Katyusha: The First Direct Acceleration of Stochastic Gradient Methods

(version 5)*

Zeyuan Allen-Zhu
zeyuan@csail.mit.edu
Princeton University / Institute for Advanced Study

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Abstract

Nesterov's momentum trick is famously known for accelerating gradient descent, and has been proven useful in building fast iterative algorithms. However, in the stochastic setting, counterexamples exist and prevent Nesterov's momentum from providing similar acceleration, even if the underlying problem is convex.

We introduce `Katyusha`, a direct, primal-only stochastic gradient method to fix this issue. It has a provably accelerated convergence rate in convex (off-line) stochastic optimization. The main ingredient is *Katyusha momentum*, a novel "negative momentum" on top of Nesterov's momentum. It can be incorporated into a variance-reduction based algorithm and speed it up, both in terms of *sequential and parallel* performance. Since variance reduction has been successfully applied to a growing list of practical problems, our paper suggests that in each of such cases, one could potentially try to give Katyusha a hug.

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1 Introduction

In large-scale machine learning, the number of data examples is usually very large. To search for the optimal solution, one often uses *stochastic gradient methods* which only require one (or a small batch of) random example(s) per iteration in order to form an *estimator* of the full gradient.

While full-gradient based methods can enjoy an *accelerated* (and optimal) convergence rate if Nesterov's momentum trick is used [35–37], theory for stochastic gradient methods are generally lagging behind and less is known for their acceleration.

At a high level, momentum is *dangerous* if stochastic gradients are present. If some gradient estimator is very inaccurate, then adding it to the momentum and moving further in this direction (for every future iteration) may hurt the convergence performance. In other words, when naively equipped with momentum, stochastic gradient methods are "very prune to error accumulation" [25] and do *not* yield accelerated convergence rates in general.[1]

In this paper, we show that at least for convex optimization purposes, such an issue can be solved with a novel "negative momentum" that can be added on top of momentum. We obtain accelerated and the first optimal convergence rates for stochastic gradient methods, and believe our new insight can potentially deepen our understanding to the theory of accelerated methods.

Problem Definition. Consider the following composite convex minimization problem

$$\min_{x\in\mathbb{R}^d}\Big\{F(x)\stackrel{\text{def}}{=}f(x)+\psi(x)\stackrel{\text{def}}{=}\frac{1}{n}\sum_{i=1}^{n}f_i(x)+\psi(x)\Big\}\ . \tag{1.1}$$

Here, $f(x)=\frac{1}{n}\sum_{i=1}^{n}f_i(x)$ is a convex function that is a finite average of n convex, smooth functions $f_i(x)$, and $\psi(x)$ is convex, lower semicontinuous (but possibly non-differentiable) function, sometimes referred to as the *proximal* function. We mostly focus on the case when $\psi(x)$ is σ-strongly convex and each $f_i(x)$ is L-smooth. (Both these assumptions can be removed and we shall discuss that later.) We look for approximate minimizers $x\in\mathbb{R}^d$ satisfying $F(x)\le F(x^*)+\varepsilon$, where $x^*\in\arg\min_x\{F(x)\}$.

Problem (1.1) arises in many places in machine learning, statistics, and operations research. All convex *regularized empirical risk minimization (ERM)* problems such as Lasso, SVM, Logistic Regression, fall into this category (see Section 1.2). Efficient stochastic methods for Problem (1.1) also lead to fast algorithms for neural nets [2, 23] as well as SVD, PCA, and CCA [4, 6, 20].

We summarize the history of stochastic gradient methods solving Problem (1.1) into three eras.

The First Era: Stochastic Gradient Descent (SGD).

Recall that stochastic gradient methods iteratively perform the following update

$$\text{stochastic gradient iteration:}\qquad x_{k+1}\leftarrow\arg\min_{y\in\mathbb{R}^d}\Big\{\frac{1}{2\eta}\|y-x_k\|_2^2+\langle\widetilde{\nabla}_k,y\rangle+\psi(y)\Big\}\ ,$$

where η is the step length and $\widetilde{\nabla}_k$ is a random vector satisfying $\mathbb{E}[\widetilde{\nabla}_k]=\nabla f(x_k)$ and is referred to as the *gradient estimator*. If the proximal function $\psi(y)$ equals zero, the update reduces to $x_{k+1}\leftarrow x_k-\eta\widetilde{\nabla}_k$. A popular choice for the gradient estimator is to set $\widetilde{\nabla}_k=\nabla f_i(x_k)$ for some random index $i\in[n]$ per iteration, and methods based on this choice are known as *stochastic gradient descent (SGD)* [14, 52]. Since computing $\nabla f_i(x)$ is usually n times faster than that of $\nabla f(x)$, SGD enjoys a low per-iteration cost as compared to full-gradient methods; however, SGD cannot converge at a rate faster than $1/\varepsilon$ even if $F(\cdot)$ is strongly convex and smooth.

[1] In practice, experimentalists have observed that momentums could sometimes help if stochastic gradient iterations are used. However, the so-obtained methods (1) sometimes fail to converge in an accelerated rate, (2) become unstable and hard to tune, and (3) have no support theory behind them. See Section 7.1 for an experiment illustrating that.

The Second Era: Variance Reduction Gives Faster Convergence.

The convergence rate of SGD can be further improved with the so-called variance-reduction technique, first proposed by Schmidt et al. [42] and followed by many others [13, 17, 23, 32, 33, 44–46, 50, 51]. In these cited results, the authors have shown that SGD converges much faster if one makes a better choice of the gradient estimator $\widetilde{\nabla}_k$ so that its variance reduces as k increases. One way to choose this estimator can be described as follows [23, 51]. Keep a *snapshot* vector $\widetilde{x} = x_k$ that is updated once every m iterations (where m is some parameter usually around $2n$), and compute the full gradient $\nabla f(\widetilde{x})$ only for such snapshots. Then, set

$$\widetilde{\nabla}_k = \nabla f_i(x_k) - \nabla f_i(\widetilde{x}) + \nabla f(\widetilde{x}) \ . \tag{1.2}$$

This choice of gradient estimator ensures that its variance approaches to zero as k grows. Furthermore, the number of stochastic gradients (i.e., the number of computations of $\nabla f_i(x)$ for some i) required to reach an ε-approximate minimizer of Problem (1.1) is only $O\big((n + \frac{L}{\sigma})\log\frac{1}{\varepsilon}\big)$. Since it is often denoted by $\kappa \stackrel{\text{def}}{=} L/\sigma$ the condition number of the problem, we rewrite the above iteration complexity as $O\big((n+\kappa)\log\frac{1}{\varepsilon}\big)$.

Unfortunately, the iteration complexities of all known variance-reduction based methods have a linear dependence on κ. It was an open question regarding how to obtain

an *accelerated* stochastic gradient method with an optimal $\sqrt{\kappa}$ dependency.

The Third Era: Acceleration Gives Fastest Convergence.

This open question was partially solved recently by the APPA [19] and Catalyst [29] reductions, both based on an outer-inner loop structure first proposed by Shalev-Shwartz and Zhang [47]. We refer to both of them as Catalyst in this paper. Catalyst solves Problem (1.1) using $O\big((n+\sqrt{n\kappa})\log\kappa\log\frac{1}{\varepsilon}\big)$ stochastic gradient iterations, through a logarithmic number of calls to a variance-reduction method.[2] However, Catalyst is still imperfect for the following reasons:

- Optimality. To the best of our knowledge, Catalyst does not match the optimal $\sqrt{\kappa}$ dependence [49] and has an extra $\log\kappa$ factor. For similar reasons, it only yields $\frac{\log^2(1/\varepsilon)}{\varepsilon^{1/2}}$ (or equivalently $\frac{\log^4 T}{T^2}$) suboptimal rates if the objective is not strongly convex, or is non-smooth; and it only yields the $\frac{\log^2(1/\varepsilon)}{\varepsilon}$ (or equivalently $\frac{\log^4 T}{T}$) suboptimal rates if the objective is both non-strongly convex and non-smooth.[3]
- Practicality. To the best of our knowledge, Catalyst is not very practical since each of its inner iterations needs to be very accurately executed. This makes the stopping criterion hard to be tuned, and makes Catalyst sometimes run slower than non-accelerated variance-reduction methods [28]. We have also confirmed this in our experiments.
- Generality. To the best of our knowledge, Catalyst has a few limitations for being a reduction-based method. It does not seem to support non-Euclidean norm smoothness (see Section 6). It does not seem to give competent parallel (i.e., mini-batch) performance (see Section 5).

