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Kappa-symmetry of superstring sigma model and generalized 10d supergravity equations

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Abstract: We determine the constraints imposed on the 10d target superspace geometry by the requirement of classical kappa-symmetry of the Green-Schwarz superstring. In the type I case we find that the background must satisfy a generalization of type I supergravity equations. These equations depend on an arbitrary vector X_a and imply the one-loop scale invariance of the GS sigma model. In the special case when X_a is the gradient of a scalar ϕ (dilaton) one recovers the standard type I equations equivalent to the 2d Weyl invariance conditions of the superstring sigma model. In the type II case we find a generalized version of the 10d supergravity equations the bosonic part of which was introduced in arXiv:1511.05795. These equations depend on two vectors X_a and K_a subject to 1st order differential relations (with the equations in the NS-NS sector depending only on the combination $X_a = \mathrm{X}_a + K_a$). In the special case of $K_a = 0$ one finds that $\mathrm{X}_a = \partial_a \phi$ and thus obtains the standard type II supergravity equations. New generalized solutions are found if K_a is chosen to be a Killing vector (and thus they exist only if the metric admits an isometry). Non-trivial solutions of the generalized equations describe K-isometric backgrounds that can be mapped by T-duality to type II supergravity solutions with dilaton containing a linear isometry-breaking term. Examples of such backgrounds appeared recently in the context of integrable η-deformations of $AdS_n \times S^n$ sigma models. The classical kappa-symmetry thus does not, in general, imply the 2d Weyl invariance conditions for the GS sigma model (equivalent to type II supergravity equations) but only weaker scale invariance type conditions.

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Contents

1 Introduction and summary

The purpose of this paper is to determine precisely which constraints the presence of the kappa-symmetry of the Green-Schwarz (GS) superstring places on the target space (super) geometry. In an influential 1985 paper [1] Witten showed that the equations of motion of 10d super Yang-Mills theory can be expressed as integrability along light-like lines and showed that this condition is closely connected to kappa-symmetry of the superparticle in the super Yang-Mills background. He also suggested that there should be a similar connection between the kappa-symmetry of the GS superstring and the supergravity equations of motion. Shortly thereafter, the type II GS superstring action in a general supergravity background was written down in [2] and it was shown that the standard on-shell superspace constraints of type IIB supergravity [3] are sufficient for the string to be kappa-symmetric. It was conjectured that these constraints are also necessary for the kappa-symmetry.

In [4] it was found that for the type I superstring the kappa-symmetry implies the basic (i.e. dimension 0 and $-\frac{1}{2}$) superspace constraints on the torsion and 3-form $H = dB$ superfields[1]

$$T_{\alpha\beta}{}^{a} = -i\gamma^{a}_{\alpha\beta}\,, \qquad H_{\alpha\beta\gamma} = 0\,, \qquad H_{a\alpha\beta} = -i(\gamma_a)_{\alpha\beta}\,. \tag{1.1}$$

Ref. [4] also argued that these constraints are enough to make the target space geometry a solution of type I supergravity.[2]

[1] There is also a constraint for the Yang-Mills sector which we will ignore here. In our notation $a, b, \ldots = 0, 1, 2, \ldots, 9$ are bosonic tangent space indices, and $\alpha, \beta, \ldots = 1, \ldots, 16$ are 10d Majorana-Weyl spinor indices.

[2] More recently [5] it was shown that (classical) BRST invariance of the pure spinor superstring [6] (which may be viewed as an analog of kappa-symmetry in this formulation) implies the basic type I and type II constraints. It was argued that these constraints are enough to put the corresponding supergravity background completely on-shell (see, however, note added in section 5).

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While in the 11d case the condition of kappa-symmetry of the supermembrane action [7] leads to a constraint on the torsion which implies that the background should satisfy the standard 11d supergravity equations of motion [8], here we will find (in disagreement with the earlier conjectures/claims) that this is not so in the 10d superstring case: the 10d supergravity equations are sufficient but not necessary for kappa-symmetry.

In the case of the type I GS superstring where the kappa-symmetry implies the basic constraints (1.1), we shall show, by solving completely the Bianchi identities for the torsion and the 3-form, that these constraints actually lead to a weaker set of equations than those of type I supergravity. These equations are similar to the conditions for 1-loop scale invariance of the GS sigma model [9], which are, in general, weaker than the Weyl invariance conditions required to define a consistent superstring theory. This is not totally surprising as the condition of classical kappa-symmetry does not take into account the dilaton term $\int d^2\xi\sqrt{g}R^{(2)}\phi(x)$ required to make the quantum 2d stress tensor traceless (see [10–13] and discussion in [9]).

Indeed, the problem in 10d compared to the 11d case is the presence of the dilaton. The dimension $\frac{1}{2}$ component of the torsion is expressed in terms of a spinor ("dilatino") superfield χ_α. If one requires that χ_α is expressed in terms of a scalar superfield ϕ (the dilaton) as

$$\chi_\alpha = \nabla_\alpha \phi \tag{1.2}$$

then the Bianchi identities for the torsion imply the standard type I supergravity equations [14]. However, if this extra assumption (1.2) (not required for kappa-symmetry) is dropped, the basic constraints (1.1) imply only the equations for a "partially off-shell" generalization of the type I supergravity equations. The solution of the constraints and Bianchi identities then depends on an arbitrary vector X_a (that replaces the dilaton gradient)[3] and the bosonic equations of motion take the form (here fermionic fields are set to zero)

$$R_{ab} + 2\nabla_{(a}X_{b)} - \frac{1}{4}H_{acd}H_b{}^{cd} = 0\,, \tag{1.3}$$

$$\nabla^c H_{abc} - 2X^c H_{abc} - 4\nabla_{[a}X_{b]} = 0\,, \tag{1.4}$$

$$\nabla^a X_a - 2X^a X_a + \frac{1}{12}H^{abc}H_{abc} = 0\,. \tag{1.5}$$

If one restricts to the special case of $X_a = \partial_a\phi$, these equations reduce to the standard type I supergravity equations of motion (or string effective equations in the NS-NS sector). The generalized equations (1.3), (1.4) coincide with the 1-loop scale invariance conditions of a bosonic sigma model $L = (G-B)_{mn}\partial_+ x^m \partial_- x^n$ provided the reparametrization and B-field gauge freedom vectors are chosen to be equal.[4] The conclusion is that the condition of classical kappa-symmetry is essentially equivalent to the one-loop scale invariance condition for

[3] As X_a is subject to a constraint on its divergence we get 8 additional bosonic fields compared to the standard type I theory. These are matched by an extra 8 fermionic components present due to the fact that the dilatino χ is now off-shell whereas in the standard type I supergravity it satisfies a Dirac equation.

[4] In the notation of [9] this means $Y_a = X_a$. This identification is a consequence of the underlying supersymmetry of the equations leading to (1.3)–(1.5). It should come out automatically if the scale invariance of the GS string is studied in the manifestly supersymmetric (superspace) form.

the type I GS sigma model. Only the stronger condition of 2d Weyl invariance (eqs. (1.3)–(1.5) with $X_a = \partial_a \phi$) is equivalent to the standard type I supergravity equations of motion.

Performing a similar analysis in the case of the type IIB GS superstring we will find that the kappa-symmetry implies the direct generalization of the basic constraints (1.1) on the torsion and 3-form[5]

$$T_{\alpha i\,\beta j}{}^{a} = -i\delta_{ij}\gamma^{a}_{\alpha\beta}\,, \qquad H_{\alpha i\,\beta j\,\gamma k} = 0\,, \qquad H_{a\,\alpha i\,\beta j} = -i\sigma^{3}_{ij}(\gamma_a)_{\alpha\beta}\,. \tag{1.6}$$

When the Bianchi identities are solved we will conclude that these constraints lead again not to the type IIB supergravity equations but to a weaker set of generalized type IIB equations involving, instead of the dilaton scalar, two vectors X_a and K_a. The corresponding bosonic equations may be written as (here as in (1.2)–(1.5) the fermionic component fields are set to zero)

$$R_{ab} + 2\nabla_{(a}X_{b)} - \frac{1}{4}H_{acd}H_b{}^{cd} + \frac{1}{128}\mathrm{Tr}(\mathcal{S}\gamma_a\mathcal{S}\gamma_b) = 0\,, \tag{1.7}$$

$$\nabla^c H_{abc} - 2X^c H_{abc} - 4\nabla_{[a}X_{b]} - \frac{1}{64}\mathrm{Tr}(\mathcal{S}\gamma_a\mathcal{S}\gamma_b\sigma^3) = 0\,, \tag{1.8}$$

$$\nabla^a X_a - 2X^a X_a + \frac{1}{12}H^{abc}H_{abc} - \frac{1}{256}\mathrm{Tr}(\mathcal{S}\gamma^a\mathcal{S}\gamma_a) = 0\,, \tag{1.9}$$

$$\gamma^a\nabla_a\mathcal{S} - \gamma^a\mathcal{S}\,(\mathrm{X}_a - \sigma^3 K_a) + \left(\frac{1}{8}\gamma^a\sigma^3\mathcal{S}\gamma^{bc} + \frac{1}{24}\gamma^{abc}\sigma^3\mathcal{S}\right)H_{abc} = 0\,. \tag{1.10}$$

They generalize the type I equations (1.3)–(1.5) to the presence of the analog of the RR field strength bispinor $\mathcal{S} = (\mathcal{S}^{\alpha i\,\beta j})$ (which includes the factor of e^{ϕ} in the standard type IIB case [15])

$$\mathcal{S} = -i\sigma^2\gamma^a\mathcal{F}_a - \frac{1}{3!}\sigma^1\gamma^{abc}\mathcal{F}_{abc} - \frac{1}{2\cdot 5!}i\sigma^2\gamma^{abcde}\mathcal{F}_{abcde}\,. \tag{1.11}$$

Combining (1.7) and (1.9) we get the following generalized "central charge" equation

$$\bar\beta^X \equiv R - \frac{1}{12}H_{abc}H^{abc} + 4\nabla^a X_a - 4X^a X_a = 0\,. \tag{1.12}$$

As was shown in [9], the relation $\partial_a\bar\beta^X = 0$ follows, in fact, from eqs. (1.7), (1.8), (1.10) so that the "dilaton equation" (1.9) is not indepedent.

In the above equations (1.7)–(1.12)

$$X_a \equiv \mathrm{X}_a + K_a\,, \tag{1.13}$$

and the vectors X_a and K_a are subject to

$$\nabla_{(a}K_{b)} = 0\,, \qquad \mathrm{X}^a K_a = 0\,, \tag{1.14}$$

$$2\nabla_{[a}\mathrm{X}_{b]} + K^c H_{abc} = 0\,. \tag{1.15}$$

[5]Here $i,j,k = 1,2$ label the two MW spinors of type IIB superspace and $(\sigma^r)_{ij}$ $(r = 1,2,3)$ are Pauli matrices. The gamma-matrices $\gamma^a_{\alpha\beta}$ and $\gamma_a^{\alpha\beta}$ are 16×16 symmetric 'Weyl blocks' of 10d Dirac matrices satisfying
$\gamma^a_{\alpha\beta}(\gamma^b)^{\beta\gamma} + \gamma^b_{\alpha\beta}(\gamma^a)^{\beta\gamma} = 2\eta^{ab}\delta_\alpha^\gamma$, see [15] for more details on our notation.

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Thus K_a satisfies the Killing vector equation, while eq. (1.15) expresses the fact that the 3-form H is isometric, i.e. the two-form potential B transforms by a gauge transformation under the isometry generated by K_a, $\mathcal{L}_K B = d(i_K B - \mathrm{X})$, where $\mathcal{L}_K = i_K\, d + d\, i_K$ is the Lie derivative. It follows from (1.14), (1.15) that not only the three-form H but also the one-form X respect the isometry, i.e.

$$\mathcal{L}_K H = 0\,, \qquad\qquad \mathcal{L}_K \mathrm{X} = 0\,. \tag{1.16}$$

Furthermore, it follows from the "Bianchi" part of the $\mathcal{S}$ equation (4.14) that the "RR" forms in (1.11) also respect the isometry

$$\mathcal{L}_K \mathcal{F}_{2n+1} = 0\,, \qquad\qquad n = 0, 1, 2\,. \tag{1.17}$$

Thus the whole bosonic background $(G, H, \mathcal{F})$ is K-isometric. This statement can, in fact, be generalized to superspace as we will show in section 4.

Assuming that the B-field may be chosen to be isometric, eq. (1.15) may be explicitly solved as [9] (here m, n are 10d coordinate indices)

$$\mathrm{X}_m = \partial_m \phi - B_{mn} K^n\,, \qquad \text{i.e.} \qquad X_m = \partial_m \phi + (G_{mn} - B_{mn}) K^n\,, \tag{1.18}$$

where ϕ is an arbitary scalar that should also satisfy the isometry condition $K^m \partial_m \phi = 0$ according to (1.14). Eqs. (1.14) and (1.15) always admit the following special solution

$$K_a = 0\,, \qquad\qquad X_a = \mathrm{X}_a = \partial_a \phi\,. \tag{1.19}$$

In that case the generalized system (1.7)–(1.10) reduces to the bosonic sector of the standard type IIB supergravity equations with ϕ being the dilaton.[6] In particular, eq. (1.10) contains both the dynamical equations and the Bianchi identities for the RR field strenghts $F_p = e^{-\phi} \mathcal{F}_p = dC_{p-1} + \ldots$ in (1.11).

