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Kaplansky-type Theorems, II

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Abstract. Let D be an integral domain with quotient field K, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. A prime ideal Q of $D[X]$ is called an upper to zero in $D[X]$ if $Q = fK[X] \cap D[X]$ for some $f \in D[X]$. In this paper, we study integral domains D such that every upper to zero in $D[X]$ contains a prime element (resp., a primary element, a t-invertible primary ideal, an invertible primary ideal).

1. Introduction

Let D be an integral domain and X be an indeterminate over D. It is well known that D is a UFD if and only if every nonzero prime ideal of D contains a nonzero prime element [12, Theorem 5]. This is the so-called *Kaplansky's theorem.* This type of theorems was studied by Anderson and Zafrullah [3] and Kim [13] to characterize GCD-domains, valuations domains, Prüfer domains, generalized GCD-domains, and PvMDs. (Definitions will be reviewed in the sequel.) In [5, Proposition 2.7], it is shown that $D[X]$ is a GWFD if and only if D is a GWFD and each upper to zero in $D[X]$ contains a primary element. This work is motivated by the results ([12, Theorem 5], [3], [13], [5, Proposition 2.7]). The purpose of this paper is to study an integral domain D such that each upper to zero in $D[X]$ contains a prime element (resp., a primary element, a t-invertible primary ideal, an invertible primary ideal). More precisely, we show that every upper to zero in $D[X]$ contains a prime element f with $c(f) = D$ if and only if D is a Bézout domain; every upper to zero in $D[X]$

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contains a primary element f with $c(f) = D$ if and only if D is a UMT-domain, each maximal ideal of D is a t-ideal, and $Cl(D[X])$ is torsion; and if D is integrally closed, then every upper to zero in $D[X]$ contains an invertible (resp., t-invertible) primary ideal if and only if D is an almost generalized GCD-domain (resp., PvMD).

We first introduce some definitions and notation. Let D be an integral domain with quotient field K, X an indeterminate over D, and $D[X]$ the polynomial ring over D. For any polynomial $f \in K[X]$, the *content* $c_D(f)$ (simply, $c(f)$) of f is the fractional ideal of D generated by the coefficients of f. An upper to zero in $D[X]$ is a prime ideal $Q_f = fK[X] \cap D[X]$ of $D[X]$, where $f \in D[X]$ is irreducible in $K[X]$. Let I be a nonzero fractional ideal I of D. Then $I^{-1} = \{x \in K | xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$, and $I_t = \bigcup\{J_v | J \subseteq I$ is a nonzero finitely generated ideal$\}$. We say that I is a *v-ideal* (resp., *t-ideal*) if $I = I_v$ (resp., $I = I_t$). A fractional ideal I of D is said to be *t-invertible* if $(II^{-1})_t = D$. A maximal t-ideal is an ideal of D maximal among proper integral t-ideals of D. Let t-Max(D) be the set of maximal t-ideals. It is easy to see that if D is not a field, then $t\text{-Max}(D) \neq \emptyset$ and $D = \bigcap_{t\text{-Max}(D)} D_P$.

An integral domain D is a *UMT-domain* if every upper to zero in $D[X]$ is a maximal t-ideal; D is a *Prüfer v-multiplication domain* (PvMD) if every nonzero finitely generated ideal of D is t-invertible; D is a *GCD-domain* if for any $0 \neq a, b \in D$, $aD \cap bD$ (equivalently, $(a, b)_v$) is principal; D is an *almost GCD-domain* (AGCD-doman) if for any $0 \neq a, b \in D$, there is a positive integer $n = n(a, b)$ such that $a^nD \cap b^nD$ is principal; D is a *generalized GCD-domain* (GGCD-domain) if $aD \cap bD$ (equivalently, $(a, b)_t$) is invertible for any $0 \neq a, b \in D$; D is an *almost GGCD-domain* (AGGCD-domain) if for $0 \neq a, b \in D$, there is a positive integer $n = n(a, b)$ such that $a^nD \cap b^nD$ is invertible; and D is a *generalized weakly factorial domain* (GWFD) if each nonzero prime ideal of D contains a primary element (a nonzero nonunit $x \in D$ is primary if xD is a primary ideal).

Let $T(D)$ be the group of t-invertible fractional t-ideals of D, and let $Prin(D)$ be its subgroup of principal fractional ideals. Then the quotient group $Cl(D) = T(D)/Prin(D)$ is an abelian group called the (t-)*class group* of D. It is known that D is a GCD-domain if and only if D is a PvMD and $Cl(D) = 0$ [6, Proposition 2]; if D is integrally closed, then D is an AGCD-domain if and only if D is a PvMD with $Cl(D)$ torsion [15, Corollary 3.8]; and D is an AGGCD-domain if and only if D is an AGCD-domain with $Cl(D)$ torsion [14, Theorem 5.1]. Any undefined terminology is standard, as in [8] or [12].