A bit less known is the work of Lan and Zhou [27], where the authors proposed a primal-dual method that also has a $\sqrt{\kappa}\log(\kappa)$ dependency. Their method is subject to the same optimality issue as Catalyst, and requires n times more storage compared with Catalyst for Problem (1.1).

In sum, it is not only desirable but also an open question to develop a *direct* and *primal-only* accelerated stochastic gradient method without using reductions or paying the extra $\log\kappa$ factor.

[2] Note that $n + \sqrt{n\kappa}$ is always less than $O(n+\kappa)$.

[3] Obtaining *optimal* rates is one of the main goals in optimization and machine learning. For instance, obtaining the optimal $1/T$ rate for online learning was a very meaningful result, even though the $\log T/T$ rate was known [21, 41].

This could have both theoretical and practical impacts to the problems that fall into the general framework of (1.1), and potentially deepen our understanding to acceleration in stochastic settings.

1.1 Our Main Results and High-Level Ideas

We develop a direct, accelerated stochastic gradient method `Katyusha` for solving Problem (1.1) in

$$\boxed{O\big((n+\sqrt{n\kappa})\log(1/\varepsilon)\big) \text{ stochastic gradient iterations (see Theorem 3.1).}}$$

This gives both optimal dependency on κ and on ε which, to the best of our knowledge, was not obtained before for stochastic gradient methods. In addition, if $F(\cdot)$ is non-strongly convex, `Katyusha` converges to an ε-minimizer in

$$\boxed{O\big(n\log(1/\varepsilon)+\sqrt{nL/\varepsilon}\cdot\|x_0-x^*\|\big) \text{ stochastic gradient iterations (see Corollary 3.12).}}$$

This gives an optimal $\varepsilon \propto \frac{n}{T^2}$ convergence rate. Again, to the best of our knowledge, was not obtained before for stochastic gradient methods. For instance, Catalyst has rate $\varepsilon \propto \frac{n\log^4 T}{T^2}$.

Our Algorithm. If ignoring the proximal term $\psi(\cdot)$ and viewing it as zero, our `Katyusha` method iteratively perform the following updates for $k=0,1,\dots$:

- $x_{k+1} \leftarrow \tau_1 z_k + \tau_2\widetilde{x} + (1-\tau_1-\tau_2)y_k$; (so $x_{k+1} = y_k + \tau_1(z_k-y_k) + \tau_2(\widetilde{x}-y_k)$)
- $\widetilde{\nabla}_{k+1} \leftarrow \nabla f(\widetilde{x}) + \nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})$ where i is a random index in $[n]$;
- $y_{k+1} \leftarrow x_{k+1} - \frac{1}{3L}\widetilde{\nabla}_{k+1}$, and
- $z_{k+1} \leftarrow z_k - \alpha\widetilde{\nabla}_{k+1}$.

Above, $\widetilde{x}$ is a snapshot point which is updated every m iterations, $\widetilde{\nabla}_{k+1}$ is the gradient estimator defined in the same way as (1.2), $\tau_1,\tau_2\in[0,1]$ are two momentum parameters, and α is a parameter that is equal to $\frac{1}{3\tau_1 L}$. The reason for keeping a sequence of three vectors (x_k,y_k,z_k) is a common ingredient that can be found in all existing accelerated methods.[4]

Our New Technique – Katyusha Momentum. The most interesting ingredient of `Katyusha` is the novel choice of x_{k+1} which is a convex combination of y_k, z_k, and $\widetilde{x}$. Our theory suggests the parameter choices $\tau_2=0.5$ and $\tau_1=\min\{\sqrt{n\sigma/L},0.5\}$ and they work well in practice too. To explain this novel combination, let us recall the classical "momentum" view of accelerated methods.

In a classical accelerated gradient method, x_{k+1} is only a convex combination of y_k and z_k (or equivalently, $\tau_2=0$ in our formulation). At a high level, z_k plays the role of "momentum" which adds a weighted sum of the gradient history into y_{k+1}. As an illustrative example, suppose $\tau_2=0$, $\tau_1=\tau$, and $x_0=y_0=z_0$. Then, one can compute that

$$y_k = \begin{cases} x_0 - \frac{1}{3L}\widetilde{\nabla}_1, & k=1;\\ x_0 - \frac{1}{3L}\widetilde{\nabla}_2 - \big((1-\tau)\frac{1}{3L}+\tau\alpha\big)\widetilde{\nabla}_1, & k=2;\\ x_0 - \frac{1}{3L}\widetilde{\nabla}_3 - \big((1-\tau)\frac{1}{3L}+\tau\alpha\big)\widetilde{\nabla}_2 - \big((1-\tau)^2\frac{1}{3L}+(1-(1-\tau)^2)\alpha\big)\widetilde{\nabla}_1, & k=3.\end{cases}$$

Since α is usually much larger than $1/3L$, the above recursion suggests that the contribution of a fixed gradient $\widetilde{\nabla}_t$ gradually increases as time goes. For instance, the weight on $\widetilde{\nabla}_1$ is increasing because $\frac{1}{3L} < \big((1-\tau)\frac{1}{3L}+\tau\alpha\big) < \big((1-\tau)^2\frac{1}{3L}+(1-(1-\tau)^2)\alpha\big)$. This is known as "momentum" which is at the heart of all accelerated first-order methods.

In `Katyusha`, we put a "magnet" around $\widetilde{x}$, where we choose $\widetilde{x}$ to be essentially "the average x_t of the most recent n iterations". Whenever we compute the next x_{k+1}, it will be attracted by the magnet $\widetilde{x}$ with weight $\tau_2=0.5$. This is a strong magnet: it ensures that x_{k+1} is not too far away

[4]One can of course rewrite the algorithm and keep track of only two vectors per iteration during implementation. This will make the algorithm statement less clean so we refrain from doing so in this paper.

from $\widetilde{x}$ so the gradient estimator remains "accurate enough". This can be viewed as a "negative momentum" component, because the magnet retracts x_{k+1} back to $\widetilde{x}$ and this can be understood as "counteracting a fraction of the positive momentum incurred from earlier iterations."

We call it the Katyusha momentum.

This summarizes the high-level idea behind our `Katyusha` method. We remark here if $\tau_1 = \tau_2 = 0$, `Katyusha` becomes almost identical to SVRG [23, 51] which is a variance-reduction based method.

1.2 Applications: Optimal Rates for Empirical Risk Minimization

Suppose we are given n feature vectors $a_1, \ldots, a_n \in \mathbb{R}^d$ corresponding to n data samples. Then, the *empirical risk minimization (ERM)* problem is to study Problem (1.1) when each $f_i(x)$ is "rank-one" structured: that is, $f_i(x) \stackrel{\text{def}}{=} g_i(\langle a_i, x\rangle)$ for some loss function $g_i : \mathbb{R} \to \mathbb{R}$. Slightly abusing notation, we also write $f_i(x) = f_i(\langle a_i, x\rangle)$. (Assuming "rank-one" simplifies the notations; all of the results stated in this subsection generalize to constant-rank structured functions $f_i(x)$.)