The above generalized type IIB equations have of course a straightforward analog in type IIA case — following from kappa-symmetry condition of type IIA GS string. Let us note also that there is a natural generalization of the notion of a supersymmetric solution to the generalized type IIB supergravity equations, namely, the one for which the component fermionic fields as well as their supersymmetry variations vanish, i.e. $\chi_{\alpha i}|_{\theta=0} = \psi^{\alpha i}_{ab}|_{\theta=0} = 0$ and $\epsilon^{\alpha i} \nabla_{\alpha i} \chi_{\beta j}|_{\theta=0} = \epsilon^{\alpha i} \nabla_{\alpha i} \psi^{\beta j}_{ab}|_{\theta=0} = 0$. The latter two equations give the generalization of the dilatino and the (integrability[7] of the) gravitino conditions respectively. Using (3.22) and (A.86) they take the form

$$\Big[(\mathrm{X}_a + \sigma^3 K_a)\gamma^a + \frac{1}{12} H_{abc}\, \sigma^3 \gamma^{abc} + \frac{1}{8} \gamma_a \mathcal{S} \gamma^a\Big] \epsilon = 0\,, \tag{1.20}$$

$$\Big[R_{ab}{}^{cd} \gamma_{cd} + \frac{1}{2} H_{ace} H_{bd}{}^{e}\, \gamma^{cd} - \nabla_{[a} H_{b]cd}\, \sigma^3 \gamma^{cd} - \nabla_{[a} \mathcal{S} \gamma_{b]}$$
$$- \frac{1}{8} (\mathcal{S} \sigma^3 \gamma_{[a} \gamma^{cd} - \gamma^{cd} \sigma^3 \mathcal{S} \gamma_{[a}) H_{b]cd} - \frac{1}{8} \mathcal{S} \gamma_{[a} \mathcal{S} \gamma_{b]}\Big] \epsilon = 0\,. \tag{1.21}$$

[6] See, e.g., appendix A in [16] where the same RR bispinor notation for RR fields is used.

[7] The Killing spinor equation itself takes the form (cf. (4.10)) $(\nabla_a + \frac{1}{8} H_{abc}\, \gamma^{bc} \sigma^3 + \frac{1}{8} \mathcal{S} \gamma_a)\epsilon = 0\,.$

These differ from the standard type IIB supersymmetry conditions only by the replacements $\nabla_a\phi \to \mathrm{X}_a + \sigma^3 K_a$ and $e^\phi F \to \mathcal{F}$ inside the RR bispinor $\mathcal{S}$. It would be interesting to find solutions to these equations with $K_a \neq 0$.

The generalized equations (1.7)–(1.15) are precisely the ones identified in [9] as being satisfied by the target space background of the so-called η-deformation [17–19] of the $AdS_5\times S^5$ superstring model.[8] The resulting picture is thus in perfect agreement with the fact that the η-model is kappa-symmetric [18] but the corresponding background does not satisfy the type IIB equations [19].[9] Further examples of solutions of the generalized type IIB equations (1.7)–(1.15) should be provided by some other η-models [21, 22], as was indeed shown in [22] for the models based on Jordanian R-matrices.

The solution (1.19) is the only possible one if the metric does not admit Killing vectors, i.e. kappa-symmetric GS sigma models with non-isometric metric must correspond to standard type IIB solutions. An example is provided by the λ-deformed model which has kappa-symmetric action [20] with the corresponding metric [23, 24] not admitting any Killing vectors: as was explicitly demonstarted in [16] in the $AdS_2 \times S^2 \times T^6$ case the corresponding λ-deformed background solves the standard type IIB equations.

It was argued in [9] that the above generalized type IIB equations imply the scale-invariance conditions for the GS sigma model. In particular, the 2nd-derivative scale-invariance conditions for the "RR" fields follow immediately upon "squaring" of the Dirac equation for $\mathcal{S}$ in (1.10).[10] Thus non-trivial solutions of the generalized equations with $K_a \neq 0$ should represent UV finite but not Weyl-invariant GS sigma models so their string theory interpretation is a priori unclear.

As follows from the analysis in [9], starting with a type IIA supergravity solution with all the fields being isometric apart from a linear term in the dilaton [25, 26], and performing the standard T-duality transformation on all the fields except the dilaton (i.e. on the GS sigma model on a flat 2d background)[11] then the resulting background should solve precisely the generalized equations (1.7)–(1.15) with K_a and X_a determined by the original dilaton and the metric.[12] The converse should also be true [9]: given a $K_a \neq 0$ solution of the generalized type IIB equations (1.7)–(1.15), its $(G, B, \mathcal{F})$ fields should be related by a T-duality transformation to the fields of the corresponding type IIA supergravity solution with the dilaton containing a linear isometry-breaking term. Thus each solution of the generalized type II system (1.7)–(1.15) can be associated with a particular solution of the standard

[8]The relation to the notation in [9] is $\mathrm{X}_a = Z_a$ and $K_a = I_a$. As in [9], we find that while the NS-NS subset of equations depends on X_a and K_a only through their sum X_a in (1.13), the two vectors enter separately in the RR equations (1.10). While in the NS-NS sector one does not need the orthogonality condition $\mathrm{X}_a K^a = 0$, this condition was, in fact, imposed in [9] once the RR fields were included (see eq. (5.37) there).

[9]This assumes that the action and kappa-symmetry transformations of the η-model are the same as those of the Green-Schwarz string. This can be shown to be the case and is also true for the λ-model of [20] upon integrating out the superalgebra-valued 2d gauge field.

[10]In general, the equations (1.7)–(1.10) are thus somewhat stronger than the scale invariance conditions, but still not sufficient to imply the Weyl invariance unless $K_a = 0$.

[11]In general, T-duality of GS sigma model on a flat 2d background should preserve its kappa-symmetry [27] and should be expected not take one out of the class of solutions of the generalized equations.

[12]In particular, the resulting K_a is then proportional to the derivative of the dilaton along the non-isometric direction [9].

type II supergravity equations. This observation may help understanding if it is possible to relate a solution of the generalized type II equations with a consistent string theory.

We shall start in section 2 with a derivation of the type I and type IIB constraints (1.1) and (1.6) on the superspace torsion and 3-form H that follow from the condition of kappa-symmetry for the GS superstring in a non-trivial background. The solution of the Bianchi identities supplemented by these basic constraints leads, as will be described in section 3, to the generalized equations of motion (1.3)–(1.5) and (1.7)–(1.15). In section 4 we shall present a superspace formulation of the equations on K_a and X_a and invariance conditions and superspace Bianchi identities for the "RR" form fields. Some concluding remarks will be made in section 5. Details of the solution of the superspace Bianchi identities will be provided in an appendix.

2 Constraints from kappa-symmetry

The classical GS superstring action in an arbitrary super-background is (in the Nambu-Goto form) [2]

$$S=\int d^2\xi\,\sqrt{-G}-\int_\Sigma B\,,\qquad\qquad G=\det G_{IJ}\,,\tag{2.1}$$

where ξ^I $(I,J=0,1)$ are worldsheet coordinates, G_{IJ} is the induced metric

$$G_{IJ}=E_I{}^aE_J{}^b\eta_{ab}\,,\qquad E_I{}^A=\partial_Iz^ME_M{}^A(z)\,,\qquad z^M=(x^m,\,\theta^\mu)\,,\tag{2.2}$$

while B is the pull-back of a superspace two-form. This action is required to be invariant under the following kappa-symmetry transformations of the coordinates z^M

$$\delta_\kappa z^ME_M{}^a=0\,,\qquad\delta_\kappa z^ME_M{}^{\alpha i}=\frac{1}{2}(1+\Gamma)^{\alpha i}{}_{\beta j}\kappa^{\beta j}\,,\qquad\Gamma=\frac{1}{2\sqrt{-G}}\varepsilon^{IJ}E_I{}^aE_J{}^b\gamma_{ab}\sigma^3\,,\tag{2.3}$$

where we have written the expressions appropriate to type IIB superspace. In the type IIA case the Pauli matrix σ^3 is replaced by Γ_{11} while in the type I case one is to keep only the $i=1$ component. The operator Γ is traceless and satisfies the projector condition $\Gamma^2=1$ so this symmetry removes half of the fermionic components.[13]

The requirement that the string action be invariant under the above transformations imposes constraints on the background. We will now determine what these basic constraints are and in the next section we will work out all their consequences. Varying the action we find

$$\delta_\kappa S=-\int d^2\xi\,\delta_\kappa z^ME_M{}^{\alpha i}\left[\sqrt{-G}\,G^{IJ}E_I{}^aE_J{}^CT_{C\alpha i}{}^b\eta_{ab}+\frac{1}{2}\varepsilon^{IJ}E_I{}^CE_J{}^BH_{BC\alpha i}\right],\tag{2.4}$$

where $T^A=dE^A+E^B\wedge\Omega_B{}^A$ is torsion and $H=dB$. Note that the term involving the super-connection $\Omega_B{}^A$ does not contribute to (2.4) due to it being valued in SO(1,9) (i.e. $\Omega_{\alpha i}{}^b=0$ and $\Omega^{ab}=\Omega^{[ab]}$), it is nevertheless convenient to write the kappa-symmetry conditions covariantly in terms of T^A rather than dE^A. Since the (pulled-back) supervielbeins

[13] The origin of kappa-symmetry is best understood via embedding a worldvolume superspace in a target superspace. This so-called superembedding formalism is reviewed in [28].

are assumed to be independent fields and since the projector Γ only involves the bosonic supervielbeins, eq. (2.4) implies the following conditions

$$H_{\beta j\gamma k\alpha i}(1+\Gamma)^{\alpha i}{}_{\delta l}=0\,, \tag{2.5}$$

$$E_I{}^a\left[\sqrt{-G}\,G^{IJ}T_{\alpha i\beta ja}-\varepsilon^{IJ}H_{a\alpha i\beta j}\right](1+\Gamma)^{\alpha i}{}_{\gamma k}=0\,, \tag{2.6}$$

$$E_I{}^aE_J{}^b\left[\sqrt{-G}\,G^{IJ}T_{\alpha iab}+\frac{1}{2}\varepsilon^{IJ}H_{\alpha iab}\right](1+\Gamma)^{\alpha i}{}_{\beta j}=0\,. \tag{2.7}$$

The third condition turns out to be implied by the first two, see eq. (3.14) in the next section.

Since the two terms in (2.5) come with different powers of the induced metric, and since the components of H cannot depend on the induced metric if H is to have a target space interpretation, the this equation implies the vanishing of the dimension $-\frac{1}{2}$ component of the 3-form H

$$H_{\alpha i\beta j\gamma k}=0\,. \tag{2.8}$$

To solve the second condition (2.6) for the dimension 0 torsion and 3-form components we parametrize these as

$$T_{\alpha i\beta j}{}^a=s^1_{ij}\gamma^b_{\alpha\beta}t_b{}^a+s^2_{ij}\gamma^{bcdef}_{\alpha\beta}t_{bcdef}{}^a+\varepsilon_{ij}\gamma^{bcd}_{\alpha\beta}t_{bcd}{}^a\,, \tag{2.9}$$

$$H_{a\alpha i\beta j}=s^3_{ij}\gamma^b_{\alpha\beta}h_{ba}+s^4_{ij}\gamma^{bcdef}_{\alpha\beta}h_{bcdefa}+\varepsilon_{ij}\gamma^{bcd}_{\alpha\beta}h_{bcda}\,, \tag{2.10}$$

where s^p_{ij} are constant symmetric matrices and t and h are tensor superfields. Tracing (2.6) with γ^a and multiplying with G_{JK} we find the condition

$$\begin{aligned}s^1_{ij}\sqrt{-G}E_K{}^bt_{ab}&-(s^3\sigma^3)_{ij}\frac{(\varepsilon G)^L{}_K\varepsilon^{IJ}}{\sqrt{-G}}E_L{}^cE_I{}^bE_{Ja}h_{bc}-3\sigma^1_{ij}\frac{(\varepsilon G)^L{}_K\varepsilon^{IJ}}{\sqrt{-G}}E_L{}^dE_I{}^bE_J{}^ch_{abcd}\\&-s^3_{ij}\varepsilon^{IJ}G_{JK}E_I{}^bh_{ab}+(s^1\sigma^3)_{ij}\varepsilon^{IJ}E_K{}^cE_I{}^bE_{Ja}t_{bc}+3\sigma^1_{ij}\varepsilon^{IJ}E_K{}^dE_I{}^bE_J{}^ct_{abcd}=0\,.\end{aligned} \tag{2.11}$$

The first three terms and last three terms here have to cancel independently since they come with different powers of the bosonic supervielbeins. The requirement that the last three terms cancel gives

$$\varepsilon^{IJ}E_I{}^bE_J{}^cE_K{}^d\left(s^3_{ij}h_{ab}\eta_{cd}+(s^1\sigma^3)_{ij}\eta_{ab}t_{cd}-3\sigma^1_{ij}t_{abcd}\right)=0\,, \tag{2.12}$$

implying that[14]

$$s^3_{ij}h_{a[b}\eta_{c]d}+(s^1\sigma^3)_{ij}(\eta_{a[b}t_{c]d}-\eta_{a[b}t_{cd]})-3\sigma^1_{ij}(t_{abcd}-t_{a[bcd]})=0\,. \tag{2.13}$$

After a little bit of algebra the solution is found to be

$$s^3=s^1\sigma^3\,,\qquad t_{ab}=-h_{ab}=-i\eta_{ab}\,,\qquad t_{abcd}=t_{[abcd]}\,, \tag{2.14}$$

[14]Note that the part anti-symmetric in $[bcd]$ vanishes trivially due to the fact that the world-sheet indices I,J,K only take two values.

where we have used the freedom to rescale the fermionic supervielbeins to normalize t_{ab}. The same freedom allows us to set $s^1_{ij} = \delta_{ij}$. The vanishing of the first three terms in (2.11) then gives also

$$h_{abcd} = h_{[abcd]} \,. \tag{2.15}$$

Tracing (2.6) with γ^{abc} and γ^{abcdef} and using the above conditions we find also that the components of t and h fields with more than two indices must vanish.

We conclude therefore that the kappa-symmetry of the type IIB GS string action implies, in addition to (2.8), the standard dimension 0 superspace constraints

$$T_{\alpha i \beta j}{}^a = -i\delta_{ij}\gamma^a_{\alpha\beta} \,, \qquad H_{a\alpha i\beta j} = -i\sigma^3_{ij}(\gamma_a)_{\alpha\beta} \,. \tag{2.16}$$

The type IIA cases can be analysed similarly. The constraints in the type I case (1.1) are obtained by keeping only the $i = j = 1$ components in the type IIB ones.

The next step is to determine the consequences of these constraints by solving the superspace Bianchi identities for the torsion and the 3-form H. This will lead us to the generalized supergravity equations described in the Introduction.

3 Generalized equations from Bianchi identities and constraints

Our aim will be to find the most general solution to the 10d superspace Bianchi identities for the torsion and 3-form consistent with the dimension $-\frac{1}{2}$ (2.8) and dimension 0 (2.16) constraints following from kappa-symmetry of the GS string. We will consider the type I and type IIB cases in parallel and present the summary of the results while details will be provided in the appendix.

