

Kato Classes for Lévy Processes

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Abstract We prove that the definitions of the Kato class through the semigroup and through the resolvent of the Lévy process in \mathbb{R}^d coincide if and only if 0 is not regular for $\{0\}$. If 0 is regular for $\{0\}$ then we describe both classes in detail. We also give an analytic reformulation of these results by means of the characteristic (Lévy-Khintchine) exponent of the process. The result applies to the time-dependent (non-autonomous) Kato class. As one of the consequences we obtain a simultaneous time-space smallness condition equivalent to the Kato class condition given by the semigroup.

Keywords Kato class · Lévy process · Lévy-Khintchine exponent · Schrödinger perturbation · Unimodal isotropic Lévy process · Subordinator · Polarity of a one point set

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1 Introduction

The Kato class plays an important role in the theory of stochastic processes and in the theory of pseudo-differential operators that emerge as generators of stochastic processes.

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The definition of the Kato class may differ according to the underlying probabilistic or analytical problem. In the first case the primary definition of the Kato condition is

$$\lim_{t \rightarrow 0^+} \left[\sup_x \mathbb{E}^x \left(\int_0^t |q(X_u)| du \right) \right] = 0. \tag{1}$$

Here q is a Borel function on the state space of the process $X = (X_t)_{t \geq 0}$. As shown in [13, section 3.2] through the Khas'minskii Lemma the condition yields sufficient local regularity of the corresponding Schrödinger (Feynman-Kac) semigroup

$$\tilde{P}_t f(x) = \mathbb{E}^x \left[\exp \left(- \int_0^t q(X_u) du \right) f(X_t) \right].$$

In particular, the existence of a density, strong continuity or strong Feller property are inherited under (1) from properties of the original semigroup $P_t f(x) = \mathbb{E}^x f(X_t)$ (for details and further results see [13, Theorems 3.10–3.12]). Moreover, if we denote by L the generator of $(P_t)_{t \geq 0}$, we expect the semigroup $(\tilde{P}_t)_{t \geq 0}$ to correspond to $L - q$ and to allow for the analysis of the Schrödinger operator $H = -L + q$ [14]. A fact that the Schrödinger operator is essentially self-adjoint and has bounded and continuous eigenfunctions is another consequence of Eq. 1, see [11, 32] and [18]. Applications of Eq. 1 to quadratic forms of Schrödinger operators are also known and we describe them shortly after Proposition 3.4.

The condition (1) can be understood as a smallness condition with respect to time. The alternative definition of the Kato condition is given by the following space smallness,

$$\lim_{r \rightarrow 0^+} \left[\sup_x \mathbb{E}^x \left(\int_0^\infty e^{-\lambda u} \mathbb{1}_{B(x,r)}(X_u) |q(X_u)| du \right) \right] = 0, \tag{2}$$

for some $\lambda > 0$ (equivalently for every $\lambda > 0$; see Lemma 3.2).

In this paper we obtain a precise description of the equivalence of Eqs. 1 and 2 for Lévy processes in \mathbb{R}^d , $d \in \mathbb{N}$. In order to formulate the result we recall that a point $x \in \mathbb{R}^d$ is said to be *regular* for a Borel set $B \subseteq \mathbb{R}^d$ if

$$\mathbb{P}^x(T_B = 0) = 1,$$

where $T_B = \inf\{t > 0 : X_t \in B\}$ is the first hitting time of B .

Theorem 1.1 *Let X be a Lévy process in \mathbb{R}^d . The conditions (1) and (2) are NOT equivalent if and only if 0 is regular for $\{0\}$.*

Complete and direct descriptions of Eqs. 1 and 2 in the case of the *compound Poisson process* are given in Proposition 3.8. When X is not a compound Poisson process and 0 is *regular for $\{0\}$* we fully describe (1) and (2) in Theorems 4.6 and 4.12. To move right away to Section 4 we recommend to read Definition 2 and Section 2.2 first. In Section 2.2 the reader will also find analytic characterization of the situation when 0 is regular for $\{0\}$.

In [11, Theorem III.1] Carmona, Masters and Simon declare that Eq. 1 can be expressed by Eq. 2 under additional assumptions on the transition density of the Lévy process. However, the general equivalence of (i) and (iii) from [11, Theorem III.1] that is claimed therein does not hold. As we show in Theorem 4.6 it fails for the Brownian motion in \mathbb{R} and for those one-dimensional unimodal Lévy processes for which $\{0\}$ is not polar. Recall that a Borel set $B \subseteq \mathbb{R}^d$ is called *polar* if

$$\mathbb{P}^x(T_B = \infty) = 1 \quad \text{for all } x \in \mathbb{R}^d.$$

For example the function $q(x) = \sum_{k=1}^{\infty} 2^k \mathbb{1}_{(k, k+2^{-k})}(x)$ satisfies (i), but fails to satisfy (iii) in [11, Theorem III.1] for such processes. The paper [11] was very influential and the mistake reappears in the literature. For instance (1) and (3) of [17, Proposition 4.5] are *not* equivalent in general.

The special character of the one-dimensional case can also be seen in [25, Remark 3.1]. In [25, Definition 3.1 and 3.2] the authors discuss the Kato class of measures for symmetric Markov processes admitting upper and lower estimates of transition density with additional integrability assumptions, see [25, Theorem 3.2].

Theorem 1.1 allows also for results on the time-dependent Kato class for Lévy processes in \mathbb{R}^d . Such a class is used for instance in [5, 7, 9, 36, 37]. See [31] for a wider discussion of the Brownian motion case, c.f. [31, Theorem 2].

Corollary 1.2 *Let X be a Lévy process in \mathbb{R}^d . For $q: \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ we have*

$$\lim_{t \rightarrow 0^+} \left[\sup_{s,x} \mathbb{E}^x \left(\int_0^t |q(s+u, X_u)| du \right) \right] = 0, \tag{3}$$

if and only if

$$\lim_{r \rightarrow 0^+} \left[\sup_{s,x} \mathbb{E}^x \left(\int_0^r \mathbb{1}_{B(x,r)}(X_u) |q(s+u, X_u)| du \right) \right] = 0. \tag{4}$$

See Section 4 for the proof. If one uses Corollary 1.2 for time-independent q , i.e., let $q: \mathbb{R}^d \rightarrow \mathbb{R}$ and put $q(u, z) = q(z)$, then the quantity in Eq. 3 coincides with Eq. 1 and we obtain the following reinforcement of Eq. 1 to a time-space smallness condition.

Theorem 1.3 *Let X be a Lévy process in \mathbb{R}^d . Then (1) holds if and only if*

$$\lim_{r \rightarrow 0^+} \left[\sup_x \mathbb{E}^x \left(\int_0^r \mathbb{1}_{B(x,r)}(X_u) |q(X_u)| du \right) \right] = 0. \tag{5}$$

In view of the equivalence of Eqs. 1 and 5 for every Lévy process (see Proposition 3.4 for other description of Eq. 1 true for Hunt processes) these conditions should be compared with Eq. 2 by its alternative form provided by Proposition 3.6 in a generality of a Hunt process, i.e.,

$$\lim_{r \rightarrow 0^+} \left[\sup_x \mathbb{E}^x \left(\int_0^t \mathbb{1}_{B(x,r)}(X_u) |q(X_u)| du \right) \right] = 0, \tag{6}$$

for some (every) fixed $t > 0$. The closeness or possible differences between Eqs. 1 and 2 are now more evident for Lévy processes through Eqs. 5 and 6.

The variety of conditions we point out is due to possible applications where one can choose a suitable version according to the knowledge about the process and derive a clear *analytic description* of the Kato condition (1). See also Theorems 4.14 and 4.15 for other conditions. For instance, in Example 1 we apply Theorem 1.1 and we make use of Eq. 6. On the other hand, by Theorem 1.1 and Eq. 2 we obtain that for a large class of subordinators (1) is equivalent to

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_0^r |q(z+x)| \frac{\phi'(z^{-1})}{z^2 \phi^2(z^{-1})} dz = 0, \tag{7}$$

where ϕ is the Laplace exponent of the subordinator. See Section 5.2 for details. This is also usual that from Eqs. 2 and 6 one learns, like through Eq. 7, about acceptable singularities of q . Schrödinger perturbations of subordinators are interesting since they exhibit peculiar properties that indicate complexity of the matter. For instance, we easily see that if q is bounded, then $\tilde{P}_t f(x) \leq c_N P_t f(x)$ for every $t \in (0, N], x \in \mathbb{R}, f \geq 0$. On the other hand, if $-q \geq 0$ is time-independent and the above inequality holds for some $N > 0$ on the level of densities, then necessarily $q \in L^\infty(\mathbb{R})$ (see [5, Corollary 3.4]). Nevertheless, perturbation techniques yield an upper bound by means of an auxiliary density for (unbounded) q from the Kato class if an appropriate 4G inequality for the transition density of the subordinator holds (see [5, Proposition 2.4]). Generators of subordinators generalize fractional derivative operators that are used in statistical physics to model anomalous subdiffusive dynamics (see [16]).

A discussion of analytic counterparts of Eq. 1 should contain the fundamental example of the standard Brownian motion in $\mathbb{R}^d, d \in \mathbb{N}$. The famous result of Aizenman and Simon [1, Theorem 4.5] says that in this case (1) is equivalent to

$$\lim_{t \rightarrow 0^+} \left[\sup_x \int_{|z-x| < \sqrt{t}} \frac{|q(z)|}{|z-x|^{d-2}} dz \right] = 0, \quad \text{for } d \geq 3, \tag{8}$$

$$\lim_{t \rightarrow 0^+} \left[\sup_x \int_{|z-x| < \sqrt{t}} |q(z)| \ln \frac{t}{|z-x|^2} dz \right] = 0, \quad \text{for } d = 2, \tag{9}$$

$$\left[\sup_x \int_{|z-x| < 1} |q(z)| dz \right] < \infty, \quad \text{for } d = 1.$$

Here we also refer to Simon [32, Proposition A.2.6], Chung and Zhao [13, Theorem 3.6], Demuth and van Casteren [14, Theorem 1.27]. The above remains true if $\ln(t/|z-x|^2)$ is replaced by $\ln(1/|z-x|)$ for $d = 2$ and if $|q(z)|$ is multiplied by $|z-x|$ for $d = 1$. In fact, the expressions in square brackets of Eqs. 1 and 8 are comparable for $d \geq 3$, while for $d = 2$ and $d = 1$ similar but slightly different results hold (see Bogdan and Szczypkowski [9], Demuth and van Casteren [14, Theorem 1.28]). We emphasise that (8) was used by Kato [20] to prove by analytic methods that the operator $-\Delta + q$ is essentially self-adjoint (see [21] for extensions to second order elliptic operators). The equivalence of Eq. 1 with Eqs. 8 and 9 follows also from Theorem 1.1 (see [38]). The one-dimensional case is also covered by Theorem 4.6 of this paper.

In what follows we present and explain our main ideas in view of the literature. A major contribution to the understanding of the subject in a general probabilistic manner is made by Zhao [38]. Zhao considers a Hunt process $X = (\Omega, \mathcal{F}_t, X_t, \vartheta_t, \mathbb{P}^x)$ with state space (S, ρ) and life-time ζ , where S is a locally compact metric space with a metric ρ (see [4]). For a strong sub-additive functional A_t of $X, t \geq 0$, he discusses relations between the following three conditions

$$\lim_{r \rightarrow 0^+} \left\{ \sup_x \mathbb{E}^x \left[\int_0^\infty \mathbb{1}_{B(x,r)}(X_t) dA_t \right] \right\} = 0, \tag{C1}$$

$$\lim_{t \rightarrow 0^+} \left[\sup_x \mathbb{E}^x (A(t)) \right] = 0, \tag{C2}$$

$$\lim_{r \rightarrow 0^+} \left\{ \sup_x \mathbb{E}^x [A(\tau_{B(x,r)})] \right\} = 0, \tag{C3}$$

in presence of three hypotheses on the process X ,

$$h_1(X) \equiv \sup_{t>0} \inf_{r>0} \sup_{x \in S} \mathbb{P}^x (\tau_{B(x,r)} > t) < 1, \tag{H1}$$

$$h_2(X) \equiv \sup_{r>0} \inf_{t>0} \sup_{x \in S} \mathbb{P}^x (\tau_{B(x,r)} < t) < 1, \tag{H2}$$

$$h_3(X) \equiv \sup_{u>0} \inf_{r>0} \sup_{\substack{x, y \in S \\ \rho(x,y) \geq u}} \mathbb{P}^y (T_{B(x,r)} < \zeta) < 1. \tag{H3}$$

Here for any Borel set B in S , T_B is the first hitting time of B , $\tau_B = T_{S \setminus B}$ is the first exit time of B (we let $\inf \emptyset = \infty$) and $B(x, r) = \{y \in S : \rho(x, y) < r\}$, $x \in S, r > 0$. We present the main theorem of Zhao [38] on Fig. 1 below; for instance, under (H3), (C3) implies (C1).

In this paper we assume that $A_t, t \geq 0$, is the additive functional of the form

$$A_t = \int_0^t |q(X_u)| du, \tag{10}$$

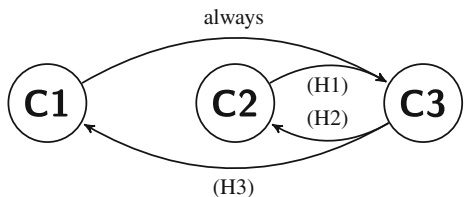
and we note that any additive functional is a strong sub-additive functional; see [38, Lemma 1]. Then (C2) coincides with Eq. 1 and as such becomes the principal object of our considerations. We explain the origin and the choice of Eq. 2 using the concept of λ -subprocess $X^\lambda, \lambda > 0$, of the process X (see [4] for the definition). We first notice that (C2) holds for X if and only if it holds for X^λ (see Remark 9 and Definition 2). A similar statement is not true in general for (C1). For the standard Brownian motion in $\mathbb{R}^d, d \geq 3$, (C2) in fact coincides with (C1), which gives rise to Eq. 8, yet for $d = 2$ or $d = 1$ the expectation in (C1) is infinite for constant non-zero q , whereas that never happens for (C2). This shows that (C1) for X is too strong for a general equivalence result. Therefore we rely on the relations of Fig. 1 for X^λ , and then (C1) results in Eq. 2. We also observe that Eq. 2 holds for X if and only if it holds for $X^{\lambda'}, \lambda' > 0$ (see Remark 9). To ultimately clarify the choice of X^λ we note that $h_1(X^\lambda) = h_1(X), h_2(X^\lambda) = h_2(X)$ and $h_3(X^\lambda) \leq h_3(X)$ (see Lemmas 2.10 and 2.11).