In such a case, Problem (1.1) becomes as

$$\text{ERM:} \quad \min_{x\in\mathbb{R}^d} \Big\{ F(x) \stackrel{\text{def}}{=} f(x) + \psi(x) \stackrel{\text{def}}{=} \tfrac{1}{n}\textstyle\sum_{i=1}^n f_i(\langle a_i, x\rangle) + \psi(x) \Big\} \ . \tag{1.3}$$

Without loss of generality, we assume each a_i has norm 1 because otherwise one can scale $f_i(\cdot)$ accordingly. As summarized for instance in [1], there are four interesting cases of ERM problems, all can be written in the form of (1.3):

Case 1: $\psi(x)$ is σ strongly convex and $f_i(x)$ is L-smooth. Examples: ridge regression, elastic net;

Case 2: $\psi(x)$ is non-strongly convex and $f_i(x)$ is L-smooth. Examples: Lasso, logistic regression;

Case 3: $\psi(x)$ is σ strongly convex and $f_i(x)$ is non-smooth. Examples: support vector machine;

Case 4: $\psi(x)$ is non-strongly convex and $f_i(x)$ is non-smooth. Examples: ℓ_1-SVM.

Known Results. For all of the four ERM cases above, accelerated stochastic methods were introduced in the literature, most notably AccSDCA [47], APCG [30], SPDC [53]. However, all known accelerated methods have suboptimal convergence rates for Case 2, 3 and 4.[5] In particular, the best known convergence rate was $\frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}}$, $\frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}}$, and $\frac{\log(1/\varepsilon)}{\varepsilon}$ respectively for Case 2, 3, and 4, and this is a factor $\log(1/\varepsilon)$ worse than the optimal rate for each of the three classes [49].

It is an open question also in the optimization community to design a stochastic gradient method with optimal convergence for such problems. In particular, Dang and Lan [16] provided an interesting attempt to remove such log factors but using a non-classical notion of convergence.[6]

Besides the log factor loss in the running time,[7] the aforementioned methods are dual-based and therefore suffer from several other issues. First, they only apply to ERM problems but not to the more general Problem (1.1). Second, they require proximal updates with respect to the Fenchel conjugate $f_i^*(\cdot)$ which is sometimes unpleasant to work with. Third, their performances cannot benefit from the implicit strong convexity in $f(\cdot)$. All of these issues together make dual-based

[5] In fact, they also have the suboptimal dependence on the condition number L/σ for Case 1.

[6] Dang and Lan work in a primal-dual $\phi(x, y)$ formulation of Problem (1.1), and produce a primal-dual pair (x, y) so that for every fixed (u, v), the expectation $\mathbb{E}[\phi(x, v) - \phi(u, y)] \le \varepsilon$. Unfortunately, to ensure x is an ε-approximate minimizer of Problem (1.1), one needs the stronger $\mathbb{E}[\max_{(u,v)} \phi(x, v) - \phi(u, y)] \le \varepsilon$ to hold.

[7] In fact, dual-based methods have to suffer from a log factor loss in the convergence rate. This is so because even for Case 1 of Problem (1.3), converting an ε-maximizer for the dual objective to the primal, one only obtains an $n\kappa\varepsilon$-minimizer on the primal objective. As a result, algorithms like APCG who directly work on the dual, algorithms like SPDC who maintain both primal and dual variables, and algorithms like RPDG [27] that are primal-like but still use dual analysis, have to suffer from a log loss in the convergence rates.

accelerated methods sometimes even outperformed by primal-only non-accelerated ones, such as SAGA [17] or SVRG [23, 51].

Our Results. `Katyusha` simultaneously closes the gap for all of the three classes of problems with the help from the optimal reductions developed in [1]. We obtain an ε-approximate minimizer for Case 2 in $O\big(n\log\frac{1}{\varepsilon}+\frac{\sqrt{nL}}{\sqrt{\varepsilon}}\big)$ iterations, for Case 3 in $O\big(n\log\frac{1}{\varepsilon}+\frac{\sqrt{n}}{\sqrt{\sigma\varepsilon}}\big)$ iterations, and for Case 4 in $O\big(n\log\frac{1}{\varepsilon}+\frac{\sqrt{n}}{\varepsilon}\big)$ iterations. In contrast, none of the existing accelerated methods can lead to such optimal rates even if the optimal reductions of [1] are used.

After this paper first appeared on arXiv, Woodworth and Srebro [49] proved the tightness of our results. They showed lower bounds $\Omega\big(n+\frac{\sqrt{nL}}{\sqrt{\varepsilon}}\big)$, $\Omega\big(n+\frac{\sqrt{n}}{\sqrt{\sigma\varepsilon}}\big)$, and $\Omega\big(n+\frac{\sqrt{n}}{\varepsilon}\big)$ for Cases 2, 3, and 4 respectively at least for small ε.[8]

1.3 Our Side Results

Parallelism / Mini-batch. Instead of using a single $\nabla f_i(\cdot)$ per iteration, for any stochastic gradient method, one can replace it with the average of b stochastic gradients $\frac{1}{b}\sum_{i\in S}\nabla f_i(\cdot)$, where S is a random subset of $[n]$ with cardinality b. This is known as the *mini-batch* technique and it allows the stochastic gradients to be computed in a distributed manner, using up to b processors.

Our `Katyusha` method trivially extends to this mini-batch setting (see Section 3). For instance, at least for $b\in\{1,2,\dots,\lceil\sqrt{n}\rceil\}$, `Katyusha` enjoys a *linear speed-up* in the parallel running time. In other words, if ignoring communication overhead,

`Katyusha` can be distributed to $b\leq\sqrt{n}$ machines with a parallel speed-up factor b.

In contrast, to the best of our knowledge, without any additional assumption, (1) non-accelerated methods such as SVRG or SAGA do not enjoy any parallel speed-up; (2) Catalyst enjoys a parallel speed-up factor of only $\sqrt{b}$.

Non-Uniform Smoothness. If each $f_i(\cdot)$ has a possibly different smooth parameter L_i and $\overline{L}=\frac{1}{n}\sum_{i=1}^n L_i$, then an naive implementation of `Katyusha` only gives a complexity that depends on $\max_i L_i$ but not $\overline{L}$. In such a case, we can select the random index $i\in[n]$ with probability proportional to L_i per iteration to slightly improve the total running time.

Furthermore, suppose $f(x)=\frac{1}{n}\sum_{i=1}^n f_i(x)$ is smooth with parameter L, it satisfies $\overline{L}\in[L,nL]$. One can ask whether or not L influences the performance of `Katyusha`. We show that, in the mini-batch setting when b is large, the total complexity becomes a function on L as opposed to $\overline{L}$.

A Precise Statement. Taking into account both the mini-batch parameter b and the non-uniform smoothness parameters L and $\overline{L}$, we show that `Katyusha` solves Problem (1.1) in

$O\Big(\big(n+b\sqrt{L/\sigma}+\sqrt{n\overline{L}/\sigma}\big)\cdot\log\frac{1}{\varepsilon}\Big)$ stochastic gradient computations (see Theorem 5.2)

Non-Euclidean Norms. If the smoothness of each $f_i(x)$ is with respect to a non-Euclidean norm (such as the well known ℓ_1 norm case over the simplex), our main result still holds. Our update on the y_{k+1} side becomes the non-Euclidean norm gradient descent, and our update on the z_{k+1} side becomes the non-Euclidean norm mirror descent. We include such details in Section 6. In contrast, to the best of our knowledge, Catalyst, AccSDCA and APCG do not work with non-Euclidean norms. SPDC can be revised to work with non-Euclidean norms, see [8].