Let us first recall the basic superspace conventions we will need. The torsion satisfies the Bianchi identity[15]

$$\nabla T^A = E^B \wedge R_B{}^A \,, \qquad T^A = \nabla E^A \equiv dE^A + E^B \wedge \Omega_B{}^A \,, \tag{3.1}$$

where $R_B{}^A$ is the curvature superfield 2-form

$$R_B{}^A = d\Omega_B{}^A + \Omega_B{}^C \wedge \Omega_C{}^A \,, \qquad \nabla R_B{}^A = 0 \,. \tag{3.2}$$

As follows from the fact that the structure group is SO(1, 9), the non-zero components of the curvature are $R_a{}^b$ and

$$\textbf{Type I:} \quad R_\alpha{}^\beta = -\frac{1}{4}R^{ab}(\gamma_{ab})^\beta{}_\alpha \,, \qquad \textbf{Type IIB:} \quad R_{\alpha i}{}^{\beta j} = -\frac{1}{4}R^{ab}\delta_{ij}(\gamma_{cd})^\beta{}_\alpha \,. \tag{3.3}$$

In components, the torsion and curvature Bianchi identities in (3.1) and (3.2) take the form

$$\nabla_{[A}T_{BC]}{}^D + T_{[AB}{}^E T_{|E|C]}{}^D = R_{[ABC]}{}^D \,, \tag{3.4}$$

[15]Our conventions are such that d acts from the right and the components of forms are defined as

$$\omega^{(n)} = \frac{1}{n!}E^{A_n} \wedge \cdots \wedge E^{A_1}\omega_{A_1\cdots A_n} \,.$$

$$\nabla_{[A} R_{BC]D}{}^E + T_{[AB}{}^F R_{|F|C]D}{}^E = 0\,. \tag{3.5}$$

A useful fact is that the curvature Bianchi identities are a consequence of the torsion Bianchi identities.[16] This means that we only need to solve the torsion Bianchi identities.

We also have to solve the Bianchi identity for the 3-form $dH = 0$, or, in components,

$$\nabla_{[A} H_{BCD]} + \frac{3}{2} T_{[AB}{}^E H_{|E|CD]} = 0\,. \tag{3.6}$$

There is some freedom in how one presents the constraints. We will write them in essentially the same form as the type II constraints of [15], which is particularly simple, rather than, for example, in the form of the type I constraints used in [14]. The details of the solution to the Bianchi identities are given in appendix.[17] We shall discuss the consequences of the Bianchi identities and constraints in order of increasing dimension of the component superfields.

Dimension -$\frac{1}{2}$. As we have seen above, kappa-symmetry of the string implies the vanishing of the dimension $-\frac{1}{2}$ component (2.8) of the 3-form, i.e.

$$\textbf{Type I:} \quad H_{\alpha\beta\gamma} = 0 \qquad \textbf{Type IIB:} \quad H_{\alpha i \beta j \gamma k} = 0\,. \tag{3.7}$$

Dimension 0. Kappa-symmetry of the string also requires the standard dimension 0 torsion and 3-form constraints (2.16)

$$\textbf{Type I:} \quad T_{\alpha\beta}{}^a = -i\gamma^a_{\alpha\beta}\,, \qquad H_{a\alpha\beta} = -i(\gamma_a)_{\alpha\beta}\,, \tag{3.8}$$

$$\textbf{Type IIB:} \quad T_{\alpha i \beta j}{}^a = -i\delta_{ij}\gamma^a_{\alpha\beta}\,, \qquad H_{a\alpha i \beta j} = -i\sigma^3_{ij}(\gamma_a)_{\alpha\beta}\,. \tag{3.9}$$

These are consistent with the dimension 0 Bianchi identity and the vanishing of the dimension $-\frac{1}{2}$ component of H.

Dimension $\frac{1}{2}$. Let us start with the type I case. For the torsion we shall require that

$$T_{\alpha[bc]} = 0\,, \tag{3.10}$$

which just serves to fix the corresponding component of the spin connection, $\Omega_\alpha{}^{bc}$. By redefining the frame fields we can also arrange that[18]

$$(\gamma^b)^{\alpha\beta} T_{\beta bc} = 0\,. \tag{3.11}$$

The torsion Bianchi identity we have to solve reads

$$T_{(\alpha\beta}{}^\delta \gamma^d_{\gamma)\delta} - \gamma^e_{(\alpha\beta} T_{\gamma)e}{}^d = 0\,, \tag{3.12}$$

[16]The proof goes as follows [29]. Taking the covariant derivative of (3.1) gives $E^B \wedge \nabla R_B{}^A = 0$ and using the fact that the indices belong to the structure group SO(1, 9) this implies $E^b \wedge \nabla R_b{}^a = 0$ and $(\gamma_{ab} E)^{\alpha i} \wedge \nabla R^{ab} = 0$. Analyzing the components of these equations it is not hard to see that they imply the curvature Bianchi identity $\nabla R_B{}^A = 0$.

[17]In appendix we analyze also a more general case when in the type I case one imposes only the torsion constraint in (1.1).

[18]Taking $E' = E + iuE^b\gamma_c T_b{}^c + ivE^b\gamma_b T_c{}^c$ gives $T'_{(bc)} = T_{(bc)} - u\gamma_{(c}\gamma_d T_{b)}{}^d - v\eta_{bc} T_a{}^a$. This implies $\gamma^b T'_{(bc)} = (1-6u)\gamma^b T_{(bc)} + (\frac{1}{2}u - v)\gamma_c T_b{}^b$ which vanishes for a suitable choice of the constants u, v.

where we used the form of the dimension 0 torsion component. With some work one can show that this, together with the Bianchi identity for the three-form, finally implies

$$\textbf{Type I:} \qquad H_{\alpha bc} = 0\,, \qquad T_{\alpha b}{}^{c} = 0\,, \qquad T_{\alpha\beta}{}^{\gamma} = 2\delta^{\gamma}_{(\alpha}\chi_{\beta)} - \gamma^{a}_{\alpha\beta}(\gamma_a\chi)^{\gamma}\,, \tag{3.13}$$

where χ_α is some MW spinor superfield.

For the type IIB case a similar analysis gives the following conditions

$$\begin{aligned}\textbf{Type IIB:} \quad & T_{\alpha i b}{}^{c} = 0\,, \qquad H_{\alpha i bc} = 0\,, \\ & T_{\alpha i\beta j}{}^{\gamma k} = \delta^{\gamma k}_{(\alpha i}\chi_{\beta j)} + (\sigma^3\delta)^{\gamma k}_{(\alpha i}(\sigma^3\chi)_{\beta j)} - \frac{1}{2}\delta_{ij}\gamma^{a}_{\alpha\beta}(\gamma_a\chi)^{\gamma k} - \frac{1}{2}\sigma^3_{ij}\gamma^{a}_{\alpha\beta}(\gamma_a\sigma^3\chi)^{\gamma k}\,,\end{aligned} \tag{3.14}$$

where $\chi_{\alpha i}$ are some two MW spinor superfields.

Dimension 1. We shall impose the standard requirement

$$T_{ab}{}^{c} = 0\,, \tag{3.15}$$

which fixes the remaining components $\Omega_c{}^{ab}$ of the spin connection. The type I torsion Bianchi identities we need to solve are then

$$-2iT_{c(\alpha}{}^{\gamma}\gamma^{d}_{\beta)\gamma} = R_{\alpha\beta c}{}^{d}\,, \qquad \nabla_{(\alpha}T_{\beta\gamma)}{}^{\delta} + T_{(\alpha\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta} - i\gamma^{a}_{(\alpha\beta}T_{|a|\gamma)}{}^{\delta} = R_{(\alpha\beta\gamma)}{}^{\delta}\,. \tag{3.16}$$

The Bianchi identity for the 3-form imposes the condition

$$T_{\alpha\beta}{}^{e}H_{cde} + 2T_{c(\alpha}{}^{\gamma}H_{\beta)\gamma d} - 2T_{d(\alpha}{}^{\gamma}H_{\beta)\gamma c} = 0\,. \tag{3.17}$$

After some algebra one obtains the solution as

$$T_{a\alpha}{}^{\delta} = \frac{1}{8}(\gamma^{bc})_{\alpha}{}^{\delta}H_{abc}\,, \qquad R_{\alpha\beta}{}^{cd} = \frac{i}{2}(\gamma_b)_{\alpha\beta}H^{bcd}\,. \tag{3.18}$$

In addition, one finds that the derivative of the spinor superfield χ in (3.13) should be given by

$$\textbf{Type I:} \qquad \nabla_\alpha\chi_\beta = \chi_\alpha\chi_\beta + \frac{i}{2}\gamma^{a}_{\alpha\beta}X_a - \frac{i}{24}\gamma^{abc}_{\alpha\beta}H_{abc}\,, \tag{3.19}$$

where X_a is some vector superfield.[19]

In the type IIB case we find, by a similar analysis,

$$\textbf{Type IIB:} \qquad T_{a\alpha i}{}^{\delta j} = \frac{1}{8}(\gamma^{bc}\sigma^3)_{\alpha i}{}^{\delta j}H_{abc} + \frac{1}{8}(\gamma_a\mathcal{S})_{\alpha i}{}^{\delta j}\,, \tag{3.20}$$

$$R_{\alpha i\beta j}{}^{cd} = \frac{i}{2}(\gamma_b\sigma^3)_{\alpha i\beta j}H^{bcd} - \frac{i}{4}(\gamma^{[c}\mathcal{S}\gamma^{d]})_{\alpha i\beta j}\,, \tag{3.21}$$

$$\begin{aligned}\nabla_{\alpha i}\chi_{\beta j} = {}& \frac{1}{2}\chi_{\alpha i}\chi_{\beta j} + \frac{1}{2}(\sigma^3\chi)_{\alpha i}(\sigma^3\chi)_{\beta j} + \frac{i}{2}\gamma^{a}_{\alpha\beta}(\delta_{ij}\mathrm{X}_a + \sigma^3_{ij}K_a) \\ & - \frac{i}{24}\sigma^3_{ij}\gamma^{abc}_{\alpha\beta}H_{abc} - \frac{i}{16}(\gamma_a\mathcal{S}\gamma^a)_{\alpha i\beta j}\,,\end{aligned} \tag{3.22}$$

[19] While as superfields χ_α and X_a are of course related, their first components will be independent fields entering the generalized equations. We will use same notation for superfields and their lowest components, with the interpretation being hopefully clear from the context.

where X_a and K_a are some vector superfields. $\mathcal{S} = (\mathcal{S}^{\alpha i,\beta j})$ is an anti-symmetric 32×32 matrix which is off-diagonal in i, j and can therefore be represented as

$$\mathcal{S} = -i\sigma^2\gamma^a\mathcal{F}'_a - \frac{1}{3!}\sigma^1\gamma^{abc}\mathcal{F}'_{abc} - \frac{1}{2\cdot 5!}i\sigma^2\gamma^{abcde}\mathcal{F}'_{abcde}\,, \tag{3.23}$$

for some p-form superfields $\mathcal{F}'_p$.[20]

Dimension $\frac{3}{2}$. The type I torsion Bianchi identities to solve at dimension $\frac{3}{2}$ are

$$-i\gamma^d_{\alpha\beta}T_{bc}{}^\beta = 2R_{\alpha[bc]}{}^d\,, \tag{3.24}$$

$$\nabla_a T_{\beta\gamma}{}^\delta - 2\nabla_{(\beta}T_{|a|\gamma)}{}^\delta + 2T_{a(\beta}{}^\epsilon T_{\gamma)\epsilon}{}^\delta - T_{\beta\gamma}{}^\epsilon T_{a\epsilon}{}^\delta - i\gamma^e_{\beta\gamma}T_{ea}{}^\delta = 2R_{a(\beta\gamma)}{}^\delta\,. \tag{3.25}$$

The first one is easily solved for the curvature as

$$R_{\alpha bcd} = \frac{i}{2}(\gamma_b\psi_{cd})_\alpha - i(\gamma_{[c}\psi_{d]b})_\alpha\,, \tag{3.26}$$

where $T_{ab}{}^\beta = \psi^\beta_{ab}$ is the gravitino field strength. Using this in (3.25) one finds after a bit of algebra that the solution is

$$\textbf{Type I:}\qquad \nabla_\alpha H_{abc} = 3i(\gamma_{[a}\psi_{bc]})_\alpha\,, \qquad\qquad i(\gamma^b\psi_{ab})_\alpha = 2\nabla_a\chi_\alpha + \frac{1}{4}(\gamma^{bc}\chi)_\alpha H_{abc}\,. \tag{3.27}$$

This solves the Bianchi identities but we must also remember the consistency conditions which follow from the equation for $\nabla_\alpha\chi_\beta$ in (3.19). Taking another spinor derivative of this equation and symmetrizing we find an expression for the spinor derivative of X_a

$$\nabla_\alpha X_a = \frac{1}{2}(\gamma^b\gamma_a\nabla_b\chi)_\alpha + (\gamma_a\gamma^b\chi)_\alpha X_b + \frac{1}{48}(\gamma_a\gamma^{bcd}\chi)_\alpha H_{bcd} + \frac{1}{8}(\gamma^{bc}\chi)_\alpha H_{abc}\,. \tag{3.28}$$

A similar analysis in the type IIB case gives the following superfield relations

$$\textbf{Type IIB:}\quad i(\gamma^b\psi_{ab})_{\alpha i} = 2\nabla_a\chi_{\alpha i} + \frac{1}{4}(\gamma^{bc}\sigma^3\chi)_{\alpha i}H_{abc}\,,$$

$$\nabla_{\alpha i}H_{abc} = 3i(\gamma_{[a}\sigma^3\psi_{bc]})_{\alpha i}\,, \tag{3.29}$$

$$R_{\alpha ibcd} = \frac{i}{2}(\gamma_b\psi_{cd})_{\alpha i} - i(\gamma_{[c}\psi_{d]b})_{\alpha i}\,, \tag{3.30}$$

$$\begin{aligned}\nabla_{\alpha i}\mathcal{S}^{\beta 1\gamma 2} = {} & \mathcal{S}^{\beta 1\gamma 2}\chi_{\alpha i} - 2\delta^{[\beta 1}_{\alpha i}(\mathcal{S}\chi)^{\gamma 2]} + 2(\gamma^a\mathcal{S})_{\alpha i}{}^{[\beta 1}(\gamma_a\chi)^{\gamma 2]} \\ & + 4i(\gamma^{ab})^{[\beta 1}{}_{\alpha i}\psi^{\gamma 2]}_{ab}\,,\end{aligned} \tag{3.31}$$

$$\begin{aligned}\nabla_{\alpha i}\mathrm{X}_a = {} & \nabla_a\chi_{\alpha i} - \frac{1}{4}(\gamma_a\gamma^b\nabla_b\chi)_{\alpha i} + \frac{1}{2}\big(\gamma_a\gamma^b(\mathrm{X}_b + \sigma^3K_b)\chi\big)_{\alpha i} \\ & + \frac{1}{8}(\gamma^{bc}\sigma^3\chi)_{\alpha i}H_{abc} + \frac{1}{96}(\gamma_a\gamma^{bcd}\sigma^3\chi)_{\alpha i}H_{bcd} + \frac{1}{16}(\gamma_a\mathcal{S}\chi)_{\alpha i}\,,\end{aligned} \tag{3.32}$$

$$\begin{aligned}\nabla_{\alpha i}K_a = {} & -\frac{1}{4}(\gamma_a\gamma^b\sigma^3\nabla_b\chi)_{\alpha i} + \frac{1}{2}\big(\gamma_a\gamma^b\sigma^3(\mathrm{X}_b + \sigma^3K_b)\chi\big)_{\alpha i} \\ & + \frac{1}{96}(\gamma_a\gamma^{bcd}\chi)_{\alpha i}H_{bcd} - \frac{1}{16}(\gamma_a\sigma^3\mathcal{S}\chi)_{\alpha i}\,.\end{aligned} \tag{3.33}$$

[20] The reason for the primes on $\mathcal{F}_p$ will become clear in the next section (the lowest components of $\mathcal{F}'_p$ and $\mathcal{F}_p$ will differ only by bilinear fermionic terms).

