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Kaluza-Klein Holography

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ABSTRACT

We construct a holographic map between asymptotically $AdS_5 \times S^5$ solutions of $10d$ supergravity and vacuum expectation values of gauge invariant operators of the dual QFT. The ingredients that enter in the construction are (i) gauge invariant variables so that the KK reduction is independent of any choice of gauge fixing; (ii) the non-linear KK reduction map from 10 to 5 dimensions (constructed perturbatively in the number of fields); (iii) application of holographic renormalization. A non-trivial role in the last step is played by extremal couplings. This map allows one to reliably compute vevs of operators dual to any KK fields. As an application we consider a Coulomb branch solution and compute the first two non-trivial vevs, involving operators of dimension 2 and 4, and reproduce the field theory result, in agreement with non-renormalization theorems. This constitutes the first *quantitative* test of the gravity/gauge theory duality away from the conformal point involving a vev of an operator dual to a KK field (which is not one of the gauged supergravity fields).

Contents

1 Introduction

Gravity/gauge theory dualities relate string theory on spacetimes that asymptote to $AdS_m \times X$, where X is a compact manifold and gauge theory residing on the conformal boundary of the AdS part of the geometry. In the initial work [1] the dual theory was a conformal field theory (CFT) and the bulk spacetime $AdS_m \times X$ (rather than asymptotic to it), but it was soon recognized that the duality can be extended to describe quantum field theories that can be obtained from the CFT by either adding new terms in the action or considering vacua where the conformal symmetry is spontaneously broken. Both of these cases are described gravitationally by a solution that is asymptotic to $AdS_m \times X$.

Despite much work however basic questions still remain. One such question that will be the subject of this paper is:

Given a ten dimensional solution that is asymptotic to $AdS_m \times X$ how does one compute the vacuum expectation values of gauge invariant operators?

Roughly speaking vevs of chiral primary operators should appear in the radial expansion of the bulk solution. However making this precise proves to be a lot more subtle that one might have anticipated, and even qualitative features are not reproduced correctly by naive methods. The answer to this question should follow from the basic AdS/CFT dictionary [2, 3]. This is indeed the case but in order to implement the idea one has to sharpen existing methods and overcome several technical issues.

To illustrate the issues involved it is instructive to consider a simple example where the physics of the solution is well understood. A class of such examples is provided by multicenter D3-brane solutions, which in the near-horizon limit correspond to the Coulomb branch of $\mathcal{N} = 4$ SYM [4]. These examples are particularly interesting because the QFT vevs are protected by a non-renormalization theorem, and the gravitational results must therefore agree exactly with those computed at weak coupling. The metric is of the well-known form

$$ds^2 = H(x_\perp)^{-1/2} dx_{||}^2 + H(x_\perp)^{1/2} dx_\perp^2 \tag{1.1}$$

where H is a harmonic function in transverse directions. For a distribution $\sigma(\vec{y})$ of D3 branes, the harmonic function reads

$$H(x_\perp) = \int d^6 y \frac{\sigma(\vec{y})}{|\vec{x}_\perp - \vec{y}|^4} = \frac{Q_0}{r^4} \left(1 + \sum_{k=1}^{\infty} \frac{Q_k Y^k}{r^k} \right) \tag{1.2}$$

where in the last equality we expanded in $r^2 = |x_\perp|^2$, Y^k are spherical harmonics and Q_k are numerical constants that depend on the distribution $\sigma(\vec{y})$. Inserting this in (1.1) and expanding in r results in a metric that is asymptotically $AdS_5 \times S^5$. The QFT vevs should be encoded in the asymptotics and the purpose of this work is to show how to unambiguously extract this information.

The solutions under discussion are special in that they are uniquely determined in terms of a harmonic function. Furthermore, the spherical harmonics appearing in (1.2) are in 1-1 correspondence with the chiral primary operators of $\mathcal{N} = 4$ SYM and the radial power at which they appear is the correct power for their coefficients to correspond to the vevs of the dual operator. This led [5] to propose that these coefficients are proportional to vevs of the corresponding operators. Although this is a well motivated proposal, it is not clear how one would generalize it to the general case where the solution is not determined by a harmonic function. Even for the case at hand there are several open questions. For instance, inserting the harmonic function in the metric leads to terms involving powers of spherical harmonics whose meaning is not clear and in general there is also a dependence on the radial coordinate used to perform the asymptotic expansion. The simplifications special to such cases (i.e. when the solution is determined by a harmonic function) will be discussed in a separate publication [6]. In this paper we strive for generality, so our starting point will be general asymptotically $AdS_5 \times S^5$ metrics and we will only use the CB solutions in order to illustrate the general procedure.

Recall that the basic dictionary of the gravity/gauge theory duality [2, 3] states that (i) there is a bulk field corresponding to each gauge invariant operator and (ii) the string partition function with bulk fields satisfying appropriate boundary conditions is equal to the generating functional of QFT correlators with the boundary conditions playing the role of sources. In particular, one could compute vevs (in the large N and large 't Hooft coupling limit) by differentiating the supergravity on-shell action once w.r.t. sources. In implementing this procedure however one finds several obstacles.

First, the relation in (ii) should be understood as a "bare relation" as both sides divergence. To make the procedure well-defined one must renormalize. This is a standard procedure on the field theory side. On the gravitational side, the corresponding procedure, denoted holographic renormalization, was developed in a series of papers [7, 8, 9, 10] (see also [11]-[16] for related work and [17] for a review)[1]. After renormalization is done, the one point functions can be computed in all generality. The answer relates the one point function to certain coefficients in the asymptotic expansion of the bulk fields. So given any

[1]The starting point in the analysis in these papers was the lower dimensional AdS gravity obtained by reducing the original theory over the compact space X. A discussion that starts from higher dimensions can be found in [18, 19].

solution of 5d gravity coupled to matter one could read off the vevs of the dual operators by looking at the asymptotics.

We would like to emphasize that the procedure of renormalization is essential for correctly extracting the vevs. To give an example where a naive prescription fails, consider the case of the CB solution corresponding to a distribution of D3 branes on a disc of radius l. The 5d metric obtained by reducing the 10d solution over the sphere has the following asymptotics,

$$ds^2 = \frac{d\hat{z}^2}{\hat{z}^2} + \frac{1}{\hat{z}^2}\left(1 - \frac{l^4}{18}\hat{z}^4 + \mathcal{O}(\hat{z}^6)\right) dx^i dx^i \tag{1.3}$$

A naive prescription for reading off vevs that is often quoted in the literature is that the vev of an operator can be obtained, up to a (non-zero) numerical constant, from the normalizable mode of the corresponding bulk field. The bulk metric is dual to the stress energy tensor so one would be tempted to identify the coefficient of the $\hat{z}^4$ term with the vev of the stress energy tensor. This is clearly incorrect since that would imply non-zero vacuum energy for the dual theory but the solution is supersymmetric so the vacuum energy should be equal to zero. Indeed the 1-point function extracted using holographic renormalization [8, 9] contains additional terms (see (5.11) below) and taking those into account one finds the expected result, zero. Such subtleties are present in all cases, including that of scalar fields. For example, for deformation flows the vevs of all operators should be equal to zero, but there are examples where the above naive prescription leads to non-zero values. Again the correct 1-point functions include additional terms so that the total result is zero (see [8, 9, 20]).

An analysis that starts from the lower dimensional gauged supergravity is sufficient if one is only interested in computing vevs for operators dual to fields of the gauged supergravity. There is however an infinity of other (half supersymmetric) gauge invariant operators which would then *ab initio* be excluded from the analysis. These operators are dual to massive KK fields. The map between KK fields and gauge invariant operators was worked out in [3] (and subsequent papers) using the computation of the KK spectrum of $AdS_5 \times S^5$ in [21]. In this paper a *linearized* analysis around $AdS_5 \times S^5$ was performed. This analysis provides an explicit map (in a *specific gauge*) between linearized solutions of the ten dimensional equations of motion and linearized solutions of the dimensionally reduced five dimensional equations. To compute the vevs however we need to know the map at the non-linear level. To illustrate this, let s^k be ten-dimensional fields and let S^k be the corresponding five dimensional fields. In general the reduction map will be non-linear and takes the form

$$S^k = s^k + \sum_{lm} \Big(J_{klm} s^l s^m + L_{klm} D_\mu s^l D^\mu s^m + \mathcal{O}[s]^3 \Big) \tag{1.4}$$

where J_{klm} and L_{klm} are numerical coefficients and we retain only terms quadratic in the fields. (We also suppress contributions on the right hand side from other scalar and non-scalar fields since they are not necessary to illustrate our point.) If S^k is dual to an operator of dimension k then we would need to extract the coefficient of order z^k to determine the operator's vev. Clearly, quadratic terms with $l + m = k$ will also contribute at the same order and therefore such non-linear terms in the KK map cannot be ignored. Similarly cubic and higher order in fluctuation terms that are of order z^k will also contribute, along with non-linear contributions involving other supergravity fields.

So to read off the vevs we need to understand the KK reduction map at the non-linear level. However, if we are interested in the vev of an operator of a given dimension, only certain non-linear terms need to be computed, namely the ones that could possibly contribute to the vev. For instance, if we are interested in computing the vev of an operator of dimension 4, we would only need to keep terms quadratic in the fields dual to operators of dimension 2. This in effect truncates the reduction to a finite number of fields. This should be contrasted with the issue of consistent truncation. When the latter is possible one keeps only the "massless" KK modes in the reduction. In our case we keep massive KK fields as well. However, only a finite subset of them contribute to the asymptotics up to a given order.

Another issue is that of gauge fixing. The analysis in [21] was done in the de Donder gauge. Generically however a given supergravity solution will not be in this gauge and finding the coordinate transformation that would bring the solution to this gauge is a complicated task. To deal with this issue we will instead develop a "gauge invariant KK reduction". Instead of fixing the gauge, we combine the fields in gauge invariant combinations. This can be done systematically in an expansion in the number of fields. Having worked out these combinations, one can relax the de Donder gauge condition by simply replacing every field by its gauge invariant generalization in all results obtained in a specific gauge.

To summarize, we argue that in order to compute the vevs we need to obtain the non-linear KK map in terms of gauge invariant variables to appropriate order in the number of fields. This procedure results in five dimensional field equations and an explicit map between 10d solutions and solutions of these 5d equations. The 5d equations can be integrated into a 5d action and from here one can obtain the 1-point functions following the procedure of holographic renormalization. There is however an additional subtlety. In some cases the five dimensional equations contain no couplings between certain fields but boundary interactions exist [22]. These boundary couplings are in fact responsible for extremal n-point functions, namely correlators involving operators of dimensions $\{k_1{=}k_2{+}\cdots{+}k_n, k_2, \ldots, k_n\}$. One must take into account these additional boundary terms when working out the holographic 1-point functions.

Combining the non-linear, gauge invariant KK reduction map with the holographic 1-point functions we finally arrive at a well defined map between the asymptotics of a 10d solution and vevs of gauge invariant operators.

This paper is organized as follows. In the next section we discuss the Coulomb branch of $\mathcal{N} = 4$ SYM. We focus on a specific case where the vevs are uniformly distributed on a disc and compute all vevs of gauge invariant operators. The challenge for the gravity/gauge theory duality is to reproduce *exactly* these vevs holographically. In sections 3, 4 and 5 we build the holographic map. In section 3 we construct gauge invariant variables; in section 4 we work out the KK map to second order in the fields and in section 5 we derive the holographic 1-point functions. In section 6 we discuss the supergravity solution dual to the CB state discussed in section 2 and use the map developed in sections 3, 4, 5 in order to compute the first two non-trivial vevs and find perfect agreement with field theory! We conclude in section 7 with a discussion of our results. Several technical details are relegated to appendices A, B and C. In appendix A we discuss the harmonic expansion of the antisymmetric gauge field; in appendix B we summarize and develop several results about spherical harmonics with $SO(4)$ symmetry and in appendix C we discuss the computation of the field equations to second order in fluctuations.

2 $\mathcal{N} = 4$ SYM on the Coulomb branch

$\mathcal{N} = 4$ SYM contains 6 scalar fields X^{i_1} in the adjoint representation of the gauge group that we take to be $SU(N)$. The Coulomb branch of $\mathcal{N} = 4$ SYM corresponds to giving a vacuum expectation value (vev) to the scalars subject to the condition $[X^{i_1}, X^{i_2}] = 0$. Upon diagonalizing the scalar fields the moduli space is parametrized by the $6(N-1)$ eigenvalues of vevs (modulo the Weyl group). In the large N limit we can approximate the eigenvalues by a continuous distribution. Notice that the Coulomb branch still preserves $\mathcal{N} = 4$ supersymmetry but the conformal symmetry and the 16 superconformal supersymmetries are broken. This implies that the vevs are protected from acquiring quantum corrections, as we explain at the end of this section. So this example represents an ideal case for a precision test of the gravity/gauge theory correspondence in a non-conformal setting. It has been known since [4] that there is a one to one correspondence between the Coulomb branch of $\mathcal{N} = 4$ SYM and multicenter D3 brane solutions. Since the vevs do not renormalize, however, one should be able to establish a strong result, namely the exact values of vevs should be reproducible by a gravitational computation.

We will consider in this paper the specific case of a uniform distribution of eigenvalues of X^1 and X^2 on a disc of radius a and vanishing vev for the remaining scalars, $\langle X^3 \rangle = \langle X^4 \rangle = \langle X^5 \rangle = \langle X^6 \rangle = 0$. Let

$$X^1 = \rho \cos\phi, \qquad X^2 = \rho \sin\phi \tag{2.1}$$

To leading order in the large N limit we may represent the eigenvalues by a uniform continuous distribution,

$$\sigma(\rho, \phi) = \frac{N}{\pi a^2}. \tag{2.2}$$

Notice that the configuration corresponding to this continuous distribution preserves an $SO(4) \times SO(2)$ symmetry of $SO(6)$.

To compare with supergravity we would like to parametrize the moduli space by vevs of composite operators. We consider the following chiral primaries (CPOs) of $\mathcal{N} = 4$ SYM

$$\mathcal{O}^{I_1} = \mathcal{N}_{I_1} C^{I_1}_{i_1 \cdots i_k} \mathrm{Tr}(X^{i_1} \cdots X^{i_k}), \tag{2.3}$$

where $\mathcal{N}_{I_1}$ is a normalization factor and C^{I_1} is a totally symmetric traceless rank k tensor of $SO(6)$ which is normalized such that $\langle C^{I_1} C^{I_2} \rangle = C^{I_1}_{i_1 \cdots i_k} C^{I_2}_{i_1 \cdots i_k} = \delta^{I_1 I_2}$. The SYM action is normalized such that the relevant propagators are

$$\left\langle X^i_a(x) X^j_b(y) \right\rangle = \frac{g^2_{YM} \delta_{ab} \delta^{ij}}{(2\pi)^2 \, |x - y|^2}, \tag{2.4}$$

where a, b are color indices.

The cases of interest are the operators which are singlets under the decomposition of $SO(6)$ into $SO(2) \times SO(4)$ since non-singlet operators have zero vev. These operators can be obtained from the explicit expression of scalar harmonics in appendix B.1 by suitably replacing x^{i_1} by X^{i_1}, compare (2.3) and (B.11). The result for the singlets is

$$\mathcal{O}^{2n} = \mathcal{N}_{2n} \frac{(-)^n}{2^n \sqrt{2n+1}} \mathrm{Tr} \left(\sum_{m=0}^{n} (-)^m \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n+m+1 \\ n+1 \end{pmatrix} \rho^{2m} R^{2(n-m)} \right). \tag{2.5}$$

where $R^2 = \sum_{i=1}^6 (X^i)^2$. The explicit expressions for the lowest dimension operators are thus

$$\mathcal{O}^2 \;=\; \mathcal{N}_2 \frac{1}{2\sqrt{3}} \mathrm{Tr}(3\rho^2 - R^2); \tag{2.6}$$

$$\mathcal{O}^4 = \mathcal{N}_4 \frac{1}{4\sqrt{5}} \mathrm{Tr}(10\rho^4 - 8\rho^2 R^2 + R^4); \tag{2.7}$$

$$\mathcal{O}^6 = \mathcal{N}_6 \frac{1}{8\sqrt{7}} \mathrm{Tr}(35\rho^6 - 45\rho^4 R^2 + 15\rho^2 R^4 - R^6). \tag{2.8}$$

To compute the vev of these operators we now use

$$\langle \mathrm{Tr}(\rho^p) \rangle = \int_0^a d\rho\rho \int_0^{2\pi} d\phi \sigma(\rho, \phi) \rho^p = \frac{2N}{(p+2)} a^p, \tag{2.9}$$

and the identity

$$(-)^n \left(\sum_{m=0}^{n} (-)^m \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} n+m+1 \\ n+1 \end{pmatrix} \right) = (n+1). \tag{2.10}$$

to arrive at

$$\langle \mathcal{O}^{2n} \rangle = \frac{\mathcal{N}_{2n} a^{2n}}{2^n \sqrt{2n+1}} N. \tag{2.11}$$

This result was derived by a tree-level computation. However it remains uncorrected both perturbatively and non-perturbatively. A quantum correction to the vev of the scalars X^i would result from a non-vanishing tadpole contribution and this would induce a correction to the effective potential. However, there are no perturbative or non-perburtative quantum corrections to the low energy (2-derivative) effective action of $N = 4$ SYM [23, 24] so the vevs of the scalars are not corrected. The only remaining issue is operator mixing. Indeed, chiral primary operators mix with certain multi-trace operators. However, this is a subleading effect[2] in $1/N$ and we are considering the leading behavior. It follows that the operators have the same vev (2.11) at strong coupling. The challenge for the AdS/CFT correspondence is to reproduce these vevs.

