Appendix A. ■

Remark 4.2: We can also interpret Lemma 4.1 as follows. One way to obtain an estimate \hat{x}_k when $\gamma_k = 1$ is simply by inverting the measurement, i.e., $\tilde{x}_k = C^{-1}y_k$. Therefore

$$\tilde{e}_k = C^{-1}v_k \text{ and } \tilde{P}_k = \mathbb{E}[\tilde{e}_k \tilde{e}_k'] = C^{-1}RC^{-1} = \bar{M}.$$

Since Kalman filter is optimal among the set of all linear filters, we must have $P_k \leq \tilde{P}_k = \bar{M}$.

For $M \geq \bar{M}$, define $k_1(M)$ and $k_2(M)$ as follows:

$$k_1(M) \triangleq \min\{t \geq 1 : h^t(\bar{M}) \not\leq M\}, \quad (6)$$

$$k_2(M) \triangleq \min\{t \geq 1 : h^t(\bar{P}) \not\leq M\}. \quad (7)$$

We sometimes write $k_i(M)$ as k_i , $i = 1, 2$ for simplicity for the rest of the paper. The following lemma shows the relationship between \bar{P} and \bar{M} as well as k_1 and k_2 .

Lemma 4.3: (1) $\bar{P} \leq \bar{M}$; (2) $k_1 \leq k_2$ whenever either k_i is finite, $i = 1, 2$.

Proof: (1) $\bar{P} = \tilde{g}(P^*) \leq \bar{M}$ where the inequality is from Lemma A.2 in Appendix A. (2) Without loss of generality, we assume k_2 is finite. If k_1 is finite, and $k_1 > k_2$, then according to their definitions, we must have

$$M \geq h^{k_1-1}(\bar{M}) \geq h^{k_1-1}(\bar{P}) \geq h^{k_2}(\bar{P})$$

which violates the definition of k_2 . Notice that we use the property that h is nondecreasing as well as $h(\bar{P}) \geq \bar{P}$ from Lemma A.1 and A.3 in Section A in the Appendix. Similarly

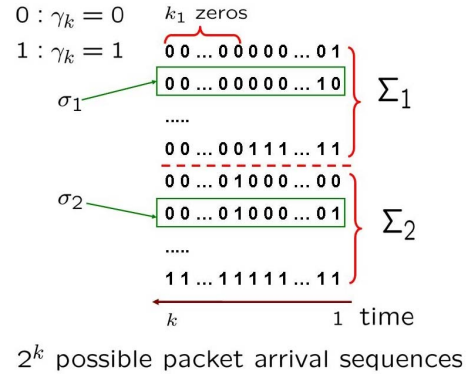


Fig. 2. $N_k \geq k_1$

we can show that k_1 cannot be infinite. Therefore we must have $k_1 \leq k_2$. ■

Lemma 4.4: Assume $P_0 \geq \bar{P}$. Then for all $k \geq 0$, $P_k \geq \bar{P}$.

Proof: We prove this by induction. Assume $P_k \geq \bar{P}$ for some $k \geq 0$. This clearly holds when $k = 0$. Let us consider P_{k+1} . If $\gamma_{k+1} = 1$, then

$$P_{k+1} = \tilde{g} \circ h(P_k) \geq \tilde{g} \circ h(\bar{P}) = \bar{P},$$

where the inequality is from Lemma A.1. If $\gamma_{k+1} = 0$, then

$$P_{k+1} = h(P_k) \geq h(\bar{P}) \geq \bar{P}.$$

The induction step is thus complete. ■

Define N_k as the number of consecutive packet drops at time k , i.e.,

$$N_k \triangleq \min\{t \geq 0 : \gamma_{k-t} = 1\}. \quad (8)$$

We have the following theorem that provides lower and upper bounds on $\Pr[P_k \leq M]$.

Theorem 4.5: Assume $\bar{P} \leq P_0 \leq \bar{M}$. For any $M \geq \bar{M}$, we have

$$1 - \Pr[N_k \geq k_1] \leq \Pr[P_k \leq M] \leq 1 - \Pr[N_k \geq k_2]. \quad (9)$$

Proof: We divide the proof into two parts. 1) Let us first prove $1 - \Pr[N_k \geq k_1] \leq \Pr[P_k \leq M]$. As $\gamma_k = 1$ or 0, there are in total 2^k possible realizations of γ_1 to γ_k as seen from Fig. 2.