Dimension 2. In the type I case the torsion Bianchi identities read

$$R_{[abc]}{}^d = 0\,, \qquad \nabla_\alpha T_{bc}{}^\beta + 2\nabla_{[b}T_{c]\alpha}{}^\beta + 2T_{[b|\alpha|}{}^\gamma T_{c]\gamma}{}^\beta + T_{bc}{}^\gamma T_{\gamma\alpha}{}^\beta = R_{bc\alpha}{}^\beta\,. \tag{3.34}$$

They determine the spinor derivative of the gravitino field strength superfield

$$\nabla_\alpha \psi^\beta_{ab} = \frac{1}{8}(\gamma^{cd})^\beta{}_\alpha\big(2\nabla_{[a}H_{b]cd} + H_{eca}H^e{}_{bd} - 2R_{abcd}\big) - \delta^\beta_\alpha \psi_{ab}\chi - \psi^\beta_{ab}\chi_\alpha + (\gamma^c\psi_{ab})_\alpha(\gamma_c\chi)^\beta\,. \tag{3.35}$$

We are finally ready to derive the equations of motion for the bosonic superfields. Contracting (3.35) with γ^a and using (3.27) gives the equations

Type I:

$$\nabla_{[a}H_{bcd]} = 0\,, \qquad R_{a[bcd]} = 0\,, \tag{3.36}$$

$$\nabla^c H_{abc} - 4\nabla_{[a}X_{b]} - 2X^c H_{abc} - 4\psi_{ab}\chi = 0\,, \quad R_{ab} + 2\nabla_{(a}X_{b)} - \frac{1}{4}H_{acd}H_b{}^{cd} = 0\,, \tag{3.37}$$

where $R_{ab} = R_{ac}{}^c{}_b$. Evaluating $\nabla_{(\alpha}\nabla_{\beta)}X_a$ and using (3.28) we find also

$$\nabla^a X_a - 2X^a X_a + \frac{1}{12}H^{abc}H_{abc} + 2i\chi\gamma^a\nabla_a\chi - \frac{i}{12}\chi\gamma^{abc}\chi\, H_{abc} = 0\,. \tag{3.38}$$

The lowest components of these superfield equations give us the generalized type I equations (1.3)–(1.5) discussed in the Introduction (where fermionic components were set to zero).

In the type IIB case one finds the fermionic equation (A.86) together with the following equations for the bosonic superfields[21]

Type IIB:

$$R_{a[bcd]} = 0\,, \qquad \nabla_{[a}H_{bcd]} = 0\,, \tag{3.39}$$

$$2\nabla_{[a}\mathrm{X}_{b]} + K^c H_{abc} + \psi_{ab}\chi = 0\,, \qquad \nabla_{(a}K_{b)} = 0\,, \tag{3.40}$$

$$K^a\mathrm{X}_a - \frac{i}{4}\chi\gamma^a\sigma^3\nabla_a\chi + \frac{i}{96}\chi\gamma^{abc}\chi\, H_{abc} = 0\,, \tag{3.41}$$

$$R_{ab} + 2\nabla_{(a}\mathrm{X}_{b)} - \frac{1}{4}H_{ade}H_b{}^{de} + \frac{1}{128}\mathrm{Tr}(\mathcal{S}\gamma_a\mathcal{S}\gamma_b) = 0\,, \tag{3.42}$$

$$\nabla^c H_{abc} - 2\mathrm{X}^c H_{abc} - 4\nabla_{[a}K_{b]} - \frac{1}{64}\mathrm{Tr}(\mathcal{S}\gamma_a\mathcal{S}\gamma_b\sigma^3) - 2\psi_{ab}\sigma^3\chi = 0\,, \tag{3.43}$$

$$\begin{aligned}\nabla^a\mathrm{X}_a - 2\mathrm{X}^a\mathrm{X}_a - 2K^aK_a + \frac{1}{12}H^{abc}H_{abc}&\\ -\frac{1}{256}\mathrm{Tr}(\mathcal{S}\gamma^a\mathcal{S}\gamma_a) + i\chi\gamma^a\nabla_a\chi - \frac{i}{24}\chi\gamma^{abc}\sigma^3\chi\, H_{abc} &= 0\,,\end{aligned} \tag{3.44}$$

$$\begin{aligned}(\gamma^a\nabla_a\mathcal{S})_{\alpha i}{}^{\beta j} - \big(\gamma^a(\mathrm{X}_a + \sigma^3K_a)\mathcal{S}\big)_{\alpha i}{}^{\beta j} + \left[\frac{1}{8}(\gamma^a\sigma^3\mathcal{S}\gamma^{bc})_{\alpha i}{}^{\beta j} + \frac{1}{24}(\gamma^{abc}\sigma^3\mathcal{S})_{\alpha i}{}^{\beta j}\right]H_{abc}&\\ + i\chi_{\alpha i}(\mathcal{S}\chi)^{\beta j} - i(\sigma^3\chi)_{\alpha i}(\sigma^3\mathcal{S}\chi)^{\beta j} + 2(\gamma^{cd}\chi)_{\alpha i}\psi^{\beta j}_{cd} - 2(\gamma^{cd}\sigma^3\chi)_{\alpha i}(\sigma^3\psi_{cd})^{\beta j} &= 0\,.\end{aligned} \tag{3.45}$$

These are the generalized type IIB equations implied by the kappa-symmetry of the GS string (generalizing (1.7)–(1.10) where fermions were set to zero). One can show that they reduce to the standard type IIB supergravity equations in the special case of $K_a = 0$.

[21] Here the covariant derivatives (e.g. in (3.40)) contain fermionic terms so, e.g., $K_m = 0$, $\mathrm{X}_m = \partial_m\phi$ is always a solution even for non-zero fermionic fields.

4 Lifting the Killing vector and IIB form fields to superspace

The generalized type IIB equations in the previous section can be formulated in a geometrical way in superspace by lifting the Killing vector field K_a and the form fields $\mathcal{F}_p$ to superspace vector field and superspace forms. We begin with the one-form with 10d coordinate components X_m and lift it to a one-form $\mathrm{X} = dz^M \mathrm{X}_M$ in superspace. We must then constrain the extra spinor component not to introduce extra degrees of freedom. This is done by imposing the constraint

$$\mathrm{X}_{\alpha i} = \chi_{\alpha i} \,. \tag{4.1}$$

The equation for $\nabla_{[a}\mathrm{X}_{b]}$ in (3.40) as well as the equation (3.32) for $\nabla_{\alpha i}\mathrm{X}_a$ and the equation for $\nabla_{(\alpha i}\chi_{\beta j)}$ in (3.22) are then all summarized by the "superspace Bianchi identity"

$$d\mathrm{X} + i_K H = 0 \qquad \Leftrightarrow \qquad \mathcal{L}_K B = d(i_K B - \mathrm{X}) \,, \tag{4.2}$$

or, in components,

$$2\nabla_{[A}\mathrm{X}_{B]} + T_{AB}{}^C \mathrm{X}_C = -K^C H_{ABC} \,. \tag{4.3}$$

This equation says that B transforms by a gauge transformation under the superisometries generated by $K^A = (K^a, \Xi^{\alpha i})$, where the Killing spinor superfield $\Xi^{\alpha i}$ is set to be

$$\Xi = \frac{i}{4}\Big(\gamma^a \nabla_a - 2\gamma^a\, \mathrm{X}_a - 2\gamma^a \sigma^3\, K_a - \frac{1}{24}\gamma^{abc}\sigma^3\, H_{abc} - \frac{1}{4}\mathcal{S}\Big)\sigma^3 \chi \,. \tag{4.4}$$

This definition together with (3.41) implies that

$$i_K \mathrm{X} = 0 \,. \tag{4.5}$$

Using (4.2) we then conclude that (super)isometries generated by K leave X invariant (the rotation matrix $L_A{}^B$ is defined below)

$$\mathcal{L}_K \mathrm{X} = 0 \,, \qquad \text{i.e.} \qquad K^C \nabla_C \mathrm{X}_A + L_A{}^B \mathrm{X}_B + i_K \Omega_A{}^B \mathrm{X}_B = 0 \,. \tag{4.6}$$

Indeed, the superspace vector field K^A satisfies the superspace Killing equation (see, for example, [30])

$$E^B L_B{}^A = \mathcal{L}_K E^A = \nabla K^A + i_K T^A - E^B i_K \Omega_B{}^A \,. \tag{4.7}$$

This equation expresses the fact that under the superisometry generated by the vector superfield K^A the frame E^A transforms by a local Lorentz transformation with the parameter $L_B{}^A = (L_b{}^a, \frac{1}{4} L_{ab} (\gamma^{ab})_{\beta j}{}^{\alpha i})$. The component form of (4.7) is

$$\nabla_B K^A + K^C T_{CB}{}^A = L_B{}^A + i_K \Omega_B{}^A \,. \tag{4.8}$$

Taking the parameter of the local Lorentz transformation to be

$$L_{ab} = \nabla_{[a} K_{b]} - i_K \Omega_{ab} \,, \tag{4.9}$$

one gets, from the (ab) component of (4.8), the standard Killing vector equation $\nabla_{(a}K_{b)} = 0$. The $(a\beta j)$-component gives the equation (3.33) for $\nabla_{\alpha i} K_a$. The $(\alpha i \beta j)$ component

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implies the equation (3.45) for $\mathcal{S}$ (except for the γ^{abcd} part), and also the equation of motion (3.43) for B, the equation (3.44) for the divergence of X_a, as well as the constraint (3.41) on $K^a\mathrm{X}_a$. To show this requires using the equation for the spinor derivative of the bosonic fields and the gravitino equation of motion. Finally, the (αib) component of (4.8) is the superspace Killing spinor equation

$$\nabla_b\Xi^{\alpha i} + \frac{1}{8}(\gamma^{cd}\sigma^3\Xi)^{\alpha i}\, H_{bcd} + \frac{1}{8}(\mathcal{S}\gamma_b\Xi)^{\alpha i} - K^c\psi_{bc}{}^{\alpha i} = 0\,, \tag{4.10}$$

and its lowest component is the usual Killing spinor equation.[22]

Finally, we can also lift to superspace the form fields appearing in the bispinor $\mathcal{S}$ in (3.23) setting there

$$\mathcal{F}'_{a_1\cdots a_n} = \mathcal{F}_{a_1\cdots a_n} + i\chi^1\gamma_{a_1\cdots a_n}\chi^2\,. \tag{4.11}$$

This works almost identically the same as for the standard type IIB supergravity theory where $\mathcal{F}_p$ are the RR field strengths multiplied by e^ϕ [15]. Imposing the following constraints on their dimension 0 and dimension $\frac{1}{2}$ components

$$\mathcal{F}_{\alpha i\beta jc} = i\sigma^1_{ij}(\gamma_c)_{\alpha\beta}\,, \qquad \mathcal{F}_{\alpha i\beta jcde} = -\sigma^2_{ij}(\gamma_{cde})_{\alpha\beta}\,, \tag{4.12}$$

$$\mathcal{F}_{\alpha i} = -i(\sigma^2\chi)_{\alpha i}\,, \qquad \mathcal{F}_{\alpha ibc} = -(\sigma^1\gamma_{bc}\chi)_{\alpha i}\,, \qquad \mathcal{F}_{\alpha ibcde} = -i(\sigma^2\gamma_{bcde}\chi)_{\alpha i}\,, \tag{4.13}$$

one can show that they satisfy the following "generalized Bianchi identities" (same as in [9] for 10d components)[23]

$$d\mathcal{F}_{2n+1} + \mathrm{X}\wedge\mathcal{F}_{2n+1} - H\wedge\mathcal{F}_{2n-1} - i_K\mathcal{F}_{2n+3} = 0\,, \qquad n = -1, 0, 1, 2\,, \tag{4.14}$$

or, in components,

$$\begin{aligned}\nabla_{[A_1}\mathcal{F}_{A_2\cdots A_{2n+2}]} &+ \frac{2n+1}{2}T_{[A_1A_2}{}^B\mathcal{F}_{|B|A_3\cdots A_{2n+2}]} - \mathrm{X}_{[A_1}\mathcal{F}_{A_2\cdots A_{2n+2}]}\\ &+\frac{(2n+1)2n}{3!}H_{[A_1A_2A_3}\mathcal{F}_{A_4\cdots A_{2n+2}]} - \frac{1}{2n+2}K^B\mathcal{F}_{BA_1\cdots A_{2n+2}} = 0\,.\end{aligned} \tag{4.15}$$

It is easy to check, using (4.2) and (4.5), that as a consequence of these generalized Bianchi identities the forms $\mathcal{F}_p$ are also invariant under the (super)isometries generated by K, i.e.

$$\mathcal{L}_K\mathcal{F}_{2n+1} = 0\,, \qquad n = 0, 1, 2\,. \tag{4.16}$$

5 Concluding remarks

In this paper we have found the equations imposed on the target space (super) geometry by the requirement that the classical Green-Schwarz superstring should be kappa-symmetric. The bosonic part of these equations are exactly the same as suggested earlier in [9]. The resulting generalization of the standard 10d supergravity equations is automatically supersymmetric as it was obtained from a superspace construction. There is also a straightforward generalization of the notion of a supersymmetric solution of the generalized equations.