3 KK reduction with gauge invariant variables

The IIB SUGRA field equations[3] for the metric and 5-form field strength are given by:

$$R_{MN} = \frac{1}{6} F_{MPQRS} F_M{}^{PQRS}, \qquad F = *F. \tag{3.1}$$

These equations admit an $AdS_5 \times S^5$ solution

$$ds_o^2 = \frac{dz^2}{z^2} + \frac{1}{z^2} dx_{||}^2 + d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\phi^2 \tag{3.2}$$

$$F^o_{\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau}, \qquad F^o_{abcde} = \epsilon_{abcde}$$

[2] The only exception is the case of extremal operators where the mixing with multitrace operators is not subleading [22]. In this paper it was argued that the supergravity fields are dual to the single trace operators so these are the relevant operators to consider.

[3] The field strength differs by a factor of 4 from the conventions in [25]. Index conventions: $M, N, ...$ are $10d$ indices, $\mu, \nu, ...$ are AdS_5 indices, $a, b, ...$ are S^5 indices. x denotes AdS coordinates and y S^5 coordinates.

We will consider here solutions that are deformations of $AdS_5 \times S^5$ such that

$$
\begin{aligned}
g_{MN} &= g^o_{MN} + h_{MN}, \\
F_{MNPQR} &= F^o_{MNPQR} + f_{MNPQR}.
\end{aligned} \tag{3.3}
$$

The fluctuations can be expanded in S^5 harmonics:

$$
\begin{aligned}
h_{\mu\nu}(x,y) &= \sum \tilde{h}^{I_1}_{\mu\nu}(x) Y^{I_1}(y) \\
h_{\mu a}(x,y) &= \sum (\tilde{B}^{I_5}_{(v)\mu}(x) Y^{I_5}_a(y) + \tilde{B}^{I_1}_{(s)\mu}(x) D_a Y^{I_1}(y)) \\
h_{(ab)}(x,y) &= \sum (\hat{\phi}^{I_{14}}_{(t)}(x) Y^{I_{14}}_{(ab)}(y) + \phi^{I_5}_{(v)}(x) D_{(a} Y^{I_5}_{b)}(y) + \phi^{I_1}_{(s)}(x) D_{(a} D_{b)} Y^{I_1}(y)) \\
h^a_a(x,y) &= \sum \tilde{\pi}^{I_1}(x) Y^{I_1}(y)
\end{aligned} \tag{3.4}
$$

and

$$
\begin{aligned}
f_{\mu\nu\rho\sigma\tau}(x,y) &= \sum 5 D_{[\mu} b^{I_1}_{\nu\rho\sigma\tau]}(x) Y^{I_1}(y) \\
f_{a\mu\nu\rho\sigma}(x,y) &= \sum (b^{I_1}_{\mu\nu\rho\sigma}(x) D_a Y^{I_1}(y) + 4 D_{[\mu} b^{I_5}_{\nu\rho\sigma]}(x) Y^{I_5}_a(y)) \\
f_{ab\mu\nu\rho}(x,y) &= \sum (3 D_{[\mu} b^{I_{10}}_{\nu\rho]}(x) Y^{I_{10}}_{[ab]}(y) - 2 b^{I_5}_{\mu\nu\rho}(x) D_{[a} Y^{I_5}_{b]}(y)) \\
f_{abc\mu\nu}(x,y) &= \sum (2 D_{[\mu} b^{I_5}_{\nu]}(x) \epsilon_{abc}{}^{de} D_d Y^{I_5}_e(y) + 3 b^{I_{10}}_{\mu\nu}(x) D_{[a} Y^{I_{10}}_{bc]}(y)) \\
f_{abcd\mu}(x,y) &= \sum (D_\mu b^{I_1}_{(s)}(x) \epsilon_{abcd}{}^e D_e Y^{I_1}(y) + (\Lambda^{I_5} - 4) b^{I_5}_\mu(x) \epsilon_{abcd}{}^e Y^{I_5}_e(y)) \\
f_{abcde}(x,y) &= \sum b^{I_1}_{(s)}(x) \Lambda^{I_1} \epsilon_{abcde} Y^{I_1}(y)
\end{aligned} \tag{3.5}
$$

Numerical constants in these expressions are inserted so as to match with the conventions of [21], see appendix A. Parentheses denote a symmetric traceless combination (i.e. $A_{(ab)} = 1/2(A_{ab}+A_{ba}) - 1/5 g_{ab} A^a_a$). $Y^{I_1}, Y^{I_5}_a, Y^{I_{14}}_{(ab)}$ and $Y^{I_{10}}_{[ab]}$ denote scalar, vector and tensor harmonics whilst Λ^{I_1} and Λ^{I_5} are the eigenvalues of the scalar and vector harmonics under (minus) the d'Alembertian. The subscripts t, v and s denote whether the field is associated with tensor, vector or scalar harmonics respectively, whilst the superscript of the harmonic label I_n derives from the number of components n of the harmonic.

Not all fluctuations are independent however. Some of the modes are diffeomorphic to each other or to the background solution, i.e. certain δh_{MN} and δf_{MNPQR} are generated by a coordinate transformation,

$$
x^{M\prime} = x^M - \xi^M. \tag{3.6}
$$

These, up to terms linear in fluctuations, are given by

$$
\begin{aligned}
\delta h_{MN} &= (D_M \xi_N + D_N \xi_M) + (D_M \xi^P h_{PN} + D_N \xi^P h_{MP} + \xi^P D_P h_{MN}); \quad (3.7) \\
\delta f_{MNPQR} &= 5 D_{[M} \xi^S F^o_{NPQR]S} + (5 D_{[M} \xi^S f_{NPQR]S} + \xi^S D_S f_{MNPQR}).
\end{aligned}
$$

The gauge parameter $\xi^M(x,y)$ can be expanded in harmonics as

$$\begin{aligned}\xi_\mu(x,y) &= \sum \xi_\mu^{I_1}(x) Y^{I_1}(y); \\ \xi_a(x,y) &= \sum (\xi_{(v)}^{I_5}(x) Y_a^{I_5}(y) + \xi_{(s)}^{I_1}(x) D_a Y^{I_1}(y)).\end{aligned} \tag{3.8}$$

In much of the previous literature this issue was dealt with by imposing a gauge fixing condition, most notably the de Donder-Lorentz gauge fixing condition

$$D^a h_{(ab)} = D^a h_{a\mu} = 0. \tag{3.9}$$

This amounts to setting to zero the coefficients $\tilde{B}_{(s)\mu}^{I_1}, \phi_{(v)}^{I_5}, \phi_{(s)}^{I_1}$ (as can be easily seen by inserting (3.4) in (3.9)). Although this gauge is a very convenient choice for deriving the spectrum, it is not very well suited for holography since generically solutions will not be in that gauge. For this reason instead of gauge fixing this symmetry we will derive gauge invariant combinations of fluctuations. This will allow us to switch easily between different gauges.

3.1 Gauge invariance at linear order

We first discuss gauge invariance at leading order, i.e. we consider the fluctuation independent terms in (3.7). These transformations map the fluctuations to the background solution. Under these transformations the coefficients in (3.4) transform as

$$\begin{aligned}&\delta \tilde{h}_{\mu\nu}^{I_1} = D_\mu \xi_\nu^{I_1} + D_\nu \xi_\mu^{I_1}, \qquad \delta \tilde{B}_{(v)\mu}^{I_5} = D_\mu \xi_{(v)}^{I_5}, \qquad \delta \tilde{B}_{(s)\mu}^{I_1} = D_\mu \xi_{(s)}^{I_1} + \xi_\mu^{I_1}, \\ &\delta \hat{\phi}_{(t)}^{I_{14}} = 0, \qquad \delta \phi_{(v)}^{I_5} = 2\xi_{(v)}^{I_5}, \qquad \delta \phi_{(s)}^{I_1} = 2\xi_{(s)}^{I_1}, \qquad \delta \tilde{\pi}^{I_1} = 2\Lambda^{I_1} \xi_{(s)}^{I_1}.\end{aligned} \tag{3.10}$$

It follows that $\hat{\phi}_{(t)}^{I_{14}}$ is gauge invariant to this order and for the rest of fields we can construct the following gauge invariant combinations

$$\begin{aligned}\hat{\pi}^{I_1} &= \tilde{\pi}^{I_1} - \Lambda^{I_1} \phi_{(s)}^{I_1} \\ \hat{B}_{(v)\mu}^{I_5} &= \tilde{B}_{(v)\mu}^{I_5} - \frac{1}{2} D_\mu \phi_{(v)}^{I_5} \\ \hat{h}_{\mu\nu}^{I_1} &= \tilde{h}_{\mu\nu}^{I_1} - D_\mu \hat{B}_{(s)\nu}^{I_1} - D_\nu \hat{B}_{(s)\mu}^{I_1}, \qquad I_1 \neq 0.\end{aligned} \tag{3.11}$$

where we define

$$\hat{B}_{(s)\mu}^{I_1} = \tilde{B}_{(s)\mu}^{I_1} - \frac{1}{2} D_\mu \phi_{(s)}^{I_1} \qquad \Rightarrow \qquad \delta \hat{B}_{(s)\mu}^{I_1} = \xi_\mu^{I_1}. \tag{3.12}$$

Note that the last formula in (3.11) is only valid for $I_1 \neq 0$ (the fields $\tilde{B}_{(s)\mu}^{I_1}$ and $\phi_{(s)}^{I_1}$ exists only for $I_1 > 0$, since Y^0 is a constant). For $I_1 = 0$, $\tilde{h}_{\mu\nu}^0$ is a deformation of the background metric and from (3.10) we see that it indeed transforms as a metric.

Similarly, the leading term in the 5-form transformation implies for the coefficients in the harmonic expansion the following transformations

$$
\begin{aligned}
&\delta b^{I_1}_{(s)} = \xi^{I_1}_{(s)}, \qquad \delta b^{I_1}_{\mu\nu\rho\sigma} = \epsilon_{\mu\nu\rho\sigma}{}^{\tau}\xi^{I_1}_{\tau}, \qquad \delta b^{I_5}_{\mu} = \frac{1}{(\Lambda^{I_5}-4)} D_\mu \xi^{I_5}_{(v)}, \\
&\delta b^{I_5}_{\mu\nu\rho} = \delta b^{I_{10}}_{\mu\nu} = 0,
\end{aligned} \tag{3.13}
$$

so that the gauge invariant combinations are

$$
\hat{b}^{I_1} = b^{I_1}_{(s)} - \frac{1}{2}\phi^{I_1}_{(s)} \tag{3.14}
$$

$$
\begin{aligned}
\hat{b}^{I_1}_{\mu\nu\rho\sigma} &= b^{I_1}_{\mu\nu\rho\sigma} - \epsilon_{\mu\nu\rho\sigma}{}^{\tau}\hat{B}^{I_1}_{(s)\tau} \\
\hat{b}^{I_5}_{(v)\mu} &= b^{I_5}_{(v)\mu} - \frac{1}{2(\Lambda^{I_5}-4)} D_\mu \phi^{I_5}_{(v)}.
\end{aligned} \tag{3.15}
$$

and the fields $b^{I_5}_{\mu\nu\rho}$ and $b^{I_{10}}_{\mu\nu}$.

3.2 Gauge invariance at quadratic order

In this subsection we will derive the gauge invariant combinations to second order in the fluctuations. The idea is the same as in the previous subsection: we insert the harmonic expansion of the fluctuations and the gauge parameter into (3.7) (which now includes all terms) and read off the transformation of each coefficient. Then we seek a quadratic modification of each field combination that is gauge invariant. One complication in this case is that because the r.h.s. of (3.7) is non-linear one needs to project onto the basic of spherical harmonics in order to extract the transformation of the coefficients. The analysis can be readily carried out in all generality but for the applications considered in this paper it is sufficient to consider only the modes that couple to scalar harmonics and their derivatives, i.e. we set to zero all fields that couple to vector and tensor harmonics (and their derivatives) (as well as $\xi^{I_5}_{(v)}$). Including these fields would result in additional terms in the gauge invariant combinations below. Since no confusion can arise we also drop the subscript (s) from relevant fields and use the condensed notation for indices and arguments: $\phi^{I_1}_{(s)} \rightarrow \phi^1, \tilde{\pi}^{I_1} \rightarrow \tilde{\pi}^1, z(k_1) \rightarrow z_1, a(k_1,k_2,k_3) \rightarrow a_{123}$ etc. Note that we consistently use the notation $\tilde{\psi}$ to denote a field in the harmonic expansion of the supergravity field; $\hat{\psi}$ to denote a field which is gauge invariant to linear order and ψ to denote the field which is gauge invariant to quadratic order.

3.2.1 Scalar fields

We first discuss the scalar fields, $\tilde{\pi}^{I_1}, \phi^{I_1}_{(s)}$ and $b^{I_1}_{(s)}$. Their transformations are given by (we suppress the linear terms determined in the previous section and factors $\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$):

$$\delta\tilde{\pi}^1 = \frac{1}{z_1}\left(2\phi^2\xi^3 d_{123} + (\frac{2}{5}\Lambda^2\xi^2\tilde{\pi}^3 + \xi^{\mu 2}D_\mu\tilde{\pi}^3)a_{123} + (\xi^2\tilde{\pi}^3 + 2\xi^{\mu 2}\tilde{B}^3_\mu)b_{123}\right); \quad (3.16)$$

$$\delta\phi^1 = \frac{1}{z_1 q_1}\left(\xi^2\phi^3 e_{123} + (\xi^{\mu 2}D_\mu\phi^3 + \frac{2}{5}\xi^2\tilde{\pi}^3)d_{213} + 2\xi^{\mu 2}\tilde{B}^3_\mu c_{123}\right); \quad (3.17)$$

$$\delta b^1 = \frac{1}{\Lambda^1 z_1}\left((\xi^{\mu 2}D_\mu b^3 + \Lambda^2 b^2\xi^3)(b_{123} + \Lambda^3 a_{123})\right), \quad (3.18)$$

where the triple overlaps $a_{123} = a(k_1,k_2,k_3), b_{123} = b(k_1,k_2,k_3)$ etc are defined in appendix B.1 and summation over (I_2, I_3) is implicit.

From these transformations one can infer quantities which are gauge invariant to quadratic order:

$$\pi^1 = \hat{\pi}^1 - \frac{1}{2z_1}\left(\left(\frac{2}{5}\Lambda^2 a_{123} + b_{123} - \frac{2\Lambda^1}{5q_1}d_{213}\right)\phi^2\hat{\pi}^3 + \left(d_{123} - \frac{\Lambda^1}{2q_1}e_{123} + \Lambda^3(\frac{1}{5}\Lambda^2 a_{123} + \frac{1}{2}b_{123} - \frac{\Lambda^1}{5q_1}d_{213})\right)\phi^2\phi^3 + 2\hat{B}^2_\mu\left(D^\mu\hat{\pi}^3 a_{123} + \hat{B}^{3\mu}(b_{123} - \frac{2\Lambda^1}{q_1}c_{123})\right)\right); \quad (3.19)$$

$$b^1 = \hat{b}^1 + \frac{1}{z_1}\left(\frac{\Lambda^3}{2\Lambda^1}\phi^2\hat{b}^3 b_{312} + \frac{1}{10q_1}d_{213}\phi^2\hat{\pi}^3 + \left(\frac{\Lambda^3}{8\Lambda^1}b_{312} + \frac{\Lambda^1}{20q_1}d_{213} + \frac{1}{8q_1}e_{123}\right)\phi^2\phi^3 + \hat{B}^2_\mu\left(\frac{1}{2q_1}\hat{B}^{3\mu}c_{123} + \frac{1}{\Lambda^1}D^\mu\hat{b}^3 b_{213}\right)\right). \quad (3.20)$$

The $I_1 = 0$ sector is special because the scalar harmonic is constant. Notice that we only have one scalar in this sector, namely $\tilde{\pi}^0$. Working out the gauge transformation yields

$$\delta\tilde{\pi}^0 = z(k)\left(2\xi^I\phi^I q(k) + \frac{2}{5}\Lambda^I\xi^I\tilde{\pi}^I + \xi^{\mu I}D_\mu\pi^I - (\xi^I\tilde{\pi}^I + 2\xi^{\mu I}B^I_\mu)\Lambda^I\right). \quad (3.21)$$

From here we obtain that the gauge invariant combination is (notice that $\tilde{\pi}^0$ was gauge invariant to leading order, i.e. $\hat{\pi}^0 = \tilde{\pi}^0$)

$$\pi^0 = \tilde{\pi}^0 + z(k)\left(\frac{3}{10}\Lambda^I\phi^I\hat{\pi}^I - \frac{1}{4}\Lambda^I(\Lambda^I + 8)\phi^I\phi^I - \hat{B}^{\mu I}D_\mu\hat{\pi}^I + \Lambda^I\hat{B}^{\mu I}\hat{B}^I_\mu\right), \quad (3.22)$$

where the summation over I is implied and $\hat{\pi}^I$ and $\hat{B}^{\mu I}$ are defined in (3.11) and (3.12).