We now restrict ourselves to the case of the Lévy process in \mathbb{R}^d . Besides being a Hunt process in \mathbb{R}^d, X is also translation invariant. We point out that (H2) holds for every Lévy process and (H1) holds if and only if X is not a compound Poisson process (see Remark 8). The case of the compound Poisson process is entirely described in Proposition 3.8. Thus, in the remaining cases, (H3) for X^λ becomes decisive for understanding the confines of the applicability of Fig. 1 to X^λ . By Proposition 2.15 the study of $h_3(X^\lambda)$ reduces to the analysis of the first hitting time of a single point set by the original Lévy process X . Namely, we consider (see also Lemma 4.2)

$$h^\lambda(x) := \mathbb{E}^0 e^{-\lambda T_{\{x\}}}, \quad x \in \mathbb{R}^d. \tag{11}$$

Eventually, by Corollary 2.16 and Remark 8 we obtain the following characterization.

Fig. 1 Zhao [38] hypotheses and conditions



Proposition 1.4 *Let X be a Lévy process in \mathbb{R}^d and $\lambda > 0$. All hypotheses (H1), (H2) and (H3) are satisfied for X^λ if and only if $\{0\}$ is polar.*

Therefore Theorem 1.1 goes much beyond the range of [38]. The reason is that in our work we also investigate all the cases that are not covered by Fig. 1. Our initial study effects in a list that classifies Lévy processes according to a non-degeneracy hypothesis (H0) and specific properties of h^λ , which is thoroughly examined by Bretagnolle [10] for one-dimensional non-Poisson Lévy processes. A full layout of our development is presented in Section 2.2. Theorem 1.1 results as a summary of Proposition 3.8 and 6 theorems of Section 4. We stress that the non-symmetric cases or those close to the compound Poisson process (without (H0)) are more delicate and require more precision.

In [38, Lemma 4] Zhao proposes sufficient conditions on X under which (H1)-(H3) are satisfied for X^λ . He uses them to re-prove the result of Aizenman and Simon [1] for $d \geq 2$. He also verifies hypotheses (H1)-(H3) directly for X in the case of Lévy processes admitting rotationally symmetric transition density with additional assumption on the behaviour of the density integrated in time [38, Lemma 5]. Finally he applies that to describe (1) for symmetric α -stable processes, $d > \alpha$, and the relativistic process. We generalize [38, Lemma 5] in Theorem 4.15.

The paper is organized as follows. In Section 2 we introduce the non-degeneracy hypothesis (H0) for a Lévy process. Next, we give a classification of Lévy processes that provides a detailed plan of our research. In the last part of Section 2 we prove results concerning hypotheses (H1)-(H3). In Section 3, for a Hunt process X , we define Kato classes $\mathbb{K}(X)$ and $\mathcal{K}(X)$ of functions q satisfying (1) and (2), respectively. We give other general descriptions of both of those classes and we establish their initial relations for Lévy processes. In Section 4 we prove the main description theorems for Lévy processes, separately under and without (H0). Section 4 ends with additional equivalence results involving the class $\mathcal{K}^0(X)$ (see (26)). In Section 5 we present a supplementary discussion on isotropic unimodal Lévy processes and subordinators. The paper finishes with examples.

2 Preliminaries

Our main focus in this paper is on a (general) Lévy process X in \mathbb{R}^d (see [29]). The characteristic exponent ψ of X defined by $\mathbb{E}^0 e^{i\langle x, X_t \rangle} = e^{-t\psi(x)}$ equals

$$\psi(x) = -i \langle x, \gamma \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x, z \rangle} - 1 - i \langle x, z \rangle \mathbb{1}_{|z| < 1} \right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma \in \mathbb{R}^d$, A is a symmetric non-negative definite matrix and ν is a Lévy measure, i.e., $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$. If $\int_{\mathbb{R}^d} (1 \wedge |z|) \nu(dz) < \infty$, then the above representation simplifies to

$$\psi(x) = -i \langle x, \gamma_0 \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left(e^{i\langle x, z \rangle} - 1 \right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma_0 = \gamma - \int_{\mathbb{R}^d} z \mathbb{1}_{|z| < 1} \nu(dz)$. Further, if $\gamma_0 = 0$, $A = 0$ and $\nu(\mathbb{R}^d) < \infty$, then X is called a compound Poisson process (see [29, Remark 27.3]). We say that X is non-Poisson if X is not a compound Poisson process. Recall that $\mathbb{E}^x F(X) = \mathbb{E}^0 F(X+x)$ for $x \in \mathbb{R}^d$ and Borel functions $F \geq 0$ on paths. In particular $h^\lambda(x) = \mathbb{E}^{(-x)} e^{-\lambda T_{[0]}}$, and thus the following holds.

Remark 1 $\{0\}$ is polar if and only if $h^\lambda(x) = 0, x \in \mathbb{R}^d$.

Remark 2 0 is regular for $\{0\}$ if and only if $h^\lambda(0) = 1$.

Remark 3 X is such that $A = 0, \gamma_0 \in \mathbb{R}^d, \int_{\mathbb{R}^d} (|x| \wedge 1) \nu(dx) < \infty$ if and only if X has finite variation on finite time intervals ([29, Theorem 21.9]). Then $\mathbb{P}^0(\lim_{s \rightarrow 0^+} s^{-1} X_s = \gamma_0) = 1$ ([33, Theorem 1]; see also [29, Theorem 43.20]).

Lemma 2.1 *Let X be non-Poisson. Then $\mathbb{P}^0(X_t = 0) = 0$ except for countably many $t > 0$.*

Proof By [29, Theorem 27.4] it suffices to consider compound Poisson process with non-zero drift. Let then ν and γ_0 be its Lévy measure and drift. According to the decomposition $\nu = \nu^d + \nu^c$ for discrete and continuous part (see [29, Chapter 5, Section 27]) we write $X_t = X_t^d + X_t^c + \gamma_0 t$. For $t > 0$, by [29, Remark 27.3] $\mathbb{P}^0(X_t^c \in dz)$ is continuous on $\mathbb{R}^d \setminus \{0\}$, therefore $\mathbb{P}^0(X_t^c \in C \setminus \{0\}) = 0$ for any countable set $C \subset \mathbb{R}^d$. By [29, Corollary 27.5 and Proposition 27.6] there is a countable set $C_{X^d} \subset \mathbb{R}^d$ such that $\mathbb{P}^0(X_t^d + \gamma_0 t = 0) > 0$ if and only if $(-\gamma_0 t) \in C_{X^d}$. Thus $\mathbb{P}^0(X_t^d + \gamma_0 t = 0) = 0$ except for countably many $t > 0$. Finally,

$$\begin{aligned} \mathbb{P}^0(X_t^d + X_t^c + \gamma_0 t = 0) &= \mathbb{P}^0(X_t^c = 0, X_t^d + \gamma_0 t = 0) \\ &\quad + \mathbb{P}^0(X_t^c = -(X_t^d + \gamma_0 t), X_t^d + \gamma_0 t \neq 0) \\ &\leq \mathbb{P}^0(X_t^d + \gamma_0 t = 0) + \mathbb{P}^0(X_t^c \in -(C_{X^d} + \gamma_0 t) \setminus \{0\}) = 0, \end{aligned}$$

except for countably many $t > 0$. □

We say that a Lévy process X is non-sticky if $\mathbb{P}^0(\tau_{\{0\}} > 0) = 0$, or equivalently that the hypothesis (H) from [10] holds. Lemma 2.1 reinforces remarks following [38, Lemma 3].

Remark 4 X is non-sticky if and only if X is non-Poisson.

If necessary we specify which Lévy process we have in mind by adding a superscript, for instance $h^{Z,\lambda}$ is the function given by Eq. 11 that corresponds to the process Z .

2.1 Non-Degeneracy Hypothesis (H0) for Lévy Processes

Before we introduce the main non-degeneracy hypothesis on a Lévy process X we recall the basic matrix notation. Let M be a matrix. We let M^* to be the transpose of M and $M(\mathbb{R}^d)$ the range of M . We call M a projection if it is symmetric and $M^2 = M$. For a subset V by V^\perp we denote the orthogonal complement of V in \mathbb{R}^d . We use the following fact.

Lemma 2.2 *If A is symmetric non-negative definite and $M^*AM = 0$, then $A(\mathbb{R}^d) \subseteq M(\mathbb{R}^d)^\perp$.*

Remark 5 Let X be a Lévy process in a linear subspace V of \mathbb{R}^d (see [29, Proposition 24.17]) and denote $d_0 = \dim(V)$. Then there exists a rotation given by a matrix $O \in \mathcal{M}_{d \times d}$ such that $Y = OX$ is a Lévy process in \mathbb{R}^{d_0} ; the correspondence between X and Y is one-to-one.

Lemma 2.3 *Let X be a Lévy process in \mathbb{R}^d and Π be a projection. If $\{0\}$ is polar for the process $Y = \Pi X$, then $\{0\}$ is polar for X .*

Proof If $X_t + x = 0$, then $Y_t + \Pi x = 0$, thus $\inf\{t > 0: X_t + x = 0\} \geq \inf\{t > 0: Y_t + \Pi x = 0\}$ and $\mathbb{P}^x(T_{\{0\}} < \infty) \leq \mathbb{P}^{\Pi x}(T_{\{0\}}^Y < \infty) = 0$. \square

Definition 1 We say that (H0) holds for X if there is no linear subspace V of \mathbb{R}^d such that

$$\dim(V) \leq \min\{1, d - 1\}, \quad A(\mathbb{R}^d) \subseteq V, \quad \nu(\mathbb{R}^d \setminus V) < \infty, \quad \text{and} \quad \gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z) \nu(dz) \in V. \quad (12)$$

We give a precise probabilistic description of (H0).

Remark 6 For $d = 1$, (H0) holds if and only if X is non-Poisson. For $d > 1$, (H0) holds if and only if X is non-Poisson and is not of the form Eq. 13 below.

Proposition 2.4 *Let $d > 1$ and X be non-Poisson. Then (H0) does not hold if and only if*

$$X = Y + Z, \quad (13)$$

and there exist a linear subspace V of \mathbb{R}^d , $\dim(V) = 1$, such that

- i) Y and Z are independent,
- ii) Y is either zero or a compound Poisson process with the Lévy measure vanishing on V ,
- iii) Z is not a compound Poisson process,
- iv) Z is supported on V .

Proof Since we assume that X is non-Poisson, if Eq. 12 holds and $\dim(V) \leq \min\{1, d - 1\}$, then $\dim(V) = 1$. We let Y to be a compound Poisson process with the Lévy measure $\nu^Y = [\nu]_{\mathbb{R}^d \setminus V}$ and let Z to be a Lévy process with the Lévy triplet $(A, \gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z) \nu(dz), [\nu]_V)$, where $[\nu]_B$ denotes the measure ν restricted to a set B . By definition $\psi = \psi^Y + \psi^Z$, hence $X = Y + Z$ and i), ii) and iii) are satisfied. The property iv) follows from [29, Proposition 24.17]. Conversely, if X is of the form (13), then its Lévy triplet is given by $A = A^Z, \gamma = \gamma^Z + \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z) \nu^Y(dz)$ and $\nu = \nu^Y + \nu^Z$. Then Eq. 12 holds since $\nu = \nu^Y$ on $\mathbb{R}^d \setminus V$. \square

The hypothesis (H0) agrees with the hypothesis (H) from [10] if $d = 1$. In particular, for $d = 1$ under (H0) we have that $\{0\}$ is essentially polar if and only if $\{0\}$ is polar. As known, in $d > 1$ $\{0\}$ is always essentially polar (see [3, Theorem 16 and Corollary 17]).

Proposition 2.5 *Let $d > 1$ and assume (H0). Then $\{0\}$ is polar.*

Proof Let V be the smallest in dimension linear subspace in \mathbb{R}^d satisfying Eq. 12. Now, let Π_1 be the projection on V and define $Y = \Pi_1 X$. Observe that by (H0) we have $\dim(V) \geq 2$. We claim that there is no one-dimensional subspace $W \subset V$ such that the projection of Y on W is a compound Poisson process. For the proof assume that there is such W and let Π_2 be the projection on W . Then $Z = \Pi_2 Y = \Pi_2 X$ is a compound Poisson process. By

[29, Proposition 11.10] we have the following consequences. First, $\Pi_2 A \Pi_2 = 0$ and by Lemma 2.2 we obtain $A(\mathbb{R}^d) \subseteq V \cap W^\perp$. Next, $\nu(\mathbb{R}^d \setminus W^\perp) = \nu \Pi_2^{-1}(\mathbb{R}^d \setminus \{0\}) < \infty$ and then $\nu(\mathbb{R}^d \setminus (V \cap W^\perp)) < \infty$. Further, since $\Pi_2 z = 0$ on $V \cap W^\perp$ we have

$$\begin{aligned} 0 &= \Pi_2 \gamma - \int_{\mathbb{R}^d} \Pi_2 z \mathbb{1}_{B(0,1)}(z) \nu(dz) \\ &= \Pi_2 \gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} \Pi_2 z \mathbb{1}_{B(0,1)}(z) \nu(dz) \\ &= \Pi_2 \left(\gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} z \mathbb{1}_{B(0,1)}(z) \nu(dz) \right). \end{aligned}$$

Thus $\gamma_1 = \gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} z \mathbb{1}_{B(0,1)}(z) \nu(dz) \in W^\perp$. Finally, by $\mathbb{R}^d \setminus (V \cap W^\perp) = (\mathbb{R}^d \setminus V) \dot{\cup} (V \setminus W^\perp)$ and by Eq. 12,

$$\gamma_1 = \left(\gamma - \int_{\mathbb{R}^d \setminus V} z \mathbb{1}_{B(0,1)}(z) \nu(dz) \right) - \int_{V \setminus W^\perp} z \mathbb{1}_{B(0,1)}(z) \nu(dz) \in V,$$

which is a contradiction, because then Eq. 12 holds with $V \cap W^\perp$ in place of V and $\dim(V \cap W^\perp) < \dim(V)$. Now, by Remark 5 we can treat Y as a process in \mathbb{R}^{d_0} , $d_0 = \dim(V) \geq 2$, and then by [10, Theoreme 4] the set $\{0\}$ is a polar set for Y as well as for X by Lemma 2.3. □

2.2 Classification of Lévy Processes

We outline our work-flow to analyze every Lévy process X .