[22] This equation (4.10) is not independent and arises by taking a spinor derivative of the $(\alpha i\beta j)$ component of (4.8), symmetrizing and using the other equations given above.

[23] The $n = -1$ case corresponds to the condition $i_K\mathcal{F}_1 = 0$ [9].

We have performed the detailed analysis for the type I and type IIB cases but the corresponding generalized type IIA equations can be written down almost immediately using the results of [15]. One open question (raised already in [9]) is whether these equations (1.7)–(1.10) can be derived from an action and should thus satisfy certain integrability conditions. Another is about possible uplift of the generalized type IIA equations to 11 dimensions and a relation to a (partially off-shell?) generalization of 11d supergravity.

Non-trivial solutions of the type II generalized equations describe backgrounds symmetric with respect to the vector K_a. Applying T-duality one then gets a type II supergravity solution with a dilaton containing a linear non-isometric term [9, 25, 26]. It would be interesting to extend the discussion in [9] to determine how more general T-dualities act on these equations. Applying T-duality to the GS sigma model [27] should transform the background fields in a way consistent with kappa-symmetry and should thus map one solution of the generalized equations to another.

To investigate the properties of the corresponding sigma models one may consider the component expansion of the type II GS superstring action in these more general backgrounds. This expansion takes the same form as in the standard type II supergravity backgrounds [15] provided one replaces the dilaton-modified RR field strengths $e^{\phi}F_p$ by $\mathcal{F}_p$ and the dilaton gradient term $\frac{i}{2}\delta_{ij}\gamma^a\partial_a\phi$ in the quartic fermion terms (appearing in the matrix T in [15]) by $\frac{i}{2}\gamma^a(\delta_{ij}\mathrm{X}_a + \sigma^3_{ij}K_a)$.

Note added. After this paper appeared in arXiv we were informed of an earlier work on the pure spinor superstring that also observed that classical BRST invariance, the analog of kappa symmetry in that formulation, is not enough to restrict the background to be a supergravity solution [32]. The relation with the condition $\chi_{\alpha i} = \nabla_{\alpha i}\phi$ and the fact that the generalized backgrounds (referred to there as "non-physical") are connected with global symmetries were commented on in section 7.3 of [33].

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A Details of solution of superspace Bianchi identities and constraints

Here we will provide details of the solution of the Bianchi identities for the torsion and the 3-form H presented in section 3. The relevant Bianchi identities are (3.4) and (3.6).

We shall start from the constraints imposed by the kappa-symmetry on the dimension $-\frac{1}{2}$ and dimension 0 components as found in section 2[24]

$$\textbf{Type I:} \qquad H_{\alpha\beta\gamma}=0\,, \qquad T_{\alpha\beta}{}^{a}=-i\gamma^{a}_{\alpha\beta}\,, \qquad H_{a\alpha\beta}=-i(\gamma_a)_{\alpha\beta}\,, \tag{A.1}$$

$$\textbf{Type IIB:} \qquad H_{\alpha i\beta j\gamma k}=0\,, \qquad T_{\alpha i\beta j}{}^{a}=-i\delta_{ij}\gamma^{a}_{\alpha\beta}\,, \qquad H_{a\alpha i\beta j}=-i\sigma^{3}_{ij}(\gamma_a)_{\alpha\beta}\,. \tag{A.2}$$

We will proceed by dimension of T and H components and at each dimension will first work out the solution of the type I Bianchi identities and then present the type IIB solution. The type IIA solution should take an almost identical form to type IIB one as is clear from the discussion in [15].

In the type I case we will be more general: we will first impose only the dimension 0 constraint on the torsion and then comment on additional conditions following from including the 3-form constraint at the end of each subsection. In that case one obtains a more general solution which contains two 3-form fields which we call g_{abc} and h_{abc}, see e.g. [31]. This more general version of type I supergravity is, of course, not directly relevant for string theory as kappa-symmetry requires the presence of the 3-form H satisfying the above constraints.

Dimension $\frac{1}{2}$. Starting with the **type I** case, at dimension $\frac{1}{2}$ we need to solve the torsion Bianchi identity

$$T_{(\alpha\beta}{}^{\delta}\gamma^{d}_{\gamma)\delta}-\gamma^{e}_{(\alpha\beta}T_{\gamma)e}{}^{d}=0\,, \tag{A.3}$$

where $T_{\alpha[bc]}=0$ and $(\gamma^b)^{\alpha\beta}T_{\alpha b}{}^{c}=0$ (see section 3) and we used the dimension zero constraint on the torsion in (A.1). Contracting with $\gamma_b^{\beta\gamma}$ this gives the equation

$$2(\gamma^d\gamma_b)_{\delta}{}^{\beta}T_{\alpha\beta}{}^{\delta}+\gamma_b^{\beta\gamma}T_{\beta\gamma}{}^{\delta}\gamma^{d}_{\alpha\delta}-20T_{\alpha b}{}^{d}=0\,. \tag{A.4}$$

Expanding in a basis of gamma matrices

$$T_{\alpha\beta}{}^{\gamma}=\gamma^{a}_{\alpha\beta}\psi^{\gamma}_{a}+\gamma^{abcde}_{\alpha\beta}\psi^{\gamma}_{abcde}\,, \tag{A.5}$$

this equation implies (using the symmetry and gamma-tracelessness of $T_{\alpha bc}$)

$$\gamma^{abcde}\psi_{abcde}=-\frac{9}{5}\gamma^a\psi_a\,, \qquad T_{\alpha ab}=\frac{4}{5}(\gamma_{(a}\psi_{b)})_\alpha-\frac{2}{25}\eta_{ab}(\gamma^c\psi_c)_\alpha \tag{A.6}$$

$$\gamma^a\gamma_{fg}\psi_a+\gamma^{abcde}\gamma_{fg}\psi_{abcde}-8\gamma_{[f}\psi_{g]}=0\,. \tag{A.7}$$

Multiplying the second equation with γ^a and using the gamma-tracelessness of $T_{\alpha ab}$ we find

$$\psi_a=-\frac{7}{8}\gamma_a\chi\,, \tag{A.8}$$

for some spinor superfield χ whose normalization we have chosen for later convenience. Using this in the above equations we find

$$T_{\alpha a}{}^{b}=0\,, \qquad \gamma^{abcde}\psi_{abcde}=\frac{63}{4}\chi\,, \qquad \gamma^{abc}\psi_{abcfg}=\frac{7}{32}\gamma_{fg}\chi-\frac{1}{4}\gamma_{[f}\gamma^{abcd}\psi_{g]abcd}\,. \tag{A.9}$$

[24]To recall, in this paper $a,b=0,1,\ldots,9$; $\alpha,\beta=1,2,\ldots,16$; $i,j=1,2$.

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Contracting the dimension $\frac{1}{2}$ Bianchi identity (A.3) with $\gamma^{\beta\gamma}_{ghabc}$ we get

$$16 \cdot 5! \gamma^f \psi_{ghabc} + \gamma^{pqrde} \gamma_{ghabc} \gamma^f \psi_{pqrde} - \frac{7}{4} \gamma^f \gamma_{ghabc} \chi = 0 \,. \tag{A.10}$$

Multiplying this equation with γ_f gives

$$\psi_{ghabc} = \frac{7}{64 \cdot 5!} \gamma_{ghabc} \chi - \frac{1}{16 \cdot 5!} \gamma^{pqrd} \gamma_{ghabc} \gamma^e \psi_{pqrde} \,. \tag{A.11}$$

This equation determines ψ_{abcde} recursively and after some algebra one finds

$$\psi_{abcde} = \frac{1}{16 \cdot 5!} \gamma_{abcde} \chi \,. \tag{A.12}$$

This completes the solution of the dimension $\frac{1}{2}$ torsion Bianchi identity. The non-vanishing torsion at dimension $\frac{1}{2}$ is thus

$$T_{\alpha\beta}{}^{\gamma} = -\frac{7}{8} \gamma^a_{\alpha\beta} (\gamma_a \chi)^\gamma + \frac{1}{16 \cdot 5!} \gamma^{abcde}_{\alpha\beta} (\gamma_{abcde} \chi)^\gamma = 2\delta^\gamma_{(\alpha} \chi_{\beta)} - \gamma^a_{\alpha\beta} (\gamma_a \chi)^\gamma \,. \tag{A.13}$$

Imposing also the dimension $\frac{1}{2}$ Bianchi identity for the 3-form[25]

$$3\nabla_{(\alpha} H_{\beta\gamma)d} - \nabla_d H_{\alpha\beta\gamma} + 3T_{(\alpha\beta}{}^{E} H_{|E|\gamma)d} - 3T_{d(\alpha}{}^{E} H_{|E|\beta\gamma)} = 0 \tag{A.14}$$

and using the dimension 0 and dimension $-\frac{1}{2}$ constraints in (A.1) we get

$$\gamma^a_{(\alpha\beta} H_{\gamma)ab} = 0 \,, \tag{A.15}$$

which implies the vanishing of the dimension $\frac{1}{2}$ component of H

$$H_{\alpha bc} = 0 \,. \tag{A.16}$$

In the **type IIB** case the torsion Bianchi identity is

$$T_{(\alpha i \beta j}{}^{\delta l} T_{\gamma k)\delta l}{}^{d} - T_{(\alpha i \beta j}{}^{e} T_{\gamma k)e}{}^{d} = 0 \,. \tag{A.17}$$

When $i = j = k$ the analysis is the same as above and we get

$$T_{\alpha i b}{}^{c} = 0 \,, \tag{A.18}$$

$$T_{\alpha 1 \beta 1}{}^{\gamma 1} = 2\delta^\gamma_{(\alpha} \chi^1_{\beta)} - \gamma^a_{\alpha\beta} (\gamma_a \chi^1)^\gamma \,, \qquad T_{\alpha 2 \beta 2}{}^{\gamma 2} = 2\delta^\gamma_{(\alpha} \chi^2_{\beta)} - \gamma^a_{\alpha\beta} (\gamma_a \chi^2)^\gamma \,. \tag{A.19}$$

The remaining components of the Bianchi identity give

$$T_{\alpha 1 \beta 1}{}^{\delta 2} \gamma^d_{\gamma\delta} + 2T_{\gamma 2(\alpha 1}{}^{\delta 1} \gamma^d_{\beta)\delta} = 0 \,, \tag{A.20}$$

and the same equation with the indices 1 and 2 interchanged.

[25] As usual, $|E|$ means that index E is not symmetrized.

From the dimension $\frac{1}{2}$ Bianchi identity for the 3-form we get[26]

$$T_{(\alpha i\beta j}{}^{\delta l}H_{\gamma k)\delta ld}=0\,. \tag{A.21}$$

When $i=j=k$ this equation implies the vanishing of the dimension $\frac{1}{2}$ component of the 3-form as before

$$H_{\alpha ibc}=0\,. \tag{A.22}$$

The other components of (A.21) give

$$T_{\alpha 1\beta 1}{}^{\delta 2}(\gamma_d)_{\gamma\delta}-2T_{\gamma 2(\alpha 1}{}^{\delta 1}(\gamma_d)_{\beta)\delta}=0\,, \tag{A.23}$$

and the same with indices 1 and 2 interchanged. Together with eq. (A.20) this leads to the vanishing of the remaining components of the torsion.

Dimension 1. The **type I** torsion Bianchi identities at dimension 1 read

$$-2iT_{c(\alpha}{}^{\gamma}\gamma^d_{\beta)\gamma}=R_{\alpha\beta c}{}^d\,, \tag{A.24}$$

$$\nabla_{(\alpha}T_{\beta\gamma)}{}^{\delta}+T_{(\alpha\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta}-i\gamma^a_{(\alpha\beta}T_{|a|\gamma)}{}^{\delta}=R_{(\alpha\beta\gamma)}{}^{\delta}\,, \tag{A.25}$$

where we used the lower dimension constraints and the fact that $T_{ab}{}^c=0$. The first equation defines the curvature in terms of the torsion and using this in the second equation we find

$$\nabla_{(\alpha}T_{\beta\gamma)}{}^{\delta}+T_{(\alpha\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta}-i\gamma^a_{(\alpha\beta}T_{|a|\gamma)}{}^{\delta}-\frac{i}{2}T_{a(\alpha}{}^{\epsilon}(\gamma_b)_{\beta|\epsilon|}(\gamma^{ab})^{\delta}{}_{\gamma)}=0\,. \tag{A.26}$$

Multiplying by $\gamma^c_{\eta\delta}$ and symmetrizing in $(\alpha\beta\gamma\eta)$ we get, using the dimension $\frac{1}{2}$ Bianchi identity,

$$T_{a(\alpha}{}^{\delta}\gamma^a_{\beta\gamma}\gamma^c_{\eta)\delta}=0\,. \tag{A.27}$$

Let us now expand $T_{a\alpha}{}^{\delta}$ in a basis of gamma matrices

$$T_{a\alpha}{}^{\delta}=\delta^{\delta}_{\alpha}f_a+(\gamma_{cd})_{\alpha}{}^{\delta}f_a{}^{cd}+(\gamma_{cdef})_{\alpha}{}^{\delta}f_a{}^{cdef}\,. \tag{A.28}$$

The first Bianchi identity (A.24) implies, using the anti-symmetry of its r.h.s. in cd that

$$f_{(ab)c}=\frac{1}{2}\eta_{c(a}f_{b)}\,,\qquad\qquad (\gamma_{cdef(a})_{\alpha\beta}f_{b)}{}^{cdef}=0\,. \tag{A.29}$$

These conditions further imply

$$f_b{}^{ab}=\frac{11}{2}f^a\,,\qquad f_a{}^{cdef}=\frac{1}{48}\delta^{[c}_a g^{def]}\,, \tag{A.30}$$

[26] If we do not impose also the 3-form Bianchi identity there exists a much more general solution

$$T_{\alpha i\beta j}{}^{\gamma k}=2\delta^{\gamma}_{(\alpha}\Lambda^{ijk}_{\beta)}-\gamma^a_{\alpha\beta}(\gamma_a\Lambda^{ijk})^{\gamma}+2i(\sigma^2\delta)_{(\alpha i}{}^{\gamma k}\psi_{\beta j)}\,,$$

where Λ^{ijk} is a spinor superfield completely symmetric in the SO(2) indices ijk and ψ is another spinor superfield.