[8] More precisely, their lower bounds for Cases 3 and 4 are $\Omega\big(\min\{\frac{1}{\sigma\varepsilon},n+\frac{\sqrt{n}}{\sqrt{\sigma\varepsilon}}\}\big)$ and $\Omega\big(\min\{\frac{1}{\varepsilon^2},n+\frac{\sqrt{n}}{\varepsilon}\}\big)$. However, since the vanilla SGD requires $O(\frac{1}{\sigma\varepsilon})$ and $O(\frac{1}{\varepsilon^2})$ iterations for Cases 3 and 4, such lower bounds are matched by combining the best between `Katyusha` and SGD.

Remark on Katyusha Momentum Weight τ_2. To provide the simplest proof, we choose $\tau_2 = 1/2$ which also works well in practice. Our proof trivially generalizes to all constant values $\tau_2 \in (0,1)$, and it could be beneficial to tune τ_2 for different datasets. However, for a stronger comparison, in our experiments we refrain from tuning τ_2: by fixing $\tau_2 = 1/2$ and without increasing parameter tuning difficulties, `Katyusha` already outperforms most of the state-of-the-arts.

In the mini-batch setting, it turns out the best theoretical choice is essentially $\tau_2 = \frac{1}{2b}$, where b is the size of the mini-batch. In other words, the larger the mini-batch size, the smaller weight we want to give to Katyusha momentum. This should be intuitive, because when $b = n$ we are almost in the full-gradient setting and do not need Katyusha momentum.

1.4 Related Work

For smooth convex minimization problems, (full) gradient descent converges at a rate $\frac{L}{\varepsilon}$ —or $\frac{L}{\sigma}\log\frac{1}{\varepsilon}$ if the objective is σ-strongly convex. This is not optimal among the class of first-order methods. Nesterov showed that the optimal rate should be $\frac{\sqrt{L}}{\sqrt{\varepsilon}}$ —or $\frac{\sqrt{L}}{\sqrt{\sigma}}\log\frac{1}{\varepsilon}$ if the objective is σ-strongly convex— and this was achieved by his celebrated accelerated (full) gradient descent method [35].

Randomized Coordinate Descent. Another way to define gradient estimator is to set $\widetilde{\nabla}_k = d\nabla_j f(x_k)$ where j is a random coordinate. This is *(randomized) coordinate descent* as opposed to stochastic gradient descent. Designing accelerated methods for coordinate descent is significantly easier than designing that for stochastic gradient descent, and has indeed been done in many previous results including [12, 30, 31, 38].[9] The state-of-the-art accelerated coordinate descent method is NUACDM [12]. Coordinate descent *cannot* be applied to solve Problem (1.1) because in our setting, only one copy $\nabla f_i(\cdot)$ is computed in a stochastic iteration.

Hybrid Accelerated and Stochastic Methods. Several recent results study hybrid methods with convergence rates that are generally *non-accelerated* and only accelerated in *extreme cases.*

- The authors of [22, 26] obtained iteration complexity of the form $O(L/\sqrt{\varepsilon}+\sigma/\varepsilon^2)$ in the presence of stochastic gradient with variance σ. These results can be interpreted as follows, if σ is very small, then one can directly apply Nesterov's accelerated gradient method and achieve $O(L/\sqrt{\varepsilon})$; or if σ is large then they match the SGD iteration complexity $O(\sigma/\varepsilon^2)$. For Problem (1.1), these algorithms do not give faster running time than `Katyusha` unless σ is very small.[10]
- Nitanda's method adds momentum to the non-accelerated variance-reduction method in a naive manner [39] and thus corresponds to this paper but *without* Katyusha momentum (i.e., $\tau_2 = 0$). The theoretical running time of [39] is always slower than this paper and cannot even outperform SVRG [23, 51] unless $\kappa > n^2$ —which is usually *false* in practice (see page 7 of [39]).[11] We have included an experiment in Section 7.1 to illustrate why Katyusha momentum is necessary.

Linear Coupling. Allen-Zhu and Orecchia proposed a framework called *linear coupling* that facilitates the design of accelerated gradient methods [11]. The simplest use of linear coupling can reconstruct Nesterov's accelerated full-gradient method [11], or to provide faster coordinate

[9]The reason behind it can be understood as follows. If a function $f(\cdot)$ is L smooth with respect to coordinate j, then a coordinate descent step $x' \leftarrow x - \frac{1}{L}\nabla_j f(x)\mathbf{e}_j$ always decreases the objective, i.e., $f(x+\frac{1}{L}\nabla_j f(x)\mathbf{e}_j) < f(x)$. In contrast, this is *false* for stochastic gradient descent, because $f(x_k - \eta\widetilde{\nabla}_k)$ may be even larger than $f(x_k)$.

[10]When σ is large, even if n is large, the iteration complexity of [22, 26] becomes $O(\sigma/\varepsilon^2)$. In this regime, almost all variance-reduction methods, including SVRG and `Katyusha`, can be shown to satisfy $\varepsilon \le O(\frac{\sqrt{\sigma}}{\sqrt{T}})$ within the first epoch, if the learning rates are appropriately chosen. Therefore, `Katyusha` and SVRG are no slower than [22, 26].

[11]Nitanda's method is usually not considered as an accelerated method, since it requires mini-batch size to be very large in order to be accelerated. If mini-batch is large then one can use full-gradient method directly and acceleration is trivial. This is confirmed by [25, Section IV.F]. In contrast, our acceleration holds even if mini-batch size is 1.

descent [12]. More careful use of linear coupling can also give accelerated methods for non-smooth problems (such as positive LP [9, 10], positive SDP [3], matrix scaling [7]) or for general non-convex problems [2]. This present paper falls into this linear-coupling framework, but our Katyusha momentum technique was not present in any of these cited results.

1.5 Roadmap

- In Section 2, we provide necessary notations and useful preliminaries .
- In Section 3, we state and prove our theorem on `Katyusha` for the strongly convex case, and apply it to non-strongly convex or non-smooth cases *using reductions*.
- In Section 4, we provide a *direct* algorithm `Katyusha`$^{\mathsf{ns}}$ for the non-strongly case.
- In Section 5, we generalize `Katyusha` to the mini-batch and non-uniform smoothness settings.
- In Section 6, we generalize `Katyusha` to the non-Euclidean norm setting.
- In Section 7, we provide an empirical evaluation to illustrate the necessity of Katyusha momentum, and the practical performance of `Katyusha` comparing to the start-of-the-arts.

2 Preliminaries

Throughout this paper (except Section 6), we denote by $\|\cdot\|$ the Euclidean norm. We denote by $\nabla f(x)$ the full gradient of function f if it is differentiable, or the subgradient if f is only Lipschitz continuous. Recall some classical definitions on strong convexity (SC) and smoothness.

