Kant on Geometry and Spatial Intuition

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Kant's philosophy of geometry can only be properly understood against the background of two more general features of his philosophical position: his fundamental dichotomy between the two basic cognitive faculties of the mind, sensibility and understanding, and his distinctive theory of space, as the "pure form of our outer sensible intuition." Kant's conception of space and time as our "pure forms of sensible intuition" is central to his general philosophical position, which he calls "formal" or "transcendental" idealism. And, although a fundamental dichotomy between the two faculties of sense and intellect precedes Kant by many centuries, and is characteristic of all forms of traditional rationalism from Plato to Leibniz, Kant's particular version of the dichotomy is entirely distinctive of him. For, in sharp contrast to all forms of traditional rationalism, Kant locates the primary seat of a priori mathematical knowledge in sensibility rather than the intellect. In particular, our pure form of outer sensible intuition—space—is the primary ground of our pure geometrical knowledge.

Kant characterizes the distinctive role of our pure intuition of space in geometry in terms of what he calls "construction in pure intuition," and he illustrates this role by examples of geometrical construction from Euclid's *Elements*. It is natural, then, to turn to recent work on diagrammatic reasoning in Euclid—originating with Ken Manders—to elucidate Kant's conception. In particular, when Kant says that spatial intuition plays a necessary role in the science of geometry, we might take him to mean that diagrammatic reasoning—in the sense of Manders and his followers—plays a necessary role. I shall argue that this kind of view of Euclidean geometry, as illuminating as it may be as an interpretation of the *Elements*, is not adequate as an interpretation of Kant and, more generally, that recent work on diagrammatic reasoning can, at best, capture only a part of what Kant's conception of geometry involves. Most importantly, it cannot explain why Kant took this conception crucially to involve a revolutionary new theory of *space*—the very (three-dimensional) space in which we, and all other physical objects, live and move and have our being.

The most fully developed diagrammatic interpretation of the *Elements* with which I am acquainted is that of Marco Panza; and, although Panza is not primarily interested in the interpretation of Kantian philosophy, he does refer to Kant (and to some of my own previous work on Kant) to motivate his interpretation of Euclid. Panza's fundamental concern, however, is to argue for an Aristotelian rather than Platonic interpretation of the

Elements, according to which the objects of Euclid's geometry are not abstract objects in the Platonic (or modern) sense—entities apprehended by the intellect alone entirely independently of the senses. The objects of Euclid's geometry rather arise by a process of Aristotelian abstraction from concrete sensible particulars: namely, from particular physical diagrams drawn on paper or a blackboard, suitably interpreted and understood.[1] I shall argue Kant's conception of the role of spatial intuition in geometry cannot involve a process of abstraction from concrete sensible particulars in this sense.

Kant, as I said, diverges from traditional rationalism in locating the seat of pure geometry in sensibility rather than the understanding, and he thereby gives a central role in geometry to what he calls "the pure productive imagination." Perhaps the most important problem facing interpretations of Kant's philosophy of geometry, then, is to explain how, for Kant, sensibility and the imagination—faculties traditionally associated with the immediate apprehension of sensible particulars—can possibly yield truly universal and necessary knowledge. For example, in a well-known passage from the Discipline of Pure Reason in its Dogmatic Employment in the first *Critique*, Kant contrasts philosophical cognition, as "*rational cognition from concepts*," with mathematical cognition, as rational cognition "from the *construction* of concepts"—and, Kant famously adds, "to *construct* a concept is to present the intuition corresponding to it a priori" (A713/B741). Kant concludes, "[philosophy] confines itself merely to universal concepts, [mathematics] can effect nothing by mere concepts, but hastens immediately to intuition, in which it considers the concept *in concreto*—not, however, empirically, but merely in an [intuition] that it presents a priori, that is, which it has constructed, and in which that which follows from the universal conditions of construction must also hold universally of the object of the constructed concept" (A715-6/B743-4).