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for some 3-form g_{abc}. Then

$$T_{a(\alpha}{}^{\delta}\gamma^a_{\beta\gamma}\gamma^c_{\eta)\delta}=0 \qquad \Rightarrow \qquad \gamma^b_{(\alpha\beta}\gamma^a_{\gamma\delta)}f_a=0 \qquad \Rightarrow \qquad f_a=0\,. \tag{A.31}$$

We therefore get

$$T_{a\alpha}{}^{\delta}=\frac{1}{8}(\gamma^{bc})_\alpha{}^\delta h_{abc}+\frac{1}{48}(\gamma_{abcd})_\alpha{}^\delta g^{bcd}\,, \tag{A.32}$$

where h_{abc} and g_{abc} are arbitrary 3-forms. The first Bianchi identity (A.24) then gives

$$R_{\alpha\beta}{}^{cd}=\frac{i}{2}(\gamma_b)_{\alpha\beta}h^{bcd}+\frac{i}{24}\gamma^{cdefg}_{\alpha\beta}g_{efg}\,. \tag{A.33}$$

The second Bianchi identity (A.25) now reads

$$\nabla_{(\alpha}T_{\beta\gamma)}{}^{\delta}+T_{(\alpha\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta}-\frac{i}{4}\gamma^a_{(\alpha\beta}(\gamma^{bc})_{\gamma)}{}^{\delta}(h_{abc}-g_{abc})=0\,. \tag{A.34}$$

Contracting the indices γ and δ and using the expression for the dimension $\frac{1}{2}$ torsion we find

$$16\nabla_{(\alpha}\chi_{\beta)}-\gamma^a_{\alpha\beta}\gamma_a^{\gamma\delta}\nabla_\gamma\chi_\delta=0 \qquad \Rightarrow \qquad \nabla_{(\alpha}\chi_{\beta)}=\frac{i}{2}\gamma^a_{\alpha\beta}X_a\,, \tag{A.35}$$

for some one-form superfield X_a. The remaining components of the Bianchi identity then give

$$\nabla_\alpha\chi_\beta=\chi_\alpha\chi_\beta+\frac{i}{2}\gamma^a_{\alpha\beta}X_a-\frac{i}{24}\gamma^{abc}_{\alpha\beta}(h_{abc}-g_{abc})\,, \tag{A.36}$$

where we used the fact that $\chi_\alpha\chi_\beta=\frac{1}{96}\gamma^{abc}_{\alpha\beta}\,\chi\gamma_{abc}\chi$.

If we finally impose the 3-form $H=dB$ Bianchi identity and the kappa-symmetry constraints on it in (A.1) we find, using the lower dimension constraints, that

$$T_{\alpha\beta}{}^{e}H_{cde}+2T_{c(\alpha}{}^{\gamma}H_{\beta)\gamma d}-2T_{d(\alpha}{}^{\gamma}H_{\beta)\gamma c}=0\,, \tag{A.37}$$

which implies that

$$h_{abc}=H_{abc}\,, \qquad\qquad g_{abc}=0\,. \tag{A.38}$$

In the **type IIB** case the dimension 1 Bianchi identities are

$$2T_{c(\alpha i}{}^{\gamma k}T_{\beta j)\gamma k}{}^{d}=R_{\alpha i\beta jc}{}^{d}\,, \tag{A.39}$$

$$\nabla_{(\alpha i}T_{\beta j\gamma k)}{}^{\delta l}+T_{(\alpha i\beta j}{}^{\epsilon m}T_{\gamma k)\epsilon m}{}^{\delta l}+T_{(\alpha i\beta j}{}^{a}T_{|a|\gamma k)}{}^{\delta l}=R_{(\alpha i\beta j\gamma k)}{}^{\delta l}\,. \tag{A.40}$$

The equations for $T_{a\beta i}{}^{\gamma i}$ and $R_{\alpha i\beta ic}{}^{d}$ with $i=1,2$ are the same as in the type I case analyzed above. This implies (here primed and unprimed quantities are independent)

$$T_{a\alpha 1}{}^{\delta 1}=\frac{1}{8}(\gamma^{bc})_\alpha{}^\delta h_{abc}+\frac{1}{48}(\gamma_{abcd})_\alpha{}^\delta g^{bcd}\,, \qquad T_{a\alpha 2}{}^{\delta 2}=\frac{1}{8}(\gamma^{bc})_\alpha{}^\delta h'_{abc}+\frac{1}{48}(\gamma_{abcd})_\alpha{}^\delta g'^{bcd}\,, \tag{A.41}$$

$$R_{\alpha 1\beta 1}{}^{cd}=\frac{i}{2}(\gamma_b)_{\alpha\beta}h^{bcd}+\frac{i}{24}\gamma^{cdefg}_{\alpha\beta}g_{efg}\,, \qquad R_{\alpha 2\beta 2}{}^{cd}=\frac{i}{2}(\gamma_b)_{\alpha\beta}h'^{bcd}+\frac{i}{24}\gamma^{cdefg}_{\alpha\beta}g'_{efg}\,, \tag{A.42}$$

$$\nabla_{\alpha 1}\chi_{\beta 1}=\chi_{\alpha 1}\chi_{\beta 1}+\frac{i}{2}\gamma^a_{\alpha\beta}X_a-\frac{i}{24}\gamma^{abc}_{\alpha\beta}(h_{abc}-g_{abc})\,, \qquad X_a\equiv \mathrm{X}_a+K_a\,, \tag{A.43}$$

$$\nabla_{\alpha 2}\chi_{\beta 2}=\chi_{\alpha 2}\chi_{\beta 2}+\frac{i}{2}\gamma^a_{\alpha\beta}X'_a-\frac{i}{24}\gamma^{abc}_{\alpha\beta}(h'_{abc}-g'_{abc})\,, \qquad X'_a\equiv \mathrm{X}_a-K_a\,. \tag{A.44}$$

Here instead of X_a and X'_a which appear as in type I case we introduced the two independent superfields X_a and K_a for later convenience.

The remaining equations to solve are

$$R_{\alpha 1\beta 2c}{}^{d} = -iT_{c\alpha 1}{}^{\gamma 2}\gamma^{d}_{\beta\gamma} - iT_{c\beta 2}{}^{\gamma 1}\gamma^{d}_{\alpha\gamma}\,, \tag{A.45}$$

$$\nabla_{\gamma 2}T_{\alpha 1\beta 1}{}^{\delta 1} - i\gamma^{a}_{\alpha\beta}T_{a\gamma 2}{}^{\delta 1} = 2R_{\gamma 2(\alpha 1\beta 1)}{}^{\delta 1}\,, \tag{A.46}$$

$$-i\gamma^{a}_{\beta\gamma}T_{a\alpha 1}{}^{\delta 1} = R_{\beta 2\gamma 2\alpha 1}{}^{\delta 1}\,, \tag{A.47}$$

$$\gamma_{(\alpha\beta}{}^{a}T_{|a|\gamma 2)}{}^{\delta 1} = 0\,, \tag{A.48}$$

together with the same equations with indices 1 and 2 interchanged. Eq. (A.47) implies

$$g_{abc} = g'_{abc} = 0\,, \qquad h'_{abc} = -h_{abc}\,, \tag{A.49}$$

while from (A.48) we get

$$T_{a\beta 2}{}^{\gamma 1} = \frac{1}{8}(\gamma_a\mathcal{S}^{21})_{\beta}{}^{\gamma}\,, \qquad T_{a\beta 1}{}^{\gamma 2} = \frac{1}{8}(\gamma_a\mathcal{S}^{12})_{\beta}{}^{\gamma}\,, \tag{A.50}$$

for some matrices $\mathcal{S}^{12}$ and $\mathcal{S}^{21}$. Eq. (A.45) now implies

$$\mathcal{S}^{12} = -(\mathcal{S}^{21})^{T}\,, \qquad R_{\alpha 1\beta 2}{}^{cd} = -\frac{i}{4}(\gamma^{[c}\mathcal{S}^{12}\gamma^{d]})_{\alpha\beta}\,. \tag{A.51}$$

Finally, eq. (A.46) gives

$$\nabla_{\alpha 2}\chi^{1}_{\beta} = -\frac{i}{16}(\gamma_a\mathcal{S}^{21}\gamma^{a})_{\alpha\beta}\,. \tag{A.52}$$

This completes the solution of the dimension 1 torsion Bianchi identities. The 3-form Bianchi identity just adds, as in the type I case, the relation

$$h_{abc} = H_{abc}\ . \tag{A.53}$$

Dimension $\frac{3}{2}$. The **type I** dimension $\frac{3}{2}$ Bianchi identities are

$$-i\gamma^{d}_{\alpha\beta}T_{bc}{}^{\beta} = 2R_{\alpha[bc]}{}^{d}\,, \tag{A.54}$$

$$\nabla_a T_{\beta\gamma}{}^{\delta} - 2\nabla_{(\beta}T_{|a|\gamma)}{}^{\delta} + 2T_{a(\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta} - T_{\beta\gamma}{}^{\epsilon}T_{a\epsilon}{}^{\delta} - i\gamma^{e}_{\beta\gamma}T_{ea}{}^{\delta} = 2R_{a(\beta\gamma)}{}^{\delta}\,. \tag{A.55}$$

Eq. (A.54) gives the dimension $\frac{3}{2}$ component of the curvature as

$$R_{\alpha bcd} = \frac{i}{2}(\gamma_b\psi_{cd})_{\alpha} - i(\gamma_{[c}\psi_{d]b})_{\alpha}\,, \tag{A.56}$$

where $\psi^{\beta}_{ab} = T_{ab}{}^{\beta}$ is the gravitino field strength. Using this in (A.55) we get

$$\begin{aligned}\nabla_a T_{\beta\gamma}{}^{\delta} - 2\nabla_{(\beta}T_{|a|\gamma)}{}^{\delta} + 2T_{a(\beta}{}^{\epsilon}T_{\gamma)\epsilon}{}^{\delta} - T_{\beta\gamma}{}^{\epsilon}T_{a\epsilon}{}^{\delta} - \frac{i}{4}(\gamma^{cd})^{\delta}{}_{(\beta}(\gamma_a\psi_{cd})_{\gamma)}\\ -\frac{i}{4}\gamma^{c}_{\beta\gamma}(\gamma_c\gamma^{b}\psi_{ba})^{\delta} - \frac{i}{2}\gamma^{b}_{\beta\gamma}\psi^{\delta}_{ba} + \frac{i}{2}\delta^{\delta}{}_{(\beta}(\gamma^{b}\psi_{ba})_{\gamma)} = 0\,.\end{aligned} \tag{A.57}$$

Contracting the indices γ and δ and using the lower dimension constraints gives

$$i(\gamma^b\psi_{ab})_\alpha = 4\nabla_a\chi_\alpha + (\gamma_a\gamma^b\nabla_b\chi)_\alpha - \frac{1}{56}(\gamma_a\gamma^{bcd})_\alpha{}^\beta\nabla_\beta\left(h_{bcd}+\frac{11}{6}g_{bcd}\right) \tag{A.58}$$
$$-\frac{1}{14}(\gamma^{bc})_\alpha{}^\beta\nabla_\beta\left(h_{abc}-\frac{1}{2}g_{abc}\right)+\frac{1}{8}(\gamma_a\gamma^{bcd}\chi)_\alpha\left(h_{bcd}+\frac{11}{6}g_{bcd}\right)+\frac{1}{2}(\gamma^{bc}\chi)_\alpha\left(h_{abc}-\frac{1}{2}g_{abc}\right).$$

Contracting (A.57) with $\gamma_e^{\beta\gamma}$ and using (A.59) we get, after some tedious algebra,

$$(\gamma^a)^{\alpha\beta}\nabla_\beta h_{abc} = -4(\gamma_{[b}\nabla_{c]}\chi)^\alpha + \frac{5}{84}(\gamma_{bc}\gamma^{def})^{\alpha\beta}\nabla_\beta g_{def} - \frac{3}{7}(\gamma_{[b}\gamma^{de})^{\alpha\beta}\nabla_\beta g_{c]de} - \frac{1}{2}(\gamma^a)^{\alpha\beta}\nabla_\beta g_{abc}$$
$$-\frac{1}{2}(\gamma_{[b}\gamma^{de}\chi)^\alpha h_{c]de} - \frac{13}{28}(\gamma_{bc}\gamma^{def}\chi)^\alpha g_{def} + \frac{95}{28}(\gamma_{[b}\gamma^{de}\chi)^\alpha g_{c]de} + 4(\gamma^a\chi)^\alpha g_{abc} + 6i\psi^\alpha_{bc}\,. \tag{A.59}$$