Let us now consider the field $\hat{\phi}_{(t)}$ associated with the tensor harmonic. Whilst this is gauge invariant to leading order, at the next order it transforms as

$$\delta\hat{\phi}^1_{(t)} = \frac{1}{z_{(t)1}}\left(\xi^2\phi^3 e^{(t)}_{123} - (\xi^{\mu 2}D_\mu\phi^3 + \frac{2}{5}\xi^2\tilde{\pi}^3 - 2\xi^{\mu 2}\tilde{B}^3_\mu)c^{(t)}_{123}\right), \quad (3.23)$$

where the normalization factor $z_{(t)}$ and overlap integrals $c^{(t)}_{123}$ etc are defined in the appendix B.1. Thus the gauge invariant combination to this order is

$$\phi^1_{(t)} = \hat{\phi}^1_{(t)} + \frac{1}{z_{(t)1}}\left((-\hat{B}^{\mu 2}\hat{B}^3_\mu + \frac{1}{5}\hat{\pi}^2\phi^3 + \frac{1}{10}\Lambda^3\phi^2\phi^3)c^{(t)}_{123} - \frac{1}{4}e^{(t)}_{123}\phi^2\phi^3\right). \quad (3.24)$$

3.2.2 Tensor fields

We now turn to the non-scalar sector. As we will see in the next section the field equations algebraically relate the field $b^{I_1}_{\mu\nu\rho\sigma}$ to the field $b^{I_1}_{(s)}$ (more precisely the field equations relate the corresponding gauge invariant combinations) so we need not discuss this field. Furthermore, $\tilde{B}^{I_1}_{(s)\mu}$ is pure gauge. It is useful however to introduce the following combination that transforms nicely up to quadratic order

$$\begin{aligned} B^1_\mu &= \hat{B}^1_\mu + \frac{1}{z_1}\Bigg(-\frac{1}{2}(\frac{1}{10}D_\mu\phi^2\hat{\pi}^3 + \hat{B}^{\nu 2}\hat{h}^3_{\mu\nu})a_{123} \\ &+ D_\mu\left(\phi^2\phi^3(-\frac{1}{8}b_{123} + \frac{1}{8q_1}e_{123} + \frac{\Lambda^3}{5q_1}d_{213}) + \frac{1}{10q_1}d_{213}\phi^2\hat{\pi}^3 + \frac{1}{2}(\frac{1}{q_1}c_{123} - a_{123})\hat{B}^2_\nu\hat{B}^{\nu 3}\right)\Bigg). \end{aligned} \tag{3.25}$$

This transforms as

$$\delta B^1_\mu = \xi^1_\mu + \frac{1}{z_1}(\xi^2\hat{B}^3_\mu b_{123} + \xi^{\nu 2}D_\nu\hat{B}^3_\mu a_{123}). \tag{3.26}$$

Now consider the KK graviton modes, $\tilde{h}^{I_1}_{\mu\nu}$. The gauge transformation reads

$$\delta\tilde{h}^1_{\mu\nu} = \frac{1}{z_1}\left((D_\mu\xi^{\lambda 2}\tilde{h}^3_{\lambda\nu} + D_\nu\xi^{\lambda 2}\tilde{h}^3_{\lambda\mu} + \xi^{\lambda 2}D_\lambda\tilde{h}^3_{\mu\nu})a_{123} + (\xi^2\tilde{h}^3_{\mu\nu} + 2D_{(\mu}\xi^2\tilde{B}^3_{\nu)})b_{123}\right). \tag{3.27}$$

From this we derive the following gauge invariant combination ($I_1 \neq 0$)

$$\begin{aligned} h^1_{\mu\nu} &= \tilde{h}^1_{\mu\nu} - D_\mu B^1_\nu - D_\nu B^1_\mu - \frac{1}{z_1}\Bigg(\frac{1}{2}(\phi^2\hat{h}^3_{\mu\nu} + \frac{1}{2}D_\mu\phi^2 D_\nu\phi^3)b_{123} \\ &+ (D_\mu\hat{B}^{\lambda 2}\hat{h}^3_{\nu\lambda} + D_\nu\hat{B}^{\lambda 2}\hat{h}^3_{\mu\lambda} + \hat{B}^{\lambda 2}D_\lambda\hat{h}^3_{\mu\nu} + D_\mu\hat{B}^{\lambda 2}D_\nu\hat{B}^3_\lambda + \hat{B}^{\lambda 2}\hat{B}^3_\lambda g^0_{\mu\nu} - \hat{B}^2_\mu\hat{B}^3_\nu)a_{123}\Bigg). \end{aligned} \tag{3.28}$$

Let us now discuss the $I_1 = 0$ case. $\tilde{h}^0_{\mu\nu}$ transforms as

$$\begin{aligned} \delta\tilde{h}^0_{\mu\nu} &= D_\mu\xi^{\lambda 0}\tilde{h}^0_{\lambda\nu} + D_\nu\xi^{\lambda 0}\tilde{h}^0_{\lambda\mu} + \xi^{\lambda 0}D_\lambda\tilde{h}^0_{\mu\nu} \\ &+ z(k)\left(D_\mu\xi^{\lambda I}\tilde{h}^I_{\lambda\nu} + D_\nu\xi^{\lambda I}\tilde{h}^I_{\lambda\mu} + \xi^{\lambda I}D_\lambda\tilde{h}^I_{\mu\nu} - \Lambda^I(\xi^I\tilde{h}^I_{\mu\nu} + 2D_{(\mu}\xi^I\tilde{B}^I_{\nu)})\right). \end{aligned} \tag{3.29}$$

We introduce

$$\begin{aligned} h^0_{\mu\nu} &= \tilde{h}^0_{\mu\nu} + \frac{1}{3}\pi^0 g^o_{\mu\nu} - z(k)\Bigg(\frac{1}{2}\Lambda^I(\phi^I\hat{h}^I_{\mu\nu} + \frac{1}{2}D_\mu\phi^I D_\nu\phi^I) \\ &+ D_\mu\hat{B}^{\lambda I}\hat{h}^I_{\nu\lambda} + D_\nu\hat{B}^{\lambda I}\hat{h}^I_{\mu\lambda} + \hat{B}^{\lambda I}D_\lambda\hat{h}^I_{\mu\nu} + D_\mu\hat{B}^{\lambda I}D_\nu\hat{B}^I_\lambda + \hat{B}^{\lambda I}\hat{B}^I_\lambda g^o_{\mu\nu} - \hat{B}^I_\mu\hat{B}^I_\nu\Bigg) \end{aligned} \tag{3.30}$$

(the term linear in π^0 was added in anticipation of the fact that it is $\tilde{h}^0_{\mu\nu} + \frac{1}{3}\tilde{\pi}^0 g^o_{\mu\nu}$ that satisfies the linearized equations of motion, see the discussion around (4.5)). Recall that this mode is a correction to the spacetime metric

$$g_{\mu\nu} = g^o_{\mu\nu} + h^0_{\mu\nu}, \tag{3.31}$$

so the combination should transform as

$$\delta g_{\mu\nu} = D^g_\mu\zeta_\nu + D^g_\nu\zeta_\mu, \tag{3.32}$$

where D^g is the covariant derivative of the corrected metric (3.31). Indeed one finds this to hold with

$$\zeta_\nu = \xi^{\mu 0} g_{\mu\nu} + z(I)(\xi^{\lambda I} D_\lambda \hat{B}^I_\nu - \Lambda^I \xi^I \hat{B}^I_\nu). \tag{3.33}$$

4 Field equations

The field equations for the coefficients in the harmonic expansion were derived in the de Donder gauge at linear order by [21] and at quadratic order in [26, 27, 28]. The gauge invariant variables derived in the previous section allow one to relax the gauge condition. Indeed notice that the gauge invariant variables when evaluated in the de Donder gauge become equal to the fields used in [21, 26, 27, 28]. It follows (and we have explicitly checked this in detail) that the field equations with no gauge condition imposed can be obtained from the results in [21, 26, 27, 28] by simply replacing each field with its gauge invariant generalization.

4.1 Linear order

In this subsection we summarize some of the results of [21] (a summary of the derivation is given in appendix C). As just mentioned, one can relax the gauge fixing condition by replacing all fields by the hatted versions given in the previous section.

The scalars satisfy the following equations

$$\begin{aligned} \Box \hat{s}^{I_1} &= k(k-4)\hat{s}^{I_1}, \quad k \geq 2, \\ \Box \hat{t}^{I_1} &= (k+4)(k+8)\hat{t}^{I_1}, \quad k \geq 0, \\ \Box \hat{\phi}^{I_{14}}_{(t)} &= k(k+4)\hat{\phi}^{I_{14}}_{(t)} \quad k \geq 2, \end{aligned} \tag{4.1}$$

where we introduce the combinations

$$\hat{s}^{I_1} = \frac{1}{20(k+2)}(\hat{\pi}^{I_1} - 10(k+4)\hat{b}^{I_1}), \qquad \hat{t}^{I_1} = \frac{1}{20(k+2)}(\hat{\pi}^{I_1} + 10k\hat{b}^{I_1}), \tag{4.2}$$

with inverse relations $\hat{b}^{I_1} = -\hat{s}^{I_1} + \hat{t}^{I_1}$, $\hat{\pi}^{I_1} = 10k\hat{s}^{I_1} + 10(k+4)\hat{t}^{I_1}$.

The remaining modes that couple to scalar spherical harmonics are the KK gravitons. They are described by transverse and traceless fields

$$\phi^{I_1}_{(\mu\nu)} = \hat{h}^{I_1}_{(\mu\nu)} - \frac{1}{(k+1)(k+3)} D_{(\mu} D_{\nu)}(\frac{2}{5}\hat{\pi}^{I_1} - 12\hat{b}^{I_1}), \quad I_1 \neq 0. \tag{4.3}$$

satisfying the equation,

$$(\Box - (k(k+4) - 2))\phi^{I_1}_{(\mu\nu)} = 0, \quad I_1 \neq 0. \tag{4.4}$$

The $I_1 = 0$ case is special in that this mode describes a deformation of the background metric. The combination that satisfies the $5d$ linearized Einstein equation is

$$h'_{\mu\nu}{}^0 = (\tilde{h}^0_{\mu\nu} + \frac{1}{3} g^o_{\mu\nu} \tilde{\pi}^0). \tag{4.5}$$

One can understand the origin of the shift by $\tilde{\pi}^0$ by considering the reduction of the $10d$ action to five dimensions. Keeping terms linear in the fluctuations, the volume of compactification manifold is

$$\int d^5y \sqrt{\det g_{ab}} = \pi^3 (1 + \frac{1}{2}\tilde{\pi}^0) \tag{4.6}$$

It follows that the reduced action is

$$S_{5d} \sim \int d^5x \sqrt{\det g_{\mu\nu}} ((1 + \frac{1}{2}\tilde{\pi}^0) R + \cdots) \tag{4.7}$$

and a Weyl transformation is required to bring the action to the Einstein frame. The transformation from $\tilde{h}^0_{\mu\nu}$ to $h'_{\mu\nu}{}^0$ is precisely this Weyl transformation.

4.2 Quadratic order

The derivation of the equations of motion to second order in fluctuations was discussed in [26, 27, 28] and is summarized in appendix C. For our applications it is sufficient to retain only the quadratic coupling to the field s^2.

4.2.1 Scalar fields

The corrected field equation for the scalar fields $\psi = \{s^2, s^4, t^0, t^2, t^4, \phi^2_{(t)}\}$ is given by

$$(\Box - m^2_\psi)\psi^I = D_{\psi 22}(\hat{s}^2)^2 + E_{\psi 22} D_\mu \hat{s}^2 D^\mu \hat{s}^2 + F_{\psi 22} D_{(\mu} D_{\nu)} \hat{s}^2 D^{(\mu} D^{\nu)} \hat{s}^2, \tag{4.8}$$

where the coefficients $D_{\psi 22}, E_{\psi 22}, F_{\psi 22}$ can be obtained from the results in appendix C and are given in table 1. The fields entering the l.h.s of this equation are the gauge invariant combinations to second order whilst the fields in the r.h.s. are the gauge invariant combinations to linear order (since the r.h.s. is quadratic in fluctuations). This follows from our general discussion and we have also explicitly checked that the terms involving $\phi^2_{(s)}$ in the second order equations (with no gauge fixing imposed) are accounted for by the $\phi^2_{(s)}$ terms in the gauge invariant combinations. When s^2 is the leading non-zero field (as it is in the application discussed in this paper) the gauge invariant combinations take the form

$$\psi = \hat{\psi} + A_{\psi s\phi} \hat{s}^2 \phi^2_{(s)} + A_{\psi\phi\phi} (\phi^2_{(s)})^2 + A_{\psi s B} D^\mu \hat{s}^2 \hat{B}^2_{(s)\mu} + A_{\psi BB} \hat{B}^{\mu 2}_{(s)} \hat{B}^2_{(s)\mu}. \tag{4.9}$$

The coefficients $A_{\psi s\phi}, A_{\psi\phi\phi}, A_{\psi sB}$ and $A_{\psi BB}$ are given in Table 1.

	s^2	s^4	t^0	t^2	t^4	$\phi^2_{(t)}$
$D_{\psi 22}$	$-\frac{4\sqrt{3}}{3}$	$-\frac{172}{5\sqrt{5}}$	$\frac{229}{75}$	$\frac{76\sqrt{3}}{25}$	$\frac{52}{\sqrt{5}}$	$\frac{48}{25}$
$E_{\psi 22}$	$\frac{\sqrt{3}}{10}$	$\frac{3}{\sqrt{5}}$	$-\frac{11}{20}$	$-\frac{3\sqrt{3}}{10}$	$-\frac{1}{\sqrt{5}}$	$-\frac{4}{5}$
$F_{\psi 22}$	$\frac{1}{12\sqrt{3}}$	$\frac{7}{9\sqrt{5}}$	$\frac{1}{60}$	$\frac{\sqrt{3}}{180}$	0	$\frac{2}{45}$
$A_{\psi s\phi}$	$\frac{7\sqrt{3}}{40}$	$\frac{7}{2\sqrt{5}}$	$-\frac{3}{40}$	$-\frac{7\sqrt{3}}{120}$	$-\frac{1}{2\sqrt{5}}$	$-\frac{1}{5}$
$A_{\psi\phi\phi}$	$-\frac{17\sqrt{3}}{160}$	$-\frac{9}{10\sqrt{5}}$	$-\frac{1}{80}$	$\frac{3\sqrt{3}}{160}$	$\frac{1}{8\sqrt{5}}$	$\frac{1}{20}$
$A_{\psi sB}$	$-\frac{\sqrt{3}}{24}$	$-\frac{3}{4\sqrt{5}}$	$-\frac{1}{48}$	$-\frac{\sqrt{3}}{120}$	0	0
$A_{\psi BB}$	$-\frac{\sqrt{3}}{32}$	$-\frac{1}{4\sqrt{5}}$	$-\frac{1}{80}$	$-\frac{\sqrt{3}}{160}$	0	$\frac{1}{20}$
$J_{\psi 22}$	$\frac{-2\sqrt{3}}{15}$	$-\frac{83}{18\sqrt{5}}$	$\frac{3}{40}$	$\frac{2\sqrt{3}}{45}$	$\frac{1}{2\sqrt{5}}$	$\frac{2}{45}$
$L_{\psi 22}$	$-\frac{1}{12\sqrt{12}}$	$-\frac{7}{18\sqrt{5}}$	$-\frac{1}{120}$	$-\frac{\sqrt{3}}{360}$	0	$-\frac{1}{90}$
$w(\psi)$	$\frac{\sqrt{8}}{3}$	$\frac{2\sqrt{3}}{5}$	$\frac{8\sqrt{5}}{\sqrt{3}}$	$\frac{4\sqrt{7}}{\sqrt{10}}$	$\frac{12}{\sqrt{70}}$	$\frac{\sqrt{15}}{4}$
$\lambda_{\Psi 22}$	$-\frac{4}{\sqrt{6}}$	0	0	0	0	0

Table 1: Coefficients in (4.8), (4.9), (4.10) and (4.11).

The field equations in (4.8) contain higher derivative terms on the r.h.s which can however be removed by the following transformation [26]

$$\Psi = w(\psi)\left(\psi + J_{\psi 22}(\hat{s}^2)^2 + L_{\psi 22} D_\mu \hat{s}^2 D^\mu \hat{s}^2\right). \tag{4.10}$$

This transformation *is* the non-linear KK map to quadratic order in the fields. It maps solutions of the 10d fields equations to solutions of the 5d field equations,

$$(\Box - m_\psi^2)\Psi = \lambda_{\Psi 22}(S^2)^2. \tag{4.11}$$

The coefficients $w(\psi), J_{\psi 22}, L_{\psi 22}$ and $l_{\Psi 22}$ are given in Table 1. We include on the r.h.s. only the terms quadratic in S^2 because these are the terms that are relevant for us. We note however that all quadratic terms (and cubic scalar couplings [27]) have been determined in the literature [26, 27, 28]. The results in Table 1 are in agreement with the results in these papers. The field equations can be integrated to a 5d action. and the constants $w(\psi)$ have been chosen such that the overall normalization agrees with the one in [8, 9]

$$S_{5d} = \frac{N^2}{2\pi^2}\int d^5x\sqrt{G}\left(\frac{1}{4}R + \frac{1}{2}G^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi + V(\Psi)\right). \tag{4.12}$$

Using the quadratic 10-dimensional supergravity action computed in [26, 29] one finds that

$$w(s^I) \;=\; \sqrt{\frac{8k(k-1)(k+2)z(k)}{(k+1)}}, \qquad w(\phi_{(t)}) = \sqrt{\frac{z_{(t)}(k^{(t)})}{8}} \tag{4.13}$$

$$w(t^I) = \sqrt{\frac{8(k+2)(k+4)(k+5)z(k)}{(k+3)}}.$$

From Table 1 we see that the only non-zero cubic term in the potential (that is also quadratic in S^2) is the cubic self coupling of S^2 and its coefficient, $-4/3\sqrt{6}$, precisely agrees with the corresponding coupling in $5d$ gauged supergravity (after matching sign conventions), compare with (2.6) of [9].