Let Σ_1 denote those packet arrival sequences of γ_1 to γ_k such that $N_k \geq k_1$. Similarly let Σ_2 denote those packet arrival sequences such that $N_k < k_1$. Let $P_k(\sigma_i)$ be the error covariance at time k when the underlying packet arrival sequence is σ_i , where $\sigma_i \in \Sigma_i$, $i = 1, 2$. Consider a particular $\sigma_2 \in \Sigma_2$. As $\gamma_{k-k_1+1} = 1$, from Lemma 4.1, $P_{k-k_1+1} \leq \bar{M}$. Therefore we have

$$P_k(\sigma_2) \leq h^{k_1-1}(P_{k-k_1+1}) \leq h^{k_1-1}(\bar{M}) \leq M,$$

where the first and second inequalities are from Lemma A.1 in Appendix A and the last inequality is from the definition of

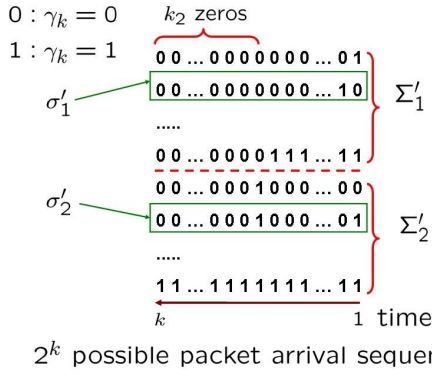


Fig. 3. $N_k \geq k_2$

k_1 . In other words, $\Pr[P_k \leq M | \sigma_2] = 1$. Therefore we have

$$\begin{aligned} \Pr[P_k \leq M] &= \sum_{\sigma \in \Sigma_1 \cup \Sigma_2} \Pr[P_k \leq M | \sigma] \Pr(\sigma) \\ &\geq \sum_{\sigma_2 \in \Sigma_2} \Pr[P_k \leq M | \sigma_2] \Pr(\sigma_2) \\ &= \sum_{\sigma_2 \in \Sigma_2} \Pr(\sigma_2) = \Pr(\Sigma_2) \\ &= 1 - \Pr(\Sigma_1) = 1 - \Pr[N_k \geq k_1]. \end{aligned}$$

where the first equality is from the total probability theorem, the second equality holds as Σ_1 and Σ_2 are disjoint, the third inequality holds as the first sum is non-negative, the rest equalities are easy to see.

2) We now prove $\Pr[P_k \leq M] \leq 1 - \Pr[N_k \geq k_2]$. Let Σ'_1 denote those packet arrival sequences of γ_1 to γ_k such that $N_k \geq k_2$ and Σ'_2 denote those packet arrival sequences such that $N_k < k_2$ (Fig. 3). Consider $\sigma'_1 \in \Sigma'_1$. Let $s(\sigma'_1) = \min\{t \geq 1 : \gamma_{k-t} = 1 | \sigma'_1\}$. As $\sigma'_1 \in \Sigma'_1$, we must have $s \geq k_2$. Consequently,

$$P_k(\sigma'_1) = h^{s(\sigma'_1)}(P_{k-s(\sigma'_1)}) \geq h^{s(\sigma'_1)}(\bar{P}),$$

where the inequality is from Lemma 4.4. Therefore we conclude $P_k(\sigma'_1) \not\leq M$ otherwise $h^{s(\sigma'_1)}(\bar{P}) \leq P_k(\sigma'_1) \leq M$ which violates the definition of k_2 . In other words, $\Pr[P_k \leq M | \sigma'_1] = 0$. Therefore we have

$$\begin{aligned} \Pr[P_k \leq M] &= \sum_{\sigma \in \Sigma_1 \cup \Sigma_2} \Pr[P_k \leq M | \sigma] \Pr(\sigma) \\ &= \sum_{\sigma'_2 \in \Sigma'_2} \Pr[P_k \leq M | \sigma'_2] \Pr(\sigma'_2) \\ &\leq \sum_{\sigma'_2 \in \Sigma'_2} \Pr(\sigma'_2) = \Pr(\Sigma'_2) \\ &= 1 - \Pr(\Sigma'_1) = 1 - \Pr[N_k \geq k_2], \end{aligned}$$

where the inequality is from the fact that $\Pr[P_k \leq M | \sigma'_2] \leq 1$ for any $\sigma'_2 \in \Sigma'_2$. ■

B. Computing $\Pr[N_k \geq k_i]$

Theorem 4.5 provides a lower and an upper bound for $\Pr[P_k \leq M]$. Both bounds involve the term $\Pr[N_k \geq k_i]$.