Exactly what, however, is a pure or non-empirical intuition corresponding to a general concept—a singular instance of this concept that is nonetheless presented purely a priori? Moreover, how can any singular instance of a general concept (no matter how it is supposed to be produced) possibly be an additional source, over and above purely conceptual representation, of universal and necessary knowledge? Immediately after the just quoted sentence defining the construction of a concept as the a priori presentation of

[1] Panza summarizes his "Aristotelian" (rather than "Platonic") interpretation of the practice of Euclidean geometry as follows: "[T]he classical interpretation of [Euclidean plane geometry], according to which it would result from contemplation of something similar to Platonic ideas, can be contrasted and replaced by another interpretation, more Aristotelian (and Kantian) in spirit, according to which [Euclidean plane geometry] is (or results from) a codified practice essentially based on the production and inspection of physical objects, like diagrams." Nothing I say is intended to undermine Panza's rich interpretation of Euclidean geometry in its (ancient Greek) intellectual context. My point is only that it cannot also serve as an interpretation of Kant himself—in his (eighteenth century) intellectual context.

the corresponding intuition, Kant says (ibid.): "For the construction of a concept we therefore require a *non-empirical* intuition, which consequently, as intuition, is a *singular* object, but which nonetheless, as the construction of a concept (a universal representation), must express universal validity, in the representation, for all possible intuitions that belong under this concept." But how, once again, can an essentially singular representation (no matter how it is supposed to be produced) possibly express such truly universal validity? Problems of precisely this kind underlie the contrary conviction, common to all traditional forms of rationalism, that mathematical knowledge must be conceptual or intellectual *as opposed to* sensible.

Kant illustrates his meaning, in the continuation of our passage, by an example of a Euclidean proof, Proposition I.32 of the *Elements*, where it is shown that the sum of the interior angles of a triangle is equal to the sum of two right angles. Given a triangle ABC one extends the side BC (in a straight line) to D and draws the line CE parallel to AB. One then notes (by Proposition I.29) that the alternate angles BAC and ACE are equal, and also that the external angle ECD is equal to the internal and opposite angle ABC. But the remaining internal angle ACB added to ACE and ECD yields the sum of two right angles (the straight line BCD), and the two angles ACE and ECD have just been shown to be equal to the first two internal angles. Therefore, the three internal angles together also equal the sum of two right angles. This construction and proof obviously has universal validity for all triangles, because the required inferences and auxiliary constructions (extending the line BC to D and drawing the parallel CE to EB) can always be carried out within Euclidean geometry, no matter what triangle ABC we start with.

It appears, in fact, that the proof-procedure of Euclid's *Elements* is paradigmatic of "construction in pure intuition" throughout Kant's discussion of mathematics in the first *Critique*—which includes a fairly complete presentation of the elementary Euclidean geometry of the triangle. In the Transcendental Aesthetic, for example, Kant presents the corresponding side-sum property of triangles—that two sides taken together are always greater than the third (Proposition I.20)—as an illustration of how geometrical propositions "are never derived from universal concepts of line and triangle, but rather from intuition, and, in fact, [are thereby derived] a priori with apodictic certainty" (A25/B39). And the Euclidean proof of this proposition proceeds, just like Proposition I.32, by auxiliary constructions and inferences starting from an arbitrary triangle ABC: we extend side BA (in a straight line) to D such that AD is equal to AC; we then draw CD and note (by Proposition I.5) that the two angles ACD and ADC are equal, so that BCD is greater than BDC; since (by Proposition I.19) the greater angle is subtended by the greater side, it follows that BD is greater than BC; but BD is equal to the sum of BA and AD (= AC). Moreover, Kant refers to the Euclidean proof of Proposition I.5 itself—that the angles at the base of an isosceles triangle are equal—in a famous passage in the second edition Preface praising the characteristic method of mathematics introduced by the "revolution in thought" effected by the Ancient Greeks (Bxi-xii); and this proof, too, proceeds by the expansion of an original (and arbitrary) triangle ABC into a more complicated figure by auxiliary constructions.

Kant's reliance on Euclid is thus very clear, and, once again, it is therefore natural to turn to recent work on the diagrammatic reasoning found in Euclid for elucidating Kant's view. In particular, we can appeal to what has been called the "schematic" character of Euclidean diagrams to explain how the appeal to sensible particulars can nonetheless result in necessary and universally valid knowledge. We observe, for example, that the specifically metrical features of the triangle used in the proof of Proposition I.32—the lengths of its particular sides and the magnitudes of its particular angles—play no role at all: it remains true for all continuous variations of these lengths and angles. Therefore, we have indeed proved a proposition valid for all particular triangles whatsoever. As we shall see, however, although the technical notion of what he calls a *schema* is indeed centrally important in Kant's theory of geometrical reasoning—and, moreover, he models this notion, in important respects, on central features of Euclid's proof-procedure—, Kant's notion of a schema (or schematic representation) is quite distinct from the diagrammatic one; and, as a result, his conception of spatial intuition goes far beyond anything envisioned in the diagrammatic approach.