Definition 2.1 (smoothness and strong convexity)**.** *For a convex function $f\colon \mathbb{R}^n \to \mathbb{R}$,*

- *f is σ-strongly convex if $\forall x, y \in \mathbb{R}^n$, it satisfies $f(y) \geq f(x) + \langle \nabla f(x), y - x\rangle + \frac{\sigma}{2}\|x-y\|^2$.*
- *f is L-smooth if $\forall x, y \in \mathbb{R}^n$, it satisfies $\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\|$.*

We also need to use the following definition of the HOOD property:

Definition 2.2 ([1])**.** *An algorithm solving the strongly convex case of Problem (1.1) satisfies the* homogenous objective decrease (HOOD) *property with time* $\mathsf{Time}(L,\sigma)$*, if for every starting point x_0, it produces an output x' satisfying $\mathbb{E}\big[F(x')\big] - F(x^*) \leq \frac{F(x_0)-F(x^*)}{4}$ in time at most* $\mathsf{Time}(L,\sigma)$*.*

The authors of [1] designed three reductions AdaptReg, AdaptSmooth, and JointAdaptRegSmooth to convert an algorithm satisfying the HOOD property to solve the following three cases:

Theorem 2.3. *Given an algorithm satisfying HOOD with* $\mathsf{Time}(L,\sigma)$ *and a starting vector x_0.*

- NonSC+Smooth. *For Problem (1.1) where $f(\cdot)$ is L-smooth,* AdaptReg *outputs x satisfying $\mathbb{E}\big[F(x)\big] - F(x^*) \leq O(\varepsilon)$ in time*

$$\textstyle\sum_{t=0}^{T-1} \mathsf{Time}(L, \frac{\sigma_0}{2^t}) \text{ where } \sigma_0 = \frac{F(x_0)-F(x^*)}{\|x_0-x^*\|^2} \text{ and } T = \log_2 \frac{F(x_0)-F(x^*)}{\varepsilon}.$$

- SC+NonSmooth. *For Problem (1.3) where $\psi(\cdot)$ is σ-SC and each $f_i(\cdot)$ is $\sqrt{G}$-Lipschitz continuous,* AdaptSmooth *outputs x satisfying $\mathbb{E}\big[F(x)\big] - F(x^*) \leq O(\varepsilon)$ in time*

$$\textstyle\sum_{t=0}^{T-1} \mathsf{Time}(\frac{2^t}{\lambda_0}, \sigma) \text{ where } \lambda_0 = \frac{F(x_0)-F(x^*)}{G} \text{ and } T = \log_2 \frac{F(x_0)-F(x^*)}{\varepsilon}.$$

- NonSC+NonSmooth. *For Problem (1.3) where each $f_i(\cdot)$ is $\sqrt{G}$-Lipschitz continuous, then* JointAdaptRegSmooth *outputs x satisfying $\mathbb{E}\big[F(x)\big] - F(x^*) \leq O(\varepsilon)$ in time*

$$\textstyle\sum_{t=0}^{T-1} \mathsf{Time}(\frac{2^t}{\lambda_0}, \frac{\sigma_0}{2^t}) \text{ where } \lambda_0 = \frac{F(x_0)-F(x^*)}{G},\ \sigma_0 = \frac{F(x_0)-F(x^*)}{\|x_0-x^*\|^2} \text{ and } T = \log_2 \frac{F(x_0)-F(x^*)}{\|x_0-x^*\|^2}.$$

We shall verify in later that `Katyusha` satisfies HOOD so the above reductions can be applied.

Algorithm 1 `Katyusha`(x_0, S, σ, L)

$m \leftarrow 2n$; ◇ *epoch length*
$\tau_2 \leftarrow \frac{1}{2}$, $\tau_1 \leftarrow \min\big\{\frac{\sqrt{m\sigma}}{\sqrt{3L}}, \frac{1}{2}\big\}$, $\alpha \leftarrow \frac{1}{3\tau_1 L}$; ◇ *parameters*
$y_0 = z_0 = \widetilde{x}^0 \leftarrow x_0$; ◇ *initial vectors*
for $s \leftarrow 0$ **to** $S-1$ **do**
  $\mu^s \leftarrow \nabla f(\widetilde{x}^s)$; ◇ *compute the full gradient once every m iterations*
  for $j \leftarrow 0$ **to** $m-1$ **do**
    $k \leftarrow (sm) + j$;
    $x_{k+1} \leftarrow \tau_1 z_k + \tau_2 \widetilde{x}^s + (1 - \tau_1 - \tau_2) y_k$;
    $\widetilde{\nabla}_{k+1} \leftarrow \mu^s + \nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x}^s)$ where i is random from $\{1, 2, \dots, n\}$;
    $z_{k+1} = \arg\min_z \big\{\frac{1}{2\alpha}\|z - z_k\|^2 + \langle \widetilde{\nabla}_{k+1}, z\rangle + \psi(z)\big\}$;
    Option I: $y_{k+1} \leftarrow \arg\min_y \big\{\frac{3L}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y\rangle + \psi(y)\big\}$;
    Option II: $y_{k+1} \leftarrow x_{k+1} + \tau_1(z_{k+1} - z_k)$ ◇ *we analyze only I but II also works*
  end for
  $\widetilde{x}^{s+1} \leftarrow \big(\sum_{j=0}^{m-1}(1+\alpha\sigma)^j\big)^{-1} \cdot \big(\sum_{j=0}^{m-1}(1+\alpha\sigma)^j \cdot y_{sm+j+1}\big)$; ◇ *compute snapshot $\widetilde{x}$*
end for
return $\widetilde{x}^S$.

3 Katyusha in the Strongly Convex Setting

We formally introduce our `Katyusha` algorithm in Algorithm 1. It follows from our high-level description in Section 1.1, and we make several remarks here behind our specific design.

- `Katyusha` *is divided into epochs each consisting of m iterations. In theory, m can be anything linear in n. We let snapshot $\widetilde{x}$ be a weighted average of y_k in the most recent epoch.*

  $\widetilde{x}$ and $\widetilde{\nabla}_k$ correspond to a standard design on variance-reduced gradient estimators, called SVRG [23, 51]. The practical recommendation is $m = 2n$ [23]. Our choice $\widetilde{\nabla}_k$ is independent from our acceleration techniques, and we expect our result continues to apply to other choices of gradient estimators. We choose $\widetilde{x}$ to be a weighted average, rather than the last or the uniform average, because it yields the tightest possible result.[12]

- *τ_1 and α are standard parameters already present in Nesterov's full-gradient method [11].*

  We choose $\alpha = 1/3\tau_1 L$ to present the simplest proof, and recall it was $\alpha = 1/\tau_1 L$ in the original Nesterov's full-gradient method. (Any α that is constant factor smaller than $1/\tau_1 L$ works in theory, and we use $1/3$ to provide the simplest proof.) In practice, like other accelerated methods, it suffices to fix $\alpha = 1/3\tau_1 L$ and only tune τ_1 and thus τ_1 is viewed as the learning rate.

- *The parameter τ_2 is our novel weight for the Katyusha momentum. Any constant in $(0, 1)$ works for τ_2, and we simply choose $\tau_2 = 1/2$ for our theoretical and experimental results.*

We state our main theorem for `Katyusha` as follows:

[12] If one uses the uniform average, in theory, the algorithm needs to restart every a number of epochs (that is, by resetting $k = 0$, $s = 0$, and $x_0 = y_0 = z_0$); we refrain from doing so because we wish to provide a simple and direct algorithm. We can also use the last iterate, then the total complexity loses a factor $\log(L/\sigma)$. In practice, it was reported that even for SVRG, choosing average works better than choosing the last iterate [13].

Theorem 3.1. *If each $f_i(x)$ is convex, L-smooth, and $\psi(x)$ is σ-strongly convex in Problem (1.1), then* $\texttt{Katyusha}(x_0, S, \sigma, L)$ *satisfies*

$$\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq \begin{cases} O\Big(\big(1+\sqrt{\sigma/(3Lm)}\big)^{-Sm}\Big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } \frac{m\sigma}{L} \leq \frac{3}{4}; \\ O\big(1.5^{-S}\big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } \frac{m\sigma}{L} > \frac{3}{4}. \end{cases}$$

In other words, choosing $m = \Theta(n)$, `Katyusha` *achieves an ε-additive error (i.e., $\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq \varepsilon$) using at most $O\big(\big(n + \sqrt{nL/\sigma}\big) \cdot \log \frac{F(x_0)-F(x^*)}{\varepsilon}\big)$ iterations.*[13]

The proof of Theorem 3.1 is included in Section 3.1 and 3.2. As discussed in Section 1.1, the main idea behind our theorem is the negative momentum that helps reduce the error occurred from the stochastic gradient estimator.