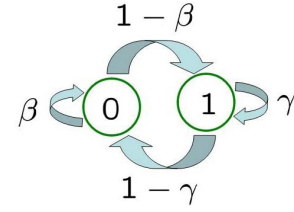


Fig. 4. Gilbert-Elliott Model

In this section, we show how we can compute $\Pr[N_k \geq k_i]$ given a packet arrival and drop model.

Let k_1 and k_2 be given (see next section for their computation and approximation). In order to compute $\Pr[N_k \geq k_i]$, we need to have a model that describes packet arrival and drop behaviors. The most commonly used models in literature are

- 1) I.I.D model: i.e., γ_k 's are independent and identically distributed Bernoulli random variables with mean γ , e.g., [4], [10].
- 2) Gilbert-Elliott model: i.e., a two state markov chain is used to describe the transition from γ_k to γ_{k+1} , e.g., [11], [12].

We give closed form solution to both models in this section.

1) *I.I.D Model*: If γ_k 's are i.i.d Bernoulli random variables with rate γ , then

$$\begin{aligned} \Pr[N_k \geq k_i] &= \Pr[\gamma_k = 0, \dots, \gamma_{k-k_i+1} = 0] \\ &= (1 - \gamma)^{k_i}. \end{aligned} \quad (10)$$

2) *Gilbert-Elliott Model*: Now consider a two state (0 or 1) markov chain that represents packet drops and arrivals (Fig. 4). Let T denote the state transition probability matrix, i.e.,

$$T = \begin{bmatrix} \beta & 1 - \beta \\ 1 - \gamma & \gamma \end{bmatrix}.$$

Let $\pi = [\pi_0 \ \pi_1]$ be the steady state distribution of the markov chain, i.e., $\pi = \pi T$. π can be computed as

$$\pi = \left[\frac{1 - \gamma}{2 - \gamma - \beta} \quad \frac{1 - \beta}{2 - \gamma - \beta} \right].$$

Let z_k be defined as

$$z_k = \begin{bmatrix} z_k^1 \\ z_k^2 \end{bmatrix} \triangleq \begin{bmatrix} \Pr[\gamma_k = 0] \\ \Pr[\gamma_k = 1] \end{bmatrix}.$$

Then z_k can be shown to satisfy the following equation

$$z_k = (T')^k z_0, \quad k \geq 1.$$

Furthermore, for k sufficiently large, $z_k \approx \pi'$, i.e.,

$$z_k^1 \approx \pi_0, \quad z_k^2 \approx \pi_1.$$

Therefore we have

$$\begin{aligned}
& \Pr[N_k \geq k_i] \\
&= \Pr[\gamma_k = 0, \dots, \gamma_{k-k_i+1} = 0] \\
&= \sum_{i=0}^1 \alpha_i \Pr[\gamma_{k-k_i} = i] \\
&= \beta^{k_i} z_{k-k_i}^1 + (1-\gamma)\beta^{k_i-1} z_{k-k_i}^2 \\
&= \frac{1-\gamma}{\beta(2-\gamma-\beta)} \beta^{k_i}
\end{aligned} \tag{11}$$

where $\alpha_i = \Pr[\gamma_k = 0, \dots, \gamma_{k-k_i+1} = 0 | \gamma_{k-k_i} = i]$.

C. $\mathbb{E}[P_k]$ as a Metric

In this section, we show that we are able to recover the results in [4] using $\Pr[P_k \leq M]$ as a metric for scalar systems. Let us consider Eqn (1) and (2) with

$$A = a > 1, Q = q > 0, C = c > 0, R = r > 0.$$

Notice that in the scalar case, the assumption that (a, c) is observable and (a, \sqrt{q}) is controllable holds trivially.