Exclusively one of the following situations holds for a Lévy process in \mathbb{R}^d .

1. (H0) holds:

- (a) $d > 1$ (then $h^\lambda(x) = 0, x \in \mathbb{R}^d$),
- (b) $d = 1$
 - (A) $h^\lambda(x) = 0, x \in \mathbb{R}$,
 - (B) $h^\lambda(0) = \liminf_{x \rightarrow 0} h^\lambda(x) < \limsup_{x \rightarrow 0} h^\lambda(x) = 1$,
 - (C) $h^\lambda(0) = \lim_{x \rightarrow 0} h^\lambda(x) = 1$.

2. (H0) does not hold:

- (a) a compound Poisson process ($d \geq 1$; then $h^\lambda(0) = 1$),
- (b) given by (13) ($d > 1$)
 - (A') $h^{Z,\lambda}(v) = 0, v \in V$,
 - (B') $h^{Z,\lambda}(0) = \liminf_{v \in V, v \rightarrow 0} h^{Z,\lambda}(v) < \limsup_{v \in V, v \rightarrow 0} h^{Z,\lambda}(v) = 1$,
 - (C') $h^{Z,\lambda}(0) = \lim_{v \in V, v \rightarrow 0} h^{Z,\lambda}(v) = 1$.

The comment in the case case 1(a) is a consequence of Proposition 2.5 and Remark 1. The partition of the case 1(b) is due to Remarks 6, 4 and [10, Théorèmes 3 and 6]. The division of the case 2 results from Remark 6. The subcases of 2(b) follow from Remark 5 and [10].

The subcases of 1(b) translate equivalently into probabilistic properties of X , see [10, Théorèmes 6, 8] and Remark 3. We have

- (A) $\{0\}$ is polar,
- (B) X has finite variation and non-zero drift,
- (C) 0 is regular for $\{0\}$.

The analytic counterpart by means of the characteristic exponent or the Lévy triplet is (see [10, Théorèmes 3, 7 and 8])

- (A) $\int_{\mathbb{R}} \operatorname{Re} \left(\frac{1}{\lambda + \psi(z)} \right) dz = \infty,$
- (B) $A = 0, \gamma_0 \neq 0$ and $\int_{\mathbb{R}} (|x| \wedge 1) \nu(dx) < \infty,$
- (C) $A \neq 0$ or (A) does not hold and $\int_{\mathbb{R}} (|x| \wedge 1) \nu(dx) = \infty.$

We could similarly reformulate 2(b) for Z , but in proofs of Theorems 4.11 and 4.12 we use the following description.

- (A') $\int_V \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(v)} \right) dv = \infty$ (dv is the one-dimensional Lebesgue measure on V),
- (B') $A^Z = 0, \gamma_0^Z \neq 0$ and $\int_V (|x| \wedge 1) \nu^Z(dx) < \infty,$
- (C') 0 is regular for $\{0\}.$

We translate (A'), (B') and (C') into X given by Eq. 13.

Lemma 2.6 $\{0\}$ is polar for X if and only if $\{0\}$ is polar for Z .

Proof If $\{0\}$ is polar for Z , then $\int_V \operatorname{Re}(1/[\lambda + \psi^Z(v)])dv = \infty$. By Lemma 2.3 to verify that $\{0\}$ is polar for X it suffices to show that it is polar for $\Pi X = \Pi(Y + Z) = \Pi Y + Z$, where Π is the projection on V . Since $\psi^{\Pi X} = \psi^{\Pi Y} + \psi^Z$ and $\psi^{\Pi Y}$ is bounded (ΠY is a compound Poisson process) we have by our assumption $\int_V \operatorname{Re}(1/[\lambda + \psi^{\Pi X}(v)])dv = \infty$. Thus Remark 5 and [10, Théorèmes 7, 3] end this part of the proof. If $\{0\}$ is not polar for Z , $\mathbb{P}^0(T_{\{x\}}^Z < \infty) > 0$ for some $x \in V$, we have for large $t > 0$

$$\mathbb{P}^0(T_{\{x\}} < \infty) \geq \mathbb{P}^0\left(Y_t = 0, T_{\{x\}} = T_{\{x\}}^Z < t\right) = \mathbb{P}^0(Y_t = 0)\mathbb{P}^0\left(T_{\{x\}}^Z < t\right) > 0.$$

□

Lemma 2.7 $\{0\}$ is not polar for X if and only if $\limsup_{x \rightarrow 0} h^\lambda(x) = 1$.

Proof If $\limsup_{x \rightarrow 0} h^\lambda(x) = 1$, then $h^\lambda(x) > 0$ for some $x \in \mathbb{R}^d$ and $\mathbb{P}^0(T_{\{x\}} < \infty) > 0$. Conversely, if $\{0\}$ is not polar for X then by Lemma 2.6 it is not polar for Z and $\limsup_{v \in V, v \rightarrow 0} h^{Z, \lambda}(v) = 1$. This implies $\limsup_{v \in V, v \rightarrow 0} \mathbb{P}^0(T_{\{v\}}^Z < t) = 1$ for every fixed $t > 0$. Thus we have for $t > 0$

$$\begin{aligned} h^\lambda(x) &\geq \mathbb{E}^0\left(Y_t = 0, T_{\{x\}}^Z < t; e^{-\lambda T_{\{x\}}}\right) = \mathbb{E}^0\left(Y_t = 0, T_{\{x\}}^Z < t; e^{-\lambda T_{\{x\}}^Z}\right) \\ &\geq \mathbb{P}^0(Y_t = 0)\mathbb{P}^0\left(T_{\{x\}}^Z < t\right) e^{-\lambda t}, \end{aligned}$$

which gives $\limsup_{x \rightarrow 0} h^\lambda(x) \geq \mathbb{P}^0(Y_t = 0)e^{-\lambda t}$. Finally, we let $t \rightarrow 0^+$.

□

Lemma 2.8 0 is regular for $\{0\}$ for X if and only if 0 is regular for $\{0\}$ for Z .

Proof We observe that the set $\{Y_s = 0 \text{ for all } s \in [0, \delta] \text{ for some } \delta > 0\}$ is of measure one with respect to \mathbb{P}^0 . On that set $T_{\{0\}} = 0$ if and only if $T_{\{0\}}^Z = 0$.

□

Corollary 2.9 For the process X of the form Eq. 13 we have

$$(A') \quad h^\lambda(x) = 0, x \in \mathbb{R}^d,$$

- (B') $h^\lambda(0) < \limsup_{x \rightarrow 0} h^\lambda(x) = 1,$
- (C') $h^\lambda(0) = \limsup_{x \rightarrow 0} h^\lambda(x) = 1,$

and

- (A') $\{0\}$ is polar;
- (B') X has finite variation and non-zero drift (see Remark 3),
- (C') 0 is regular for $\{0\}$.

The last observation facilitates a discussion of (H3) in the next subsection.

Remark 7 For a non-Poisson Lévy process we have $\limsup_{x \rightarrow 0} h^\lambda(x) = 1$ or $h^\lambda(x) = 0, x \in \mathbb{R}^d$.

2.3 Hypotheses (H1)-(H3)

We start with a general case of a Hunt process X on S with life-time ζ . In the proofs of Lemmas 2.10 and 2.11 all objects corresponding to X^λ , the λ -subprocess of X , are indicated with a bar, e.g., $\bar{T}_B = \inf\{t > 0: X_t^\lambda \in B\}$.

Lemma 2.10 *Let $\lambda > 0$. We have $h_1(X^\lambda) = h_1(X)$ and $h_2(X^\lambda) = h_2(X)$.*

Proof Recall that $\inf \emptyset = \infty$. For any Borel set B in S and $t > 0$ we have $\{\bar{\tau}_B > t\} = \{\tau_B > t\} \times [0, \infty) \dot{\cup} \{\tau_B \leq t\} \times [0, \tau_B)$ and $\{\bar{\tau}_B < t\} = \{\tau_B < t\} \times (\tau_B, \infty)$. Thus,

$$\bar{\mathbb{P}}^x(\bar{\tau}_B > t) = \mathbb{P}^x(\tau_B > t) + \mathbb{E}^x(\tau_B \leq t; 1 - e^{-\lambda\tau_B}) \leq \mathbb{P}^x(\tau_B > t) + 1 - e^{-\lambda t},$$

and

$$\begin{aligned} \bar{\mathbb{P}}^x(\bar{\tau}_B < t) &= \mathbb{E}^x(\tau_B < t; e^{-\lambda\tau_B}) = \mathbb{P}^x(\tau_B < t) + \mathbb{E}^x(\tau_B < t; e^{-\lambda\tau_B} - 1) \\ &\geq \mathbb{P}^x(\tau_B < t) + e^{-\lambda t} - 1. \end{aligned}$$

Since we may change $\sup_{t>0}$ with $\limsup_{t \rightarrow 0^+}$, $h_1(X) \leq h_1(X^\lambda) \leq h_1(X) + \lim_{t \rightarrow 0^+} (1 - e^{-\lambda t})$ and since we may replace $\inf_{t>0}$ with $\liminf_{t \rightarrow 0^+}$, $h_2(X) \geq h_2(X^\lambda) \geq h_2(X) + \lim_{t \rightarrow 0^+} (e^{-\lambda t} - 1)$. This ends the proof. \square

Lemma 2.11 *Let $\lambda > 0$. We have $h_3(X^\lambda) \leq h_3(X)$, more precisely*

$$h_3(X^\lambda) = \sup_{u>0} \inf_{r>0} \sup_{\substack{x, y \in S \\ \rho(x, y) \geq u}} \mathbb{E}^y(T_{B(x, r)} < \zeta; e^{-\lambda T_{B(x, r)}}).$$

Proof For any Borel set B in S we have $\{\bar{T}_B < \bar{\zeta}\} = \{T_B < \zeta\} \times (T_B, \infty)$. This results in $\bar{\mathbb{P}}^y(\bar{T}_B < \bar{\zeta}) = \mathbb{E}^y(T_B < \zeta; e^{-\lambda T_B})$. \square

Now, let $S = \mathbb{R}^d$ be the Euclidean space and $\zeta = \infty$. The following lemmas and corollary address the question whether $h_3(X^\lambda) = \sup_{u>0} \inf_{r>0} \sup_{|x-y| \geq u} \mathbb{E}^y e^{-\lambda T_{B(x, r)}} < 1$.

Lemma 2.12 *Let $x \in \mathbb{R}^d$ be fixed. Then*

$$\lim_{r \rightarrow 0^+} T_{\bar{B}(x, r)} = T_{\{x\}} \quad \mathbb{P}^0 \text{ a.s.} \tag{14}$$

Proof Fix $x \in \mathbb{R}^d$. Define the stopping times $T_r = T_{\overline{B}(x,r)}$ and $T = \lim_{r \rightarrow 0^+} T_r, r > 0$. Obviously, $T_r \leq T \leq T_{\{x\}}$. It suffices to consider (14) on the set $\{T < \infty\}$, otherwise both sides of Eq. 14 are infinite. Since T_r is non-increasing in $r > 0$ we have by the quasi-left continuity $\lim_{r \rightarrow 0^+} X_{T_r} = X_T$ a.s. on $\{T < \infty\}$. On the other hand, by the right continuity we have $X_{T_r} \in \overline{B}(x, r)$ and thus $\lim_{r \rightarrow 0^+} X_{T_r} = x$ a.s. on $\{T < \infty\}$. Finally, $X_T = x$ and consequently $T \geq T_{\{x\}}$ a.s. on $\{T < \infty\}$. \square

Lemma 2.13 *Let $\tau_n = \tau_{B(0,n)}$. Then $\lim_{n \rightarrow \infty} \tau_n = \infty$ \mathbb{P}^0 a.s.*

Proof Denote $\tau = \lim_{n \rightarrow \infty} \tau_n$. Since τ_n is non-decreasing, by the quasi-left continuity $X_{\tau_n} \xrightarrow{n \rightarrow \infty} X_\tau$ a.s. on $\{\tau < \infty\}$. On $\{\tau < \infty\}$ for $n \geq |X_\tau| + 1$ by the right continuity we have $|X_{\tau_n}| \geq |X_\tau| + 1$, which is a contradiction; it shows that a.s $\tau < \infty$ does not occur. \square

Lemma 2.14 *Let $\lambda > 0$. Then*

$$\sup_{u>0} \inf_{r>0} \sup_{|x|\geq u} \mathbb{E}^0 e^{-\lambda T_{\overline{B}(x,r)}} = \sup_{x \neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}} . \tag{15}$$

Proof Let $f_r(x) = \mathbb{E}^0 e^{-\lambda T_{\overline{B}(x,r)}}, r \geq 0, x \in \mathbb{R}^d$, where $\overline{B}(x, 0) = \{x\}$. Notice that $f_r(x) \geq f_0(x)$. Therefore

$$a := \sup_{u>0} \inf_{r>0} \sup_{|x|\geq u} f_r(x) \geq \sup_{u>0} \inf_{r>0} \sup_{|x|\geq u} f_0(x) = \sup_{u>0, |x|\geq u} f_0(x) = \sup_{x \neq 0} f_0(x) \geq 0 . \tag{16}$$

It suffices to prove the reverse inequality in the case $a \neq 0$, otherwise (15) holds by Eq. 16. Thus let $a \in (0, 1]$. Then for $\varepsilon > 0$ there is $u > 0$ such that for all $r > 0$ we have $\sup_{|x|\geq u} f_r(x) > a - \varepsilon$. Hence, there is a sequence $\{x_n\}$ such that $f_{1/n}(x_n) > a - \varepsilon$ and $|x_n| \geq u$. We will show that $\{x_n\}$ is bounded. For $r \in (0, 1], m \in \mathbb{N}$ and $|x| \geq m + 2$, we have $T_{\overline{B}(x,r)} \geq \tau_m$ thus by Lemma 2.13 and the dominated convergence theorem there is m_0 such that

$$\sup_{|x|\geq m_0+2} f_r(x) \leq \mathbb{E}^0 e^{-\lambda \tau_{m_0}} \leq a - \varepsilon .$$

This proves that $m_0 + 2 \geq |x_n| \geq u > 0$ for every n . We let $y \neq 0$ to be the limit point of $\{x_n\}$. Observe that for every $r > 0$ there is n such that $B(x_n, 1/n) \subseteq B(y, r)$, which implies $T_{\overline{B}(y,r)} \leq T_{\overline{B}(x_n,1/n)}$ and $f_r(y) \geq f_{1/n}(x_n) > a - \varepsilon$. Finally, by Lemma 2.12 and the dominated convergence theorem we obtain

$$\sup_{x \neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}} \geq \mathbb{E}^0 e^{-\lambda T_{\{y\}}} = \lim_{r \rightarrow 0} \mathbb{E}^0 e^{-\lambda T_{\overline{B}(y,r)}} = \lim_{r \rightarrow 0} f_r(y) \geq a - \varepsilon .$$

This ends the proof since $\varepsilon > 0$ was arbitrary. \square

We continue discussing (H1)-(H3) for a Lévy process X in \mathbb{R}^d . Remark 4 and [38, Lemmas 2 and 3] ensure the following.