Remark 3.2. Because $m = 2n$, each iteration of `Katyusha` computes only 1.5 stochastic gradients $\nabla f_i(\cdot)$ in the amortized sense, the same as non-accelerated methods such as SVRG [23].[14] Therefore, the per-iteration cost of `Katyusha` is dominated by the computation of $\nabla f_i(\cdot)$, the proximal update in Line 10 of Algorithm 1, plus an overhead $O(d)$. If $\nabla f_i(\cdot)$ has at most $d' \leq d$ non-zero entries, this overhead $O(d)$ is improvable to $O(d')$ using a sparse implementation of `Katyusha`.[15]

For ERM problems defined in Problem (1.3), the amortized per-iteration complexity of `Katyusha` is $O(d')$ where d' is the sparsity of feature vectors, the same as the per-iteration complexity of SGD.

3.1 One-Iteration Analysis

In this subsection, we first analyze the behavior of `Katyusha` in a single iteration (i.e., for a fixed k). We view y_k, z_k and x_{k+1} as fixed in this section so the only randomness comes from the choice of i in iteration k. We abbreviate in this subsection by $\widetilde{x} = \widetilde{x}^s$ where s is the epoch that iteration k belongs to, and denote by $\sigma_{k+1}^2 \stackrel{\text{def}}{=} \|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|^2$ so $\mathbb{E}[\sigma_{k+1}^2]$ is the variance of the gradient estimator $\widetilde{\nabla}_{k+1}$ in this iteration.

Our first lemma lower bounds the expected objective decrease $F(x_{k+1}) - \mathbb{E}[F(y_{k+1})]$. Our $\mathsf{Prog}(x_{k+1})$ defined below is a non-negative, classical quantity that would be a lower bound on the amount of objective decrease if $\widetilde{\nabla}_{k+1}$ were equal to $\nabla f(x_{k+1})$ [11]. However, since the variance σ_{k+1}^2 is non-zero, this lower bound must be compensated by a negative term that depends on $\mathbb{E}[\sigma_{k+1}^2]$.

Lemma 3.3 (proximal gradient descent)**.** *If*

$$y_{k+1} = \arg\min_y \Big\{ \frac{3L}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \ , \qquad \textit{and}$$

$$\mathsf{Prog}(x_{k+1}) \stackrel{\text{def}}{=} -\min_y \Big\{ \frac{3L}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \geq 0 \ ,$$

we have (where the expectation is only over the randomness of $\widetilde{\nabla}_{k+1}$)

$$F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] \geq \mathbb{E}\big[\mathsf{Prog}(x_{k+1})\big] - \frac{1}{4L}\mathbb{E}\big[\sigma_{k+1}^2\big] \ .$$

[14] The claim "SVRG or `Katyusha` computes 1.5 stochastic gradients" requires one to store $\nabla_i f(\widetilde{x})$ in the memory for each $i \in [n]$, and this costs $O(dn)$ space in the most general setting. If one does not store $\nabla_i f(\widetilde{x})$ in the memory, then each iteration of SVRG or `Katyusha` computes 2.5 stochastic gradients for the choice $m = 2n$.

[15] This requires to defer a coordinate update to the moment it is accessed. Update deferral is a standard technique used in sparse implementations of all stochastic gradient methods, including SVRG, SAGA, APCG [17, 23, 30].

Proof.

$$
\begin{aligned}
\mathsf{Prog}(x_{k+1}) &= -\min_y \Big\{ \frac{3L}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \\
&\overset{①}{=} -\Big(\frac{3L}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&= -\Big(\frac{L}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \nabla f(x_{k+1}), y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&\quad + \Big(\langle \nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle - L\|y_{k+1} - x_{k+1}\|^2 \Big) \\
&\overset{②}{\leq} -\Big(f(y_{k+1}) - f(x_{k+1}) + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) + \frac{1}{4L}\|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|^2 \enspace .
\end{aligned}
$$

Above, ① is by the definition of y_{k+1}, and ② uses the smoothness of function $f(\cdot)$, as well as Young's inequality $\langle a, b\rangle - \frac{1}{2}\|b\|^2 \leq \frac{1}{2}\|a\|^2$. Taking expectation on both sides we arrive at the desired result. □

The following lemma provides a novel upper bound on the expected variance of the gradient estimator. Note that all known variance reduction analysis for convex optimization, in one way or another, upper bounds this variance essentially by $4L \cdot (f(\widetilde{x}) - f(x^*))$, the objective distance to the minimizer (c.f. [17, 23]). The recent result of Allen-Zhu and Hazan [1] upper bounds it by the point distance $\|x_{k+1} - \widetilde{x}\|^2$ for non-convex objectives, which is tighter if $\widetilde{x}$ is close to x_{k+1} but unfortunately not enough for the purpose of this paper.

In this paper, we upper bound it by the tightest possible quantity which is essentially $2L \cdot \big(f(\widetilde{x}) - f(x_{k+1})\big) \ll 4L \cdot \big(f(\widetilde{x}) - f(x^*)\big)$. Unfortunately, this upper bound needs to be compensated by an additional term $\langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle$, which could be positive but we shall cancel it using the introduced Katyusha momentum.

Lemma 3.4 (variance upper bound)**.**

$$
\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|^2\big] \leq 2L \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big) \enspace .
$$

Proof. Each $f_i(x)$, being convex and L-smooth, implies the following inequality which is classical in convex optimization and can be found for instance in Theorem 2.1.5 of the textbook of Nesterov [36].

$$
\|\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\|^2 \leq 2L \cdot \big(f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle \nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big)
$$

Therefore, taking expectation over the random choice of i, we have

$$
\begin{aligned}
\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|^2\big] &= \mathbb{E}\Big[\big\|\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big) - \big(\nabla f(x_{k+1}) - \nabla f(\widetilde{x})\big)\big\|^2\Big] \\
&\overset{①}{\leq} \mathbb{E}\Big[\big\|\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big\|^2\Big] \\
&\overset{②}{\leq} 2L \cdot \mathbb{E}\big[f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle \nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big] \\
&= 2L \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big) \enspace .
\end{aligned}
$$

Above, ① is because for any random vector $\zeta \in \mathbb{R}^d$, it holds that $\mathbb{E}\|\zeta - \mathbb{E}\zeta\|^2 = \mathbb{E}\|\zeta\|^2 - \|\mathbb{E}\zeta\|^2$; ② follows from the first inequality in this proof. □

The next lemma is a classical one for proximal mirror descent.

Lemma 3.5 (proximal mirror descent)**.** *Suppose $\psi(\cdot)$ is σ-SC. Then, fixing $\widetilde{\nabla}_{k+1}$ and letting*

$$
z_{k+1} = \arg\min_z \Big\{ \frac{1}{2}\|z - z_k\|^2 + \alpha\langle \widetilde{\nabla}_{k+1}, z - z_k\rangle + \alpha\psi(z) - \alpha\psi(z_k) \Big\} \enspace ,
$$

it satisfies for all $u \in \mathbb{R}^d$*,*

$$\alpha\langle\widetilde{\nabla}_{k+1}, z_{k+1}-u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \le -\frac{1}{2}\|z_k - z_{k+1}\|^2 + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\|z_{k+1}-u\|^2 \enspace .$$

Proof. By the minimality definition of z_{k+1}, we have that

$$z_{k+1} - z_k + \alpha\widetilde{\nabla}_{k+1} + \alpha g = 0$$

where g is *some* subgradient of $\psi(z)$ at point $z = z_{k+1}$. This implies that for every u it satisfies

$$0 = \big\langle z_{k+1} - z_k + \alpha\widetilde{\nabla}_{k+1} + \alpha g, z_{k+1} - u\big\rangle \enspace .$$