It is important to note that the transformation (4.10) gives an explicit map between solutions of the ten dimensional equation and solutions of the five dimensional equation and vice versa, i.e. any solution of the five dimensional theory specified by the action (4.12) can be uplifted to a ten dimensional solution. We emphasize that this map is valid irrespectively of whether there is a consistent truncation since we keep all KK modes.

4.2.2 Tensor fields

Let us consider first the graviton. The quadratic correction to the gravitational equation is obtained in appendix C:

$$(L_E + 4)h^0_{\mu\nu} = \frac{1}{12}\left(-\frac{2}{9}D_\mu D_\rho D_\sigma \hat{s}^2 D_\nu D^\rho D^\sigma \hat{s}^2 - \frac{16}{3}(D_\mu D_\nu D_\rho \hat{s}^2 D^\rho \hat{s}^2 + D_\mu D_\rho \hat{s}^2 D_\nu D^\rho \hat{s}^2)\right.$$
$$\left. + \frac{364}{9}D_\mu \hat{s}^2 D_\nu \hat{s}^2 + g^o_{\mu\nu}\left(-\frac{8}{9}D_\rho D_\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2 + 20 D_\rho \hat{s}^2 D_\rho \hat{s}^2 - \frac{512}{9}s^2\right)\right), \quad (4.14)$$

where L_E is the linearized Einstein operator (C.20). This equation contains higher derivative interactions. Just as in the case of scalars one can remove them by considering the following transformation

$$G_{\mu\nu} = h^0_{\mu\nu} - \frac{1}{12}\left(\frac{2}{9}D_\mu D^\rho \hat{s}^2 D_\nu D_\rho \hat{s}^2 - \frac{10}{3}\hat{s}^2 D_\mu D_\nu \hat{s}^2 + (\frac{10}{9}(D\hat{s}^2)^2 - \frac{32}{9}(\hat{s}^2)^2)g^o_{\mu\nu}\right), \quad (4.15)$$

as can be verified using (C.21). In terms of these variables the field equation becomes

$$R_{\mu\nu}[G] = 2(T_{\mu\nu} - \frac{1}{3}G_{\mu\nu}T^\sigma_\sigma) \quad (4.16)$$

where $R_{\mu\nu}$ is the Ricci tensor of $G_{\mu\nu}$ and

$$T_{\mu\nu} = \partial_\mu S^2 \partial_\nu S^2 - G_{\mu\nu}(\frac{1}{2}(\partial S^2)^2 + V(S^2)). \quad (4.17)$$

The equation (4.16) is indeed the field equation for $G_{\mu\nu}$ derived from (4.12) and $T_{\mu\nu}$ is the corresponding matter stress energy tensor, where keeping with our approximations we only retain terms quadratic in S^2.

Now let us briefly describe the equations determining the KK gravitons. The traceless part of the ten-dimensional field is given by

$$h^{I_1}_{(\mu\nu)} = \phi^{I_1}_{(\mu\nu)} + \psi^{I_1}_{(\mu\nu)} + \frac{1}{(k+1)(k+3)} D_{(\mu} D_{\nu)} (\frac{2}{5}\pi^{I_1} - 12 b^{I_1}), \tag{4.18}$$

where all fields are the appropriate combinations which are gauge invariant to quadratic order. $\phi^{I_1}_{(\mu\nu)}$ is transversal but $\psi^{I_1}_{\mu\nu}$ is not; indeed its defining equation is

$$D^\mu D^\nu \psi^{I_1}_{\mu\nu} = Z^{I_1}[s], \tag{4.19}$$

where $Z^{I_1}[s]$ is quadratic in the field s^2 and is determined by the quadratic corrections to the trace of the Einstein equation in the AdS directions. The equation for the transversal field is then

$$\begin{aligned} (\Box + 2 - k(k+4))\phi^{I_1}_{(\mu\nu)} &= (2L_E + 8 + k(k+4))\psi^{I_1}_{(\mu\nu)} + Z^{I_1}_{(\mu\nu)}; \\ &= (2L_E + 8 + k(k+4))\psi^{(t)I_1}_{(\mu\nu)}, \end{aligned} \tag{4.20}$$

where $Z^{I_1}_{(\mu\nu)}$ is again quadratic in the field s^2 and follows from the corrections to the $(\mu\nu)$ Einstein equation. $\psi^{(t)I_1}_{(\mu\nu)}$ is a transversal field which is quadratic in s^2 and contains up to six derivatives. We have verified that that the right hand side of the equation can be written in the latter form; this follows from the detailed structure of $Z^{I_1}_{(\mu\nu)}$ and $\psi^{I_1}_{\mu\nu}$. It is then immediately manifest that if one removes the higher derivative terms in the equation by defining the five dimensional field as $\Phi^{I_1}_{\mu\nu} = \phi^{I_1}_{(\mu\nu)} - \psi^{(t)I_1}_{(\mu\nu)}$ then this five dimensional field satisfies the free field equation. This is in agreement with the result of [27] which found the corresponding cubic coupling to vanish and implies that the five-dimensional field must vanish to the order to which we work. As we discuss later, there is no physical content in these fields (to the order to which we work), so we suppress explicit details of the (rather complicated) KK reduction map.

5 Holographic 1-point functions

5.1 Generalities

The KK reduction discussed in the previous sections provides an explicit map between ten dimensional solutions and five dimensional solutions as well as an associated five dimensional action for gravity coupled to massless and massive KK modes. If one would consider this problem in full generality the resulting action would involve an infinity of fields. For determining the holographic 1-point functions, however, we are only interested in the near boundary behavior of the solutions (as is reviewed below). The near boundary expansion effectively decouples all but a finite number of fields, the number of which depends on the dimension of the operator whose 1-point function one is computing; the higher the dimension, the greater the number of fields switched on.

Starting from a five dimensional action there is a well developed method for computing holographic 1-point functions, namely holographic renormalization [7, 9], see [17] for a review. Recall that the basic formula expressing the AdS/CFT correspondence [2, 3] relates the on-shell supergravity action with prescribed boundary conditions for all fields to the generating functional of QFT correlators with the boundary fields playing the role of sources[4],

$$\langle \exp\left(-S_{\mathrm{QFT}}[G_{(0)}] - \int d^4x \sqrt{G_{(0)}}\mathcal{O}(x)\Phi_{(0)}(x)\right) = \exp(-S_{SG}[G_{(0)}, \Phi_{(0)}]) \tag{5.1}$$

where $G_{(0)}, \Phi_{(0)}$ are the fields parameterizing the boundary values of the bulk metric G and of other bulk fields denoted collecting by Φ.

Naively, both sides of this relation diverge: the l.h.s. suffers from (the well known QFT) UV divergences and the r.h.s. suffers from IR divergences (due to the infinite volume of the spacetime). The divergences on the l.h.s may be dealt with by standard renormalization. The infinities on the r.h.s. are dealt by holographic renormalization. Namely, one adds a number of boundary counterterms that cancel all possible infinities that can arise in the on-shell action. Holographic 1-point functions in the presence of sources are then obtained by computing in full generality the first variation of the renormalized on-shell supergravity action. This leads to relations between the 1-point functions and certain coefficients in the near-boundary expansion of the bulk fields. This relation effectively replaces (5.1) since higher point functions can be computing by further differentiating the 1-point functions w.r.t. sources.

The near-boundary expansion of the bulk metric $G_{\mu\nu}$ and scalar field Φ^k, where k is the dimension of the dual operator, take the form

$$\begin{aligned} ds_5^2 &= \frac{dz^2}{z^2} + \frac{1}{z^2}\left(G_{(0)ij}(x) + z^2 G_{(2)ij}(x) + z^4(G_{(4)ij}(x) + \log z^2 h_{(4)ij}(x))\right) dx^i dx^j; \\ \Phi^2(x,z) &= z^2\left(\log z^2 \Phi^2_{(0)}(x) + \tilde{\Phi}^2_{(0)}(x) + \cdots\right); \\ \Phi^k(x,z) &= z^{(4-k)}\Phi^k_{(0)}(x) + \cdots + z^k \Phi^k_{(2k-4)}(x) + \cdots, \qquad k > 2. \end{aligned} \tag{5.2}$$

In these expressions the boundary fields $G_{(0)ij}, \Phi^2_{(0)}, \Phi^k_{(0)}$ parametrize the Dirichlet boundary conditions and are also the field theory sources for the QFT stress energy tensor and

[4] We work with Euclidean signature.

operators of dimension 2 and k, respectively. The near-boundary analysis determines all coefficients in these expansions except the ones corresponding to the normalizable modes, namely $G_{(4)ij}, \tilde{\Phi}^2_{(0)}, \Phi^k_{(2k-4)}$. These are related to 1-point functions, as we review below.

5.1.1 Radial Hamiltonian formalism

The structure of the 1-point functions is most transparent in the radial Hamiltonian formalism [16, 10] (see [13, 14, 15] for earlier work). So before giving the explicit relation between the 1-point functions and the coefficients of the asymptotic solutions we digress to explain this relation. Let us define a radial canonical momentum for each field as

$$\pi = \frac{\partial L}{\partial \Phi'} \tag{5.3}$$

where L is the Lagrangian and prime denotes differential w.r.t. $r = -\log z$. A covariant version of the near boundary expansion in (5.2) is provided by the expansion of momenta in eigenfunctions of the dilatation operator,

$$\delta_D = \int_{z=\epsilon} d^d x \left(2\gamma_{ij} \frac{\delta}{\delta \gamma_{ij}} + \sum_k (k-4)\Phi^k \frac{\delta}{\delta \Phi^k} \right) = -z\frac{\partial}{\partial z}(1 + O(z)), \tag{5.4}$$

where γ_{ij} is the induced metric at the regulating surface $z = \epsilon$. The last equality follows from the leading asymptotics in (5.2). The near boundary expansion (5.2) now translates into the following expansions

$$\begin{aligned} \pi^i_j(x,\epsilon) &= \sqrt{\gamma}(\pi_{(0)}{}^i_j + \cdots + \pi_{(4)}{}^i_j + \tilde{\pi}_{(4)}{}^i_j \log \epsilon^2 + \cdots); \\ \pi^k(x,\epsilon) &= \sqrt{g}(\pi^k_{(4-k)} + \cdots + \pi^k_{(k)} + \tilde{\pi}^k_{(k)} \log \epsilon^2 + \cdots), \qquad k \geq 2 \end{aligned} \tag{5.5}$$

where π^i_j and π^k are the radial momenta for the bulk metric and the scalar field Φ^k, respectively, and each term in this expansion transforms as indicated by its index

$$\delta_D \pi_{(n)} = -n\pi_{(n)}, \tag{5.6}$$

except for $\pi_{(4)}{}^i_j, \pi^2_{(2)}, \pi^k_{(k)}$ which transform inhomogeneously, with the inhomogeneous term being equal to minus two times the coefficient of the logarithmic term, namely

$$\delta_D \pi_{(4)}{}^i_j = -4\pi_{(4)}{}^i_j - 2\tilde{\pi}_{(4)}{}^i_j, \qquad \delta_D \pi^k_{(k)} = -k\pi^k_{(k)} - 2\tilde{\pi}^k_{(k)}. \tag{5.7}$$

One advantage of the momenta expansion (5.5) over the near boundary expansion of the bulk fields (5.2) is that the momentum coefficients $\pi_{(n)}{}^i_j, \pi^k_{(n)}$ are covariant w.r.t. 4d diffeomorphisms that respect the regulating hypersurface $z = \epsilon$ whereas the coefficients in (5.2) are not.

The coefficients in the momentum expansions can be obtained by inserting the expansions in Hamilton's equations. This leads to a number of equations obtained by collecting all terms with the same weight. One then solves these equations iteratively and each of them algebraically determines one of the coefficients in the expansion in terms of coefficients with lower weight. This determines all coefficients except $\pi_{(4)}{}^i_j$ and $\pi^k_{(k)}$, just as the asymptotic analysis of the bulk equations determines all coefficients in (5.2) except for the normalizable modes.

The renormalized 1-point functions are now simply given by the coefficient of the right dimension

$$\begin{aligned} \langle T_{ij} \rangle &= \pi_{(4)ij} \\ \langle \mathcal{O}^k \rangle &= \pi^k_{(k)} \end{aligned} \tag{5.8}$$

Following [16] one can show that there is a one to one correspondence between the momentum coefficients and the coefficients in (5.2). In particular,

$$\pi^2_{(2)} = \frac{N^2}{2\pi^2}\left(2\tilde{\phi}_{(0)}\right), \qquad \pi^k_{(k)} = \frac{N^2}{2\pi^2}\left((2k-4)\phi_{(2k-4)} + \text{lower}\right) \tag{5.9}$$

where the factor $N^2/2\pi^2$ is due to the overall factor in (4.12) and "lower" indicates terms with index less than $(2k-4)$. These terms are local functions of the sources so they are not important in computation of n-point functions for $n > 1$ (they lead to contact terms). They are important in the computation of vevs in cases where the solution describes a deformation flow [8, 20]. The specific example we discuss in this paper however is a vev flow so we need not specify them.

5.2 $5d$ supergravity fields

The part of the $5d$ action involving the metric and the field S^2 is same as the sector of gauged supergravity analyzed in [8, 9] (where S^2 was called Φ) (see also [15, 10]). Thus, the results for the 1-point functions carry over unchanged. The result for $\langle \mathcal{O}^2 \rangle$ is as given above

$$\langle \mathcal{O}^2 \rangle = \frac{N^2}{2\pi^2}\left(2\tilde{S}^2_{(0)}\right), \tag{5.10}$$

and for the stress energy tensor

$$\begin{aligned} \langle T_{ij} \rangle &= \frac{N^2}{2\pi^2}\Bigg(G_{(4)ij} + \frac{1}{3}\tilde{S}^2_{(0)}G_{(0)ij} + \frac{1}{8}[\mathrm{Tr}G^2_{(2)} - (\mathrm{Tr}G_{(2)})^2]G_{(0)ij} \\ &- \frac{1}{2}(G^2_{(2)})_{ij} + \frac{1}{4}G_{(2)ij}\mathrm{Tr}G_{(2)} + \frac{3}{2}h_{(4)ij} + (\frac{2}{3}S^2_{(0)} - \tilde{S}^2_{(0)})S^2_{(0)}G_{(0)ij}.\Bigg). \end{aligned} \tag{5.11}$$

5.3 KK modes

We now move to the one point functions of the other fields $S^4, S^6, T^0, T^2, T^6, \Phi^2_{(t)}$. From the last row of Table 1 we see that the cubic couplings (relevant for us) vanish for all of them so to the order to which we are working their field equations are just free field equations. One would therefore naively conclude that the one-point functions are simply given by (5.8)-(5.9). It turns out however that this is not true and there is an additional subtlety.

The 1-point functions (5.8) were derived starting from a $5d$ action but in principle one should really evaluate on-shell the $10d$ action. In the majority of cases this distinction does not make any difference in practice, but there is a distinguished class of additional finite (non-local) boundary terms that one obtains from reducing the 10d action. These boundary terms are relevant for the computation of extremal correlators, namely n-point functions of 1/2 BPS operators whose dimensions are $\{k_1{=}k_2+\cdots+k_n, k_2, \ldots, k_n\}$. These correlators are special in that they factorize into a product of 2-point functions.

The bulk couplings associated with all extremal 3-point functions were shown to be zero in [26] (a result we reproduced here for the coupling $(S^2)^2 S^4$, see Table 1). The 3-point functions of the corresponding operators however are non-zero. It was shown in [22] in a specific example involving the dilaton and the t-field that even though the bulk contribution to the three point function vanishes, there are additional boundary contributions which lead to the correct 3-point function. By supersymmetry, the same should apply to all other extremal 3-point functions. It was further conjectured that these results generalize to all extremal n-point correlators. We refer to [30] for further discussion and references on this issue.

These results should follow from holographic renormalization by starting from the $10d$ action, requiring that the variational problem is well posed and then KK reducing the action with boundary terms. Recall that in the $5d$ context all boundary terms, including counterterms, are uniquely fixed by the requirement that the variational problem is well posed with chosen boundary conditions [31]. We leave a detailed derivation for future work. Here we will instead use known results in order to fix the form of the 1-point functions.

Covariance under $4d$ diffeomorphisms implies that the 1-point function should be a function of the coefficients in the momentum expansion (5.5). Furthermore the dimensions should match between the l.h.s. and the r.h.s. and the result for extremal n-point functions should factorize into products of 2-point functions. This uniquely fixes the form of the 1-point functions. For concreteness we discuss the 1-point functions of $\mathcal{O}^4$ and $\mathcal{O}^6$ but the generalization is obvious. Thus

$$\langle \mathcal{O}^4 \rangle = \pi^4_{(4)} + a_{422}(\pi^2_{(2)})^2 \tag{5.12}$$

$$\langle \mathcal{O}^6 \rangle = \pi^6_{(6)} + a_{642}\pi^4_{(4)}\pi^2_{(2)} + a_{633}(\pi^3_{(3)})^2 + a_{6222}(\pi^2_{(2)})^3 \tag{5.13}$$

where $a_{422}, a_{642}, a_{633}, a_{6222}$ are numerical constants that we show how to compute in the next subsection. Let us explain the structure of these 1-point functions. The leading term $\pi^k_{(k)}$ is the term discussed above in (5.8). Observe that the non-linear terms are possible only when the weight of the operator in l.h.s. can be written as a sum of weights of other operators, which is also the condition for extremal correlators. One could also consider adding terms involving π^k_n with $n < k$. Such terms can only possibly contribute to the coincident limit of correlators (since π^k_n with $n < k$ is local in the sources) or to vevs of solutions describing deformation flows, so they are not important for our analysis.