In the Axioms of Intuition (the principles of pure understanding corresponding to the categories of quantity—unity, plurality, and totality), Kant considers the Euclidean construction of a triangle in general from any three lines such that two taken together are greater than the third (Proposition I.22: the restriction is obviously necessary because of what has just been proved in Proposition I.20). This makes it clear, in my view, that the construction in pure intuition of the concept of a triangle in general, for Kant, just *is* the Euclidean construction demonstrated in Proposition I.22—where, in Kant's words, "I have here the mere function of the productive imagination, which can draw the lines greater or smaller, and thereby allow them to meet at any and all arbitrary angles" (A164-5/B205). Moreover, in the chapter on the Schematism of the Pure Concepts of the Understanding, Kant carefully distinguishes the general *schema* of a "pure sensible concept" (i.e., a mathematical concept) from any particular *image* falling under this concept which may be produced by the general schema (A140/B179-180): "I call [the] representation of a general procedure of the imagination for providing a concept with its image the schema of this concept." Kant then illustrates this idea, once again, with the example of a triangle:

> In fact, schemata rather than images of objects are what lie at the basis of our pure sensible concepts. No image at all would ever be adequate to the concept of a triangle in general. For it would never attain the universality of the concept, which makes it hold for all triangles, whether right-angled, acute-angled, and so on, but would always be limited to only a part of this sphere. The schema of the triangle can never exist anywhere but in thought, and it signifies a rule of synthesis of the imagination with respect to pure figures in space. (A140-1/B180)

This "rule of synthesis," therefore, appears to be nothing more nor less than the Euclidean construction of an arbitrary triangle considered in the Axioms of Intuition as a "mere [universal] function of the productive imagination."

More generally, then, we can take the Euclidean constructions corresponding to the fundamental geometrical concepts (line, circle, triangle, and so on) as what Kant means by the *schemata* of such concepts. We can understand the schema of the concept of triangle as a function or constructive operation which takes three arbitrary lines (such that two together are greater than the third) as input and yields the triangle constructed out of these three lines as output (in accordance with Proposition I.22); we can understand the schema of the concept of circle as a function which takes an arbitrary point and line segment as input and yields the circle with the given point as center and the given line segment as radius as output (in accordance with Postulate 3; compare A234/B287); and so on.[2] Such constructive operations have all the generality or universality of the corresponding concepts: they yield, with appropriate inputs, *any and all* instances of these concepts. Unlike general concepts themselves, however, the outputs of a schema are indeed singular or individual representations—particular instances, or what Kant calls images, which fall under the concept in question. The outputs of a schema, therefore, are not conceptual or logical entities like propositions or truth-values.

This last point is crucial for understanding why Kant takes pure mathematics essentially to involve non-discursive or non-conceptual cognitive resources which, nonetheless, possess all the universality and necessity of purely conceptual thought.

[2] Corresponding to the concept of line (segment) in Euclid are two such constructive operations: Postulate 1, which takes two arbitrary points as input and yields the line segment connecting them as output, and Postulate 2, which takes two arbitrary line segments as input and yields the extension of the first by the second as output.

Characteristic of conceptual thinking, for Kant, is the logical procedure of *subsumption*, whether of an individual under a general concept or of a less general concept (species) under a more general concept (genus). Characteristic of mathematical reasoning, by contrast, is the procedure of *substitution*—by which, as we would now put it, an object is inserted into the argument place of a function, yielding another object that can be inserted into the argument places of further functions, and so on. Reasoning by substitution is therefore essentially *iterative*, and it is precisely such iterative thinking, for Kant, that underlies both pure geometry (in the guise of Euclidean proof) and the more general calculative manipulation of magnitudes in algebra and arithmetic.