Remark 8 Clearly (H1) does not hold for any compound Poisson process.

(H1) holds for every non-Poisson Lévy process X with $h_1(X) = 0$.

(H2) holds for every Lévy process X with $h_2(X) = 0$.

Proposition 2.15 *Let X be a Lévy process in \mathbb{R}^d and $\lambda > 0$. For h^λ defined in Eq. 11 we have*

$$h_3(X^\lambda) = \sup_{x \neq 0} h^\lambda(x).$$

Proof By Lemma 2.11, $\overline{B}(x, r/2) \subseteq B(x, r) \subseteq \overline{B}(x, r)$ and Lemma 2.14

$$\begin{aligned} h_3(X^\lambda) &= \sup_{u>0} \inf_{r>0} \sup_{|x-y| \geq u} \mathbb{E}^y(T_{B(x,r)} < \infty; e^{-\lambda T_{B(x,r)}}) \\ &= \sup_{u>0} \inf_{r>0} \sup_{|x-y| \geq u} \mathbb{E}^0(e^{-\lambda T_{\overline{B}(x-y,r)}}) \\ &= \sup_{x \neq 0} \mathbb{E}^0 e^{-\lambda T_{\{x\}}}. \end{aligned}$$

□

By Proposition 2.15, Remarks 7 and 1 we obtain an improvement of [38, Lemma 4].

Corollary 2.16 *Let X be non-Poisson and $\lambda > 0$. Then (H3) holds for X^λ if and only if $\{0\}$ is polar for X . If this is the case, then we have $h_3(X^\lambda) = 0$.*

3 Kato Class

Let X be a Hunt process in \mathbb{R}^d . For $t \geq 0$ we define the transition kernel $P_t(x, dz)$ and the corresponding transition operator P_t by

$$P_t(x, B) = \mathbb{P}^x(X_t \in B), \quad P_t f(x) = \int_{\mathbb{R}^d} f(z) P_t(x, dz).$$

Moreover, for $\lambda \geq 0$ and $t \in (0, \infty]$ we let

$$G_t^\lambda(x, B) = \int_0^t e^{-\lambda s} P_u(x, B) du, \quad G_t^\lambda f(x) = \int_{\mathbb{R}^d} f(z) G_t^\lambda(x, dz) = \int_0^t e^{-\lambda u} P_u f(x) du,$$

to be the (truncated) λ -potential kernel and the (truncated) λ -potential operator G_t^λ , respectively. We simplify the notation by putting $G^\lambda(x, dz) = G_\infty^\lambda(x, dz)$ and $G^\lambda = G_\infty^\lambda$.

Definition 2 Let $q : \mathbb{R}^d \rightarrow \mathbb{R}$. We write $q \in \mathbb{K}(X)$ if Eq. 1 holds, i.e.,

$$\lim_{t \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}^d} G_t^0 |q|(x) \right] = 0. \tag{17}$$

We write $q \in \mathcal{K}(X)$ if Eq. 2 holds for some (every) $\lambda > 0$, i.e.,

$$\lim_{r \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G^\lambda(x, dz) \right] = 0. \tag{18}$$

If the process X is understood from the context we will write in short \mathbb{K}, \mathcal{K} for $\mathbb{K}(X), \mathcal{K}(X)$. In the next two lemmas we show that the definition of \mathcal{K} is consistent. The first one is an apparent reinforcement of Eqs. 2 and 18.

Lemma 3.1 For all $\lambda \geq 0, t \in (0, \infty]$,

$$\left[\sup_{x,y \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G_t^\lambda(y, dz) \right] \leq \left[\sup_{x \in \mathbb{R}^d} \int_{B(x,2r)} |q(z)| G_t^\lambda(x, dz) \right], \quad r > 0.$$

Proof Let $T = T_{\overline{B}(x,r)}$. The strong Markov property leads to

$$\begin{aligned} & \mathbb{E}^y \left(\int_0^\infty e^{-\lambda s} \mathbb{1}_{(0,t](s)} \mathbb{1}_{B(x,r)}(X_s) |q(X_s)| ds \right) \\ &= \mathbb{E}^y \left(T < \infty; \int_T^\infty e^{-\lambda s} \mathbb{1}_{(0,t](s)} \mathbb{1}_{B(x,r)}(X_s) |q(X_s)| ds \right) \\ &\leq \mathbb{E}^y \left(T < \infty; e^{-\lambda T} \int_0^\infty e^{-\lambda u} \mathbb{1}_{(0,t](u)} \mathbb{1}_{B(x,r)}(X_u \theta_T) |q(X_u \theta_T)| du \right) \\ &= \mathbb{E}^y \left(T < \infty; e^{-\lambda T} \mathbb{E}^{X_T} \left(\int_0^\infty e^{-\lambda u} \mathbb{1}_{(0,t](u)} \mathbb{1}_{B(x,r)}(X_u) |q(X_u)| du \right) \right), \end{aligned}$$

where θ denotes the usual shift operator. By the right continuity $X_T \in \overline{B}(x, r)$ and $B(x, r) \subseteq B(X_T, 2r)$ on $\{T < \infty\}$. Thus eventually

$$\begin{aligned} \int_{B(x,r)} |q(z)| G_t^\lambda(y, dz) &\leq \mathbb{E}^y \left(T < \infty; e^{-\lambda T} \mathbb{E}^{X_T} \left(\int_0^\infty e^{-\lambda u} \mathbb{1}_{(0,t](u)} \mathbb{1}_{B(X_T,2r)}(X_u) |q(X_u)| du \right) \right) \\ &\leq \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[\int_0^\infty e^{-\lambda u} \mathbb{1}_{(0,t](u)} \mathbb{1}_{B(x,2r)}(X_u) |q(X_u)| du \right] \\ &= \sup_{x \in \mathbb{R}^d} \int_{B(x,2r)} |q(z)| G_t^{\lambda_0}(x, dz). \end{aligned}$$

□

Lemma 3.2 If Eqs. 2 or 18 holds for some $\lambda_0 > 0$, then it holds for every $\lambda > 0$.

Proof Clearly, by the resolvent formula (see [4, Chapter 1, (8.10)]) it suffices to consider the measure $A \mapsto \int \mathbb{1}_A(z) G^{\lambda_0} G^\lambda(x, dz) = \iint \mathbb{1}_A(z) G^{\lambda_0}(y, dz) G^\lambda(x, dy)$. We have

$$\begin{aligned} \int_{B(x,r)} |q(z)| G^{\lambda_0} G^\lambda(x, dz) &= \int_{\mathbb{R}^d} \left(\int_{B(x,r)} |q(z)| G^{\lambda_0}(y, dz) \right) G^\lambda(x, dy) \\ &\leq \lambda^{-1} \left[\sup_{x,y \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G^{\lambda_0}(y, dz) \right]. \end{aligned}$$

This ends the proof due to Lemma 3.1. □

Now, we give alternative characterisations of $\mathbb{K}(X)$ and $\mathcal{K}(X)$. We easily observe that

$$e^{-\lambda t} G_t^0(x, dz) \leq G_t^\lambda(x, dz) \leq G_t^0(x, dz). \tag{19}$$

Lemma 3.3 For $\lambda > 0$ and $t \in [1/\lambda, \infty]$ we have

$$(1 - e^{-1}) \sup_x [G_t^\lambda |q|(x)] \leq \sup_x [G_{1/\lambda}^0 |q|(x)] \leq e \sup_x [G_t^\lambda |q|(x)].$$

Proof Actually, the upper bound holds pointwise as follows,

$$G_{1/\lambda}^0 |q|(x) = \int_0^{1/\lambda} P_u |q|(x) du \leq e \int_0^{1/\lambda} e^{-\lambda u} P_u |q|(x) du \leq e G_t^\lambda |q|(x).$$

We prove the lower bound,

$$\begin{aligned} G^\lambda |q|(x) &\leq \sum_{k=0}^\infty e^{-k} \int_{k/\lambda}^{(k+1)/\lambda} P_{k/\lambda} P_{u-k/\lambda} |q|(x) du = \sum_{k=0}^\infty e^{-k} P_{k/\lambda} \left(\int_0^{1/\lambda} P_u |q|(\cdot) du \right) (x) \\ &\leq (1 - e^{-1})^{-1} \sup_{z \in \mathbb{R}^d} \left[\int_0^{1/\lambda} P_u |q|(z) du \right]. \end{aligned}$$

□

Here is a conclusion from Eq. 19 and Lemma 3.3.

Proposition 3.4 *The following conditions are equivalent to $q \in \mathbb{K}(X)$.*

- i) $\lim_{t \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}^d} G_t^\lambda |q|(x) \right] = 0$ for some (every) $\lambda \geq 0$.
- ii) $\lim_{\lambda \rightarrow \infty} \left[\sup_{x \in \mathbb{R}^d} G_t^\lambda |q|(x) \right] = 0$ for some (every) $t \in (0, \infty)$.

For resolvent operators R^λ , $\lambda > 0$, of a strongly continuous contraction semigroup on a Banach space we have $\lim_{\lambda \rightarrow \infty} \lambda R^\lambda \phi = \phi$. Thus $\lim_{\lambda \rightarrow \infty} R^\lambda \phi = 0$ in the norm for every element ϕ of the Banach space. For a Markov process the counterparts of the resolvent operators are the λ -potential operators G_∞^λ .

Proposition 3.4 extends the equivalence of (i) and (ii) of [11, Theorem III.1] from a subclass of Lévy processes to any Hunt process. Similar result is proved in [24, Lemma 3.1] where authors discuss the Kato class of measures for Markov processes possessing transition densities that satisfy the Nash type estimate (see [25] for the symmetric case). In Lemma 3.7 we also show that the uniform local integrability of V ([11, Theorem III.1]) is necessary for $V \in \mathbb{K}(X)$ for any Lévy process X in \mathbb{R}^d .

We briefly explain the role of Proposition 3.4. For the Brownian motion, as mentioned in [26] (see also [34]), by Stein’s interpolation theorem the inequality $\sup_{x \in \mathbb{R}^d} [G^\lambda |q|(x)] \leq \gamma$ leads to $\| |q|^{1/2} \phi \|_2^2 \leq \gamma (\| \nabla \phi \|_2^2 + \lambda \| \phi \|_2^2)$, $\phi \in C_c^\infty(\mathbb{R}^d)$ (a partial reverse result is proved in [1, Theorem 4.9]). For a counterpart of such implication for other processes see remarks preceding [17, Theorem 4.10]. The latter inequality with $\gamma < 1$ allows to define a self-adjoint Schrödinger operator in the sense of quadratic forms, cf. [27, Theorem 3.17], the analogue of Kato-Rellich theorem.

We use Lemma 3.1 to get a better insight into the result of Lemma 3.3.

Lemma 3.5 *For $t \in (0, \infty)$ we have $G_t^0(x, dz) \leq e G^{1/t}(x, dz)$ and*

$$(1 - e^{-1}) \sup_{x \in \mathbb{R}^d} \left[\int_{B(x,r)} |q(z)| G^{1/t}(x, dz) \right] \leq \sup_{x \in \mathbb{R}^d} \left[\int_{B(x,2r)} |q(z)| G_t^0(x, dz) \right], \quad r > 0.$$

Proof For a fixed $y \in \mathbb{R}^d$ by Lemma 3.3 with $\tilde{q}(z) = q(z)\mathbb{1}_{B(y,r)}(z)$ we have

$$\begin{aligned} (1 - e^{-1}) \int_{B(y,r)} |q(z)|G^{1/t}(y, dz) &= (1 - e^{-1})G^{1/t}|\tilde{q}|(y) \\ &\leq \sup_{x \in \mathbb{R}^d} \int_0^t P_s|\tilde{q}|(x)ds = \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\tilde{q}(z)|G_t^0(x, dz) = \sup_{x \in \mathbb{R}^d} \int_{B(y,r)} |q(z)|G_t^0(x, dz). \end{aligned}$$

Thus, by Lemma 3.1 we obtain

$$(1 - e^{-1}) \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |q(z)|G^{1/t}(y, dz) \leq \sup_{x \in \mathbb{R}^d} \int_{B(x,2r)} |q(z)|G_t^0(x, dz).$$

□

The following is the aftermath of Eq. 19 and Lemma 3.5.