Contracting (A.57) with $(\gamma^{ef})_\delta{}^\gamma$ gives

$$\nabla_\alpha h_{abc} = 3i(\gamma_{[a}\psi_{bc]})_\alpha + \frac{1}{60}(\gamma_{abc}\gamma^{def})_\alpha{}^\beta\nabla_\beta g_{def} - \frac{3}{20}(\gamma_{[ab}\gamma^{de})_\alpha{}^\beta\nabla_\beta g_{c]de} - \frac{3}{10}(\gamma_{[a}\gamma^d)_\alpha{}^\beta\nabla_\beta g_{bc]d}$$
$$+\frac{1}{10}\nabla_\alpha g_{abc} - \frac{2}{15}(\gamma_{abc}\gamma^{def}\chi)_\alpha g_{def} + \frac{6}{5}(\gamma_{[ab}\gamma^{de}\chi)_\alpha g_{c]de} + \frac{12}{5}(\gamma_{[a}\gamma^d\chi)_\alpha g_{bc]d} - \frac{4}{5}\chi_\alpha g_{abc}\,. \tag{A.60}$$

Using this in (A.57) it finally becomes

$$-\frac{5}{3}(\gamma^{abcd})_{(\gamma}{}^\delta\langle\nabla_{\beta)}g^{bcd} + 2\chi_{\beta)}g^{bcd}\rangle - (\gamma^{cd})_{(\gamma}{}^\delta\langle\nabla_{\beta)}g^{acd} + 2\chi_{\beta)}g^{acd}\rangle = 0\,, \tag{A.61}$$

where we use the angle-brackets to denote the gamma-traceless part, e.g.,

$$\langle\nabla_\alpha g_{abc}\rangle = \nabla_\alpha g_{abc} + \frac{1}{21\cdot 16}(\gamma_{abc}\gamma^{def})_\alpha{}^\beta\nabla_\beta g_{def} - \frac{1}{14}(\gamma_{[ab}\gamma^{de})_\alpha{}^\beta\nabla_\beta g_{c]de} - \frac{1}{2}(\gamma_{[a}\gamma^d)_\alpha{}^\beta\nabla_\beta g_{bc]d}\,. \tag{A.62}$$

This equation is easily shown to imply

$$\langle\nabla_\alpha g_{abc} + 2\chi_\alpha g_{abc}\rangle = 0\,. \tag{A.63}$$

Using this in the expressions (A.59) and (A.60) they become

$$i(\gamma^b\psi_{ab})_\alpha = 2\nabla_a\chi_\alpha + \frac{1}{4}(\gamma^{bc}\chi)_\alpha h_{abc} + \frac{1}{84}(\gamma^{abcd})_\alpha{}^\beta\nabla_\beta g_{bcd} - \frac{17}{168}(\gamma^{abcd}\chi)_\alpha g_{bcd}\,, \tag{A.64}$$

$$\nabla_\alpha h_{abc} = 3i(\gamma_{[a}\psi_{bc]})_\alpha + \frac{11}{21\cdot 32}(\gamma_{abc}\gamma^{def})_\alpha{}^\beta\nabla_\beta g_{def} - \frac{1}{7}(\gamma_{[ab}\gamma^{de})_\alpha{}^\beta\nabla_\beta g_{c]de} - \frac{1}{4}(\gamma_{[a}\gamma^d)_\alpha{}^\beta\nabla_\beta g_{bc]d}$$
$$-\frac{15}{7\cdot 16}(\gamma_{abc}\gamma^{def}\chi)_\alpha g_{def} + \frac{17}{14}(\gamma_{[ab}\gamma^{de}\chi)_\alpha g_{c]de} + \frac{5}{2}(\gamma_{[a}\gamma^d\chi)_\alpha g_{bc]d} - \chi_\alpha g_{abc} \tag{A.65}$$

This completes the solution of the torsion Bianchi identity.

It remains to analyze the consequence of the constraint (A.36) on $\nabla_\alpha\chi_\beta$ found at dimension one. To do this we take a symmetrized spinor derivative of this equation which gives

$$2T_{\alpha\gamma}{}^D\nabla_D\chi_\beta + 2R_{\alpha\gamma\beta}{}^\delta\chi_\delta + 4\nabla_{(\alpha}\chi_{\gamma)}\chi_\beta$$
$$-4\chi_{(\alpha}\nabla_{\gamma)}\chi_\beta + 2i\gamma^a_{\beta(\alpha}\nabla_{\gamma)}X_a + \frac{i}{6}\gamma^{abc}_{\beta(\alpha}(\nabla_{\gamma)}h_{abc} - \nabla_{\gamma)}g_{abc}) = 0\,. \tag{A.66}$$

Using the above expressions we find the equation

$$2\gamma^a_{\beta(\alpha}\nabla_{\gamma)}X_a + \gamma^a_{\alpha\gamma}(\gamma_a\gamma^b\chi)_\beta X_b - 2\gamma^a_{\beta(\alpha}\nabla_a\chi_{\gamma)} - \frac{1}{2}\gamma^a_{\alpha\gamma}(\gamma_a\gamma^b\nabla_b\chi)_\beta$$

$$+\frac{1}{96}\gamma^a_{\alpha\gamma}(\gamma_a\gamma^{def})_\beta{}^\delta\nabla_\delta g_{def}+\frac{1}{48}\gamma^a_{\alpha\gamma}(\gamma_a\gamma^{bcd}\chi)_\beta h_{bcd}-\frac{1}{4}\gamma^a_{\beta(\alpha}(\gamma^{bc}\chi)_{\gamma)}h_{abc}-\frac{1}{32}\gamma^a_{\alpha\gamma}(\gamma_a\gamma^{bcd}\chi)_\beta g_{bcd}$$
$$-\frac{1}{8}\gamma^a_{\alpha\gamma}(\gamma^{de}\chi)_\beta g_{ade}+\frac{5}{8}\gamma^a_{\beta(\alpha}(\gamma^{de}\chi)_{\gamma)}g_{ade}+\frac{1}{6}\gamma^{abc}_{\beta(\alpha}\chi_{\gamma)}g^{abc}+\frac{1}{48}\gamma^{abcde}_{\alpha\gamma}(\gamma_{de}\chi)_\beta g_{abc}=0\,. \quad \text{(A.67)}$$

Contracting this with $\gamma_d^{\alpha\gamma}$ gives, after a bit of algebra,

$$\nabla_\alpha X_a=\frac{1}{2}(\gamma^b\gamma_a\nabla_b\chi)_\alpha+(\gamma_a\gamma^b\chi)_\alpha X_b+\frac{1}{96}(\gamma_a\gamma^{bcd})_\alpha{}^\beta\nabla_\beta g_{bcd}+\frac{1}{48}(\gamma_a\gamma^{bcd}\chi)_\alpha h_{bcd}$$
$$+\frac{1}{8}(\gamma^{bc}\chi)_\alpha h_{abc}-\frac{7}{96}(\gamma_a\gamma^{bcd}\chi)_\alpha g_{bcd}-\frac{1}{16}(\gamma^{bc}\chi)_\alpha g_{abc}\,. \quad \text{(A.68)}$$

It is not hard to show that this solves (A.67). This completes the solution of the torsion Bianchi identity in the type I case.

Imposing the Bianchi identity for the 3-form gives no new constraints beyond what follows from the constraints found at dimension one, i.e. $h_{abc}=H_{abc}$ and $g_{abc}=0$.

In the **type IIB** case the dimension $\frac{3}{2}$ Bianchi identities are

$$T_{\alpha i\beta j}{}^dT_{bc}{}^{\beta j}=2R_{\alpha i[bc]}{}^d\,, \quad \text{(A.69)}$$
$$\nabla_aT_{\beta i\gamma j}{}^{\delta k}-2\nabla_{(\beta i}T_{|a|\gamma j)}{}^{\delta k}+2T_{a(\beta i}{}^{\epsilon l}T_{\gamma j)\epsilon l}{}^{\delta k}$$
$$-T_{\beta i\gamma j}{}^{\epsilon l}T_{a\epsilon l}{}^{\delta k}-i\delta_{ij}\gamma^e_{\beta\gamma}T_{ea}{}^{\delta k}=2R_{a(\beta i\gamma j)}{}^{\delta k}\,. \quad \text{(A.70)}$$

The first gives again the dimension $\frac{3}{2}$ component of the curvature as

$$R_{\alpha ibcd}=\frac{i}{2}(\gamma_b\psi_{cd})_{\alpha i}-i(\gamma_{[c}\psi_{d]b})_{\alpha i}\,. \quad \text{(A.71)}$$

Eq. (A.70) with $i=j=k$ is the same as in the type I case and the solution is therefore (note that in the type IIB case $g_{abc}=0$ and $h_{abc}=H_{abc}$)

$$\nabla_{\alpha i}H_{abc}=3i(\gamma_{[a}\sigma^3\psi_{bc]})_{\alpha i}\,, \quad \text{(A.72)}$$
$$i(\gamma^b\psi_{ab})_{\alpha i}=2\nabla_a\chi_{\alpha i}+\frac{1}{4}(\gamma^{bc}\sigma^3\chi)_{\alpha i}H_{abc}\,. \quad \text{(A.73)}$$

The remaining components of the Bianchi identity are

$$-2\nabla_{(\beta 1}T_{|a|\gamma 1)}{}^{\delta 2}-T_{\beta 1\gamma 1}{}^{\epsilon 1}T_{a\epsilon 1}{}^{\delta 2}-i\gamma^e_{\beta\gamma}T_{ea}{}^{\delta 2}=0\,, \quad \text{(A.74)}$$
$$-2\nabla_{(\beta 2}T_{|a|\gamma 1)}{}^{\delta 2}+T_{a\gamma 1}{}^{\epsilon 2}T_{\beta 2\epsilon 2}{}^{\delta 2}+R_{\gamma 1a\beta 2}{}^{\delta 2}=0\,, \quad \text{(A.75)}$$

and the same with indices 1 and 2 interchanged. Eq. (A.75) gives

$$\nabla_{\alpha 2}\mathcal{S}^{\beta 1\gamma 2}=\delta^\gamma_\alpha\mathcal{S}^{\beta 1\delta 2}\chi^2_\delta+\mathcal{S}^{\beta 1\gamma 2}\chi^2_\alpha-\mathcal{S}^{\beta 1\delta 2}\gamma^a_{\alpha\delta}(\gamma_a\chi^2)^\gamma-2i(\gamma^{ab})^\gamma{}_\alpha\psi^{\beta 1}_{ab}\,,$$
$$\nabla_{\alpha 1}\mathcal{S}^{\beta 1\gamma 2}=\delta^\beta_\alpha\mathcal{S}^{\delta 1\gamma 2}\chi^1_\delta+\mathcal{S}^{\beta 1\gamma 2}\chi^1_\alpha-\mathcal{S}^{\delta 1\gamma 2}\gamma^a_{\alpha\delta}(\gamma_a\chi^1)^\beta+2i(\gamma^{cd})^\beta{}_\alpha\psi^{\gamma 2}_{cd}\,. \quad \text{(A.76)}$$

Eq. (A.74) is then automatically satisfied.

It remains to analyze the consequences of the dimension one conditions on $\nabla_{\alpha i}\chi_{\beta j}$ in (A.43) and (A.44). Applying $\nabla_{\gamma k}$ and symmetrizing the derivatives we get, for $i=j=k$, the same condition as in the type I case but now not for one X_a but two vectors $\mathrm{X}_a\pm K_a$

$$\nabla_{\alpha 1}(\mathrm{X}_a+K_a)=\frac{1}{2}(\gamma^b\gamma_a\nabla_b\chi)_{\alpha 1}+(\gamma_a\gamma^b\chi)_{\alpha 1}(\mathrm{X}_b+K_b)$$

$$+\frac{1}{48}(\gamma_a\gamma^{bcd}\sigma^3\chi)_{\alpha 1}H_{bcd}+\frac{1}{8}(\gamma^{bc}\sigma^3\chi)_{\alpha 1}H_{abc}\,, \tag{A.77}$$

$$\nabla_{\alpha 2}(\mathrm{X}_a-K_a)=\frac{1}{2}(\gamma^b\gamma_a\nabla_b\chi)_{\alpha 2}+(\gamma_a\gamma^b\chi)_{\alpha 2}(\mathrm{X}_b-K_b)$$
$$+\frac{1}{48}(\gamma_a\gamma^{bcd}\sigma^3\chi)_{\alpha 2}H_{bcd}+\frac{1}{8}(\gamma^{bc}\sigma^3\chi)_{\alpha 2}H_{abc}\,. \tag{A.78}$$

The remaining equations involve $\nabla_{(\gamma 1}\nabla_{\alpha 1)}\chi_{\beta 2}$ and $\nabla_{(\gamma 1}\nabla_{\alpha 2)}\chi_{\beta 1}$ giving

$$-i\gamma^a_{\alpha\gamma}\nabla_a\chi_{\beta 2}+T_{\alpha 1\gamma 1}^{\delta 1}\nabla_{\delta 1}\chi_{\beta 2}+\frac{1}{4}R_{\alpha 1\gamma 1}{}^{cd}(\gamma_{cd}\chi)_{\beta 2}-\frac{i}{8}(\gamma_a\nabla_{(\gamma 1}\mathcal{S}^{12}\gamma^a)_{\alpha)\beta}=0\,, \tag{A.79}$$

$$\frac{1}{4}R_{\gamma 1\alpha 2}{}^{cd}(\gamma_{cd}\chi)_{\beta 1}-\frac{i}{16}(\gamma_a\nabla_{\gamma 1}\mathcal{S}^{21}\gamma^a)_{\alpha\beta}+\nabla_{\alpha 2}\chi_{\gamma 1}\chi_{\beta 1}-\chi_{\gamma 1}\nabla_{\alpha 2}\chi_{\beta 1}$$
$$+\frac{i}{2}\gamma^a_{\gamma\beta}\nabla_{\alpha 2}(\mathrm{X}_a+K_a)-\frac{i}{24}\gamma^{abc}_{\gamma\beta}\nabla_{\alpha 2}H_{abc}=0\,, \tag{A.80}$$

and the same with indices 1 and 2 interchanged. Using the results derived so far it is easy to check that the first equation is automatically satisfied while the second determines the remaining spinor derivatives of $\mathrm{X}_a\pm K_a$

$$\nabla_{\alpha 2}(\mathrm{X}_a+K_a)=\nabla_a\chi_{\alpha 2}+\frac{1}{8}(\gamma^{bc}\sigma^3\chi)_{\alpha 2}H_{abc}+\frac{1}{8}(\gamma^a\mathcal{S}\chi)_{\alpha 2}\,, \tag{A.81}$$

$$\nabla_{\alpha 1}(\mathrm{X}_a-K_a)=\nabla_a\chi_{\alpha 1}+\frac{1}{8}(\gamma^{bc}\sigma^3\chi)_{\alpha 1}H_{abc}+\frac{1}{8}(\gamma^a\mathcal{S}\chi)_{\alpha 1}\,. \tag{A.82}$$

This completes the solution of the dimension $\frac{3}{2}$ torsion Bianchi identities. Imposing the Bianchi identity for the 3-form gives no new constraints.