Kant's conception of the essentially non-conceptual character of geometrical reasoning is thus especially sensitive to the circumstance that, in Euclid's formulation of geometry, the iterative application of initial constructive operations represents the existential assumptions we would express by explicit quantificational statements in modern formulations following Hilbert. Thus, for example, whereas Hilbert represents the infinite divisibility of a line by an explicit quantificational axiom stating that between any two points there exists a third, Euclid represents the same idea by showing how to construct a bisection function for any given line segment (Proposition I.10): our ability to iterate this construction indefinitely then represents the *infinite divisibility* of the same segment. More generally, Euclid constructs all the points in his plane by the iterative application of three initial constructive operations to any given pair of points: connecting any two points by a straight line segment (Postulate 1), extending any given line segment by another given line segment (Postulate 2), constructing a circle with any point as center and any given line segment as radius (Postulate 3). This constructive procedure yields all points constructible by straight-edge and compass, which, of course, comprise only a small (denumerable) subset of the full two-dimensional continuum whose existence is explicitly postulated by Hilbert.[3] In this sense, the existential assumptions needed for Euclid's particular proof-procedure—the very assumptions needed to justify all the auxiliary constructions needed along the way—are given by Skolem functions for the existential quantifiers we would use in formulating a Hilbert-style axiomatization in modern quantificational logic, where (in Euclid) all such Skolem functions are explicitly

[3] More precisely, we can represent all the points constructible by straight-edge and compass construction in the Euclidean plane by the Cartesian product of a square-root extension field of the rationals with itself, whereas the full set of points generated by a true (second order) continuity axiom is of course represented by $\mathbf{R}^2$, where **R** is the real numbers. An important intermediate case, studied by Tarski, uses a (first order) continuity *schema*, and is represented by a Cartesian product over any real closed field: see Tarski (1959).

constructed by finite iterations of the three initial constructive operations laid down in the first three postulates.

Following Leibniz, Kant takes the discursive structure of the understanding or intellect to be delimited by the logical forms of traditional subject-predicate logic. In explicit opposition to Leibniz, however, Kant takes these logical forms to be strictly limited to essentially finitary representations: there are, for Kant, no Leibnizean "complete concepts" comprising within themselves (that is, within their defining sets of characteristics [*Merkmale*] or partial concepts [*Teilbegriffe*]) an infinite manifold of further conceptual representations. But mathematical representations (such as the representation of space) can and do contain an infinite manifold of further (mathematical) representations within themselves (as in the representation of infinite divisibility). So such representations, for Kant, are not and cannot be conceptual. Of course, we now have an entirely different conception of logic from Kant's, one that is much more powerful than anything either he, or even Leibniz, ever envisioned. Nevertheless, we can still understand Kant's fundamental insight, from our own point of view, if we observe that no infinite mathematical structure (such as either the space of Euclidean geometry or the number series) can possibly be represented within *monadic* quantificational logic. Such infinite structures, in modern logic, are represented by the use of nested sequences of universal and existential quantifiers using *polyadic* logic. These same representations, from Kant's point of view, are instead made possible by the iterative application of constructive functions in the "productive imagination," where, as we have seen, Skolem functions for the existential quantifiers we would use in our formulations are rather explicitly constructed.

We now see, from Kant's point of view, why mathematical thinking essentially involves what he calls "the pure productive imagination" and why, accordingly, this type of thinking essentially exceeds the bounds of purely conceptual, purely intellectual thought. My first problem with using diagrammatic reasoning—in the sense of Manders and his followers—to interpret Kant's notion of "construction in pure intuition," therefore, is that it does not square with Kant's understanding of the relationship between conceptual thought and sensible intuition. It does not square, in particular, with his developed view of the relationship between general (geometrical) concepts, their corresponding general schemata, and the particular sensible images (particular geometrical figures) which then result by applying these schemata. Whereas diagrammatic accounts of the generality of geometrical propositions—following Manders—begin with particular concrete diagrams and then endeavor to explain how we can abstract from their irrelevant particular features (specific lengths of sides and angles, say) by an appropriate process of continuous variation, Kant rather begins with general concepts as conceived within the Leibnizean (logical) tradition and then shows how to "schematize" them sensibly by means of an intellectual act or function of "the pure productive imagination." Both the general concepts in question and their corresponding general schemata are pure rather than empirical representations; and a particular concrete figure occurs, as it were, only accidentally for Kant, at the end of a process of intellectual determination. Nothing like Aristotelian abstraction from concrete sensible particulars plays any role at all.