Proposition 3.6 $q \in \mathcal{K}(X)$ if and only if

$$\lim_{r \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |q(z)|G_t^\lambda(x, dz) \right] = 0,$$

for some (all) $t \in (0, \infty)$, $\lambda \geq 0$.

The above truncation in time is useful when the distribution $\mathbb{P}^x(X_s \in dz)$ is well estimated only for $s \in (0, t]$ near every $x \in \mathbb{R}^d$. See [19], [12, Theorems 2.4 and 3.1] for such estimates. In view of [25, (A2.3), Lemmas 4.1 and 4.3] Proposition 3.6 can also be regarded as an extension or counterpart of [25, Theorem 3.1]. We use Proposition 3.6 in Example 1 below.

Remark 9 Let $\lambda > 0$. Then $\mathbb{K}(X) = \mathbb{K}(X^\lambda)$ and $\mathcal{K}(X) = \mathcal{K}(X^\lambda)$.

Lemma 3.7 Let X be a Lévy process in \mathbb{R}^d . Assume that there are $t > 0$ and $0 \leq M < \infty$ such that for all $x \in \mathbb{R}^d$,

$$G_t^0|q|(x) = \int_0^t P_u|q|(x) du \leq M.$$

Then there is a constant $0 \leq M' < \infty$ independent of q such that

$$\sup_x \int_{B(x,1)} |q(z)| dz \leq M'. \tag{20}$$

Proof Let $\varphi \in C_0(\mathbb{R}^d)$ be such that $\varphi \geq 0$, $\varphi = 1$ on $B(0, 1)$ and $\int_{\mathbb{R}^d} \varphi(x) dx = N < \infty$. For $x_0 \in \mathbb{R}^d$ we have, for $h \leq t$,

$$\begin{aligned} MN &\geq \int_0^h \int_{\mathbb{R}^d} P_u |q|(x) \varphi(x_0 - x) dx du = \int_0^h \int_{\mathbb{R}^d} \mathbb{E}^0 |q|(X_u + x) |\varphi(x_0 - x)| dx du \\ &= \int_0^h \mathbb{E}^0 \left[\int_{\mathbb{R}^d} |q|(X_u + x) |\varphi(x_0 - x)| dx \right] du = \int_0^h \mathbb{E}^0 \left[\int_{\mathbb{R}^d} |q|(z) |\varphi(X_u + x_0 - z)| dz \right] du \\ &= \int_0^h \int_{\mathbb{R}^d} |q|(z) P_u \varphi(x_0 - z) dz du \geq \int_0^h \int_{B(x_0, 1)} |q|(z) P_u \varphi(x_0 - z) dz du \\ &\geq (\varepsilon/2) \int_{B(x_0, 1)} |q|(z) dz, \end{aligned}$$

where $0 < \varepsilon \leq h$ is such that $\|P_u \varphi - \varphi\|_\infty \leq 1/2$ for $u \leq \varepsilon$ (see [29, Theorem 31.5]). \square

Here $C_0(\mathbb{R}^d)$ denotes the set of continuous functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\lim_{|x| \rightarrow \infty} f(x) = 0$. We write $q \in L^1_{unif}(\mathbb{R}^d)$ if Eq. 20 holds. By $B(\mathbb{R}^d)$ we denote the set of bounded (Borel) functions on \mathbb{R}^d . We collect basic properties of $\mathbb{K}(X)$ and $\mathcal{K}(X)$ for a Lévy process X in \mathbb{R}^d .

Proposition 3.8 *We have*

1. $\mathcal{K} \subseteq \mathbb{K} \subseteq L^1_{unif}(\mathbb{R}^d)$ for every Lévy process,
2. $B(\mathbb{R}^d) \subseteq \mathbb{K}$ for every Lévy process,
3. $B(\mathbb{R}^d) \subseteq \mathcal{K}$ for every non-Poisson Lévy process,
4. $\mathcal{K} = \{0\}$ and $\mathbb{K} = B(\mathbb{R}^d)$ for every compound Poisson process.

Proof The inclusion $\mathbb{K} \subseteq L^1_{unif}(\mathbb{R}^d)$ follows from Lemma 3.7. To complete 1. we let $q \in \mathcal{K}(X)$, which reads as (C1) for X^λ , $\lambda > 0$. By Remark 8 and Lemma 2.10, (H2) holds for X^λ and thus the result of Zhao on Fig. 1 implies that (C2) holds for X^λ , i.e., $q \in \mathbb{K}(X^\lambda) = \mathbb{K}(X)$ (see Remark 9). Plainly, 2. holds. Now, let X be non-Poisson. By Lemma 2.1 we get $P_t(\{0\}) = 0$ for almost all $t > 0$ and consequently $G^\lambda(\{0\}) = 0$. Further, since $G^\lambda(dx)$ is a finite measure, for $q \in B(\mathbb{R}^d)$ we have

$$\lim_{r \rightarrow \infty} \sup_{x \in \mathbb{R}^d} \int_{B_r} |q(x+z)| G^\lambda(dz) \leq \lim_{r \rightarrow 0^+} G^\lambda(B_r) \sup_{x \in \mathbb{R}^d} |q(x)| = G(\{0\}) \sup_{x \in \mathbb{R}^d} |q(x)| = 0,$$

and 3. holds. Finally, if X is a compound Poisson process, then $G^\lambda(\{0\}) \geq (\lambda + \nu(\mathbb{R}^d))^{-1} > 0$ and for every $r > 0$

$$\sup_{x \in \mathbb{R}^d} \int_{B_r} |q(x+z)| G^\lambda(dz) \geq \sup_{x \in \mathbb{R}^d} |q(x)| (\lambda + \nu(\mathbb{R}^d))^{-1}.$$

Hence $q \in \mathcal{K}$ if and only if $q \equiv 0$. Moreover,

$$\sup_{x \in \mathbb{R}^d} \int_0^t P_u |q|(x) du \geq \sup_{x \in \mathbb{R}^d} |q(x)| \int_0^t e^{-\nu(\mathbb{R}^d)u} du,$$

which proves 4. \square

4 Main Theorems

In this section we consider a Lévy process X in \mathbb{R}^d and we pursue according to the cases of Section 2.2. Before that, we prove Corollary 1.2 directly from Theorem 1.1.

Proof of Corollary 1.2 Consider a Lévy process Y in $\mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d$ defined by $Y_t = (t, X_t), t \geq 0$, where X is an arbitrary Lévy process in $\mathbb{R}^d, d \geq 1$. Observe that for $(s, x) \in \mathbb{R}^{d+1}$ and a Borel set $B \subseteq \mathbb{R}^{d+1}$ we have $\mathbb{P}^{(s,x)}(Y_u \in B) = \mathbb{E}^x[\mathbb{1}_B(s + u, X_u)], u \geq 0$. Since for Y 0 is not regular for $\{0\}$ Theorem 1.1 applies to Y . Finally, we use (2) taking into account that $\mathbb{1}_{B_{d+1}((s,x),r)}(s + u, X_u)$, where $B_{d+1}(x, r)$ denotes a ball in \mathbb{R}^{d+1} , can be replaced with $\mathbb{1}_{[0,r)}(u)\mathbb{1}_{B(x,r)}(X_u)$ and that $e^{-\lambda u}$ is comparable with one for $u \in [0, r)$. \square

4.1 Under (H0)

In this subsection we consider a Lévy process X satisfying (H0).

Theorem 4.1 *For $d > 1$ or $d = 1$ under (A) we have $\mathcal{K}(X) = \mathbb{K}(X)$.*

Proof By Proposition 3.8 we concentrate on $\mathbb{K}(X) \subseteq \mathcal{K}(X)$. Let $q \in \mathbb{K}(X) = \mathbb{K}(X^\lambda), \lambda > 0$. This reads as (C2) for X^λ . Since X is non-Poisson, by Remark 8 and Lemma 2.10 the hypothesis (H1) holds for X^λ . To obtain (C1) for X^λ , that is to prove $q \in \mathcal{K}(X)$, it remains to verify (H3) for X^λ . In view of Corollary 2.16 it suffices to justify that $\{0\}$ is a polar set. For $d > 1$ this is assured by Proposition 2.5. For $d = 1$ it is our assumption. \square

From now on in this subsection we discuss the case of $d = 1$. For simplicity we recall from [10, Théorèmes 7, 1, 5, 6 and 8] the following facts.

Lemma 4.2 *Let $d = 1$ and $\int_{\mathbb{R}} \text{Re} \left(\frac{1}{\lambda + \psi(z)} \right) dz < \infty, \lambda > 0$. Then $G^\lambda(dz)$ has a bounded density $G^\lambda(z) = k^\lambda h^\lambda(z), z \in \mathbb{R}$, with respect to the Lebesgue measure which is continuous on $\mathbb{R} \setminus \{0\}$. Further, $G^\lambda(z)$ is continuous at 0 if and only if 0 is regular for $\{0\}$ (i.e. $h^\lambda(0) = 1$), and then $0 < h^\lambda(z) \leq 1$ for $z \in \mathbb{R}$.*

We investigate the properties of $G_t^\lambda(dz), \lambda > 0, t \in (0, \infty)$.

Lemma 4.3 *Let $d = 1$ and $\int_{\mathbb{R}} \text{Re} \left(\frac{1}{\lambda + \psi(z)} \right) dz < \infty, \lambda > 0$. Then $G_t^\lambda(dz)$ has a bounded density $G_t^\lambda(z)$ with respect to the Lebesgue measure which is lower semi-continuous on $\mathbb{R} \setminus \{0\}$.*

Proof According to Lemma 4.2 we define $F^\lambda(z) := G^\lambda(z)$ on $\mathbb{R} \setminus \{0\}$ and $F^\lambda(0) := \limsup_{z \rightarrow 0} F^\lambda(z)$. Then $F^\lambda(z)$ is a density of $G^\lambda(dz)$. Since $G_t^\lambda(B) \leq G^\lambda(B)$ and $G_t^\lambda(B) = G^\lambda(B) - e^{-\lambda t} \int_{\mathbb{R}} G^\lambda(B - z)P_t(dz)$, $G_t^\lambda(dx)$ is absolutely continuous and its density $G_t^\lambda(x)$ can be chosen as

$$G_t^\lambda(x) := F^\lambda(x) - e^{-\lambda t} \int_{\mathbb{R}} F^\lambda(x - z)P_t(dz). \tag{21}$$

To prove the lower semi-continuity of G_t^λ we observe that for $x_0 \in \mathbb{R} \setminus \{0\}$,

$$G_t^\lambda(x) = F^\lambda(x) - e^{-\lambda t} \left(\int_{\mathbb{R} \setminus \{x_0\}} F^\lambda(x - z) P_t(dz) + F^\lambda(x - x_0) P_t(\{x_0\}) \right).$$

Then by continuity of F^λ on $\mathbb{R} \setminus \{0\}$ and the bounded convergence theorem

$$\begin{aligned} \liminf_{x \rightarrow x_0} G_t^\lambda(x) &= F^\lambda(x_0) - e^{-\lambda t} \left(\int_{\mathbb{R} \setminus \{x_0\}} \lim_{x \rightarrow x_0} F^\lambda(x - z) P_t(dz) + \limsup_{x \rightarrow x_0} F^\lambda(x - x_0) P_t(\{x_0\}) \right) \\ &= G_t^\lambda(x_0). \end{aligned}$$

□

Theorem 4.4 *For $d = 1$ under (B) we have*

$$\mathcal{K}(X) = \mathbb{K}(X) = \left\{ q : \limsup_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{B(x,r)} |q(z)| dz = 0 \right\}.$$

Proof Without loss of generality we may and do assume that $\gamma_0 > 0$. Due to Proposition 3.8 and Lemma 4.2 (boundedness of the function G^λ) it remains to prove $\mathbb{K}(X) \subseteq \{q : \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{B(x,r)} |q(z)| dz = 0\}$. By Remark 3 we get $\mathbb{P}^0(\lim_{u \rightarrow 0^+} u^{-1} X_u = \gamma_0) = 1$. Hence, there is $\varepsilon > 0$ such that $\mathbb{P}^0(|X_u - \gamma_0 u| < \gamma_0 u) \geq 1/2$ for $u \leq \varepsilon$. This implies that for $t \leq \varepsilon$,

$$G_t^\lambda((0, 2\gamma_0 t]) = \int_0^t e^{-\lambda u} \mathbb{P}^0(X_u \in (0, 2\gamma_0 t]) du \geq \int_0^t e^{-\lambda u} \mathbb{P}^0(|X_u - \gamma_0 u| < \gamma_0 u) du \geq \frac{1 - e^{-\lambda t}}{2\lambda}.$$

Hence, $\sup_{z \in (0, 2\gamma_0 t]} G_t^\lambda(z) \geq \frac{1 - e^{-\lambda t}}{\lambda t} \frac{1}{4\gamma_0} \geq \frac{1 - e^{-\lambda \varepsilon}}{\lambda \varepsilon} \frac{1}{4\gamma_0}$. Since $G_t^\lambda(z)$ is lower semi-continuous on $\mathbb{R} \setminus \{0\}$ there exist $0 < a_t < b_t \leq 2\gamma_0 \varepsilon$ such that $G_t^\lambda(z) \geq \frac{1 - e^{-\lambda \varepsilon}}{\lambda \varepsilon} \frac{1}{8\gamma_0}$ for $z \in (a_t, b_t)$. Now, let $q \in \mathbb{K}(X)$. We obtain for $t \leq \varepsilon$,

$$\int_{\mathbb{R}} |q(x + z)| G_t^\lambda(dz) \geq \frac{1 - e^{-\lambda \varepsilon}}{8\lambda \varepsilon \gamma_0} \int_{a_t}^{b_t} |q(x + z)| dz.$$

Thus,

$$0 = \limsup_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{a_t}^{b_t} |q(x + z)| dz \geq \limsup_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{B(x,r)} |q(z)| dz.$$