From Lemma A.4 in Appendix A, we can write $\mathbb{E}[P_k]$ as

$$\begin{aligned}
\mathbb{E}[P_k] &= \int_0^{\bar{M}} (1 - \Pr[P_k \leq M]) dM \\
&\quad + \int_{\bar{M}}^{\infty} (1 - \Pr[P_k \leq M]) dM.
\end{aligned}$$

Using the fact $0 \leq \Pr[P_k \leq M] \leq 1$, we have

$$\mathbb{E}[P_k] \geq \int_{\bar{M}}^{\infty} (1 - \Pr[P_k \leq M]) dM$$

and

$$\mathbb{E}[P_k] \leq \bar{M} + \int_{\bar{M}}^{\infty} (1 - \Pr[P_k \leq M]) dM.$$

From Theorem 4.5, we know that when $M \geq \bar{M}$,

$$1 - \Pr[N_k \geq k_1] \leq \Pr[P_k \leq M] \leq 1 - \Pr[N_k \geq k_2].$$

Since in [4], i.i.d packet drop model is used, from Eqn (10), we have $\Pr[N_k \geq k_i] = (1-\gamma)^{k_i}$, $i = 1, 2$. Therefore we obtain

$$\begin{aligned}
\int_{\bar{M}}^{\infty} (1-\gamma)^{k_2(M)} dM \leq \mathbb{E}[P_k] \leq \int_{\bar{M}}^{\infty} (1-\gamma)^{k_1(M)} dM \\
+ \bar{M}. \tag{12}
\end{aligned}$$

Recall that $k_1(M) = \min\{t \geq 1 : h^t(\bar{M}) \not\leq M\}$ and

$$\begin{aligned}
h^t(\bar{M}) &= a^{2t}\bar{M} + q(1 + a^2 + \dots + a^{2t-2}) \\
&= c_1 a^{2t} - c_2,
\end{aligned}$$

where

$$c_1 = \bar{M} + \frac{q}{a^2-1}, c_2 = \frac{q}{a^2-1}.$$

Therefore for any $t \geq 1$,

$$k_1(M) = t, \text{ if } c_1 a^{2t-2} - c_2 \leq M < c_1 a^{2t} - c_2.$$

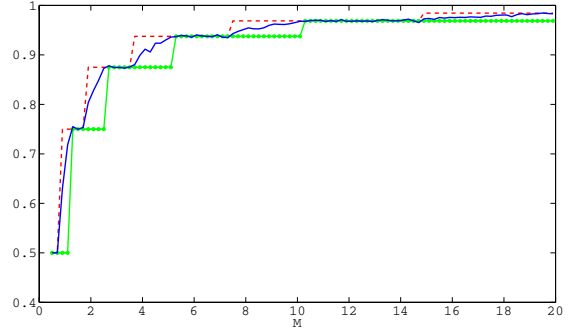


Fig. 5. y_k is sent

From Eqn (12)

$$\begin{aligned}
\mathbb{E}[P_k] &\leq \bar{M} + \int_{\bar{M}}^{\infty} (1-\gamma)^{k_1(M)} dM \\
&= \bar{M} + \sum_{t=1}^{\infty} \int_{c_1 a^{2t-2} - c_2}^{c_1 a^{2t} - c_2} (1-\gamma)^t dM \\
&= \bar{M} + \sum_{t=1}^{\infty} c_1 (1 - \frac{1}{a^2}) (a^2 - \gamma a^2)^t.
\end{aligned}$$

Clearly $\mathbb{E}[P_k]$ converges if $a^2 - \gamma a^2 < 1$, i.e., $\gamma > 1 - \frac{1}{a^2}$. Similarly from Eqn (12)

$$\begin{aligned}
\mathbb{E}[P_k] &\geq \int_0^{\infty} (1-\gamma)^{k_2(M)} dM \\
&= \sum_{t=1}^{\infty} \int_{c'_1 a^{2t-2} - c_2}^{c'_1 a^{2t} - c_2} (1-\gamma)^t dM \\
&= \sum_{t=1}^{\infty} c'_1 (1 - \frac{1}{a^2}) (a^2 - \gamma a^2)^t.
\end{aligned}$$

where $c'_1 = \bar{P} + \frac{q}{a^2-1}$. Hence $\mathbb{E}[P_k]$ diverges if $a^2 - \gamma a^2 \geq 1$, i.e., $\gamma \leq 1 - \frac{1}{a^2}$. We therefore conclude that

$$\lambda_c = 1 - \frac{1}{a^2}$$

which is exactly the same as Eqn (5) for scalar systems.

V. EXAMPLE

Consider Eqn (1) and (2) with

$$A = 1.4, C = 1, Q = 0.2, R = 0.5, \gamma = 0.5.$$

We run a monte carlo simulation for demonstrating the main results in Section IV. Fig. 5 plots the result, where the red dashed curve is the upper bound, green dotted curve is the lower bound, and the blue solid curve is the actual value of $\Pr[P_k \leq M]$ measured as the relative frequency of $P_k \leq M$. We can see from Fig. 5 that the lower and upper bounds that we have derived in Eqn (9) provide tight approximation of $\Pr[P_k \leq M]$.