5.4 Extremal couplings

In this subsection we will compute the coefficients a_{422}, a_{6222}. The coefficients a_{633} and a_{642} could be computed in similar way but we will not need their explicit values in this paper. To obtain the coefficients a_{422}, a_{6222} we compute the 3- and 4-point functions starting from (5.12) and then fix the coefficients so that the numerical factors agree with the the computation in free field theory. Note that the dependence on the coordinates is guaranteed to be correct by the structure of (5.12) (as should become clear shortly).

We start with the computation of a_{422}. By definition

$$\langle \mathcal{O}^k(x)\mathcal{O}^k(y)\rangle = -\left.\frac{\delta\langle \mathcal{O}^k(x)\rangle}{\delta\phi^k_{(0)}(y)}\right|_{\phi^k_{(0)}=0} = -\left.\frac{\delta\pi^k_{(k)}(x)}{\delta\phi^k_{(0)}(y)}\right|_{\phi^k_{(0)}=0} \tag{5.14}$$

$$\begin{aligned}\langle \mathcal{O}^4(x_1)\mathcal{O}^2(x_2)\mathcal{O}^2(x_3)\rangle &= \left.\frac{\delta^2\langle \mathcal{O}^4(x_1)\rangle}{\delta\phi^2_{(0)}(x_2)\delta\phi^2_{(0)}(x_3)}\right|_{\phi^2_{(0)}=0} \\ &= \left.\left(\frac{\delta^2\pi^4_{(4)}(x_1)}{\delta\phi^2_{(0)}(x_2)\delta\phi^2_{(0)}(x_3)} + 2a_{422}\left(\frac{\delta\pi^2_{(2)}(x_1)}{\delta\phi^2_{(0)}(x_2)}\right)\left(\frac{\delta\pi^2_{(2)}(x_1)}{\delta\phi^2_{(0)}(x_3)}\right)\right)\right|_{\phi^2_{(0)}=0}\end{aligned} \tag{5.15}$$

To evaluate these expressions we need to know $\pi^k_{(k)}$ to linear order in $\phi^k_{(0)}$ and $\pi^4_{(4)}$ to quadratic order in $\phi^2_{(0)}$. Recall that $\pi^k_{(k)}$ is proportional to the vev part of the solution (see (5.9)), so in order to compute 2- and 3-point functions we need to solve bulk equations expanded around the background solution (which in the current context is just AdS) to linear and quadratic order, respectively, and then extract the coefficient of order z^k. For extremal couplings the cubic coupling to S^2 is zero, $\lambda_{422} = 0$, so the bulk equation for S^4

continues to be $\Box S^4 = 0$ and the solution does not acquire dependence on $\phi^2_{(0)}$ so the second variation of $\pi^{(4)}_4$ w.r.t. $\phi^2_{(0)}$ is zero[5]. Therefore only the last term in (5.15) contributes and using (5.14) we see that the extremal 3-point function is a product of 2-point functions

$$\langle \mathcal{O}^4(x_1)\mathcal{O}^2(x_2)\mathcal{O}^2(x_3)\rangle = 2a_{422}\langle \mathcal{O}^2(x_1)\mathcal{O}^2(x_2)\rangle\langle \mathcal{O}^2(x_1)\mathcal{O}^2(x_3)\rangle \tag{5.16}$$

Since this 3-point function does not renormalize one can compute it via free fields, which allows us to fix the proportionality constant a_{422}.

In the large N limit (i.e. dropping non-planar contributions) the free field computations for the 2- and (extremal) 3-point functions yield

$$\begin{aligned}
\left\langle \mathcal{O}^k(x)\mathcal{O}^k(y)\right\rangle &= \mathcal{N}_k^2\lambda^k\frac{k}{(2\pi)^{2k}\,|x-y|^{2k}}, &(5.17)\\
\left\langle \mathcal{O}^{k_1}(x_1)\mathcal{O}^{k_2}(x_2)\mathcal{O}^{k_3}(x_3)\right\rangle &= \mathcal{N}_{k_1}\mathcal{N}_{k_2}\mathcal{N}_{k_3}\frac{\lambda^{k_1}}{N}\frac{k_1k_2k_3\langle C_{k_1}C_{k_2}C_{k_3}\rangle}{(2\pi)^{2k_1}\,|x_1-x_2|^{k_1}\,|x_1-x_3|^{k_1}}, &(5.18)
\end{aligned}$$

where the operators are defined in (2.3) and in the extremal 3-point function $k_1 = k_2 + k_3$. The normalization factors $\mathcal{N}_k$ are chosen such that the 2-point function of (5.17) agrees with the supergravity results; in particular given that for $k \neq 2$ the supergravity result is

$$\left\langle \mathcal{O}^{I_1}(x)\mathcal{O}^{I_2}(y)\right\rangle = \frac{N^2}{2\pi^2}\left(\frac{\Gamma(k+1)}{\pi^2\Gamma(k-2)}\frac{(2k-4)}{k}\frac{\delta^{I_1I_2}}{|x-y|^{2k}}\right), \tag{5.19}$$

whilst for $k = 2$ the result is

$$\left\langle \mathcal{O}^2(x)\mathcal{O}^2(y)\right\rangle = \frac{N^2}{2\pi^2}\left(\frac{2\delta^{I_1I_2}}{\pi^2\,|x-y|^4}\right). \tag{5.20}$$

The normalizations are thus

$$\begin{aligned}
\mathcal{N}_{I_1} &= \frac{N}{\lambda^{\frac{1}{2}k}}\frac{2^k\pi^{k-2}}{k}\sqrt{\frac{\Gamma(k+1)(2k-4)}{2\Gamma(k-2)}}; \qquad k\neq 2, &(5.21)\\
\mathcal{N}_2 &= 2\sqrt{2}\frac{N}{\lambda}.
\end{aligned}$$

Inserting (5.17) in (5.16) and comparing with (5.18) leads to

$$a_{422} = \frac{2\mathcal{N}_4}{\mathcal{N}_2^2N}\langle C_4C_2C_2\rangle = \frac{3\mathcal{N}_4}{\sqrt{5}\mathcal{N}_2^2N}, \tag{5.22}$$

where in the last equality we have used the explicit value of the triple overlap $\langle C_4C_2C_2\rangle$. This can be obtained from the formulae (B.13)-(B.14) by computing the overlap of $Y^{(4,0)}$ and $(Y^{(2,0)})^2$ using the explicit expressions (B.9).

[5] In the non-extremal case, the bulk equation reads $\Box S^k = \lambda_{klm}S^lS^m$ and the r.h.s. induces a correction to S^k proportional to $z^k(\phi^l_{(0)}\phi^m_{(0)})$ so the second variation of $\pi^{(k)}_k$ w.r.t. $\phi^l_{(0)}$ and $\phi^m_{(0)}$ is non-zero yielding the 3-point function, see [17] for a detailed discussion.

The computation of the coupling a_{6222} is analogous. The bulk quartic coupling was shown to be zero in [32] and thus the only contribution to the extremal 4-point function comes from the last term in (5.13)

$$\langle \mathcal{O}^6(x_1)\mathcal{O}^2(x_2)\mathcal{O}^2(x_3)\mathcal{O}^2(x_4)\rangle = 6a_{64222}\prod_{k=2}^{4}\langle \mathcal{O}^2(x_1)\mathcal{O}^2(x_k)\rangle \tag{5.23}$$

The free field result for extremal 4-point functions (in the planar limit) is

$$\langle \mathcal{O}^{k_1}(x_1)\mathcal{O}^{k_2}(x_2)\mathcal{O}^{k_3}(x_3)\mathcal{O}^{k_4}(x_4)\rangle = (\prod_i \mathcal{N}_i)\frac{\lambda^{k_1}}{N}\frac{k_1k_2k_3k_4\langle C_{k_1}C_{k_2}C_{k_3}C_{k_4}\rangle}{(2\pi)^{2k_1}\,|x_1-x_2|^{2k_2}\,|x_1-x_3|^{2k_3}\,|x_1-x_4|^{2k_4}}, \tag{5.24}$$

where $k_1 = k_2 + k_3 + k_4$, which fixes the proportionality constant to be

$$a_{6222} = \frac{\mathcal{N}_6}{\mathcal{N}_2^3 N}\langle C^4C^2C^2C^2\rangle = \frac{\mathcal{N}_6}{\mathcal{N}_2^3 N}\frac{3\sqrt{3}}{5\sqrt{7}} \tag{5.25}$$

where $\langle C^4C^2C^2C^2\rangle$ was computed using the following integral formula valid for extremal overlaps

$$\int Y^{I_1}Y^{I_2}Y^{I_3}Y^{I_4} = \frac{\pi^3}{(k_1+1)(k_1+2)2^{k_1-1}}\langle C^{I_1}C^{I_2}C^{I_3}C^{I_3}\rangle, \tag{5.26}$$

along with the explicit forms of the spherical harmonics.

5.4.1 Summary

To summarize we have shown that 1-point functions of the operators $\mathcal{O}^4, \mathcal{O}^6$ are [6]

$$\begin{aligned}
\langle \mathcal{O}^4\rangle &= \pi^4_{(4)} + \frac{3\mathcal{N}_4}{\sqrt{5}\mathcal{N}_2^2 N}(\pi^2_{(2)})^2 \\
\langle \mathcal{O}^6\rangle &= \pi^6_{(6)} + a_{642}\pi^4_{(4)}\pi^2_{(2)} + a_{633}(\pi^3_{(3)})^2 + \frac{\mathcal{N}_6}{\mathcal{N}_2^3 N}\frac{3\sqrt{3}}{5\sqrt{7}}(\pi^2_{(2)})^3
\end{aligned} \tag{5.27}$$

Notice that although we have used the 3- and 4-point functions on AdS to fix the couplings a_{422} and a_{6222}, these 1-point functions hold for *any* solution of the bulk field equations.

Notice also that these 1-point functions are compatible with the (conjectured) structure of near-extremal correlators. Recall that near-extremal correlators have weights $k_1 = k_2 + \cdots k_n - 2m$ with $0 \le m \le n-2$. These correlators are conjectured (and checked through order g^2) to be sums of terms each of which factors into products of lower-point correlators [33]. When $m = 1$ the correlator is called next-to-extremal and factorizes into a 3-point function and $(n-2)$ 2-point functions. One can easily check that this structure emerges from (5.27) after using the fact that the bulk coupling vanishes. (To be more precise, the

[6] The non-linear terms in these relations may have the interpretation as operator mixing between single trace and multi-trace operators.

bulk coupling is known to vanish up to quartic order and is conjectured to vanish to all orders). For example the next-to-extremal correlator $\langle \mathcal{O}^4 \mathcal{O}^2 \mathcal{O}^2 \mathcal{O}^2 \rangle$ is given by

$$\begin{aligned} \langle \mathcal{O}^4(x_1)\mathcal{O}^2(x_2)\mathcal{O}^2(x_3)\mathcal{O}^2(x_4)\rangle &= 2a_{422}\left(\frac{\delta\pi^2_{(2)}(x_1)}{\delta\phi^2_{(0)}(x_2)\delta\phi^2_{(0)}(x_3)}\right)\left(\frac{\delta\pi^2_{(2)}(x_1)}{\delta\phi^2_{(0)}(x_4)}\right)+\cdots \\ &= 2a_{422}\left(\langle \mathcal{O}^4(x_1)\mathcal{O}^2(x_2)\mathcal{O}^2(x_3)\rangle\langle \mathcal{O}^2(x_1)\mathcal{O}^2(x_4)\rangle+\cdots\right) \end{aligned} \tag{5.28}$$

where the dots indicate permutation in x_2, x_3, x_4.

6 Coulomb branch solution

6.1 Continuous distributions of D3 branes

It is intuitively clear that the Coulomb branch of $\mathcal{N} = 4$ SYM should be described by multi-center D3 brane solutions. Solutions describing N separated D3 branes solve the field equations that follow from the bulk supergravity action coupled to the worldvolume action of N (separated) D3-branes. It will be important for us to keep track of all normalizations factors so we set the stage by first reviewing some standard material. The bulk action is normalized as

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}(R+\cdots), \qquad 2\kappa^2 = (2\pi)^7(\alpha')^4 g_s^2, \tag{6.1}$$

and the worldvolume theory is given by

$$S_{\rm source} = \sum_{a=1}^{N}\int d^{10}x\int d^4\sigma^a \mathcal{L}_{DBI}(\sigma^a)\delta(x^M - X^M(\sigma^a)), \tag{6.2}$$

where the Lagrangian for each D-brane is normalized as

$$\mathcal{L}_{DBI}(\sigma^a) = T_3(\sqrt{\det(\gamma+2\pi\alpha' F)}+\cdots), \qquad T_3 = \frac{1}{(2\pi)^3(\alpha')^2 g_s}, \tag{6.3}$$

where $\gamma_{ij} = \partial_i X^M \partial_j X^N g_{MN}$ is the induced metric and derivatives are with respect to the worldvolume coordinates σ^i. The D3-brane solutions take the form

$$ds^2 = H(x_\perp)^{-1/2}dx^2_{||} + H(x_\perp)^{1/2}dx^2_\perp \tag{6.4}$$

$$F = \frac{1}{4}(dH^{-1}\wedge\omega_{||} - *_\perp d_\perp H) \tag{6.5}$$

where $\omega_{||}$ is the volume form in the worldvolume directions, $*_\perp$ and $d_\perp$ refer to the Hodge star and exterior derivative in the flat overall transverse directions and H is a harmonic function. We are interested in the case of a uniform distribution of N D3-branes on a two dimensional disc of radius l. Approximating the distribution as a continuum distribution with density

$$\rho(r) = \frac{N}{\pi l^2}\theta(l^2 - r^2), \tag{6.6}$$

where r is the radial coordinate in the plane of the distribution, the solution for the harmonic function is (see for example [34, 35])

$$H = \frac{L^4}{\pi l^2}\int_{r'\leq l} d^2r' \frac{1}{(\vec{x}_\perp - \vec{r'})^4} = -\frac{L^4}{2l^2y^2}\left(\frac{r^2 - l^2 + y^2}{\sqrt{(r^2+l^2+y^2)^2 - 4r^2l^2}} - 1\right) \tag{6.7}$$

where $L^4 = 4\pi g_s N(\alpha')^2$, $\vec{y}$ are coordinates in the four dimensional space transverse to the distribution of the D3 branes and $\vec{r}$ lies in the plane of the distribution.

A change of coordinates

$$y = \bar{r}\sin\bar{\theta}, \qquad r = \sqrt{l^2 + \bar{r}^2}\cos\bar{\theta} \tag{6.8}$$

brings the solution into the form

$$\begin{aligned}
ds^2 &= \frac{\bar{r}^2\zeta}{L^2}\left(dx_{||}^2 + \frac{L^4 d\bar{r}^2}{\bar{r}^4\lambda^6}\right) + \frac{L^2}{\zeta}\left(\zeta^2 d\bar{\theta}^2 + \sin^2\bar{\theta}d\Omega^2 + \lambda^6\cos^2\bar{\theta}d\phi^2\right) \\
F &= L^{-4}\left(\bar{r}^3(1 + \frac{l^2}{2\bar{r}^2}\sin^2\bar{\theta})d\bar{r} + \frac{1}{4}l^2\bar{r}^2\sin 2\bar{\theta}d\bar{\theta}\right)\wedge\omega_{||} \\
&+ L^4\sin^3\bar{\theta}\cos\bar{\theta}\frac{1}{\zeta^4}\left(\lambda^6(1 + \frac{l^2}{2\bar{r}^2}\sin^2\bar{\theta})d\bar{\theta} - \frac{l^2}{4\bar{r}^3}\sin 2\bar{\theta}d\bar{r}\right)\wedge d\Omega_3\wedge d\phi
\end{aligned} \tag{6.9}$$

where

$$\zeta^2 = 1 + \frac{l^2}{\bar{r}^2}\sin^2\bar{\theta}, \qquad \lambda^6 = 1 + \frac{l^2}{\bar{r}^2} \tag{6.10}$$

Now note that if we rescale the four dimensional coordinates as

$$x_{||} \to L^2 x_{||} \tag{6.11}$$

the metric has an overall L^2 factor whilst the five form has an overall L^4 factor. These factors combined with the prefactor of (6.1) result in the overall normalization factor of the five dimensional action (4.12); we can therefore suppress the L prefactors in the rest of this section.