□

Lemma 4.5 *Let 0 be regular for $\{0\}$. There is $0 < M_{G^\lambda} < \infty$ such that*

$$G^\lambda(x) \leq M_{G^\lambda} G^\lambda(y), \quad x, y \in \mathbb{R}, \quad |x - y| \leq 1. \tag{22}$$

Further, $G_t^\lambda(x)$ given by Eq. 21 is continuous on \mathbb{R} and

$$G_t^\lambda(x) \leq G^\lambda(x)(\lambda t + \|P_t f - f\|_\infty), \quad f(x) = h^\lambda(-x) \in C_0(\mathbb{R}).$$

Proof Let F^λ be defined as in the proof of Lemma 4.3. By Lemma 4.2 the functions G^λ and F^λ are equal and continuous on \mathbb{R} . Further, Lemma 2.13 implies that the function

$h^\lambda(x) = G^\lambda(x)/k^\lambda = \mathbb{E}^0 e^{-\lambda T_{|x|}}$ is in $C_0(\mathbb{R})$. Since $h^\lambda(x+y) \geq h^\lambda(x)h^\lambda(y)$, $x, y \in \mathbb{R}$ (see remarks after [10, Lemma 2]), we get

$$\frac{G^\lambda(x-z)}{G^\lambda(x)} = \frac{h^\lambda(x-z)}{h^\lambda(x)} \geq h^\lambda(-z).$$

By positivity and continuity of h^λ we obtain (22) with $M_{G^\lambda} = \sup_{|z| \leq 1} 1/[h^\lambda(z)] < \infty$. Eventually, by Eq. 21,

$$\begin{aligned} G_t^\lambda(x) &= G^\lambda(x) \left(1 - e^{-\lambda t} + e^{-\lambda t} \int_{\mathbb{R}} \left(1 - \frac{G^\lambda(x-z)}{G^\lambda(x)} \right) P_t(dz) \right) \\ &\leq G^\lambda(x) \left(\lambda t + \int_{\mathbb{R}} (h^\lambda(0) - h^\lambda(-z)) P_t(dz) \right). \end{aligned}$$

□

Theorem 4.6 For $d = 1$ under (C) we have $\mathcal{K}(X) \subsetneq \mathbb{K}(X)$,

$$\mathcal{K}(X) = \left\{ q : \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_{B(x,r)} |q(z)| dz = 0 \right\},$$

and

$$\mathbb{K}(X) = L^1_{unif}(\mathbb{R}) = \left\{ q : \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)| dz < \infty \right\}.$$

Proof For $\mathcal{K}(X)$ we just observe that $G^\lambda(z)$ is bounded and $G^\lambda(z) \geq \varepsilon > 0$ if $|z| \leq 1$. Now, we describe $\mathbb{K}(X)$. The condition $q \in L^1_{unif}(\mathbb{R})$ is necessary by Lemma 3.7. We show that it is sufficient. Let $\lambda > 0$ and denote $c_t = \lambda t + \|P_t f - f\|_\infty$, where $f(x) = h^\lambda(-x) = \mathbb{E} e^{-\lambda T_{|-x|}}$. By Lemma 4.5

$$\begin{aligned} \int_{\mathbb{R}} |q(x+z)| G_t^\lambda(dz) &\leq c_t \int_{\mathbb{R}} |q(x+z)| G^\lambda(z) dz = c_t \sum_{k=-\infty}^\infty \int_{k-1/2}^{k+1/2} |q(x+z)| G^\lambda(z) dz \\ &\leq c_t M_{G^\lambda} \sum_{k=-\infty}^\infty G^\lambda(k) \int_{k-1/2}^{k+1/2} |q(x+z)| dz \\ &\leq c_t M_{G^\lambda} \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)| dz \sum_{k=-\infty}^\infty G^\lambda(k) \\ &\leq c_t (M_{G^\lambda})^2 \lambda^{-1} \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)| dz. \end{aligned} \tag{23}$$

Since $f \in C_0(\mathbb{R})$ we get $c_t \rightarrow 0$ as $t \rightarrow 0^+$ (see [29, Theorem 31.5]). □

4.2 Without (H0)

In this subsection we assume that (H0) does not hold. In view of Proposition 3.8 we assume that $d > 1$ and X is given by Eq. 13. We use results of Section 4.1 and analyze the cases (A'), (B') and (C').

Theorem 4.7 Under (A') we have $\mathcal{K}(X) = \mathbb{K}(X)$.

Proof Following the proof of Theorem 4.1 it remains to show that $\{0\}$ is polar for the process X . This is assured by Corollary 2.9. \square

We proceed to the remaining cases. The transition kernel of X equals

$$P_t(dx) = P_t^Z * \sum_{n=0}^{\infty} e^{-tv^Y(\mathbb{R}^d)} \frac{t^n (v^Y)^{*n}}{n!}(dx).$$

The characteristic exponent ψ of X can be written as $\psi = \psi^Y + \psi^Z$. We note that $\psi^Z(z) = \psi^Z(v)$ for $z = v + w \in \mathbb{R}^d$, $v \in V$, $w \in V^\perp$. For $\lambda > 0$, $t \in (0, \infty]$ and $n \in \mathbb{N}$ we define

$$G_t^{Z,\lambda,n}(dv) := \int_0^t u^n e^{-\lambda u} P_u^Z(dv) du.$$

We investigate n -moment λ -potentials $G_t^{Z,\lambda,n}(dv) := G_\infty^{Z,\lambda,n}(dv)$ and truncated λ -potentials $G_t^{Z,\lambda}(dv) := G_t^{Z,\lambda,0}(dv)$ of Z . We also write $G^{Z,\lambda}(dv) = G_\infty^{Z,\lambda,0}(dv)$ for λ -potentials of Z . The measures $G^{Z,\lambda}$, $G_t^{Z,\lambda}$, $G^{Z,\lambda,n}$ are concentrated on V . Observe that

$$G^\lambda(dx) = \sum_{n=0}^{\infty} \frac{1}{n!} G^{Z,\lambda+v^Y(\mathbb{R}^d),n} * (v^Y)^{*n}(dx). \tag{24}$$

We reformulate Lemmas 4.3 and 4.5 in view of Remark 5. We write $C_0(V)$ for the set of continuous functions $f : V \rightarrow \mathbb{R}$ such that $\lim_{v \in V, |v| \rightarrow \infty} f(v) = 0$.

Lemma 4.8 *Let $\int_V \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(v)} \right) dv < \infty$, $\lambda > 0$. Then $G_t^{Z,\lambda}(dv)$ has a bounded density $G_t^{Z,\lambda}(v)$ with respect to the Lebesgue measure on V which is lower semi-continuous on $V \setminus \{0\}$. If f is regular for $\{0\}$ for Z then there is $0 < M_{G^{Z,\lambda}} < \infty$ such that*

$$G^{Z,\lambda}(v) \leq M_{G^{Z,\lambda}} G^{Z,\lambda}(v'), \quad v, v' \in V, \quad |v - v'| \leq 1,$$

$G_t^{Z,\lambda}(v)$ is continuous on V and

$$G_t^{Z,\lambda}(v) \leq G^{Z,\lambda}(v)(\lambda t + \|P_t^Z f - f\|_\infty), \quad f(v) \in C_0(V).$$

Lemma 4.9 *Let $\int_V \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(v)} \right) dv < \infty$, $\lambda > 0$. Then $G^{Z,\lambda,n}(dv)$ has a density $G^{Z,\lambda,n}(v)$ with respect to the Lebesgue measure on V , and*

$$G^{Z,\lambda,n}(v) \leq \frac{n!}{\lambda^n} \int_V \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(u)} \right) du. \tag{25}$$

Proof By Remark 5 we assume that $V = \mathbb{R}$ and we observe that the Fourier transform of $G^{Z,\lambda,n}$ equals

$$\int_0^\infty t^n e^{-\lambda t} e^{-t\psi^Z(\xi)} dt = \frac{n!}{[\lambda + \psi^Z(\xi)]^{n+1}}, \quad \xi \in \mathbb{R}.$$

Since $\operatorname{Re}(1/z) = \operatorname{Re}(\bar{z})/|z|^2$ and $\operatorname{Re}[\psi] \geq 0$ we obtain

$$\frac{1}{|\lambda + \psi^Z(\xi)|^{n+1}} \leq \lambda^{-n+1} \frac{1}{|\lambda + \psi^Z(\xi)|^2} \leq \lambda^{-n} \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(\xi)} \right).$$

This implies that the Fourier transform is integrable and (25) follows by the inversion formula. \square

Lemma 4.10 *Let $\int_V \operatorname{Re} \left(\frac{1}{\lambda + \psi^Z(v)} \right) dv < \infty, \lambda > 0$. Then*

$$\sup_{x \in \mathbb{R}^d} \left(\int_{B(0,r)} |q(x+z)| G^\lambda(dz) \right) \leq \sup_{x \in \mathbb{R}^d} \left(\int_{B(0,r) \cap V} |q(x+v)| dv \right) C [1 + v^Y(\mathbb{R}^d)/\lambda],$$

where dv is the one-dimensional Lebesgue measure on V and $C = \int_V \operatorname{Re} (1/[\lambda + v^Y(\mathbb{R}^d) + \psi^Z(u)]) du$.

Proof By Eqs. 24 and 25 we have

$$\begin{aligned} \int_{B(0,r)} |q(x+z)| G^\lambda(dz) &= \sum_{n=0}^\infty \frac{1}{n!} \int_{\mathbb{R}^d} \left(\int_V \mathbb{1}_{B(0,r)}(v+w) |q(x+v+w)| G^{Z, \lambda + v^Y(\mathbb{R}^d), n}(dv) \right) \\ &\quad \times (v^Y)^{*n}(dw) \\ &\leq \sup_{x, w \in \mathbb{R}^d} \left(\int_V \mathbb{1}_{B(0,r)}(v+w) |q(x+v+w)| dv \right) \sum_{n=0}^\infty C \left(\frac{v^Y(\mathbb{R}^d)}{\lambda + v^Y(\mathbb{R}^d)} \right)^n, \end{aligned}$$

and

$$\begin{aligned} \sup_{x, w \in \mathbb{R}^d} \left(\int_V \mathbb{1}_{B(0,r)}(v+w) |q(x+v+w)| dv \right) &= \sup_{x, w \in \mathbb{R}^d} \left(\int_{B(-w,r) \cap V} |q(x+v)| dv \right) \\ &= \sup_{x \in \mathbb{R}^d, w \in V} \left(\int_{B(-w,r) \cap V} |q(x+v)| dv \right) \\ &= \sup_{x \in \mathbb{R}^d} \left(\int_{B(0,r) \cap V} |q(x+v)| dv \right), \end{aligned}$$

where the last equality follows by the translation invariance of the Lebesgue measure on V . This ends the proof. □

Theorem 4.11 *Under (B') we have*

$$\mathcal{K}(X) = \mathbb{K}(X) = \left\{ q : \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0 \right\},$$

where dv is the one-dimensional Lebesgue measure on V .

Proof Lemma 4.10 gives $\{q : \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0\} \subseteq \mathcal{K}(X)$. By Proposition 3.8 it suffices to show $\mathbb{K}(X) \subseteq \{q : \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0\}$. Since for $t > 0$ and $x \in \mathbb{R}^d$ we have

$$\int_0^t P_u |q|(x) du \geq \int_0^t \int_{\mathbb{R}^d} |q(x+z)| e^{-uv^Y(\mathbb{R}^d)} P_u^Z(dz) du = \int_{\mathbb{R}^d \cap V} |q(x+v)| G_t^{Z, v^Y(\mathbb{R}^d)}(dv),$$

the inclusion follows by adapting the proof of Theorem 4.4 to the one-dimensional process Z with the support of Lemma 4.8 and Remark 3. □

Theorem 4.12 *Under (C') we have $\mathcal{K}(X) \subsetneq \mathbb{K}(X)$,*

$$\mathcal{K}(X) = \left\{ q: \lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0 \right\},$$

and

$$\mathbb{K}(X) = \left\{ q: \sup_{x \in \mathbb{R}^d} \int_{B(0,1) \cap V} |q(x+v)| dv < \infty \right\},$$

where dv is the one-dimensional Lebesgue measure on V .