2. Kaplansky-type theorems for uppers to zero

Let D be an integral domain with quotient field K, $D^* = D \setminus \{0\}$, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D.

Lemma 2.1(4, Lemma 2.1). *If $f \in D[X] \setminus D$, then*

(1) *$fK[X] \cap D[X] = fD[X]$ if and only if $c(f)_v = D$;*

(2) *if f is a product of primary elements in $D[X]$, then $fK[X] \cap D[X] = fD[X]$.*

It is well known that D is a UFD if and only if every nonzero prime ideal of D contains a nonzero prime element of D [12, Theorem 5].

Theorem 2.2. *Every upper to zero in $D[X]$ contains a prime element if and only if D is a GCD-domain.*

Proof. ($\Rightarrow$) For any $0 \neq a, b \in D$, let $f = aX + b$. Then $Q_f = fK[X] \cap D[X]$ is an upper to zero in $D[X]$, and so Q_f contains a prime element g. Note that $\mathrm{ht}(Q_f) = 1$; so $Q_f = gD[X]$, and hence $c(g)_v = D$ by Lemma 2.1 and $f = ug$ for some $u \in K$ (actually $u \in D$). Thus, $(a, b)_v = c(f)_v = uc(g)_v = uD$.

($\Leftarrow$) Suppose that D is a GCD-domain, and let $h \in D[X]$ be such that $Q_h = hK[X] \cap D[X]$ is an upper to zero in $D[X]$. Recall that a GCD-domain is integrally closed and $c(h)^{-1}$ is principal, say, $c(h)^{-1} = aD$. Thus, ah is a prime element, because $Q_h = hc(h)^{-1}[X]$ [8, Corollary 34.9]. □

Corollary 2.3. *Every upper to zero in $D[X]$ contains a prime element f with $c(f) = D$ if and only if D is a Bézout domain.*

Proof. Let $a, b \in D$ be nonzero, and assume that $Q_g = gK[X] \cap D[X]$, where $g = aX + b$, contains a prime element f with $c(f) = D$. Then $g = uf$ for some $u \in K$, and thus $(a, b) = c(g) = uD$, which means that D is a Bézout domain. Conversely, assume that D is a Bézout domain, and let Q be an upper to zero in $D[X]$. Then Q contains a prime element f by Theorem 2.2, and since D is a Bézout domain, $c(f) = aD$ for some $a \in D$. But, since f is a prime element, $aD = D$, and thus $c(f) = D$. □

Let S be a multiplicative subset of D. We say that S is an *almost splitting* (resp., *almost g^d-splitting*) set if, for each $0 \neq r \in D$, there is an integer $n = n(r) \geq 1$ such that $r^n = st$ for some $s \in S$ and $t \in D$ with $(s', t)_v = D$ (resp., $(s', t) = D$) for all $s' \in S$. Recall that D is a *quasi-AGCD-domain* if D^* is an almost splitting set in $D[X]$. The next theorem appears in [4, Theorem 2.4], which is a motivation for this paper.

Theorem 2.4. *The following statements are equivalent.*

(1) *Every upper to zero in $D[X]$ contains a primary element.*

(2) *D is a quasi-AGCD-domain.*

(3) *D is a UMT-domain and $Cl(D[X])$ is torsion.*

Following [2], an integral domain D is called an *almost Bézout domain* (AB-domain) if, for each $a, b \in D$, there is an integer $n \geq 1$ such that (a^n, b^n) is principal. Obviously, if D is integrally closed, then D is an AB-domain if and only if D is a Prüfer domain with $Cl(D)$ torsion. It is known that D is an AB-domain if and only if D is an AGCD domain and each maximal ideal of D is a t-ideal [2, Corollary 5.4]. So it is natural to call D a *quasi-AB-domain* if D is a quasi-AGCD-domain whose maximal ideals are t-ideals. Clearly, a quasi-AB-domain is a quasi-AGCD-domain, but not vice versa (for example, if D is a GCD-domain, then $D[X]$ is a GCD-domain (hence a quasi-AGCD-domain) but not a quasi-AB-domain). However, if D has (Krull) dimension one, then a quasi-AGCD-domain is a quasi-AB-domain.