6.2 Asymptotic expansion

We now wish to expand the metric near the boundary. A systematic way to do this is to use Gaussian normal coordinates centered at the boundary of AdS_5 and then expand all fields using the radial coordinate as a small parameter. This radial axial gauge can be reached by the charge of coordinates

$$\begin{aligned} \frac{l}{\bar{r}} &= z(1 + a_1 z^2 + a_2 z^4 + O[z]^6) \\ \bar{\theta} &= \theta + b_1 z^2 + b_2 z^4 + O[z]^6 \end{aligned} \tag{6.12}$$

where

$$\begin{aligned} a_1 &= \frac{1}{2^3}(2 - \sin^2\theta), \quad a_2 = \frac{1}{2^8}\cos^2\theta(5 + 11\cos 2\theta) \\ b_1 &= \frac{1}{2^3}\cos\theta\sin\theta, \quad b_2 = \frac{5}{2^9}(-\sin 2\theta + \sin 4\theta) \end{aligned} \tag{6.13}$$

The metric then takes the form

$$\begin{aligned} ds^2 &= \frac{dz^2}{z^2} + \frac{l^2}{z^2}(1 + \alpha_1 z^2 + \alpha_2 z^4) dx_{||}^2 \\ &+ d\theta^2(1 + \beta_1 z^2 + \beta_2 z^4) + \sin^2\theta d\Omega_3^2(1 + \gamma_1 z^2 + \gamma_2 z^4) + \cos^2\theta d\phi^2(1 + \delta_1 z^2 + \delta_2 z^4) \end{aligned} \tag{6.14}$$

where the coefficients α_1, α_2 etc. depend on the angular coordinate θ. By scaling

$$z \to zl \tag{6.15}$$

the leading metric becomes a unit radius $AdS_5 \times S^5$ and factors of l appear in the fluctuations. These factors can be easily reinstated in the final formulae so for simplicity we set $l = 1$ for now. Using the explicit form of spherical harmonics in (B.9) and (B.18) the deviation of the metric from $AdS_5 \times S^5$ can be rewritten as in (3.4) with the following coefficients (valid up to terms of order z^4),

$$\begin{aligned} &\tilde{h}^0_{ij}(z) = \frac{1}{32} z^2 \delta_{ij}, \quad \tilde{h}^2_{ij}(z) = \sqrt{12}\left(-\frac{1}{4} + \frac{23}{320} z^2\right)\delta_{ij}, \quad \tilde{h}^4_{ij}(z) = -\frac{3\sqrt{5}}{20} z^2 \delta_{ij}, \\ &\hat{\phi}^2_{(t)}(z) = \frac{3}{160} z^4, \quad \phi^2_{(s)}(z) = \sqrt{12}(\frac{1}{8} z^2 - \frac{1}{256} z^4), \quad \phi^4_{(s)}(z) = \frac{\sqrt{5}}{32} z^4 \\ &\tilde{\pi}^0(z) = \frac{1}{8} z^4, \quad \tilde{\pi}^2(z) = \sqrt{12}(z^2 - \frac{17}{64} z^4), \quad \tilde{\pi}^4(z) = \frac{3\sqrt{5}}{2} z^4. \end{aligned} \tag{6.16}$$

Similarly, from the expansion of the five form we obtain

$$\begin{aligned} b^2_{(s)} &= -\frac{\sqrt{3}}{8} z^2 + \frac{31\sqrt{3}}{1280} z^4 \\ b^4_{(s)} &= -\frac{39\sqrt{5}}{640} z^4 \end{aligned} \tag{6.17}$$

There are several comments in order here. Firstly, the solution is not in the de Donder gauge, as one can see from the fact that the scalar fields $\phi^2_{(s)}$ and $\phi^4_{(s)}$ are non-zero. Secondly, the expansion contains many more non-zero fields that one would naively expect. In particular, there are non-zero KK gravitons, $\tilde{h}^2_{ij}$ and $\tilde{h}^4_{ij}$ (which are dual to the operators of the schematic form, Tr$F_+F_-X^k$ for $k=2,4$ (see Table 7 of [30])), scalar field $\phi^2_{(t)}$ (that couples to a tensor harmonic and is dual to the operator Tr$\lambda\lambda\bar{\lambda}\bar{\lambda}X^2$) scalar fields t^0, t^2, t^4 (that are dual to Tr$F_+^2F_-^2X^k$ for $k=0,2,4$) and scalar fields s^2, s^4 (that are dual to the operators TrX^k, $k=2,4$). However, we know that in the CB flow only the operators TrX^k get a vev. So what is the meaning of the values of the additional fields?

To answer this question we should apply our map to obtain the gauge invariant five dimensional fields. As a first step we need to construct gauge invariant combinations. Using (3.19)-(3.20)-(3.22) and (3.24) and the definition of t^k, s^k we get

$$t^0 = -\frac{1}{128}z^4, \qquad t^2 = -\frac{\sqrt{3}}{160}z^4, \qquad t^4 = -\frac{3\sqrt{5}}{160}z^4, \tag{6.18}$$
$$s^2 = \frac{\sqrt{3}}{4}z^2 - \frac{\sqrt{3}}{160}z^4, \qquad s^4 = \frac{37\sqrt{5}}{160}z^4, \qquad \phi^2_{(t)} = 0. \tag{6.19}$$

The five-dimensional fields are obtained from these by the KK reduction formula (4.10) yielding

$$T^0 = T^2 = T^4 = \Phi^2_{(t)} = 0, \qquad S^2 = \frac{1}{\sqrt{6}}(z^2 - \frac{1}{6}z^4), \qquad S^4 = 0. \tag{6.20}$$

We thus see that that all additional scalar fields are equal to zero! The same is also true for the KK gravitons but we do not give the details here. We would like to emphasize, however, that a non-zero answer for these fields would not be a problem for the duality. The only cases where it would problematic is if the non-zero values correspond to a source or a vev. Note that all additional fields correspond to irrelevant operators and a non-zero source would not be consistent with AdS asymptotics. Furthermore, the corresponding vev part would appear at much higher power of z. To understand the (possibly non-zero) values note that the $5d$ equations are schematically of the form

$$(\Box + m^2_\Phi)\Phi = \lambda_{\Phi 22}(S^2)^2 + \cdots \tag{6.21}$$

where Φ denotes collectively the fields other than s^2, s^4 and the metric that are turned on. Any non-zero value for these fields would simply be induced by interaction terms – such non-zero fields just reflect the non-linear structure of gravity. In our case, it turns out that the couplings $\lambda_{\Phi 22}$ are zero, so the fields Φ had to zero to this order.

The fields that are important to understand at each order z^k are the ones which correspond to operators whose vevs can receive a contribution from the asymptotics at this order. In our case, these are the fields S^2, S^4 and the metric $G_{\mu\nu}$. The solution we discuss can be reduced to five dimensions using a "consistent truncation ansatz" (see [34, 36]). The reduced model involves the metric and S^2. The expression for S^2 in (6.20) exactly agrees with the asymptotic expansion of the $5d$ solution, compare with (5.2) of [9] (and use $\Phi = -S^2$). We will return to the metric momentarily.

The expression for S^4 in (6.20) is new information. The fact that it is zero comes out of non-trivial cancellations and at first sight is surprising since the vev of the dual operator is non-zero. From Table 1 we see that the coupling of S^4 to $(S^2)^2$ is zero; this is an example of an extremal coupling

$$\Box S^4 = 0. \tag{6.22}$$

In this case however the vanishing of the coupling only explains the absence of logarithmic terms in the asymptotic expansion of S^4. Logarithmic terms in the asymptotic expansion are related to conformal anomalies. Such conformal anomalies due to 3-point functions are possible when the couplings are extremal, see section 2 of [37]. They are however proportional to the sources so they evaluate to zero on the Coulomb branch, in agreement with the absence of logarithmic terms in the asymptotic expansion. In other words, the vanishing of the extremal couplings is *required* by the AdS/CFT correspondence and the structure of the conformal anomaly. Equation (6.22) allows for a homogeneous solution that is proportional to z^4 and one might have anticipated that the homogeneous term would be non-zero, since the vev of the dual operator is non-zero. We will resolve this issue in the next subsection.

We now return to the spacetime metric. We see from (6.16) that the metric is corrected at the normalizable mode order. More precisely the combination that diagonalizes the field equations to linear order is $h'_{\mu\nu} = (\tilde{h}^0_{\mu\nu} + \frac{1}{3}\tilde{\pi}^0 g^o_{\mu\nu})$ and in our explicit solution this is given by

$$h'_{zz} = \frac{1}{24}z^2, \qquad h'_{ij} = \frac{7}{96}z^2\delta_{ij}. \tag{6.23}$$

Naively this would imply that the dual stress energy tensor is non-zero. The solution, however, is supersymmetric so the vev of the stress energy tensor must be equal to zero. (We discussed this point earlier in the introduction.) As mentioned above the 10d solution can be reduced to five dimensions using a consistent truncation ansatz. The asymptotics of the $5d$ metric were given in the introduction in (1.3). As mentioned there, despite the non-zero coefficient of the $\hat{z}^4$ term, the vev of T_{ij} is zero because of additional contributions to the 1-point function. This does not immediately resolve the issue however because the metric in (6.23) does not agree with the metric in (1.3) (when both written in the same gauge)!

The issue here is that $h'_{\mu\nu}$ is not the correct $5d$ metric. Firstly, $h'_{\mu\nu}$ does not transform correctly, i.e. as a five-dimensional metric. As derived in section 3.2.2 the combination which transforms properly is $h^0_{\mu\nu}$ in (3.30). Evaluating this formula for the case at hand gives

$$h^0_{zz} = -\frac{11}{48}z^2, \qquad h^0_{ij} = \frac{19}{192}z^2\delta_{ij}. \tag{6.24}$$

The five-dimensional metric is now obtained by using the non-linear KK map in (4.15). The resulting five-dimensional metric (including the background term) is given by

$$ds^2 = (1 - \frac{13}{144}z^4)\frac{dz^2}{z^2} + \frac{1}{z^2}(1 - \frac{19}{576}z^4)dx^2_{||}. \tag{6.25}$$

This metric is not in the same gauge as the metric in (1.3). To correct for that we change coordinates as $z = \hat{z}(1 + \frac{13}{1152}\hat{z}^4)$ and the metric becomes

$$ds^2 = \frac{d\hat{z}^2}{\hat{z}^2} + \frac{1}{\hat{z}^2}(1 - \frac{1}{18}\hat{z}^4)dx^2_{||}, \tag{6.26}$$

which precisely agrees with (1.3) (after reinstating the factors of l)! Note that this coordinate change does not affect any other fields to the order we work to.

6.3 Comparison with field theory

Given the asymptotic expansions of the five dimensional fields we can now read off the vevs for the corresponding corresponding operators. These must agree with the field theory results discussed in section 2 because, as we explained there, $\mathcal{N} = 4$ supersymmetry forbids any quantum corrections, so this computation is a test of the gravity/gauge theory duality.

Using the formula (5.10) we see that the vev of the $\Delta = 2$ operator is given in terms of the normalizable mode $\tilde{S}^2_0(x)$ of the supergravity field S^2 as

$$\langle \mathcal{O}^2 \rangle_{SUGRA} = \frac{N^2}{2\pi^2}(2\tilde{S}^2_0(x)) = \frac{N^2}{\sqrt{6}\pi^2}l^2, \tag{6.27}$$

where we use the explicit result for our solution $\tilde{S}^2_0 = 1/\sqrt{6}$ and we reinstated the factors of l. This 1-point function was previously derived in [8]. To compare with the field theory expression (set $n = 1$ in (2.11)),

$$\langle \mathcal{O}^2 \rangle_{QFT} = \frac{\mathcal{N}_2 a^2}{2\sqrt{3}}N \tag{6.28}$$

we need to normalize operators and the action in the same way. The normalization of the operators was given in (5.21). Expanding the source action (6.2) to leading order in α' and taking into account the rescaling of the $R^{3,1}$ coordinates in (6.11) the kinetic term for the scalar fields is normalized as

$$T_3 L^4 \int d^4\sigma (\frac{1}{2}(\partial X)^2) = \left(\frac{\lambda}{2\pi^2}\right)\frac{1}{g_s}\int d^4\sigma(\frac{1}{2}(\partial X)^2). \tag{6.29}$$

On the other hand, the field theory computation was done with canonically normalized scalars. It follows that the radius a of the distribution in field theory (with canonically normalized scalars) is related to the radius l of distribution of D3 branes by

$$a = \sqrt{\frac{\lambda}{2\pi^2}} l. \tag{6.30}$$

Using the normalization $\mathcal{N}_2$ in (5.21) and (6.30) we find precise agreement between the supergravity and field theory computations!

Next consider the $\Delta = 4$ operator. Given the result of the previous subsection that the normalizable mode of the corresponding supergravity field vanishes, the vev (5.27) receives contributions only from the term quadratic in $\pi^2_{(2)}$:

$$\langle \mathcal{O}^4 \rangle = \frac{3\mathcal{N}_4}{\sqrt{5}\mathcal{N}_2^2 N} (\langle \mathcal{O}^2 \rangle)^2 = \frac{\mathcal{N}_4 a^4}{2^2 \sqrt{5}} N \tag{6.31}$$

where we used (5.8) and (6.28). This is precisely the correct field theory vev!

Now let us consider the $\Delta = 6$ operator. The vanishing of $\pi^4_{(4)}$ and $\pi^3_{(3)}$ means that in this case the formula (5.27) reduces to just

$$\langle \mathcal{O}^6 \rangle = \pi^6_{(6)} + \frac{\mathcal{N}_6}{\mathcal{N}_2^3 N} \frac{3\sqrt{3}}{5\sqrt{7}} (\langle \mathcal{O}^2 \rangle)^3 \tag{6.32}$$

The latter of these terms evaluates to

$$\frac{1}{5} \left(\frac{\mathcal{N}_6 a^6}{2^3 \sqrt{7}} N \right) = \frac{1}{5} \langle \mathcal{O}^6 \rangle_{QFT} \tag{6.33}$$

Given that in the $\Delta = 4$ case, there was no contribution to the vev from the bulk supergravity field one might have wondered whether the same was true in the $\Delta = 6$ case, and indicative of a more general result. However, (6.33) only accounts for one fifth of the field theory result, and there must therefore be an additional contribution from the supergravity field dual to $\mathcal{O}^6$. To verify this one would have to extend our supergravity computations to one order higher, including quartic couplings, to extract the normalizable mode of the supergravity field S^6. Note that the structure of the 1-point functions is such that the terms non-linear in momenta always give a contribution that is proportional to the QFT vev. It is a curious fact that up to at least $\mathcal{O}^{10}$ (which is as far as we explicitly checked) the proportionality coefficient is a rational number, despite the fact that intermediate formulae contain square roots.

The vev of the stress energy tensor was already computed in [8] (using (5.11)) and, as noted earlier, was found to be zero, in agreement with the fact that the solution is supersymmetric. We have thus succeeded in showing that the vevs of all operators up to dimension $\Delta = 4$ are correctly reproduced by supergravity!

7 Conclusions

We have developed in this paper a systematic method for constructing the holographic map between the asymptotics of a ten dimensional solution and the 1-point functions of the dual QFT. Our main goal was to develop an unambiguous method that can, at least in principle, always be carried out. The main elements entering our construction are (i) the development of a gauge invariant version of KK reduction; (ii) construction of the KK map to non-linear order and (iii) application of holographic renormalization, including a proper treatment of extremal couplings.

One-point functions can be derived rigorously starting from a 5d action via holographic renormalization. Our strategy for obtaining the 1-point functions dual to general KK fields was thus to reduce the field equations over the compact manifold and then use holographic renormalization. Recall that holographic renormalization relates the vevs to coefficients in the asymptotic expansion of the 5d solution. So to compute the vevs starting from a 10d solution one has to understand quantitatively how the solution is reduced to five dimensions at the non-linear level. The point is that non-linear terms can give a contribution at exactly the same order (in a radial expansion) as linear terms. However, 1-point functions of operators of a given dimension can only receive contribution from non-linear terms involving fields dual to operators of lower dimension.

The KK reduction map is constructed by first computing the fluctuation equations around $AdS_5 \times S^5$ to a certain order in the fluctuation fields and then finding the field transformation that removes the higher derivative terms from these equations. This field transformation *is* the KK reduction map (to this order) and the resulting field equations *are* the 5d field equations.

We would like to contrast our procedure with the procedure of "consistent truncuation". In the latter one only keeps certain (typically low lying) modes in the KK reduction and then has to prove that the dynamics of these modes decouple from the rest. The existence of such a truncation is highly non-trivial and only holds for special compactifications. In the AdS/CFT correspondence consistent truncation maps to the closure of a subset of operators of the CFT under OPEs. In our discussion we keep all KK modes, so there is never an issue of consistency. We are however interested only in the asymptotic expansion of the resulting field equations. This effectively decouples all but a finite number of fields at each order in the expansion. In particular, for computing the vev of an operator of a given dimension only the fields dual to operators of the same or lower dimensions need to be kept.

The need for a gauge invariant KK reduction stems from the fact that the KK reduction is most efficiently done in a specific gauge, the de Donder gauge, but in general explicit solutions will not be - and many known interesting solutions are not - in this gauge. Reaching this gauge would require finding a transformation that in general is not easy to obtain (at the non-linear level). Thus instead of gauge fixing the diffeomorphisms we construct gauge invariant variables. This allows us to immediately lift results derived in one gauge to another gauge: one can first obtain the KK map in the de Donder gauge and then relax the gauge condition by simply replacing all fields by their gauge invariant generalization. The construction of the gauge invariant variables can be done systematically in the number of fields and we have done so up to second order in the fields.

A final subtlety involves extremal couplings. Extremal correlators involve operators with the dimension of one of them equal to the sum of the dimensions of all the others. Such correlators are non-zero and are believed not to renormalize. A naive computation in supergravity however would give zero because the corresponding bulk coupling vanishes. It was argued in [22] that precisely in these cases there are additional boundary terms, originating from the higher derivative terms in the fluctuation equations, that one should take into account when evaluating the on-shell action and these yield the correct answer. In holographic renormalization one effectively replaces the on-shell action by renormalized 1-point functions in the presence of sources. These 1-point functions are valid for *any* solution of the field equations and higher n-point functions can be computed by further differentiating w.r.t. sources. Additional boundary terms in the $5d$ action, beyond the ones implied by the bulk 5d action via the variational problem, would manifest as additional contributions to the renormalized 1-point functions. The form of the additional terms is uniquely fixed by general principles. This leaves a few numerical coefficients to be determined and these can be easily computed by comparing the extremal correlators computed in weakly coupled $\mathcal{N} = 4$ SYM and in supergravity.

Combining these elements one obtains a well-defined holographic map. In our discussion we focused on solutions of IIB supergravity that involve only the metric and the self-dual five-form and asymptote to $AdS_5 \times S^5$ but the discussion readily generalizes to include all other fields or more generally to any theory with solutions that asymptote to $AdS_m \times X$, for some m and any compact manifold X.