At this point, using the equality $\langle z_{k+1} - z_k, z_{k+1} - u\rangle = \frac{1}{2}\|z_k - z_{k+1}\|^2 - \frac{1}{2}\|z_k - u\|^2 + \frac{1}{2}\|z_{k+1}-u\|^2$, as well as the inequality $\langle g, z_{k+1} - u\rangle \ge \psi(z_{k+1}) - \psi(u) + \frac{\sigma}{2}\|z_{k+1}-u\|^2$ which comes from the strong convexity of $\psi(\cdot)$, we can write

$$\begin{aligned}
&\alpha\langle\widetilde{\nabla}_{k+1}, z_{k+1}-u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&= -\langle z_{k+1} - z_k, z_{k+1} - u\rangle - \langle \alpha g, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&\le -\frac{1}{2}\|z_k - z_{k+1}\|^2 + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\|z_{k+1}-u\|^2 \enspace .
\end{aligned}$$

□

The following lemma combines Lemma 3.3, Lemma 3.4 and Lemma 3.5 all together, using the special choice of x_{k+1} which is a convex combination of y_k, z_k and $\widetilde{x}$:

Lemma 3.6 (coupling step 1)**.** *If* $x_{k+1} = \tau_1 z_k + \tau_2 \widetilde{x} + (1-\tau_1-\tau_2) y_k$*, where* $\tau_1 \le \frac{1}{3\alpha L}$ *and* $\tau_2 = \frac{1}{2}$*,*

$$\begin{aligned}
&\alpha\langle\nabla f(x_{k+1}), z_k - u\rangle - \alpha\psi(u) \\
&\le \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 f(x_{k+1}) - \tau_2\langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\Big) \\
&\quad + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1}-u\|^2\big] + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\psi(y_k) - \frac{\alpha}{\tau_1}\psi(x_{k+1}) \enspace .
\end{aligned}$$

Proof. We first apply Lemma 3.5 and get

$$\begin{aligned}
&\alpha\langle\widetilde{\nabla}_{k+1}, z_k-u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&= \alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle + \alpha\langle\widetilde{\nabla}_{k+1}, z_{k+1}-u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&\le \alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2 + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\|z_{k+1}-u\|^2 \enspace .
\end{aligned} \tag{3.1}$$

By defining $v \stackrel{\text{def}}{=} \tau_1 z_{k+1} + \tau_2\widetilde{x} + (1-\tau_1-\tau_2)y_k$, we have $x_{k+1} - v = \tau_1(z_k - z_{k+1})$ and therefore

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2\Big] = \mathbb{E}\Big[\frac{\alpha}{\tau_1}\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{1}{2\tau_1^2}\|x_{k+1}-v\|^2\Big] \\
&= \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1}-v\rangle - \frac{1}{2\alpha\tau_1}\|x_{k+1}-v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{①}{\le} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1}-v\rangle - \frac{3L}{2}\|x_{k+1}-v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{②}{\le} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{1}{4L}\sigma_{k+1}^2\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{③}{\le} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{1}{2}\big(f(\widetilde{x}) - f(x_{k+1}) - \langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big)\Big) \\
&\quad + \frac{\alpha}{\tau_1}\Big(\tau_1\psi(z_{k+1}) + \tau_2\psi(\widetilde{x}) + (1-\tau_1-\tau_2)\psi(y_k) - \psi(x_{k+1})\Big)\Big] \enspace .
\end{aligned} \tag{3.2}$$

Above, ① uses our choice $\tau_1 \le \frac{1}{3\alpha L}$, ② uses Lemma 3.3, ③ uses Lemma 3.4 together with the convexity of $\psi(\cdot)$ and the definition of v. Finally, noticing that $\mathbb{E}[\langle \widetilde{\nabla}_{k+1}, z_k - u\rangle] = \langle \nabla f(x_{k+1}), z_k - u\rangle$ and $\tau_2 = \frac{1}{2}$, we obtain the desired inequality by combining (3.1) and (3.2). $\square$

The next lemma simplifies the left hand side of Lemma 3.6 using the convexity of $f(\cdot)$, and gives an inequality that relates the objective-distance-to-minimizer quantities $F(y_k) - F(x^*)$, $F(y_{k+1}) - F(x^*)$, and $F(\widetilde{x}) - F(x^*)$ to the point-distance-to-minimizer quantities $\|z_k - x^*\|^2$ and $\|z_{k+1} - x^*\|^2$.

Lemma 3.7 (coupling step 2)**.** *Under the same choices of τ_1, τ_2 as in Lemma 3.6, we have*

$$0 \le \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - F(x^*)) - \frac{\alpha}{\tau_1}\big(\mathbb{E}\big[F(y_{k+1})\big] - F(x^*)\big) + \frac{\alpha\tau_2}{\tau_1}\big(F(\widetilde{x}) - F(x^*)\big) + \frac{1}{2}\|z_k - x^*\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - x^*\|^2\big] \ .$$

Proof. We first compute that

$$\begin{aligned}
&\alpha\big(f(x_{k+1}) - f(u)\big) \overset{①}{\le} \alpha\langle \nabla f(x_{k+1}), x_{k+1} - u\rangle \\
&= \alpha\langle \nabla f(x_{k+1}), x_{k+1} - z_k\rangle + \alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
&\overset{②}{=} \frac{\alpha\tau_2}{\tau_1}\langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\langle \nabla f(x_{k+1}), y_k - x_{k+1}\rangle + \alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \\
&\overset{③}{\le} \frac{\alpha\tau_2}{\tau_1}\langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(f(y_k) - f(x_{k+1})) + \alpha\langle \nabla f(x_{k+1}), z_k - u\rangle \ .
\end{aligned}$$

Above, ① uses the convexity of $f(\cdot)$, ② uses the choice that $x_{k+1} = \tau_1 z_k + \tau_2\widetilde{x} + (1-\tau_1-\tau_2)y_k$, and ③ uses the convexity of $f(\cdot)$ again. By applying Lemma 3.6 to the above inequality, we have

$$\begin{aligned}
&\alpha\big(f(x_{k+1}) - F(u)\big) \le \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - f(x_{k+1})) \\
&+ \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 f(x_{k+1})\Big) + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - u\|^2\big] - \frac{\alpha}{\tau_1}\psi(x_{k+1})
\end{aligned}$$

which implies

$$\begin{aligned}
&\alpha\big(F(x_{k+1}) - F(u)\big) \le \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - F(x_{k+1})) \\
&\quad + \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 F(x_{k+1})\Big) + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - u\|^2\big] \ .
\end{aligned}$$

After rearranging and setting $u = x^*$, the above inequality yields

$$0 \le \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - F(x^*)) - \frac{\alpha}{\tau_1}\big(\mathbb{E}\big[F(y_{k+1}) - F(x^*)\big]\big) + \frac{\alpha\tau_2}{\tau_1}\big(F(\widetilde{x}) - F(x^*)\big) + \frac{1}{2}\|z_k - x^*\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - x^*\|^2\big] \ .$$

$\square$

3.2 Proof of Theorem 3.1

We are now ready to combine the analyses across iterations, and derive our final Theorem 3.1. Our proof next requires a careful telescoping of Lemma 3.7 together with our specific parameter choices.