Proof The condition postulated for the description of $\mathcal{K}(X)$ is sufficient by Lemma 4.10. Next, by Remark 5 and Lemma 4.2 the λ -potential kernel of Z , that is $G^{Z,\lambda}(dv) = G^{Z,\lambda,0}(dv)$, has a density $G^{Z,\lambda}(v)$ with respect to the Lebesgue measure on V , such that $G^{Z,\lambda}(v) \geq \varepsilon > 0$ if $v \in B(0, 1) \cap V$ (ε may depend on λ). Thus,

$$\int_{B(0,r)} |q(x+z)| G^\lambda(dz) \geq \int_{B(0,r) \cap V} |q(x+v)| G^{Z,\lambda+v^Y(\mathbb{R}^d)}(dv) \geq \varepsilon \int_{B(0,r) \cap V} |q(x+v)| dv,$$

which proves the necessity. Further, the necessity of the condition proposed to describe $\mathbb{K}(X)$ follows from Remark 5, Lemma 3.7 and

$$\begin{aligned} \int_0^t P_u |q|(x) du &\geq \int_0^t \int_{\mathbb{R}^d \cap V} |q(x+v)| e^{-uv^Y(\mathbb{R}^d)} P_u^Z(dv) du \\ &\geq e^{-tv^Y(\mathbb{R}^d)} \int_0^t \int_{\mathbb{R}^d \cap V} |q(x+v)| P_u^Z(dv) du. \end{aligned}$$

For the sufficiency we partially follow the proof of Theorem 4.6. Note that $\int_0^t u^n e^{-\lambda u} P_u^Z(dv) du \leq t^n G_t^{Z,\lambda}(dv)$ which gives

$$G_t^\lambda(dx) \leq \sum_{n=0}^\infty \frac{t^n}{n!} G_t^{Z,\lambda+v^Y(\mathbb{R}^d)} * (v^Y)^{*n}(dx).$$

Thus by Lemma 4.8 and adaptation of Eq. 23 we have with $c_t = (\lambda + v^Y(\mathbb{R}^d))t + \|P_t^Z f - f\|_\infty$,

$$\begin{aligned} \int_{\mathbb{R}^d} |q(x+z)| G_t^\lambda(dz) &\leq \sum_{n=0}^\infty \frac{t^n}{n!} \int_{\mathbb{R}^d} \left(\int_V |q(x+v+w)| G_t^{Z,\lambda+v^Y(\mathbb{R}^d)}(dv) \right) (v^Y)^{*n}(dw) \\ &\leq \left(c_t \left(M_{G^{Z,\lambda+v^Y(\mathbb{R}^d)}} \right)^2 (\lambda + v^Y(\mathbb{R}^d))^{-1} \sup_{x \in \mathbb{R}^d} \int_{B(0,1) \cap V} |q(x+v)| dv \right) \\ &\quad \sum_{n=0}^\infty \frac{t^n}{n!} \int_{\mathbb{R}^d} (v^Y)^{*n}(dw), \end{aligned}$$

which ends the proof. □

4.3 Zero-Potential Kernel

In the previous sections and subsections we have already used measures G_t^λ , $\lambda \geq 0$, $t \in (0, \infty]$. Below we present additional sufficient assumptions on a Lévy process X under

which $G^0 = G_\infty^0$ can be used to describe $\mathbb{K}(X)$. The condition we want to analyze now is $q \in \mathcal{K}^0(X)$ defined by

$$\lim_{r \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}^d} \int_{B(0,r)} |q(z+x)| G^0(dz) \right] = 0. \tag{26}$$

Since $G^\lambda(dz) \leq G^0(dz)$, Eq. 26 implies $q \in \mathcal{K}(X)$ and thus $\mathcal{K}^0(X) \subseteq \mathcal{K}(X) \subseteq \mathbb{K}(X)$ by Proposition 3.8. Our aim is to obtain the equivalence, i.e., the implication from $q \in \mathbb{K}(X)$ to Eq. 26, and this is the subcase of $\mathcal{K}(X) = \mathbb{K}(X)$. We will assume that X is transient and $\{0\}$ is polar (in Theorem 4.15 polarity follows implicitly from other assumptions). The transience is necessary, otherwise $G^0(dz)$ is locally unbounded (see [29, Theorem 35.4]) and non-zero constant functions do not belong to $\mathcal{K}^0(X)$, which shows $\mathcal{K}^0(X) \subsetneq \mathbb{K}(X)$. The polarity of $\{0\}$ assures $\mathcal{K}(X) = \mathbb{K}(X)$. Moreover, if $\{0\}$ is not polar, the class $\mathbb{K}(X)$ is explicitly described by our previous theorems. Both, transience and polarity of $\{0\}$ are to some extent encoded in the characteristic exponent ψ (see [29, Remark 37.7] and Section 2.2). Finally, we note that $q \in \mathcal{K}^0(X)$ is equivalent to (C1) and $q \in \mathbb{K}(X)$ to (C2). Thus according to Fig. 1 and Remark 8, we focus on showing (H3) for X .

Remark 10 If X is transient, then we have

$$\lim_{r \rightarrow 0^+} \mathbb{P}^0(T_{\bar{B}(x,r)} < \infty) = \mathbb{P}^0(T_{\{x\}} < \infty), \quad x \in \mathbb{R}^d. \tag{27}$$

Such statement is not true in general, but here it follows from $\mathbb{P}^0(T_{\bar{B}(x,r)} < \infty) = \mathbb{P}^0(T_{\bar{B}(x,r)} < \infty, T_{\{x\}} < \infty) + \mathbb{P}^0(T_{\bar{B}(x,r)} < \infty, T_{\{x\}} = \infty)$, Lemma 2.12 and $\lim_{t \rightarrow \infty} |X_t| = \infty$ \mathbb{P}^0 a.s.

We say that a measure $G^0(dz)$ tends to zero at infinity if $\lim_{|x| \rightarrow \infty} \int_{\mathbb{R}^d} f(z+x) G^0(dz) = 0$ for all $f \in C_c(\mathbb{R}^d)$ (i.e., f is continuous with compact support). Under certain assumptions on the group of the Lévy process [29, Definition 24.21] $G^0(dz)$ tends to zero for every transient X if $d \geq 2$. The case $d = 1$ is more complicated. See [29, Exercise 39.14] and Remark 13.

Lemma 4.13 *Let X be transient. If $G^0(dz)$ tends to zero at infinity then*

$$h_3(X) = \sup_{x \neq 0} \mathbb{P}^0(T_{\{x\}} < \infty).$$

Proof The statement follows by the same proof as for Proposition 2.15 but with $\lambda = 0$ and a version of Lemma 2.14 for $\lambda = 0$. To prove the latter one we also repeat its proof with functions f_r extended to $\lambda = 0$, i.e., $f_r(x) = \mathbb{P}^0(T_{\bar{B}(x,r)} < \infty)$ up to a moment when $a > 0$ and a sequence $\{x_n\}$ such that $f_{1/n}(x_n) > a - \varepsilon$ are chosen. The rest of the proof easily applies with Eq. 27 in place of Lemma 2.12 as soon as we can show that $\{x_n\}$ is bounded. To this end assume that the sequence is unbounded. Since $f_r(x) = \mathbb{P}^y(T_{\bar{B}(x+y,r)} < \infty)$, $r > 0$, $y \in \mathbb{R}^d$, for $r \in (0, 1]$ and $|x - x_n| < 1$ we have

$$a - \varepsilon < f_r(x_n) = \mathbb{P}^{-x}(T_{\bar{B}(x_n-x,r)} < \infty) \leq \mathbb{P}^{-x}(T_{\bar{B}(0,2)} < \infty) = f_2(x), \tag{28}$$

Next, by [29, Theorem 42.8] there is a finite measure ρ supported on $\overline{B}(0, 2)$ (see also [29, Definition 42.1]) such that for any $g \in C_c(\mathbb{R}^d)$ satisfying $\mathbb{1}_{B(0,1)} \leq g$ we get

$$\int_{\mathbb{R}^d} g(x_n - x) f_2(x) dx = \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} g(x_n + w - v) G^0(dv) \right] \rho(dw) \xrightarrow{n \rightarrow \infty} 0,$$

since $G^0(dv)$ tends to zero at infinity. This contradicts (28) and ends the proof. □

Theorem 4.14 *Let X be transient, $\{0\}$ be polar and $G^0(dz)$ tend to zero at infinity. Then $q \in \mathbb{K}(X)$ if and only if Eq. 26 holds, i.e., $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$.*

In the next result we improve [38, Lemma 5] and we cover some cases when $G^0(dz)$ may not tend to zero at infinity.

Theorem 4.15 *Let X be transient and let $G^0(dz)$ have a density $G^0(z)$ with respect to the Lebesgue measure which is unbounded and bounded on $|z| \geq r$ for every $r > 0$. Then $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$.*

Proof We note that the polarity of $\{0\}$ follows by our assumptions (see [29, Theorems 41.15 and 43.3]). By [29, Proposition 42.13 and Definition 42.9] for $r > 0$ we have

$$\mathbb{P}^x(T_{B(0,r)} < \infty) = \int_{\overline{B}(0,r)} G^0(y - x) m_{B(0,r)}(dy), \quad x \in \mathbb{R}^d.$$

Next, for $u > 0$, $|x| \geq u$ and $0 < r < u/2$ we obtain,

$$\mathbb{P}^x(T_{B(0,r)} < \infty) \leq \left[\sup_{|y| \geq u/2} G^0(y) \right] C(B(0, r)),$$

where $C(\cdot)$ stands for 0-order capacity. By [29, Proposition 42.10 and (42.20)] and Remark 10 we have $\lim_{r \rightarrow 0^+} C(B(0, r)) = C(\{0\})$ (see also [28, Proposition 8.4]). This gives

$$\begin{aligned} h_3(X) &= \sup_{u>0} \inf_{r>0} \sup_{|x| \geq u} \mathbb{P}^x(T_{B(0,r)} < \infty) \leq \sup_{u>0} \left[\sup_{|y| \geq u/2} G(y) \right] \inf_{0 < r < u/2} C(B(0, r)) \\ &= \sup_{u>0} \left[\sup_{|y| \geq u/2} G(y) \right] C(\{0\}). \end{aligned}$$

Finally, since $\{0\}$ is polar, by [29, Theorem 42.19] we have $C(\{0\}) = 0$ and so (H3) holds with $h_3(X) = 0$. □

5 Further Discussion and Applications

In this section we give additional results for isotropic unimodal Lévy processes concerning (the implication) $\mathcal{K}(X) \subseteq \mathbb{K}(X)$, we apply general results to a subclass of subordinators and we present examples.

We recall from [6] the definition of weak scaling. Let $\underline{\theta} \in [0, \infty)$ and ϕ be a non-negative non-zero function on $(0, \infty)$. We say that ϕ satisfies the *weak lower scaling condition* (at infinity) if there are numbers $\underline{\alpha} \in \mathbb{R}$ and $\underline{c} \in (0, 1]$, such that

$$\phi(\eta\theta) \geq \underline{c}\eta^{\underline{\alpha}}\phi(\theta) \quad \text{for } \eta \geq 1, \quad \theta > \underline{\theta}.$$

In short we say that ϕ satisfies $WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$ and write $\phi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$. Similarly, we consider $\bar{\theta} \in [0, \infty)$. The *weak upper scaling condition* holds if there are numbers $\bar{\alpha} \in \mathbb{R}$ and $\bar{C} \in [1, \infty)$ such that

$$\phi(\eta\theta) \leq \bar{C}\eta^{\bar{\alpha}}\phi(\theta) \quad \text{for } \eta \geq 1, \quad \theta > \bar{\theta}.$$

In short, $\phi \in WUSC(\bar{\alpha}, \bar{\theta}, \bar{C})$.

5.1 Isotropic Unimodal Lévy Processes

A measure on \mathbb{R}^d is called isotropic unimodal, in short, unimodal, if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with a radial non-increasing density (such measures may have an atom at the origin). A Lévy process X is called (isotropic) unimodal if all of its one-dimensional distributions $P_t(dx)$ are unimodal. Unimodal pure-jump Lévy processes are characterized in [35] by isotropic unimodal Lévy measures $\nu(dx) = \nu(x)dx = \nu(|x|)dx$. The distribution of X_t has a radial non-increasing density $p(t, x)$ on $\mathbb{R}^d \setminus \{0\}$, and atom at the origin, with mass $\exp[-t\nu(\mathbb{R}^d)]$ (no atom if ψ is unbounded).

For a continuous non-decreasing function $\phi : [0, \infty) \rightarrow [0, \infty)$, such that $\phi(0) = 0$, we let $\phi(\infty) = \lim_{s \rightarrow \infty} \phi(s)$ and we define the generalized left inverse $\phi^- : [0, \infty) \rightarrow [0, \infty]$,

$$\phi^-(u) = \inf\{s \geq 0 : \phi(s) = u\} = \inf\{s \geq 0 : \phi(s) \geq u\}, \quad 0 \leq u < \infty,$$

with the convention that $\inf \emptyset = \infty$. The function is increasing and càglàd where finite. Notice that $\phi(\phi^-(u)) = u$ for $u \in [0, \phi(\infty)]$ and $\phi^-(\phi(s)) \leq s$ for $s \in [0, \infty)$. Moreover, by the continuity of ϕ we have $\phi^-(\phi(s) + \varepsilon) > s$ for $\varepsilon > 0$ and $s \in [0, \infty)$. We also define $f^*(u) = \sup_{|x| \leq u} |f(x)|$ for $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

In view of general results for Schrödinger perturbations [8, Theorem 3] and the so-called 3G type inequalities [7, (40) and Corollary 11] it is desirable to have the following results which extend [14, Theorem 1.28] and [9, Proposition 4.3] (see also [8, Remark 2]).

Proposition 5.1 *Let X be unimodal. For $t_0 \in (0, \infty]$, $r > 0$ and $0 < t < t_0$,*

$$\sup_{x \in \mathbb{R}^d} \int_0^t P_u |q|(x) du \leq \left(1 + \frac{t}{|B(0, 1/2)|r^d G_{t_0}^0(r)} \right) \left[\sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G_{t_0}^0(z-x) dz \right],$$

where $G_{t_0}^0(z) = \int_0^{t_0} p(u, z) du$, $z \in \mathbb{R}^d$, and $G_{t_0}^0(r) = G_{t_0}^0(x)$, $|x| = r$.

Proof We use [9, Lemma 4.2] with $k(x) = \int_0^t p(u, x) du$ and $K(x) = G_{t_0}^0(x)$. □

In what follows we assume that $d \geq 3$ and that the Lévy-Khintchine exponent ψ is unbounded. Then since X is (isotropic) unimodal by [29, Theorem 37.8] it is transient and the measure $G^0(dz)$ has a radially non-increasing density $G^0(z)$. This density is unbounded (see [29, Theorems 43.9 and 43.3]). Thus Theorem 4.15 applies and $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$. Under additional assumptions we investigate this relations.

Remark 11 Below we use the result of [15, Theorem 3] which says that if X is unimodal and $d \geq 3$ we always have $G^0(x) \leq C/(|x|^d \psi^*(|x|^{-1}))$, $x \in \mathbb{R}^d$, for some $C > 0$. If additionally $\psi \in WLSC(\underline{\alpha}, \underline{\theta}, \underline{c})$, $\underline{\alpha} > 0$, then $c/(|x|^d \psi^*(|x|^{-1})) \leq G^0(x)$ for $|x|$ small enough and some $c > 0$.