Let us summarize the steps involved in the construction of the map:

1. Expand the solution (using a radial coordinate z as a small parameter) up to certain order and write the deviation from $AdS_m \times X$ in terms of harmonics of X.

2. If the solution is not in the de Donder gauge, combine the fluctuations in gauge

invariant variables.

3. Use the KK map to obtain the asymptotic expansion of the corresponding lower dimensional fields.

4. Insert the coefficients of the asymptotic expansion in the renormalized 1-point functions to obtain the vevs.

The asymptotic expansion of the 10d solution in general will contain non-zero terms for many coefficients. The only ones that carry physical information are the ones that have the correct leading radial behavior to correspond to normalizable or non-normalizable modes. The former give a contribution to the vev of the dual operator and the latter to the coefficient of the deformation of the QFT Lagrangian by the dual operator. We should note however that the remaining coefficients will generically contribute to vevs of higher dimension operators via non-linear contributions to the holographic map.

The higher the dimension of the dual operator, the higher the order needed in the gauge invariant variables and the KK map. So although this work solves the problem of computing the vevs in principle, in practice the method becomes cumbersome to carry out when sufficiently high dimension operators are involved (but since the procedure is algorithmic one could in principle computerize it). In this paper we explicitly worked out the map to the first non-trivial order. This is sufficient to compute vevs of operators up to dimension 4 and thus covers all relevant and marginal operators in four dimensions. As noted in the introduction, more efficient methods may be available when the solution has special properties. In this paper we mainly aimed at settling the issue of principle in full generality.

To illustrate the general procedure we analyzed a solution that corresponds to a particular point on the Coulomb branch of $\mathcal{N} = 4$ SYM. This is an interesting example because the vevs are protected by supersymmetry and therefore the supergravity dual must reproduce them *exactly*. The vevs corresponding to fields in gauged supergravity were previously computed in [8, 9]. Here we computed in addition the vev of the operator of dimension 4 and exact agreement with the quantum field theory values was found! This constitutes the first non-trivial quantitative test of gravity/gauge theory duality away from the fixed point that involves a vev of an operator dual to a KK field.

In the discussion so far we focused on how to compute vevs starting from a given 10d solution. The (inverse of the) holographic map can be used to see how spacetime is reconstructed from QFT data. In particular, we see from our discussion that the vevs of operators dual to KK modes provide a harmonic resolution of the compact space. From a more general viewpoint, notice that in the radial Hamiltonian formulation of holographic renormalization the vevs are associated with the radial canonical momenta conjugate to the sources. The holographic map therefore maps the field theory data to the phase space of the gravitational theory. It follows that these data are sufficient to uniquely determine the bulk solution, even though the explicit formulae only provide an asymptotic solution up to a certain order.

One could thus "holographically engineer" duals of interesting quantum field theories by starting from the field theory vevs and using the holographic map. For this procedure to yield a smooth geometry, the vevs should clearly be large compared to the string scale. Even if this condition is satisfied, there is still no guarantee that a smooth geometry would emerge. For instance, it is well known that a necessary condition for a *smooth supergravity dual* is that the conformal anomaly of the theory at the UV fixed point should satisfy $c = a$ in the large N and λ limit [11]. Let us also note that even if a given theory *has* a smooth dual geometry, in practice it may not be easy to sum the asymptotic solution into this smooth solution. New tools that capture global issues of the correspondence may be needed to properly analyze this problem. It would be very interesting to explore this line of thought further.

In many cases it is clear from the construction of the supergravity solution (with AdS asymptotics) what the corresponding gauge theory dual is. For instance this is the case if the solution is obtained via a near-horizon limit from another solution. Yet there are many other cases where solutions with AdS asymptotics have been obtained by directly solving the supergravity equations and there is no physical argument that would identify the dual theory. The work presented here should be useful both in verifying the gauge theory duals in cases where a proposed identification is available and also for extracting the gauge theory dual in other cases.

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A Harmonic expansion of the antisymmetric tensor

We expand the antisymmetric tensor as

$$
\begin{aligned}
a_{\mu\nu\rho\sigma}(x,y) &= \sum \tilde{b}^{I_1}_{\mu\nu\rho\sigma}(x)Y^{I_1}(y) \\
a_{\mu\nu\rho a}(x,y) &= \sum (\tilde{b}^{I_5}_{(v)\mu\nu\rho}(x)Y^{I_5}_a(y) + \tilde{b}^{I}_{(s)\mu\nu\rho}(x)D_aY^{I_1}(y)) \\
a_{\mu\nu ab}(x,y) &= \sum (\tilde{b}^{I_{10}}_{(t)\mu\nu}(x)Y^{I_{10}}_{[ab]}(y) + \tilde{b}^{I_5}_{(v)\mu\nu}(x)D_{[a}Y^{I_5}_{b]}(y)) \\
a_{\mu abc}(x,y) &= \sum (\tilde{b}^{I_5}_{(v)\mu}(x)\epsilon_{abc}{}^{de}D_dY^{I}_e(y) + \tilde{b}^{I_{10}}_{(t)\mu}(x)D_{[a}Y^{I_{10}}_{bc]}(y)) \\
a_{abcd}(x,y) &= \sum (b^{I_1}_{(s)}(x)\epsilon_{abcd}{}^{e}D_eY^{I}(y) + b^{I_5}_{(v)}(x)\epsilon_{abcd}{}^{e}Y^{I_5}_e(y))
\end{aligned} \tag{A.1}
$$

Gauge transformations act on the 4-form as follows

$$
\delta a_{MNPQ} = 4D_{[M}c_{NPQ]} \tag{A.2}
$$

The antisymmetric tensor parameter has the following expansion

$$
c_{\mu\nu\rho}(x,y) = \sum c^{I_1}_{\mu\nu\rho}(x)Y^{I_1}(y) \tag{A.3}
$$

$$
\begin{aligned}
c_{\mu\nu a}(x,y) &= \sum (c^{I_5}_{(v)\mu\nu}(x)Y^{I_5}_a(y) + c^{I_1}_{(s)\mu\nu}(x)D_aY^{I_1}(y)) \\
c_{\mu ab}(x,y) &= \sum (c^{I_{10}}_{(t)\mu}(x)Y^{I_{10}}_{[ab]}(y) + c^{I_5}_{(v)\mu}(x)D_{[a}Y^{I_5}_{b]}(y)) \\
c_{abc}(x,y) &= \sum (c^{I_5}_{(v)}(x)\epsilon_{abc}{}^{de}D_dY^{I_5}_e(y) + c^{I_{10}}_{(t)}(x)D_{[a}Y^{I_{10}}_{bc]}(y))
\end{aligned} \tag{A.4}
$$

This implies the following gauge transformations for the fields,

$$
\begin{gathered}
\delta\tilde{b}^{I_1}_{\mu\nu\rho\sigma} = 4D_{[\mu}c^{I_1}_{\nu\rho\sigma]}, \qquad \delta\tilde{b}^{I_5}_{(v)\mu\nu\rho} = 3D_{[\mu}c^{I_5}_{(v)\nu\rho]}, \qquad \delta\tilde{b}^{I_1}_{(s)\mu\nu\rho} = -c^{I_1}_{\mu\nu\rho} \\
\delta\tilde{b}^{I_{10}}_{(t)\mu\nu} = 2D_{[\mu}c^{I_{10}}_{(t)\nu]}, \qquad \delta\tilde{b}^{I_5}_{(v)\mu\nu} = 2D_{[\mu}\tilde{c}^{I_5}_{(v)\nu]} + c^{I_5}_{(v)\mu\nu} \\
\delta\tilde{b}^{I_5}_{(v)\mu} = D_\mu c^{I_5}_{(v)}, \qquad \delta\tilde{b}^{I_{10}}_{(t)\mu} = D_\mu c^{I_{10}}_{(t)} - 3c^{I_{10}}_{(t)\mu}, \qquad \delta b^{I_5}_{(v)} = -(\Lambda^{I_5}-4)c^{I_5}_{(v)}
\end{gathered} \tag{A.5}
$$

It follows that the following combinations are gauge invariant,

$$
b^{I_1}_{\mu\nu\rho\sigma} = \tilde{b}^{I_1}_{\mu\nu\rho\sigma} + 4D_{[\mu}\tilde{b}^{I_1}_{(s)\nu\rho\sigma]} \tag{A.6}
$$

$$
\begin{aligned}
b^{I_5}_{(v)\mu\nu\rho} &= \tilde{b}^{I_5}_{(v)\mu\nu\rho} - \frac{3}{2}D_{[\mu}\tilde{b}^{I_5}_{(v)\nu\rho]} \\
b^{I_{10}}_{(t)\mu\nu} &= \tilde{b}^{I_{10}}_{(t)\mu\nu} + \frac{2}{3}D_{[\mu}\tilde{b}^{I_{10}}_{(t)\nu]} \\
b^{I_5}_{(v)\mu} &= \tilde{b}^{I_5}_{(v)\mu} + \frac{1}{(\Lambda^{I_5}-4)}D_\mu b^{I_5}_{(v)}
\end{aligned}
$$

Indeed the field strength

$$
f_{MNPSR} = 5D_{[M}a_{NPRS]} \tag{A.7}
$$

can be expressed in terms of these modes.

The gauge used in [21],

$$D^a a_{aMNP} = 0 \tag{A.8}$$

amounts to setting to zero

$$\tilde{b}^{I_1}_{(s)\mu\nu\rho} = \tilde{b}^{I_5}_{(v)\mu\nu} = \tilde{b}^{I_{10}}_{(t)\mu} = b^{I_5}_{(v)} = 0. \tag{A.9}$$

Our normalizations are such that the gauge invariant variables evaluated in this gauge agree with the parametrization in [21].

B Spherical harmonics

The defining equations for the spherical harmonics are

$$\begin{aligned}
\Box_y Y^{I_1} &= \Lambda^{I_1} Y^{I_1}, \qquad \Lambda^{I_1} = -k(k+4), \quad k = 0,1,2,... \\
\Box_y Y^{I_5}_a &= \Lambda^{I_5} Y^{I_5}_a, \qquad \Lambda^{I_5} = -(k^2+4k-1), \quad k = 1,2,... \\
\Box_y Y^{I_{14}}_{(ab)} &= \Lambda^{I_{14}} Y^{I_{14}}_{(ab)}, \qquad \Lambda^{I_{14}} = -(k^2+4k-2), \quad k = 2,3,... \\
\Box_y Y^{I_{10}}_{[ab]} &\equiv \Lambda^{I_{10}} Y^{I_{10}}_{[ab]}, \qquad \Lambda^{I_{10}} = -(k^2+4k-2), \quad k = 1,2,... \\
D^a Y^{I_5}_a &= D^a Y^{I_{14}}_{(ab)} = D^a Y^{I_{10}}_{[ab]} = 0.
\end{aligned} \tag{B.1}$$

Useful identities for the scalar harmonics include

$$\begin{aligned}
D^a D_{(a} D_{b)} Y^I &= 4(1+\frac{\Lambda^I}{5}) D_a Y^I; \\
\Box_y D_{(a} D_{b)} Y^I &= (10+\Lambda^I) D_{(a} D_{b)} Y^I; \\
\Box_y D_a Y^I &= (\Lambda^I + 4) D_a Y^I.
\end{aligned} \tag{B.2}$$

B.1 Spherical harmonics with $SO(4)$ symmetry

We introduce the following coordinates on S^5

$$ds^2 = d\theta^2 + \sin^2\theta d\Omega_3^2 + \cos^2\theta d\phi^2. \tag{B.3}$$

The differential equation (B.1) for the scalar harmonics is separable. Imposing $SO(4)$ symmetry implies that the spherical harmonics depend only on θ and ϕ. The general solution can then be expressed in terms of a hypergeometric functions,

$$Y^{(k,m)}(\theta,\phi) = c_{(n,m)} y^k_m(\theta) e^{im\phi} \tag{B.4}$$

where $c_{(n,m)}$ is a normalization constant and the function $y^k_m(\theta)$ is given by

$$y^k_m(x) = x^{|m|} {}_1F_2(-\frac{1}{2}(k-|m|), 2+\frac{1}{2}(k+|m|), 1+|m|; x^2) \tag{B.5}$$

with $x = \cos\theta$ (there are also a second solution with leading behavior $x^{-|m|}$ but this solution does not reduces to a finite polynomial for any choice of the quantum numbers). The hypergeometric function reduces to a finite polynomial when either the first or second argument is zero or a negative integer. This leads to the following cases

$$(k = 2l, \quad m = 2n), \qquad (k = 2l+1, \quad m = 2n+1) \qquad n \in [-l, l], \; l \in Z^+ \tag{B.6}$$

with

$$\begin{aligned} y_{2n}^{2l}(x) &= x^{2|n|}{}_1F_2(-l+|n|, 2+l+|n|, 1+2|n|; x^2) \\ y_{2n+1}^{2l+1}(x) &= x^{|2n+1|}{}_1F_2(-l+|n|, 3+l+|n|, 2+2|n|; x^2) \end{aligned} \tag{B.7}$$

Particularly relevant for us here are harmonics that are also $SO(2)$ symmetric which are given by

$$Y^{(2l,0)}(\theta,\phi) = \frac{(-)^l}{2^l\sqrt{2l+1}}\left(\sum_{m=0}^{l}(-)^m \begin{pmatrix} l \\ m \end{pmatrix}\begin{pmatrix} l+m+1 \\ l+1 \end{pmatrix}(\cos\theta)^{2m}\right). \tag{B.8}$$

The lowest harmonics are therefore

$$\begin{aligned} Y^{(2,0)} &= \frac{1}{2\sqrt{3}}(3\cos^2\theta - 1), \\ Y^{(4,0)} &= \frac{1}{4\sqrt{5}}(10\cos^4\theta - 8\cos^2\theta + 1), \\ Y^{(6,0)} &= \frac{1}{8\sqrt{7}}(35\cos^5\theta - 45\cos^4\theta + 15\cos^2\theta - 1) \end{aligned} \tag{B.9}$$

The overall normalization in (B.8) has been chosen so that the harmonics are normalized as in [26], i.e.

$$\int Y^{I_1}Y^{I_2} = z(k)\delta^{I_1I_2}, \qquad z(k) = \frac{\pi^3}{2^{k-1}(k+1)(k+2)} \tag{B.10}$$

Recall that the scalar harmonics can be represented as

$$Y^{I_1} = C^{I_1}_{i_1\cdots i_k}x^{i_1}\cdots x^{i_k} \tag{B.11}$$

where x^{i_n} are Cartesian coordinates on S^5 and $C^I_{i_1\cdots i_k}$ is a totally symmetric traceless rank k tensor of $SO(6)$. The normalization in (B.10) corresponds to delta function normalization for the C^I's, i.e.

$$\langle C^{I_1}C^{I_2}\rangle \equiv C^{I_1}_{i_1\cdots i_k}C^{I_2 i_1\cdots i_k} = \delta^{I_1I_2}. \tag{B.12}$$

For the scalar harmonics we use the following definitions:

$$\int D_{(a}D_{b)}Y^{I_1}D_{(a}D_{b)}Y^{I_2} = z(k)q(k)\delta^{I_1I_2}$$

$$\int Y^{I_1} Y^{I_2} Y^{I_3} = a(k_1, k_2, k_3)\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle \tag{B.13}$$
$$\int Y^{I_1} D_a Y^{I_2} D^a Y^{I_3} = b(k_1, k_2, k_3)\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$$
$$\int D^{(a}D^{b)} Y^{I_1} D_a Y^{I_2} D_b Y^{I_3} = c(k_1, k_2, k_3)\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$$
$$\int Y^{I_1} D^{(a}D^{b)} Y^{I_2} D_a D_b Y^{I_3} = d(k_1, k_2, k_3)\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$$
$$\int D^{(a}D^{b)} Y^{I_1} (2 D_a D_c Y^{I_2} D_{(c} D_{b)} Y^{I_3} + D_c Y^{I_2} D_c D_{(a} D_{b)} Y^{I_3}) = e(k_1, k_2, k_3)\langle \mathcal{C}^{I_1}\mathcal{C}^{I_2}\mathcal{C}^{I_3}\rangle$$

where

$$q(k) = \Lambda^I (4 + \frac{4}{5}\Lambda^I), \qquad a(k_1, k_2, k_3) = \frac{\pi^3}{(\frac{1}{2}\Sigma + 2)! 2^{\frac{1}{2}(\Sigma - 2)}} \frac{k_1! k_2! k_3!}{\alpha_1! \alpha_2! \alpha_3!} \tag{B.14}$$

and $\Sigma = k_1 + k_2 + k_3$, $\alpha_1 {=} \frac{1}{2}(k_2 + k_3 - k_1)$ etc. One can derive explicit formulae that express $b(k_1, k_2, k_3), c(k_1, k_2, k_3), d(k_1, k_2, k_3), e(k_1, k_2, k_3)$ in terms of $a(k_1, k_2, k_3)$ by use partial integrations. Useful identities include:

$$\begin{aligned} d(k_2, k_1, k_3) + c(k_1, k_2, k_3) + \frac{q(k_1)}{\Lambda^{I_1}} b(k_2, k_1, k_3) &= 0; \\ b(k_2, k_1, k_3) + b(k_1, k_2, k_3) + \Lambda^{I_3} a(k_1, k_2, k_3) &= 0. \end{aligned}$$

We further have

$$D_{(\theta} D_{\theta)} Y^2_0 = \frac{6}{5}(1 - 2\cos 2\theta), \qquad D_{(\theta} D_{\theta)} Y^4_0 = \frac{12}{5}(2 + \cos 2\theta - 5\cos 4\theta). \tag{B.15}$$

For the tensor harmonics we use the following definitions

$$\int Y^{I_1}_{ab} D_a Y^{I_2} D_b Y^{I_3} = c^{(t)}(k_1^{(t)}, k_2, k_3); \tag{B.16}$$
$$\int Y^{I_1}_{ab} (2 D_a D_c Y^{I_2} D_{(c} D_{b)} Y^{I_3} + D_c Y^{I_2} D_c D_{(a} D_{b)} Y^{I_3}) = e^{(t)}(k_1^{(t)}, k_2, k_3).$$

The normalization of the spherical harmonic is defined as

$$\int Y^{I_1}_{ab} Y^{I_2}_{ab} = z_{(t)}(k^{(t)}) \delta^{I_1 I_2}. \tag{B.17}$$

The only tensor harmonic of relevance here is

$$Y^{(2,0)}_{\theta\theta} = -3, \qquad Y^{(2,0)}_{\phi\phi} = \cos^2\theta(-3 + 15\cos^2\theta), \qquad Y^{(2,0)}_{\psi^a\psi^a} = \sin^2\theta(2 - 5\cos^2\theta) \tag{B.18}$$

where ψ^a are the coordinates on S^3; we thus do not need to discuss the tensor harmonics more generally.