Corollary 2.5. *The following statements are equivalent.*

(1) *Every upper to zero in $D[X]$ contains a primary element f with $c(f) = D$.*

(2) *D is a UMT-domain, each maximal ideal of D is a t-ideal, and $Cl(D[X])$ is torsion.*

(3) *D is a quasi-AB-domain.*

Proof. (1) $\Rightarrow$ (2). By Theorem 2.4, D is a UMT-domain and $Cl(D[X])$ is torsion. Assume that there is a maximal ideal which is not a t-ideal. Then there is an $f \in D[X]$ such that $c(f)_v = D$ but $c(f) \subsetneq D$. Let $f = f_1^{e_1} \cdots f_n^{e_n}$ be the prime factorization of f in $K[X]$ (note that $K[X]$ is a UFD). Then $fD[X] = fK[X] \cap D[X] = (f_1^{e_1}K[X] \cap D[X]) \cap \cdots \cap (f_n^{e_n}K[X] \cap D[X])$ by Lemma 2.1 and each $f_i^{e_i}K[X] \cap D[X]$ is a Q_i-primary ideal, where $Q_i = f_iK[X] \cap D[X](1 \leq i \leq n)$. Since each Q_i is an upper to zero in $D[X]$, Q_i contains a primary element g_i with $c(g_i) = D$. Clearly, each $g_i^{e_i} \in f_i^{e_i}K[X] \cap D[X]$, and so if we set $g := g_1^{e_1} \cdots g_n^{e_n}$, then $g \in fD[X]$ and $c(g) = D$. Thus, $c(f) = D$, a contradiction.

(2) $\Rightarrow$ (1). Let Q be an upper to zero in $D[X]$. Since D is a UMT-domain, Q is t-invertible. Also, since $Cl(D[X])$ is torsion, there is an integer $n \geq 1$ such that $(Q^n)_t = fD[X]$ for some $f \in D[X]$. Note that f is primary, and since Q is a maximal t-ideal, $c(f)_t = D$. Thus, f is a primary element with $c(f) = D$, because each maximal ideal is a t-ideal.

(2) $\Leftrightarrow$ (3). This follows from Theorem 2.4. □

It is naturally asked that it follows from the definition that if D is a quasi-AB-domain, then D^* is an almost g^d-splitting set in $D[X]$. However, $(a, X) \neq D[X]$ for any nonunit $a \in D$. Hence D^* cannot be an almost g^d-splitting set in $D[X]$.

Corollary 2.6. *The following statements are equivalent for an integrally closed domain D.*

(1) *Every upper to zero in $D[X]$ contains a primary element f with $c(f) = D$.*

(2) *D is a Prüfer domain and $Cl(D)$ is torsion.*

(3) *D is a quasi-AB-domain.*

(4) *D is an AB-domain.*

Proof. (1) $\Leftrightarrow$ (2). Note that an integrally closed domain is a Prüfer domain if and only if it is a UMT-domain whose maximal ideals are t-ideals. Also, if D is integrally closed, then $Cl(D[X]) = Cl(D)$ ([7, Theorem 3.6]). Thus, the result follows from Corollary 2.5.

(1) $\Leftrightarrow$ (3). This follows from Corollary 2.5.

(2) $\Leftrightarrow$ (4). This is clear. □

Corollary 2.7. *If D is a quasi-AB-domain, then each overring R of D is a quasi-AB-domain. In particular, if R is integrally closed, then R is a Prüfer domain with torsion class group.*

Proof. Let Q be an upper to zero in $R[X]$. Then there is an $f \in K[X]$ such that $Q = fK[X] \cap R[X]$, and hence $Q \cap D[X] = fK[X] \cap D[X]$ is an upper to zero in $D[X]$. By Corollary 2.5, there is a primary element $g \in Q \cap D[X]$ such that $c_D(g) = D$. Clearly, $g \in Q$ and $c_R(g) = R$; in particular, Q is a maximal t-ideal of $R[X]$ [9, Theorem 1.4]. Note that, since g is a primary element of $D[X]$, there exist some $u \in K$ and an integer $n \geq 1$ such that $g = uf^n$. Hence $\sqrt{gR[X]} = fK[X] \cap R[X]$, and thus g is a primary element of $R[X]$ [5, Lemma 2.1]. Thus, R is a quasi-AB-domain by Corollary 2.5. In particular, if R is integrally closed, then R is a Prüfer domain with torsion class group by Corollary 2.6. □

It is well known that if D is integrally closed, then D is a UMT-domain if and only if D is a PvMD [9, Proposition 3.2]. Also, it is known that D is a Krull domain if and only if every nonzero prime (t-)ideal contains a t-invertible prime ideal [11, Theorem3.6] and D is a GGCD-domain if and only if each upper to zero in $D[X]$ is invertible [1, Theorem 15].