My second problem, however, is even more serious. For we also need to connect Kant's conception of geometrical reasoning with space and time—not with particular spatial figures drawn on paper or a blackboard but with space and time themselves. And it precisely here, as I have intimated, that Kant also engages Newton's conception of space (and time) as it figures in his controversy with Leibniz. Space, for Newton, is a great ontological receptacle, as it were, for both all possible geometrical figures and all possible material objects, and Kant's theory of space as a pure form of intuition is supposed to be an alternative, as we shall see, to precisely this Newtonian conception.

It is centrally important to Kant's philosophy of geometry that all possible objects of human sense-perception, all objects of what Kant calls *empirical* intuition, must necessarily conform to the a priori principles of mathematics established in *pure* intuition (A165-6/B206): "The synthesis of spaces and times, as the essential form of all intuition, is that which, at the same time, makes possible the apprehension of appearance, and thus every outer experience, [and] therefore all cognition of the objects thereof; and what mathematics in its pure employment demonstrates of the former necessarily holds also of the latter." In order to appreciate the role pure geometry plays in our perception of empirical objects, then, we need explicitly to connect the functions of the pure productive imagination expressed in the construction of geometrical concepts with the Kantian forms of pure intuition (space and time), as they are described in the "metaphysical expositions" of space and time in the Transcendental Aesthetic.

Kant (in his controversy with Eberhard in 1790) distinguishes between "geometrical" and "metaphysical" space. The former is the space of pure Euclidean geometry, which, for Kant, is gradually or successively constructed by the two fundamental operations of drawing a straight line and constructing a circle in accordance with Postulates 1-3. The latter, by contrast, is the space of our pure outer sensible intuition—which, for Kant, is the "pure form" of all empirical intuition of any and all physical objects that may exist in this space. "Metaphysical space" is thus that described in the "metaphysical exposition" of space in the Transcendental Aesthetic. So let us now turn to the first two argument of this metaphysical exposition, which are intended to show, in particular, that space is a necessary a priori representation that precedes all empirical perception—not a representation that can be in any way abstracted from our empirical perception of (outer) spatial objects.

The first argument attempts to show that space is an a priori rather than empirical representation by arguing that all perception of outer (empirical) objects in space presupposes the representation of space:

> Space is no empirical concept that has been derived from outer experiences. For, in order that certain sensations are related to something outside me (that is, to something in another place in space than the one in which I find myself), and, similarly, in order that I be able to represent them as outside of and next to one another—and thus not merely as different but as in different places—the representation of space must already lie at the basis. Therefore, the representation of space cannot be obtained from the relations of outer appearance through experience; rather, this outer experience is itself only possible in the first place by means of the representation in question. (A23/B38)

This argument emphasizes that space as the form of outer sense enables us to represent objects as outer precisely by representing them as spatially external to the perceiving subject, so that the space in question contains the point of view from which the objects of outer sense are perceived and around which the objects of outer sense are arranged. Empirical spatial intuition or perception occurs when an object spatially external to the point of view of the subject affects this subject—along a spatial line of sight, as it were—so as to produce a corresponding sensation; and it is in this sense, therefore, that the pure form of (spatial) sensible intuition expresses the manner in which we are affected by (outer) spatial objects. Let us call this structure "perspectival space."

The second argument goes on to claim that space is a necessary a priori representation, which functions as a condition of the possibility of all outer experience:

> Space is a necessary a priori representation, which lies at the basis of all outer intuition. One can never make a representation [of the supposed fact] that there is no space, although one can very well think that no objects are to be found therein. It must therefore be viewed as the condition of the possibility of appearances, not as a determination depending on them, and is an a priori representation, which necessarily lies at the basis of outer appearances. (A24/B38-9)

The crux of this argument is that one cannot represent outer objects without space, whereas one can think this very same space as entirely empty of such objects. What exactly does it mean, however, to represent space as empty of outer objects, and in what precise context do we succeed in doing this? A very natural suggestion is that we think space as empty of outer (empirical) objects just when we are doing pure geometry. This would accord very well, in particular, with the concluding claim that space thereby functions as a necessary a priori condition of the possibility of outer appearances, for they would then be necessarily subject to the a priori necessary science of pure geometry.