Dimension 2. The **type I** Bianchi identities at dimension two read

$$R_{[abc]}{}^d=0\,,\qquad \nabla_\alpha T_{bc}{}^\delta+2\nabla_{[b}T_{c]\alpha}{}^\delta+2T_{[b|\alpha|}{}^\beta T_{c]\beta}{}^\delta+T_{bc}{}^\beta T_{\beta\alpha}{}^\delta=R_{bc\alpha}{}^\delta\,. \tag{A.83}$$

Using the above results for the lower dimensional components the latter becomes

$$\nabla_\alpha\psi^\delta_{ab}=\frac{1}{4}(\gamma^{cd})_\alpha{}^\delta\Big[R_{ab}{}^{cd}-\nabla_{[a}h_{b]cd}+\frac{1}{2}h_{ac}{}^e h_{bde}-\frac{1}{8}g_{ac}{}^e g_{bde}+\frac{1}{8}\eta_{c[a}g_{b]ef}g_d{}^{ef}-\frac{1}{48}\eta_{ac}\eta_{bd}g_{efg}g^{efg}\Big]$$
$$+\frac{1}{48}(\gamma^{cdef})_\alpha{}^\delta\left[h_{abc}g_{def}-2\eta_{c[a}\nabla_{b]}g_{def}+3\eta_{c[a}h_{b]d}{}^g g_{efg}\right]-\frac{1}{192}(\gamma^{cdefgh})_\alpha{}^\delta\eta_{c[a}g_{b]de}g_{fgh}$$
$$+\frac{1}{128}(\gamma_{abcdef})_\alpha{}^\delta g^{cd}{}_g g^{efg}-\psi_{ab}{}^\delta\chi_\alpha-\delta^\delta_\alpha\psi_{ab}\chi+(\gamma^c\psi_{ab})_\alpha(\gamma_c\chi)^\delta\,. \tag{A.84}$$

Multiplying this with $\gamma^b_{\beta\delta}$ and using the dimension $\frac{3}{2}$ constraint on the gamma-trace of ψ_{ab} in (A.64) as well as the other lower dimension constraints gives some of the equations of motion. Let us also use the Bianchi identitiy for the 3-form H which, as we have seen, lead to $g_{abc}=0$, $h_{abc}=H_{abc}$. We then obtain the equations of motion (3.36) and (3.37). The final equation of motion comes from evaluating $\nabla_{(\alpha}\nabla_{\beta)}X_a$ using the consequences of the dimension $\frac{3}{2}$ constraint. Setting $g_{abc}=0$ this gives the equation (3.38) for the divergence of X_a.

In the **type IIB** case the dimension 2 Bianchi identities are

$$R_{[abc]}{}^d=0\,,\qquad \nabla_{\alpha i}T_{bc}{}^{\delta j}+2\nabla_{[b}T_{c]\alpha i}{}^{\delta j}+2T_{[b|\alpha i|}{}^{\beta k}T_{c]\beta k}{}^{\delta j}+T_{bc}{}^{\beta k}T_{\beta k\alpha i}{}^{\delta j}=R_{bc\alpha i}{}^{\delta j}\,. \tag{A.85}$$

The latter gives

$$\nabla_{\alpha i}\psi_{ab}^{\delta j} = -\frac{1}{4}\sigma^3_{ij}(\gamma^{cd})_\alpha{}^\delta\nabla_{[a}H_{b]cd} + \frac{1}{4}(\gamma_{[a}\nabla_{b]}\mathcal{S})_{\alpha i}{}^{\delta j} + \frac{1}{8}\delta_{ij}(\gamma^{cd})_\alpha{}^\delta H_{ace}H_{bd}{}^e - \frac{1}{4}R_{ab}{}^{cd}\delta_{ij}(\gamma_{cd})^\delta{}_\alpha$$
$$+\frac{1}{32}(\gamma^{cd}\sigma^3\gamma_{[a}\mathcal{S})_{\alpha i}{}^{\delta j}H_{b]cd} - \frac{1}{32}(\gamma_{[a}\mathcal{S}\gamma^{cd}\sigma^3)_{\alpha i}{}^{\delta j}H_{b]cd} - \frac{1}{32}(\gamma_{[a}\mathcal{S}\gamma_{b]}\mathcal{S})_{\alpha i}{}^{\delta j}$$
$$-\frac{1}{2}\delta_{ij}\delta^\delta_\alpha\psi_{ab}\chi - \frac{1}{2}\sigma^3_{ij}\delta^\delta_\alpha\psi_{ab}\sigma^3\chi - \frac{1}{2}\psi^{\delta j}_{ab}\chi_{\alpha i} - \frac{1}{2}(\sigma^3\psi_{ab})^{\delta j}(\sigma^3\chi)_{\alpha i}$$
$$+\frac{1}{2}(\gamma^c\psi_{ab})_{\alpha i}(\gamma_c\chi)^{\delta j} + \frac{1}{2}(\gamma^c\sigma^3\psi_{ab})_{\alpha i}(\gamma_c\sigma^3\chi)^{\delta j}\,. \tag{A.86}$$

Multiplying this with $\gamma^a_{\beta\delta}$ and using the dimension $\frac{3}{2}$ constraint (A.73) on $\gamma^a\psi_{ab}$ as well as the lower dimension constraints we get

$$\delta_{ij}\gamma^a_{\alpha\beta}\nabla_b\mathrm{X}_a + \frac{1}{2}\delta_{ij}\gamma^c_{\alpha\beta}K^aH_{abc} - \frac{1}{4}\sigma^3_{ij}\gamma^c_{\alpha\beta}(\nabla^aH_{abc} - 2\mathrm{X}^aH_{abc})$$
$$+\sigma^3_{ij}\gamma^a_{\alpha\beta}\nabla_bK_a + \frac{1}{6}\sigma^3_{ij}\gamma^{cde}_{\alpha\beta}\nabla_{[b}H_{cde]} - \frac{1}{8}(\gamma_b\nabla_a\mathcal{S}\gamma^a)_{\alpha i\beta j} + \frac{1}{8}(\gamma_b\mathcal{S}\gamma^a)_{\alpha i\beta j}\mathrm{X}_a$$
$$+\frac{1}{8}(\gamma_b\mathcal{S}\sigma^3\gamma^a)_{\alpha i\beta j}K_a - \frac{1}{8}\delta_{ij}\gamma^a_{\alpha\beta}H_{acd}H_b{}^{cd} - \frac{1}{4}R_{ab}{}^{cd}\delta_{ij}(\gamma^a\gamma_{cd})_{\beta\alpha}$$
$$+\frac{1}{192}(\gamma_b\mathcal{S}\gamma^{cde}\sigma^3)_{\alpha i\beta j}H_{cde}+\frac{1}{16}(\gamma^c\mathcal{S}\gamma^d\sigma^3)_{\alpha i\beta j}H_{bcd}-\frac{1}{64}(\gamma^{cd}\sigma^3\gamma_b\mathcal{S}\gamma^a)_{\alpha i\beta j}H_{acd}$$
$$-\frac{1}{64}(\gamma_a\mathcal{S}\gamma_b\mathcal{S}\gamma^a)_{\alpha i\beta j} - \frac{i}{8}(\gamma_b\mathcal{S}\chi)_{\alpha i}\chi_{\beta j} - \frac{i}{8}(\gamma_b\mathcal{S}\sigma^3\chi)_{\alpha i}(\sigma^3\chi)_{\beta j} - \frac{1}{2}\delta_{ij}\gamma^a_{\alpha\beta}\psi_{ab}\chi$$
$$-\frac{1}{2}\sigma^3_{ij}\gamma^a_{\alpha\beta}\psi_{ab}\sigma^3\chi - \frac{1}{4}(\gamma_b\psi_{cd})_{\alpha i}(\gamma^{cd}\chi)_{\beta j} + \frac{1}{4}(\gamma_b\sigma^3\psi_{cd})_{\alpha i}(\gamma^{cd}\sigma^3\chi)_{\beta j} = 0\,, \tag{A.87}$$

which implies the equations (3.39) and (3.42)–(3.45).

In addition, we have the consistency conditions that come from applying two symmetrized spinor derivatives to a dimension 1 superfield and using the dimension $\frac{3}{2}$ constraints. Doing this on the equations for the spinor derivative of H_{abc} and $\mathcal{S}$ gives nothing new, but from the equations for the derivative of X_a and K_a we get

$$-\frac{i}{4}(\gamma_a(1+\sigma^3))_{\alpha i\beta j}\Big[\nabla^b(\mathrm{X}_b+K_b)-2(\mathrm{X}^b+K^b)(\mathrm{X}_b+K_b)+\frac{1}{12}H^{bcd}H_{bcd}-\frac{1}{256}\mathrm{Tr}(\mathcal{S}\gamma^b\mathcal{S}\gamma_b)$$
$$+i\chi\gamma^b(1+\sigma^3)\nabla_b\chi - \frac{i}{24}\chi\gamma^{bcd}(1+\sigma^3)\chi\, H_{bcd}\Big] + \ldots = 0\,, \tag{A.88}$$
$$-\frac{i}{4}(\gamma_a(1-\sigma^3))_{\alpha i\beta j}\Big[\nabla^b(\mathrm{X}_b-K_b)-2(\mathrm{X}^b-K^b)(\mathrm{X}_b-K_b)+\frac{1}{12}H^{bcd}H_{bcd}-\frac{1}{256}\mathrm{Tr}(\mathcal{S}\gamma^b\mathcal{S}\gamma_b)$$
$$+i\chi\gamma^b(1-\sigma^3)\nabla_b\chi + \frac{i}{24}\chi\gamma^{bcd}(1-\sigma^3)\chi\, H_{bcd}\Big] + \ldots = 0\,, \tag{A.89}$$

where the ellipsis denotes terms that vanish upon use of the other equations of motion. These give the remaining equations of motion for the bosonic superfields (3.40) and (3.41). The Bianchi identity for the 3-form gives no new constraints.

Dimension $\frac{5}{2}$. The highest component of the **type I** torsion Bianchi identity reads

$$\nabla_{[a}T_{bc]}{}^\alpha - T_{[ab}{}^\beta T_{c]\beta}{}^\alpha = 0\,. \tag{A.90}$$

It gives the Bianchi identity for the gravitino field strength

$$\nabla_{[a}\psi^{\alpha}_{bc]} + \frac{1}{8}(\gamma^{de}\psi_{[ab})^{\alpha}\, h_{c]de} - \frac{1}{48}(\gamma_{def[a}\psi_{bc]})^{\alpha}\, g^{def} = 0\,. \tag{A.91}$$

The condition coming from the symmetrized spinor derivative of ψ_{ab} using the dimension $\frac{3}{2}$ constraint on $\nabla_\alpha\psi_{ab}$ gives an equation for $\nabla_\alpha R_{ab}{}^{cd}$. The latter is more easily obtained from the curvature Bianchi identity (3.5) and reads

$$\begin{aligned}\nabla_\alpha R_{ab}{}^{cd} =& -i(\gamma_{[a}\nabla_{b]}\psi^{cd})_\alpha - i(\gamma^{[c}\nabla^{d]}\psi_{ab})_\alpha - \frac{i}{8}(\gamma^{ef}\gamma_{[a}\psi^{cd})_\alpha h_{b]ef} - \frac{i}{8}(\gamma_{ef}\gamma^{[c}\psi_{ab})_\alpha h^{d]ef}\\ &+ i(\gamma_e\psi^{[c}{}_{[a})_\alpha h_{b]}{}^{d]e} - \frac{i}{48}(\gamma_{efg[a}\gamma_{b]}\psi^{cd})_\alpha g^{efg} - \frac{i}{16}(\gamma^{cdefg}\psi_{ab})_\alpha g_{efg}\\ &- \frac{i}{16}(\gamma^{ef[c}\psi_{ab})_\alpha g^{d]}{}_{ef} - \frac{i}{4}(\gamma_{ef[a}\psi_{b]}{}^{[c})_\alpha g^{d]ef} + \frac{i}{12}(\gamma^{efg}\psi_{[a}{}^{[c})_\alpha \delta^{d]}_{b]} g_{efg}\,. \end{aligned}\tag{A.92}$$

Similarly, in the **type IIB** case we find

$$\nabla_{[a}\psi^{\alpha i}_{bc]} + \frac{1}{8}(\gamma^{de}\sigma^3\psi_{[ab})^{\alpha i}\, H_{c]de} + \frac{1}{8}(\mathcal{S}\gamma_{[a}\psi_{bc]})^{\alpha i} = 0\,, \tag{A.93}$$

$$\begin{aligned}\nabla_{\alpha i} R_{ab}{}^{cd} =& -i(\gamma_{[a}\nabla_{b]}\psi^{cd})_{\alpha i} - i(\gamma^{[c}\nabla^{d]}\psi_{ab})_{\alpha i} - \frac{i}{8}(\gamma^{ef}\gamma_{[a}\sigma^3\psi^{cd})_{\alpha i} H_{b]ef} - \frac{i}{8}(\gamma_{ef}\gamma^{[c}\sigma^3\psi_{ab})_{\alpha i} H^{d]ef}\\ &+ i(\gamma_e\sigma^3\psi^{[c}{}_{[a})_{\alpha i} H_{b]}{}^{d]e} + \frac{i}{8}(\gamma_{[a}\mathcal{S}\gamma_{b]}\psi^{cd})_{\alpha i} + \frac{i}{8}(\gamma^{[c}\mathcal{S}\gamma^{d]}\psi_{ab})_{\alpha i} + \frac{i}{4}(\gamma_{[a}\mathcal{S}\gamma^{[c}\psi_{b]}{}^{d]})_{\alpha i}\\ &- \frac{i}{4}(\gamma^{[c}\mathcal{S}\gamma_{[a}\psi_{b]}{}^{d]})_{\alpha i}\,. \end{aligned}\tag{A.94}$$

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