Theorem 2.8. *If D is integrally closed, then*

(1) *every upper to zero in $D[X]$ contains a t-invertible primary ideal if and only if D is a PvMD;*

(2) *every upper to zero in $D[X]$ contains an invertible primary ideal if and only if D is an almost generalized GCD-domain.*

Proof. (1) ($\Rightarrow$) Let Q be an upper to zero in $D[X]$, and let I be a t-invertible primary t-ideal contained in Q. Since $\text{ht}(Q) = 1$, we have $\sqrt{I} = Q$. Let $N_v = \{f \in D[X] | c(f)_v = D\}$, and suppose $Q \cap N_v = \emptyset$. Then $I_{N_v} \subseteq Q_{N_v} \subsetneq D[X]_{N_v}$. Since I is t-invertible, I_{N_v} is invertible (cf. [10, Proposition 2.1(3)]), and hence I_{N_v} is principal [10, Theorem 2.14]. So $Q_{N_v} = \sqrt{I_{N_v}}$ is a maximal t-ideal [5, Lemma 2.1]. This is contrary to the fact that $\text{Max}(D[X]_{N_v}) = t\text{-Max}(D[X]_{N_v}) = \{P[X]_{N_v} | P \in t\text{-Max}(D)\}$ [10, Propositions 2.1 and 2.2]. So $Q \cap N_v \neq \emptyset$, and thus Q is a maximal t-ideal [9, Theorem 1.4]. Thus, D is a PvMD.

($\Leftarrow$) Let Q be an upper to zero in $D[X]$. Then Q is a maximal t-ideal, because a PvMD is a UMT-domain. Thus, Q is a t-invertible prime (hence primary) t-ideal [9, Proposition 1.4].

(2) ($\Rightarrow$) We first note that D is a PvMD by (1). Let $0 \neq a, b \in D$, and put $f = aX + b$. Then $Q_f = fK[X] \cap D[X]$ is an upper to zero in $D[X]$, and so Q_f contains an invertible primary ideal A. It is easy to see that $Q_f = fc(f)^{-1}[X]$ [8, Corollary 34.9] and $A = ((Q_f)^n)_t$ for some positive integer n. Note that $((Q_f)^n)_t = f^n c(f^n)^{-1}[X]$ and $c(f^n)^{-1} = (c(f)^n)^{-1} = ((a,b)^n)^{-1}$. Thus, $(a^n, b^n)_t$ is invertible, because $(a,b)_t$ is t-invertible by (1), and so $(((a,b)^n)^{-1})^{-1} = ((a,b)^n)_t = (a^n, b^n)_t$ [2, Lemma 3.3].

($\Leftarrow$) Let $Q_g = gK[X] \cap D[X]$, where $g \in D[X]$, be an upper to zero in $D[X]$. Note that $Q_g = gc(g)^{-1}[X]$ [8, Corollary 34.9], because D is integrally closed. Note also that, since D is an almost GGCD-domain, there is a positive integer m such that $(c(g)^m)_t = c(g^m)_t$ is invertible by (1), [8, Proposition 34.8], and [14, Theorem 3.2]. Thus $(Q_g^m)_t = g^m K[X] \cap D[X] = g^m c(g^m)^{-1}[X]$ is an invertible primary ideal. $\square$

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References

[1] D. D. Anderson, T. Dumitrescu, and M. Zafrullah, *Quasi-Schreier domains, II*, Comm. Algebra, **35**(2007), 2096-2104.

[2] D. D. Anderson and M. Zafrullah, *Almost Bézout domains*, J. Algebra, **142**(1991), 285-309.

[3] D. D. Anderson and M. Zafrullah, *On a theorem of Kaplansky*, Boll. Un. Mat. Ital. A (7), **8**(1994), 397-402.

[4] D. F. Anderson and G. W. Chang, *Almost splitting sets in integral domains II*, J. Pure Appl. Algebra, **208**(2007), 351-359.

[5] D. F. Anderson, G. W. Chang, and J. Park, *Generalized weakly factorial domains*, Houston J. Math., **29**(2003), 1-13.

[6] A. Bouvier, *Le groupe des classes d'un anneau intègre*, 107ème Congres des Sociétés Savantes, Brest, 1982, fasc. IV, 85-92.

[7] S. Gabelli, *On divisorial ideals in polynomial rings over Mori domains*, Comm. Algebra, **15**(1987), 2349-2370.

[8] R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.

[9] E. Houston and M. Zafrullah, *On t-invertibility, II*, Comm. Algebra, **17**(1989), 1955-1969.

[10] B. G. Kang, *Prüfer v-multiplication domains and the ring* $R[X]_{N_v}$, J. Algebra, **123**(1989), 151-170.

[11] B. G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra, **124**(1989), 284-299.

[12] I. Kaplansky, Commutative Rings, rev. ed., Univ. of Chicago, Chicago, 1974.

[13] H. Kim, *Kaplansky-type theorems*, Kyungpook Math. J., **40**(2000), 9-16.

[14] R. Lewin, *Almost generalized GCD-domains*, Lecture Notes in Pure and Appl. Math., Marcel Dekker, **189**(1997), 371-382.

[15] M. Zafrullah, *A general theory of almost factoriality*, Manuscripta Math., **51**(1985), 29-62.