VI. CONCLUSION

In this paper, we study the problem of state estimation of a discrete time process over a packet dropping network based on a modified Kalman filter. We consider a probabilistic metric on the error covariance matrix, i.e., $\Pr[P_k \leq M]$, and we derive lower and upper bounds for $\Pr[P_k \leq M]$. We also recover the result for scalar systems in [4].

There are many interesting directions for continuing this work, which include: study closed loop system performance from a probabilistic angle; look at distributed and cooperative control problems over packet dropping networks; and experimentally evaluate the theory developed in the paper.

APPENDIX

A. Supporting Lemmas

Lemma A.1: For any $0 \leq X \leq Y$,

$$\begin{aligned} h(X) &\leq h(Y), \quad g(X) \leq g(Y), \quad \tilde{g}(X) \leq \tilde{g}(Y), \\ \tilde{g}(X) &\leq X, \quad h \circ \tilde{g}(X) = g(X), \quad g(X) \leq h(X). \end{aligned}$$

Proof: $h(X) \leq h(Y)$ holds as $h(X)$ is affine in X . Proof for $g(X) \leq g(Y)$ can be found in Lemma 1-c in [4]. As \tilde{g} is a special form of g by setting $A = I$ and $Q = 0$, we immediately obtain $\tilde{g}(X) \leq \tilde{g}(Y)$. Next we have

$$\tilde{g}(X) = X - XC'[CXC' + R]^{-1}CX \leq X$$

and

$$\begin{aligned} h \circ \tilde{g}(X) &= A(X - XC'[CXC' + R]^{-1}CX)A' + Q \\ &= g(X). \end{aligned}$$

Finally we have

$$g(X) = h(X) - AXC'[CXC' + R]^{-1}CXA' \leq h(X). \quad \blacksquare$$

Lemma A.2: For any $X \geq 0$, $\tilde{g}(X) \leq \bar{M}$.

Proof: For any $t > 0$, we have

$$\tilde{g}(t\bar{M}) = \frac{t}{t+1}\bar{M} \leq \bar{M}.$$

For all $X \geq 0$, since $\bar{M} > 0$, it is clear that there exists $t_1 > 0$ such that $t_1\bar{M} > X$. Therefore

$$\tilde{g}(X) \leq \tilde{g}(t_1\bar{M}) \leq \bar{M}. \quad \blacksquare$$

Lemma A.3: $\bar{P} \leq h(\bar{P})$.

Proof:

$$h(\bar{P}) = h \circ \tilde{g}(P^*) = g(P^*) = P^* \geq \tilde{g}(P^*) = \bar{P},$$

where the first and the last equality are from the definition of \bar{P} , the third equality is from the definition of P^* . The rest equality and inequality are from Lemma A.1. \blacksquare

Lemma A.4: Let X be a continuous random variable defined on $[0, \infty)$ and let $F(x) = \Pr[X \leq x]$. Then

$$\mathbb{E}[X] = \int_0^\infty [1 - F(x)]dx.$$

Proof: See Lemma (4) in [13], page 93. \blacksquare

B. When C Is Not Full Rank

Assume C is not full rank. Since (A, C) is observable, there exists $2 \leq r \leq n$ such that $[C; CA; \dots; CA^{r-1}]'$ is full rank. In this section, we consider the special case when $r = 2$, and in particular, we assume $[C; CA]^{-1}$ exists. The idea readily extends to other cases.

Unlike the case when C^{-1} exists, and y_k is sent across the network, here we assume that the previous measurement y_{k-1} is sent along with y_k . This only requires that the sensor has a buffer that stores y_{k-1} . Then if $\gamma_k = 1$, both y_k and y_{k-1} are received. Thus we can use the following linear estimator to generate \hat{x}_k

$$\hat{x}_k = A \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}.$$

The corresponding error covariance can be calculated as

$$P_k = AM_1A' + Q$$

where

$$M_1 = \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} CQC' + R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} CA \\ C \end{bmatrix}^{-1}.$$

Since Kalman filter is optimal among the set of all linear estimators, we conclude that

$$\bar{P} \leq P_k = AM_1A' + Q \triangleq \bar{M} \text{ if } \gamma_k = 1.$$

Therefore we obtain the same results as in Section IV with the new \bar{M} .

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