What is the precise relationship between the a priori structure attributed to space in the first argument (perspectival space) and that attributed to space in the second (the structure of pure geometry)? It is natural, in the first place, to view the former structure as itself a priori and geometrical, since it does not depend at all on the particular (empirical) outer objects actually perceived from any particular point of view. On the contrary, this perspectival structure is invariant under all changes in both the objects perceived and the point of view from which they are perceived, and, in this sense, it thereby expresses the *form* rather than the *matter* or *content* of outer intuition. Moreover, and in the second place, these possible changes in perspective themselves constitute what we now take to be a geometrical object: namely, a *group* of (Euclidean) motions or transformations, comprising all possible translations of our initial point of view through space and all possible rotations of the perspective associated with this point of view around the given point. In particular, any perceptible spatial object, located anywhere in space, can thereby be made accessible by an appropriate sequence of such translations and rotations starting from any initial point of view and associated perspective.

But there is a clear connection between this (modern, group-theoretical) geometrical structure and geometry in Kant's sense; for, as Kant himself emphasizes, the two fundamental Euclidean constructions of drawing a line and constructing a circle are generated precisely by translations and rotations—as we generate a line segment by the motion (translation) of a point and then rotate this segment (in a given plane) around one of its endpoints.[4] On the present interpretation, therefore, it is precisely this relationship

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[4] Kant states this most explicitly in his (1790) controversy with Eberhard (Ak. 20, 410-11): "[I]t is very correctly said that 'Euclid assumes the possibility of drawing a straight line and describing a circle without proving it'—which means without proving this possibility *through inferences*. For *description*, which takes place a priori through the imagination in accordance with a rule and is called construction, is itself the proof of the possibility of the object. . . . However, that the possibility of a straight line and a circle can be proved, not *mediately* through inferences, but only immediately through the construction of these concepts (which is in no way empirical), is due to the circumstance that among all constructions (presentations determined in accordance with a rule in a priori intuition) some must still be *the first*—namely, the *drawing* or describing (in thought) of a straight line and the *rotating* of such a line around a fixed point—where the

between perspectival space and geometrical space that links Kant's theory of space as a form of outer intuition or perception to his conception of pure mathematical geometry in terms of the successive execution of Euclidean constructions in the pure imagination. It is in precisely this way, in particular, that Kant is now is a position to claim that pure mathematical geometry is necessarily applicable to all possible objects of empirical perception—that "[t]he synthesis of spaces and times, as the essential form of all intuition, is that which, at the same time, makes possible the apprehension of appearance, and thus every outer experience, [and] therefore all cognition of the objects thereof; and what mathematics in its pure employment demonstrates of the former necessarily holds also of the latter."

The urgent need to establish this result places Kant in a completely different intellectual world from that inhabited by Euclid, Plato, and Aristotle. For it is characteristic of the new view of mathematics arising in the seventeenth century that pure mathematical geometry is taken to be the foundation for all knowledge of physical reality. Pure mathematical geometry, beginning with Descartes, is taken to describe—and to describe exactly—the most fundamental features of matter itself; and, in this sense, physical space and geometrical space (that is, Euclidean space) are now taken to be identical. Kant's own understanding of this idea, as I have suggested, is framed by the controversy between Newton and Leibniz—where both took the geometrization of nature to be a now established fact, but they reacted to this fact in radically different ways. Newton understood the situation quite literally: "mathematical" space and time—"true" or "absolute" space and time—constitute the fundamental ontological framework of all reality. Even God himself is necessarily spatial and temporal (existing always and everywhere), and all physical or material objects are then created and "moved," as Newton puts it, within God's "boundless and uniform sensorium."

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latter cannot be derived from the former, nor can it be derived from any other construction of the concept of a magnitude." [Compare A234/B287: "Now a postulate in mathematics is the practical proposition that contains nothing but the synthesis by which we first give an object to ourselves and generate its concept—e.g., to describe a circle with a given line from a given point on a plane—and such a proposition cannot be proved, because the procedure it requires is precisely that by which we generate the concept of such a figure." And, more generally, compare (A162-3/B203-4): "I can represent no line to myself, no matter how small, without drawing it in thought, that is, gradually generating all its parts from a point. . . . . On this successive synthesis of the productive imagination in the generation of figures is based the mathematics of extension (geometry), together with its axioms, which express the conditions of a priori sensible intuition under which alone the schema of a pure concept of outer appearance can arise."] Straight lines and circles thereby appear as what we call the *orbits* (confined to any two dimensional plane) of the Euclidean group of rigid motions in space. As explained in Friedman (2000a), an advantage of this reading is that it then allows us to connect Kant's theory of pure geometrical intuition with the later discussions of Hermann von Helmholtz and Henri Poincaré (who were self-consciously influenced by Kant)—although there can of course be no question of attributing an explicit understanding of the group-theoretical approach to geometry to Kant himself.