C Field equations up to second order

We discuss in this appendix the derivation of the field equations up to second order in fluctuations.

The linearized equations read

$$\begin{aligned} E^{(1)}_{MN} &\equiv R^{(1)}_{MN} + \frac{4}{3!} h^{KL} F^{o}_{MKM_1M_2M_3} F^{o\;\;M_1M_2M_3}_{NL} \\ &- \frac{1}{3!}(f_{MM_1M_2M_3M_4} F^{o\;M_1M_2M_3M_4}_{N} + f_{NM_1M_2M_3M_4} F^{o\;M_1M_2M_3M_4}_{M}) = 0 \end{aligned} \tag{C.1}$$

$$E^{(1)}_{M_1\dots M_5} \equiv (f - f^*)_{M_1\dots M_5} + \frac{1}{2} h^L_L F^o_{M_1\dots M_5} - 5 h^K_{[M_1} F^o_{M_2\dots M_5]K} = 0 \tag{C.2}$$

where

$$R^{(1)}_{MN} = D_K h^K_{MN} - \frac{1}{2} D_M D_N h^L_L, \qquad h^K_{MN} = \frac{1}{2}(D_M h^K_N + D_N h^K_M - D^K h_{MN}). \tag{C.3}$$

Projecting these equations onto the various harmonics leads to the following equations[7]

$$\begin{aligned} (E^{(1)}_{ab})|_{D_{(a}D_{b)}Y^{I_1}} = 0 &\Rightarrow (\frac{1}{2}\hat{h}^{\sigma I_1}_{\sigma} + \frac{3}{10}\hat{\pi}^{I_1}) = 0; \\ (E^{(1)}_{ab})|_{Y^{I_{14}}_{ab}} = 0 &\Rightarrow \frac{1}{2}(\Box + \Lambda^{I_{14}} - 2)\hat{\phi}^{I_{14}}_{(t)} = 0; \\ (E^{(1)\,a}_{\;\;\;a})|_{Y^{I_1}} &\Rightarrow \frac{1}{10}((\Box + \Lambda^{I_1} - 32)\hat{\pi}^{I_1} + 80\Lambda^{I_1}\hat{b}^{I_1} + \Lambda^{I_1}(\hat{h}^{\sigma I_1}_{\sigma} + \frac{3}{5}\hat{\pi}^{I_1})) = 0; \\ (E^{(1)}_{\mu\nu\rho\sigma a})|_{D_a Y^{I_1}} = 0 &\Rightarrow (\hat{b}^{I_1}_{\mu\nu\rho\sigma} + \epsilon_{\mu\nu\rho\sigma}{}^{\tau} D_\tau \hat{b}^{I_1}) = 0; \\ (E^{(1)}_{\mu\nu\rho\sigma\tau})|_{Y^{I_1}} = 0 &\Rightarrow (5D_{[\mu}\hat{b}^{I_1}_{\nu\rho\sigma\tau]} - \epsilon_{\mu\nu\rho\sigma}{}^{\tau}(\frac{1}{2}\hat{h}^{\sigma I_1}_{\sigma} + \Lambda^{I_1}\hat{b}^{I_1} - \frac{1}{2}\hat{\pi}^{I_1})) = 0. \end{aligned} \tag{C.4}$$

These equations lead to the scalar field equations quoted in section 4.1 upon elimination of $\hat{b}_{\mu\nu\rho\sigma}$ and $\hat{h}^{\sigma}_{\sigma}$ and then diagonalization.

We now move to the quadratic order. The field equations are

$$E^{(1)}_{MN} = T^{(2)}_{MN}, \qquad E^{(1)}_{M_1\dots M_5} = T^{(2)}_{M_1\dots M_5} \tag{C.5}$$

where the quadratic corrections are given by [27]

$$\begin{aligned} T^{(2)}_{M_1\dots M_5} &= -\frac{1}{2} h^L_L f^*_{M_1\dots M_5} + 5 h^K_{[M_1} f^*_{M_2\dots M_5]K} \\ &- \frac{5}{2} h^L_L h^K_{[M_1} F^o_{M_2\dots M_5]K} + (\frac{1}{8}(h^L_L)^2 + \frac{1}{4} h^{ML} h_{ML}) F^o_{M_1\dots M_5} + 10 h^{K_1}_{[M_1} h^{K_2}_{M_2} F^o_{M_3M_4M_5]K_1K_2} \end{aligned} \tag{C.6}$$

$$\begin{aligned} T^{(2)}_{MN} &= -R^{(2)}_{MN} \\ &+ \frac{4}{3!} h^{KL} h^S_L F^o_{MNM_1M_2M_3} F^{o\;\;M_1M_2M_3}_{NS} + \frac{2\cdot 3}{3!} h^{K_1S_1} h^{K_2S_2} F^o_{MK_1K_2M_1M_2} F^{o\;\;\;\;M_1M_2}_{NS_1S_2} \\ &- \frac{4}{3!} h^{KS}(f_{MKM_1M_2M_3} F^{o\;\;M_1M_2M_3}_{NS} + f_{NKM_1M_2M_3} F^{o\;\;M_1M_2M_3}_{MS}) + \frac{1}{3!} f_{MM_1\dots M_4} f_N{}^{M_1\dots M_4} \end{aligned} \tag{C.7}$$

[7] In comparing with [21] one should note that we expand in harmonics $h_{\mu\nu}$ rather than $h'_{\mu\nu}$ (compare (2.5)-(2.7) with our (3.4)).

where

$$R^{(2)}_{MN} = -D_K(h^K_L h^L_{MN}) + \frac{1}{2}D_N(h_{KL}D_M h^{KL}) + \frac{1}{2}h^K_{MN}D_K h^L_L - h^L_{MK}h^K_{NL} \tag{C.8}$$

These quantities were computed to second order in the field s in the de Donder gauge in [26], by substituting the linear solution of the field equations

$$\begin{aligned} h^{I_1}_{\mu\nu} &= U(k)s^{I_1}g^o_{\mu\nu} + W(k)D_{(\mu}D_{\nu)}s^{I_1}; \\ h^{I_1}_{\alpha\beta} &= V(k)s^{I_1}g_{\alpha\beta}, \qquad b^{I_1} = -s^{I_1}; \\ V(k) &= -\frac{5}{3}U(k) = 2k, \qquad W(k) = \frac{4}{k+1}. \end{aligned} \tag{C.9}$$

As discussed in the main text, the resulting field equations will be applicable to other gauge choices provided that one replaces each field by the corresponding gauge invariant field. In particular, for the field "s^{I_1}" must denote the appropriate gauge invariant field. For computing the quadratic corrections, however, it is sufficient to use the field $\hat{s}^{I_1}$ which is gauge invariant to linear order, since the difference between this field and the gauge invariant field is itself quadratic in fluctuations.

C.1 Scalar fields

To compute the corrected scalar equations we will need to use the following components of (C.6) and (C.7):

$$\begin{aligned} Q^1_1 &\equiv \frac{1}{5}(T^{(2)}_{ab})|_{D_{(a}D_{b)}Y^1}; \\ &= \frac{1}{20q_1z_1}\left((c_{123} + d_{231} + d_{321})T_{23} + 32c_{123}D_\mu\hat{s}^2D^\mu\hat{s}^3\right), \\ T_{23} &= (3V_2V_3 + 5U_2U_3)\hat{s}^2\hat{s}^3 + W_2W_3D^{(\mu}D^{\nu)}\hat{s}^2D_{(\mu}D_{\nu)}\hat{s}^3. \end{aligned} \tag{C.10}$$

The notation in the first line implies the projection of the tensor (which is quadratic in spherical harmonics) onto the spherical harmonic. (Note that the factor of five is included in the definition so as to match the conventions of [26]). Similarly one has

$$\begin{aligned} Q^1_2 &\equiv \frac{1}{5}(T^{(2)\ a}_a)|_{Y^1} = \frac{1}{20z_1}\left(10S_{123} + T_{23}(b_{123} - 2f_3a_{123}) + 32D_\mu\hat{s}^2D^\mu\hat{s}^2b_{123}\right), \\ S_{123} &= V_3U_2a_{123}D^\mu(\hat{s}^2D_\mu\hat{s}^3) + W_2V_3a_{123}D_\mu(D^{(\mu}D^{\nu)}\hat{s}^2D_\nu\hat{s}^2) \\ &\quad -V_2V_3b_{213}\hat{s}^2\hat{s}^3 - 8(a_{123}f_2f_3\hat{s}^2\hat{s}^3 + b_{123}D^\mu\hat{s}^2D_\nu\hat{s}^3) - a_{123}V_2(64f_3 + 80V_3)\hat{s}^2\hat{s}^3. \end{aligned} \tag{C.11}$$

We also need

$$(T^{(2)}_{\mu\nu\rho\sigma a})|_{D_aY^1} \equiv -\epsilon_{\mu\nu\rho\sigma\tau}Q^{\tau 1}_3, \tag{C.12}$$

$$
\begin{aligned}
Q_3^{\tau 1} &= -\frac{1}{f_1 z_2}\left((U_2+3V_2)\hat{s}^2 D^\tau \hat{s}^3 + W_2 D^{(\tau}D^{\rho)}\hat{s}^2 D_\rho \hat{s}^3\right) b_{213}; \\
(T^{(2)}_{\mu\nu\rho\sigma\tau})|_{Y^1} &\equiv Q_4^1 \epsilon_{\mu\nu\rho\sigma\tau}, \\
Q_4^1 &= -\frac{1}{4z_1}\left(T_{23} - (16V_2 f_3 + 40V_2V_3)\hat{s}^2\hat{s}^3\right) a_{123}.
\end{aligned} \tag{C.13}
$$

For the scalar field coupling to the tensor harmonic we need

$$
\begin{aligned}
Q^{I_{14}}_{(t)} &\equiv (T^{(2)}_{ab})|_{Y^{I_{14}}_{ab}}, \\
&= \frac{1}{4z_{(t)1}}\left(d^{(t)}_{123}T_{23} + 32c^{(t)}_{123}D_\mu \hat{s}^2 D^\mu \hat{s}^3\right).
\end{aligned} \tag{C.14}
$$

Then the corrected equations of motion are written in terms of quantities just defined as

$$
\begin{aligned}
(\Box - k(k-4))s^{I_1} &= \frac{1}{2(k+2)}\left((k+4)(k+5)Q_1 + Q_2 + (k+4)(D_\mu Q_3^\mu + Q_4)\right)^{I_1}; \\
(\Box - (k+8)k(k+4))t^{I_1} &= \frac{1}{2(k+2)}\left(k(k-1)Q_1 + Q_2 - k(D_\mu Q_3^\mu + Q_4)\right)^{I_1},
\end{aligned} \tag{C.15}
$$

whilst the corrected equation for the scalar $\phi^{I_{14}}_{(t)}$ is

$$
(\Box - k(k+4))\phi^{I_{14}}_{(t)} = 2Q^{I_{14}}_{(t)}. \tag{C.16}
$$

C.2 Tensor fields

To compute the correction to the metric and KK gravitons we we need $T^{(2)}_{\mu\nu}$. For brevity we will include only the terms of interest here, namely $\hat{s}^2$. The curvature contribution to $(T^{(2)}_{\mu\nu})|_{Y^I}$ is then

$$
\begin{aligned}
&\frac{1}{z_I}\Big(-\frac{4}{9}a_{I22}D_\mu D_\rho D_\sigma \hat{s}^2 D_\nu D^\rho D^\sigma \hat{s}^2 - \frac{32}{3}a_{I22}D^\rho \hat{s}^2 D_\rho D_\mu D_\nu \hat{s}^2 - \frac{8}{9}b_{I22}D_\mu D_\rho \hat{s}^2 D_\nu D^\rho \hat{s}^2 \\
&\quad + \frac{8}{9}a_{I22}(D^\rho D^\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2)g^o_{\mu\nu} + (\frac{40}{9}b_{I22} - 32a_{I22})\hat{s}^2 D_\mu D_\nu \hat{s}^2 \\
&\quad - \frac{136}{9}a_{I22}(D_\mu \hat{s}^2 D^\nu \hat{s}^2) + \frac{32}{9}(a_{I22} - b_{I22})(\hat{s}^2)^2 g^o_{\mu\nu}\Big)
\end{aligned} \tag{C.17}
$$

whilst the field strength contribution to $(T^{(2)}_{\mu\nu})|_{Y^I}$ is

$$
\begin{aligned}
&\frac{1}{z_I}\Big(g^o_{\mu\nu}(\frac{32}{9}a_{I22}(2(\hat{s}^2)^2 - D_\rho D_\sigma \hat{s}^2 D^\rho D^\sigma \hat{s}^2) - 4b_{I22}(D_\rho \hat{s}^2 D^\rho \hat{s}^2)) - \frac{64}{3}a_{I22}\hat{s}^2 D_\mu D_\nu \hat{s}^2 \\
&\quad + 8D_\mu \hat{s}^2 D_\nu \hat{s}^2 b_{I22}\Big).
\end{aligned} \tag{C.18}
$$

These lead to the following equation for the graviton,

$$
(L_E + 4)h^0_{\mu\nu} = T^{(2)}_{\mu\nu}|_{Y^0} - g^o_{\mu\nu}(\frac{5}{3}Q_2^0 + 8Q_4^0)|_{Y^0} \tag{C.19}
$$

where the linearized Einstein operator is defined as usual by

$$L_E\lambda_{\mu\nu} = \frac{1}{2}(-\Box\lambda_{\mu\nu} + D_\rho D_\mu \lambda^\rho_\nu + D_\rho D_\nu \lambda^\rho_\mu - D_\mu D_\nu \lambda^\rho_\rho). \tag{C.20}$$

The term proportional to Q_2 and Q_4 arise when eliminating $\Box\tilde{\pi}^0$ and $\tilde{h}^{\sigma 0}_\sigma$ from the equation.

The following identities prove useful:

$$\begin{aligned}
L_E((D^\rho D^\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2) g^o_{\mu\nu}) &= -\frac{1}{2}\Box(D^\rho D^\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2) g^o_{\mu\nu} - 3D_\mu D_\nu D_\rho D_\sigma \hat{s}^2 D^\rho D^\sigma \hat{s}^2 \\
&\quad -3D_\mu D_\rho D_\sigma \hat{s}^2 D_\nu D^\rho D^\sigma \hat{s}^2 \\
L_E(D_\mu D_\rho \hat{s}^2 D_\nu D^\rho \hat{s}^2) &= 2D^\rho D_\mu \hat{s}^2 D_\rho D_\nu \hat{s}^2 - D_\mu D^\rho D^\sigma \hat{s}^2 D_\nu D_\rho D_\sigma \hat{s}^2 \\
&\quad +g^o_{\mu\nu}(D^\rho D^\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2) - 9D^\rho \hat{s}^2 D_\mu D_\nu D_\rho \hat{s}^2 - 7D_\mu \hat{s}^2 D_\nu \hat{s}^2 \\
&\quad +12\hat{s}^2 D_\mu D_\nu \hat{s}^2 - (D^\rho \hat{s}^2 D_\rho \hat{s}^2) g^o_{\mu\nu}, \qquad \text{(C.21)} \\
L_E((D^\rho \hat{s}^2 D_\rho \hat{s}^2) g^o_{\mu\nu}) &= 8(D^\rho \hat{s}^2 D_\rho \hat{s}^2) g^o_{\mu\nu} - g^o_{\mu\nu}(D^\rho D^\sigma \hat{s}^2 D_\rho D_\sigma \hat{s}^2) \\
&\quad -3D^\rho \hat{s}^2 D_\mu D_\nu D_\rho \hat{s}^2 - 3D^\rho D_\mu \hat{s}^2 D_\rho D_\nu \hat{s}^2; \\
L_E(\hat{s}^2 D_\mu D_\nu \hat{s}^2) &= -3D_\mu \hat{s}^2 D_\nu \hat{s}^2 - (D^\rho \hat{s}^2 D_\rho \hat{s}^2) g^o_{\mu\nu} + D_\mu D^\rho \hat{s}^2 D_\nu D_\rho \hat{s}^2; \\
L_E((\hat{s}^2)^2 g^o_{\mu\nu}) &= 4(\hat{s}^2)^2 g^o_{\mu\nu} - (D^\rho \hat{s}^2 D_\rho \hat{s}^2) g^o_{\mu\nu} - 3\hat{s}^2 D_\mu D_\nu \hat{s}^2 - 3D_\mu \hat{s}^2 D_\nu \hat{s}^2.
\end{aligned}$$

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