Corollary 5.2 *Let $d \geq 3$, X be unimodal with $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c})$, $\underline{\alpha} > 0$. There exist constants $C = C(d, \underline{\alpha}, \underline{c})$ and $b = (d, \underline{\alpha}, \underline{c})$ such that for any $0 < t < 1/\psi^*(\underline{\theta}/b)$ and $q : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$\sup_{x \in \mathbb{R}^d} \int_0^t P_u |q|(x) du \leq C \sup_{x \in \mathbb{R}^d} \int_{B(x, 1/(\psi^*)^-(1/t))} |q(z)| G^0(z-x) dz.$$

Proof We let $t_0 = \infty$ in Proposition 5.1. For $0 < t < \infty$ we take $r = 1/(\psi^*)^-(1/t) > 0$. Since $\psi^*(r^{-1}) = 1/t$ by [15, Theorem 3] $r^d G^0(r) \geq c/\psi^*(r^{-1}) = ct$ if $1/(\psi^*)^-(1/t) \leq b/\underline{\theta}$ for some constant $c > 0$. The last holds if $t < 1/\psi^*(\underline{\theta}/b)$. \square

Lemma 5.3 *Let $d \geq 3$, X be unimodal and $\psi \in \text{WLSC}(\underline{\alpha}, \underline{\theta}, \underline{c}) \cap \text{WUSC}(\overline{\alpha}, \underline{\theta}, \overline{C})$, $\underline{\alpha}, \overline{\alpha} \in (0, 2)$. Then there exist constants $c = c(d, \underline{\alpha}, \overline{\alpha}, \underline{c}, \overline{C})$ and $a = (d, \underline{\alpha}, \overline{\alpha}, \underline{c}, \overline{C})$ such that for any $0 < t < 1/\psi^*(\underline{\theta}/a)$ and $q : \mathbb{R}^d \rightarrow \mathbb{R}$,*

$$\sup_{x \in \mathbb{R}^d} \int_0^t P_u |q|(x) du \geq c \sup_{x \in \mathbb{R}^d} \int_{B(x, 1/(\psi^*)^-(1/t))} |q(z)| G^0(z-x) dz.$$

Proof Let $x \in \mathbb{R}^d$ be such that $|x| < 1/(\psi^*)^-(1/t)$, which gives $1/\psi^*(|x|^{-1}) \leq t$. Further, since $t < 1/\psi^*(\underline{\theta}/a)$ implies $1/(\psi^*)^-(1/t) < a/\underline{\theta}$ we get $|x| < a/\underline{\theta}$ and also $u \psi^*(\underline{\theta}/a) < 1$ if $u < 1/\psi^*(|x|^{-1})$. Then [6, Theorem 21 and Lemma 17] ($r_0 = a$) yield

$$\int_0^t p(u, x) du \geq \int_0^{1/\psi^*(|x|^{-1})} p(u, x) du \geq c^* \int_0^{1/\psi^*(|x|^{-1})} \frac{u \psi^*(|x|^{-1})}{|x|^d} du = \frac{c^*}{2|x|^d \psi^*(|x|^{-1})}.$$

Finally, we apply [15, Theorem 3] to obtain

$$\int_0^t p(u, x) du \geq c G^0(x), \quad \text{for } |x| < 1/(\psi^*)^-(1/t).$$

\square

5.2 Subordinators

Let X be a subordinator (without killing) with the Laplace exponent ϕ . Then ϕ is a Bernstein function (in short BF) with zero killing term. Two important subclasses of BF are special Bernstein functions (SBF) and complete Bernstein functions (CBF). We refer the reader to [30] for definitions and an overview. Since the cases when ϕ is bounded (equivalently X is a compound Poisson process) or when X has a non-zero drift γ_0 , are completely described by Theorems 3.8 and 4.4, we assume that

(S1) ϕ is unbounded (X is non-Poisson) and $\gamma_0 = 0$.

Note that for $d = 1$ if a Lévy process is non-Poisson and $A = 0$, $\gamma_0 = 0$, $\int_{\mathbb{R}} (|x| \wedge 1) \nu(dx) < \infty$, then we are in the case (A) of Section 2.2 (see Remark 6). Thus by Theorem 4.1 the following is true for subordinators.

Remark 12 If X satisfies (S1), then $\{0\}$ is polar and $\mathcal{K}(X) = \mathbb{K}(X)$.

We impose further assumptions on the exponent ϕ to study $G^\lambda(dz)$, $\lambda \geq 0$, and describe its behaviour near the origin:

- (S2) $a + \phi \in \text{SBF}$ for some $a \geq 0$ (see [30, Remark 11.21]),
- (S3) $\frac{\phi'}{\phi^2} \in \text{WUSC}(-\beta, \bar{\theta}, \bar{C})$, $\beta > 0$.

We shall mention that (S2) is always satisfied if $\phi \in \text{CBF}$. Indeed, if $\phi \in \text{CBF}$, then $a + \phi \in \text{CBF}$, $a \geq 0$, and $\text{CBF} \subset \text{SBF}$.

Remark 13 Recall that X is a subordinator without killing, i.e., $\phi \in \text{BF}$ with zero killing term. Note that $U(dz) = G^a(dz)$ is a potential kernel of (possibly killed) subordinator $S = X^a$, see [30, (5.2)]. The Laplace exponent of S equals $a + \phi$, thus by [30, Theorem 11.3, formulas (11.9) and Corollary 11.8] we have

- (a) under (S2), the measure $G^a(dz)$ is absolutely continuous with respect to the Lebesgue measure if and only if $\nu(0, \infty) = \infty$ (X is non-Poisson) or $\gamma_0 > 0$,
- (b) under (S1) and (S2), the density $G^a(z)$ of $G^a(dz)$ satisfies: $G^a(z) = 0$ on $(-\infty, 0]$, $G^a(z)$ is finite, positive and non-increasing on $(0, \infty)$, and $\lim_{z \rightarrow 0^+} G^a(z) = \infty$,
- (c) under (S2) with $a = 0$, $G^0(dz)$ tends to zero if and only if $\int_1^\infty xv(dx) = \infty$.

We already know by Remark 12 that G^a , $a > 0$, describes $\mathbb{K}(X)$ by Eq. 18. We extend this observation to $a = 0$.

Proposition 5.4 *Assume (S1) and (S2) with $a = 0$. Then $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$, that is $q \in \mathbb{K}(X)$ if and only if*

$$\lim_{r \rightarrow 0^+} \left[\sup_{x \in \mathbb{R}} \int_0^r |q(z+x)|G^0(z)dz \right] = 0.$$

Proof Obviously X is transient and by Remark 13 the result of Theorem 4.15 applies. □

Lemma 5.5 *Assume (S1), (S2) and (S3) and let $a \geq 0$ be chosen according to (S2). Then the density $G^a(z)$ of $G^a(dz)$ satisfies*

$$G^a(z) \approx \frac{\phi'(z^{-1})}{z^2\phi^2(z^{-1})}, \quad 0 < z \leq 1.$$

Proof The Laplace transform of $G^a(z)$ is given by $\Phi = 1/[a + \phi]$. Note that

$$\Phi' = \frac{\phi'}{\phi^2} \left[\frac{\phi}{a + \phi} \right]^2 \approx \frac{\phi'}{\phi^2} \quad \text{on} \quad [1, \infty).$$

Thus by [6, Remark 3] $\Phi' \in \text{WUSC}(-\beta, \bar{\theta} \vee 1, \bar{C}/c)$, $c = [\phi(1)/[a + \phi(1)]]^2$. Next, [6, Lemma 5] and a version of Lemma 13 from [6] imply $G^a(z) \approx z^{-2}\Phi'(z^{-1}) \approx z^{-2}\phi'(z^{-1})/\phi^2(z^{-1})$ as $z \rightarrow 0^+$ (see also [22, Proposition 3.4]). The result extends to $z \in (0, 1]$ by the regularity of both sides of the estimate. □

Lemma 5.5, Remark 12 and Proposition 5.4 imply the following result.

Proposition 5.6 *Let X be a subordinator satisfying (S1), (S2) and (S3). Then $q \in \mathbb{K}(X)$ if and only if Eq. 7 holds.*

5.3 Examples

We refer the reader to [1, 11, 38] and [25] for basic examples of the Brownian motion, the relativistic process, symmetric α -stable processes and relativistic α -stable processes. We proceed towards our examples.

Example 1 Denote $A_1 = \{2^n : n \in \mathbb{Z}\}$ and

$$f(s) = \mathbb{1}_{(0,1]}(s) s^{-\alpha} + e^m \mathbb{1}_{(1,\infty)}(s) e^{-ms^\beta} s^{-\delta}, \quad s > 0,$$

where $m > 0, \beta \in (0, 1], \delta > 0$ and $\alpha \in (0, 2)$. Define a Lévy measure in \mathbb{R} as

$$\nu(dz) = \sum_{y \in A_1} f(|y|) (\delta_y(dz) + \delta_{-y}(dz)). \tag{29}$$

Let X be a Lévy process with $A = 0, \gamma = 0$ and (an infinite symmetric) ν given by Eq. 29. Then X is a recurrent process, $\psi(z)$ is a real valued function comparable with $|z|^2 \wedge |z|^\alpha$ (see [19, Example 4] and [29, Corollary 37.6]). Further, if $\alpha \in (1, 2)$ Theorem 4.6 applies and describes both $\mathcal{K}(X)$ and $\mathbb{K}(X)$. If now $\alpha \in (0, 1]$ by Theorem 4.1 we obtain $\mathcal{K}(X) = \mathbb{K}(X)$. By [23, Theorem 2.5] there are constants $c_1, c_2 \in (0, 1)$ such that $p(t, x) \geq c_1 t^{-1/\alpha}$ on $|x| \leq c_2 t^{1/\alpha}, t \in (0, 1]$. Then for some $c > 0$

$$\int_0^1 p(u, x) du \geq c H(|x|), \quad |x| \leq c_2/2.$$

where

$$H(r) = \begin{cases} r^{\alpha-1}, & 0 < \alpha < 1, \\ \ln(r^{-1}), & \alpha = 1. \end{cases}$$

Moreover, by [19, Example 4] there is $c_3 > 0$ so that $p(t, x) \leq c_3 t^{-1/\alpha} (1 \wedge t |x|^{-\alpha})$ on $|x| \leq 1, t \in (0, 1]$. Thus, if $\alpha \in (1/2, 1]$, there exists a constant $c > 0$ such that

$$\int_0^1 p(u, x) du \leq c H(|x|), \quad |x| \leq 1/2.$$

Finally, by Proposition 3.6 for $\alpha \in (1/2, 1]$ we have $q \in \mathcal{K}(X) = \mathbb{K}(X)$ if and only if

$$\lim_{r \rightarrow 0^+} \int_{B(x,r)} |q(z)| H(|z-x|) dz = 0.$$

We note that this considerations superficially resemble the results of [25] (see especially [25, Definition 3.2]). We explain why [25] cannot be applied in this example if $\alpha \leq 1$. Let $f(t, x)$ be a function that is non-increasing on $x \in (0, 1]$ for every fixed $t \in (0, 1]$. If $p(t, x) \leq f(t, x)$ by the lower bound for p and monotonicity of f we have $f(t, x) \geq c_4 t^{-1/\alpha} (1 \wedge t 2^{\alpha k}), x \in (2^{-k-1}, 2^{-k}]$. Then for $n(t) = (1/\alpha) \log_2(1/t)$ we obtain

$$\int_0^1 f(t, x) dx \geq c_4 t^{-1/\alpha} \sum_{k=0}^{n(t)} 2^{(\alpha-1)k-1} \xrightarrow{t \rightarrow 0^+} \infty, \quad \text{if } \alpha \in (0, 1].$$

Finally, if the upper bound assumption [25, (A2.3)] holds, i.e., $p(t, x) \leq t^{-1/\beta} \Phi_2(t^{-1/\beta} |x|) = f(t, x)$ for some $\beta > 0$, we have

$$\int_0^{t^{-1/\beta}} \Phi_2(z) dz = \int_0^1 f(t, x) dx \xrightarrow{t \rightarrow 0^+} \infty, \quad \text{if } \alpha \in (0, 1],$$

which contradicts with the integrability assumption in [25, (A2.3)].

In fact, we have $p(s, x) \leq c_3 t^{-1/\alpha} \Phi_2(t^{-1/\alpha} |x|)$ for $|x| \leq 1, t \in (0, 1]$ with $\Phi_2(r) = 1 \wedge r^{-\alpha}$, which is a precise estimate for $x \in A_1$ and $|x| \leq 1$, and the integrability condition for Φ_2 holds only if $\alpha \in (1, 2)$.

Example 2 Let $\psi(x, y) = |x|^2 + iy$ that is $X_t = (B_t, t)$, where B_t is the standard Brownian motion in \mathbb{R}^d (see [2, 10.4 and Example 13.30]). We note that in this case the transition kernel is not absolutely continuous but the potential kernel is. Then $q \in \mathbb{K}(X)$ reads as

$$\lim_{t \rightarrow 0^+} \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}} \int_0^t \int_{\mathbb{R}^d} |q(z + x, u + y)| u^{-d/2} e^{-|z|^2/(4u)} dz du = 0,$$

and by Corollary 1.2 holds if and only if

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}} \int_0^r \int_{B(0,r)} |q(z + x, u + y)| u^{-d/2} e^{-|z|^2/(4u)} dz du = 0.$$

Now we discuss in detail subordinators. Since functions ϕ presented below are unbounded CBF with zero drift term, see [30, Chapter 16: No 2 and 59, Proposition 7.1], they satisfy (S1) and (S2). The assumption (S3) can be easily checked. The first example covers the case of α -stable subordinator, $\alpha \in (0, 1)$, and the inverse Gaussian subordinator.

Example 3 Let $\phi(u) = \delta[(u + m)^\alpha - m^\alpha], \delta > 0, m \geq 0, \alpha \in (0, 1)$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_x^{x+r} |q(z)|(z - x)^{\alpha-1} dz = 0.$$

Example 4 Let $\phi(u) = \ln(1 + u^\alpha)$, where $\alpha \in (0, 1]$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_x^{x+r} |q(z)| \frac{dz}{(z - x) \ln^2(z - x)} = 0.$$

Example 5 Let $\phi(u) = \frac{u}{\ln(1 + u^\alpha)}$, where $\alpha \in (0, 1)$. Then $q \in \mathbb{K}(X)$ if and only if

$$\lim_{r \rightarrow 0^+} \sup_{x \in \mathbb{R}} \int_x^{x+r} |q(z)| |\ln(z - x)| dz = 0.$$

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