For Leibniz, by contrast, the entire physical world described by the new mathematical science (including the space in which bodies move) is a secondary appearance or phenomenon of an underlying metaphysical reality of simple substances or monads—substances which, at this level, are not spatial at all but rather have only purely internal properties and no external relations. And this point, in turn, is closely connected with the fact that Leibniz self-consciously adheres to the idea that purely intellectual knowledge is essentially logical. For, although Leibniz appears to have envisioned some sort of extension of Aristotelian logic capable of embracing the new algebraic methods of his calculus, there is no doubt that the traditional subject-predicate structure of this logic pervades his monadic metaphysics: it is precisely because ultimate metaphysical reality is essentially intellectual in the logical sense that the entire sensible world, including space, is a merely secondary reality or phenomenon. Thus, although Leibniz, like everyone else in the period, holds that there are exact mathematical laws governing the sensible and material world (the phenomenal world), he reintroduces a new kind of necessary gap—a new kind of *Platonic* gap—between reality as known by the intellect (noumenal reality) and this sensible world.

Kant's philosophy of transcendental idealism is also based on a fundamental dichotomy between reality as thought by the pure understanding alone and the phenomenal world in space and time given to our senses. But Kant sharply differs from Leibniz in two crucial respects. First, mathematical knowledge, for Kant, is sensible rather than purely intellectual: indeed, mathematics is the very paradigm of rational and objective sensible knowledge, resulting from the schematism of specifically mathematical concepts in our pure forms of sensible intuition. Second, and as a consequence, we can only have (theoretical) knowledge, for Kant, of precisely this sensible (phenomenal) world: the noumenal reality thought by the pure understanding alone remains forever unknowable from a theoretical point of view, and we can only have purely practical knowledge of its inhabitants (God and the soul) via moral experience. Indeed, it is precisely this necessary limitation of all theoretical knowledge to the sensible or phenomenal world that ultimately results from Kant's doctrine of the schematism of the pure concepts of the understanding—which, as Kant sees it, Leibniz entirely missed.

Kant's philosophy of geometry—seen against the background of his more general transcendental idealism—combines central insights of both Leibniz and Newton. For, in the first place, Kant's emphasis on the perceptual and intuitive aspects of geometry (and mathematics more generally) corresponds to Newton's approach, in contrast to the logico-algebraic approach of Leibniz. And, in the second place, Kant's sharp distinction between the faculties of intellect and sensibility, together with his parallel sharp distinction between logical or discursive and mathematical or intuitive reasoning, arises precisely against the background of the Leibnizean conception of the pure intellect, and it is aimed, more specifically, at Leibniz's view that pure mathematics (including geometry) is, in Kant's sense, analytic—depending only on relations of conceptual containment within the traditional logic of concepts. Nevertheless, Kant accepts Leibniz's characterization of the pure intellect in terms of the traditional logic of concepts, and Kant's point about pure mathematics, against Leibniz, is simply that the pure intellect, characterized in this way, is not, after all, adequate to the task. It is for precisely this reason, in Kant's view, that the pure understanding must be applied to, or schematized in terms of, a second rational faculty modelled on Newtonian absolute space—no longer conceived along the lines of Newton's divine sensorium, but rather as a pure form of our (human) faculty of sensibility.

We are now in a position, finally, to appreciate the deeper explanation for why no process of Aristotelian abstraction from concrete sensible particulars can play any role in our knowledge of Euclidean geometry for Kant. All perception of concrete sensible particulars—including particular diagrams drawn on paper or a blackboard—presuppose that space as our pure form of outer sensible intuition is already in place, and, in particular, that it already possesses the structure of Euclidean geometry. Space acquires this structure, for Kant, precisely from the pure acts or operations of the productive imagination (the application of general geometrical constructions or Kantian schemata), resulting in what he calls "[t]he synthesis of spaces and times, as the essential form of all intuition." All the constructive procedures of Euclidean geometry must therefore have already taken place, in the space of *pure* intuition, in order for any perception of concrete physical objects in *empirical* intuition to be possible—including any perception of concrete physical diagrams.