Proof of Theorem 3.1. Define $D_k \overset{\text{def}}{=} F(y_k) - F(x^*)$, $\widetilde{D}^s \overset{\text{def}}{=} F(\widetilde{x}^s) - F(x^*)$, and rewrite Lemma 3.7:

$$0 \le \frac{(1-\tau_1-\tau_2)}{\tau_1}D_k - \frac{1}{\tau_1}D_{k+1} + \frac{\tau_2}{\tau_1}\mathbb{E}\big[\widetilde{D}^s\big] + \frac{1}{2\alpha}\|z_k - x^*\|^2 - \frac{1+\alpha\sigma}{2\alpha}\mathbb{E}\big[\|z_{k+1} - x^*\|^2\big] \ .$$

At this point, let us define $\theta = 1+\alpha\sigma$ and multiply the above inequality by θ^j for each $k = sm+j$. Then, we sum up the resulting m inequalities for all $j = 0, 1, \dots, m-1$:

$$0 \le \mathbb{E}\Big[\frac{(1-\tau_1-\tau_2)}{\tau_1}\sum_{j=0}^{m-1} D_{sm+j}\cdot\theta^j - \frac{1}{\tau_1}\sum_{j=0}^{m-1} D_{sm+j+1}\cdot\theta^j\Big] + \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j$$
$$+ \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\big[\|z_{(s+1)m}-x^*\|^2\big] \enspace .$$

Note that in the above inequality we have assumed all the randomness in the first $s-1$ epochs are fixed and the only source of randomness comes from epoch s. We can rearrange the terms in the above inequality and get

$$\mathbb{E}\Big[\frac{\tau_1+\tau_2-(1-1/\theta)}{\tau_1}\sum_{j=1}^{m} D_{sm+j}\cdot\theta^j\Big] \le \frac{(1-\tau_1-\tau_2)}{\tau_1}\Big(D_{sm}-\theta^m\mathbb{E}\big[D_{(s+1)m}\big]\Big)$$
$$+ \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \enspace .$$

Using the special choice that $\widetilde{x}^{s+1} = \big(\sum_{j=0}^{m-1}\theta^j\big)^{-1}\cdot\sum_{j=0}^{m-1} y_{sm+j+1}\cdot\theta^j$ and the convexity of $F(\cdot)$, we derive that $\widetilde{D}^{s+1} \le \big(\sum_{j=0}^{m-1}\theta^j\big)^{-1}\cdot\sum_{j=0}^{m-1} D_{sm+j+1}\cdot\theta^j$. Substituting this into the above inequality, we get

$$\frac{\tau_1+\tau_2-(1-1/\theta)}{\tau_1}\theta\mathbb{E}\big[\widetilde{D}^{s+1}\big]\cdot\sum_{j=0}^{m-1}\theta^j \le \frac{(1-\tau_1-\tau_2)}{\tau_1}\Big(D_{sm}-\theta^m\mathbb{E}\big[D_{(s+1)m}\big]\Big)$$
$$+ \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \enspace . \quad (3.3)$$

We consider two cases next.

Case 1. Suppose $\frac{m\sigma}{L} \le \frac{3}{4}$. In this case, we choose $\alpha = \frac{1}{\sqrt{3m\sigma L}}$ and $\tau_1 = \frac{1}{3\alpha L} = m\alpha\sigma = \frac{\sqrt{m\sigma}}{\sqrt{3L}} \in [0, \frac{1}{2}]$ for `Katyusha`. It implies $\alpha\sigma \le 1/2m$ and therefore the following inequality holds:

$$\tau_2(\theta^{m-1}-1) + (1-1/\theta) = \frac{1}{2}((1+\alpha\sigma)^{m-1}-1) + (1-\frac{1}{1+\alpha\sigma}) \le (m-1)\alpha\sigma + \alpha\sigma = m\alpha\sigma = \tau_1 \enspace .$$

In other words, we have $\tau_1+\tau_2-(1-1/\theta) \ge \tau_2\theta^{m-1}$ and thus (3.3) implies that

$$\mathbb{E}\Big[\frac{\tau_2}{\tau_1}\widetilde{D}^{s+1}\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{(s+1)m} + \frac{1}{2\alpha}\|z_{(s+1)m}-x^*\|^2\Big]$$
$$\le \theta^{-m}\cdot\Big(\frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{sm} + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2\Big) \enspace .$$

If we telescope the above inequality over all epochs $s = 0, 1, \dots, S-1$, we obtain

$$\mathbb{E}\big[F(\widetilde{x}^S)-F(x^*)\big] = \mathbb{E}\big[\widetilde{D}^S\big] \overset{①}{\le} \theta^{-Sm}\cdot O\Big(\widetilde{D}^0 + D_0 + \frac{\tau_1}{\alpha m}\|x_0-x^*\|^2\Big)$$
$$\overset{②}{\le} \theta^{-Sm}\cdot O\Big(1+\frac{\tau_1}{\alpha m\sigma}\Big)\cdot(F(x_0)-F(x^*))$$
$$\overset{③}{=} O((1+\alpha\sigma)^{-Sm})\cdot\big(F(x_0)-F(x^*)\big) \enspace . \quad (3.4)$$

Above, ① uses the fact that $\sum_{j=0}^{m-1}\theta^j \geq m$ and $\tau_2 = \frac{1}{2}$; ② uses the strong convexity of $F(\cdot)$ which implies $F(x_0) - F(x^*) \geq \frac{\sigma}{2}\|x_0 - x^*\|^2$; and ③ uses our choice of τ_1.

Case 2. Suppose $\frac{m\sigma}{L} > \frac{3}{4}$. In this case, we choose $\tau_1 = \frac{1}{2}$ and $\alpha = \frac{1}{3\tau_1 L} = \frac{2}{3L}$ as in `Katyusha`. Our parameter choices help us simplify (3.3) as (noting $(\tau_1 + \tau_2 - (1 - 1/\theta))\theta = 1$)

$$2\mathbb{E}\big[\widetilde{D}^{s+1}\big] \cdot \sum_{j=0}^{m-1}\theta^j \leq \widetilde{D}^s \cdot \sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm} - x^*\|^2 - \frac{\theta^m}{2\alpha}\mathbb{E}\big[\|z_{(s+1)m} - x^*\|^2\big] \ .$$

Since $\theta^m = (1+\alpha\sigma)^m \geq 1 + \alpha\sigma m = 1 + \frac{2\sigma m}{3L} \geq \frac{3}{2}$, the above inequality implies

$$\frac{3}{2}\mathbb{E}\big[\widetilde{D}^{s+1}\big] \cdot \sum_{j=0}^{m-1}\theta^j + \frac{9L}{8}\mathbb{E}\big[\|z_{(s+1)m} - x^*\|^2\big] \leq \widetilde{D}^s \cdot \sum_{j=0}^{m-1}\theta^j + \frac{3L}{4}\|z_{sm} - x^*\|^2 \ .$$

If we telescope this inequality over all the epochs $s = 0, 1, \ldots, S-1$, we immediately have

$$\mathbb{E}\Big[\widetilde{D}^S \cdot \sum_{j=0}^{m-1}\theta^j + \frac{3L}{4}\|z_{Sm} - x^*\|^2\Big] \leq \big(\frac{2}{3}\big)^S \cdot \Big(\widetilde{D}^0 \cdot \sum_{j=0}^{m-1}\theta^j + \frac{3L}{4}\|z_0 - x^*\|^2\Big) \ .$$

Finally, since $\sum_{j=0}^{m-1}\theta^j \geq m$ and $\frac{\sigma}{2}\|z_0 - x^*\|^2 \leq F(x_0) - F(x^*)$ owing to the strong convexity of $F(\cdot)$, we conclude that

$$\mathbb{E}\big[F(\widetilde{x}^S) - F(x^*)\big] \leq O\big(1.5^{-S}\big) \cdot \big(F(x_0) - F(x^*)\big) \ . \tag{3.5}$$

Combining (3.4) and (3.5) we finish the proof of Theorem 3.1. □

3.3 Corollaries on Non-Smooth or Non-SC Problems

In this section we apply reductions from Theorem 2.3 to translate our Theorem 3.1 into optimal algorithms also for non-strongly convex objectives and/or non-smooth objectives. To begin with, it is an immediate corollary of Theorem 3.1 that `Katyusha` satisfies the HOOD property:

Corollary 3.8. `Katyusha` *satisfies the HOOD property with* $T(L,\sigma) = O\big(n + \frac{\sqrt{nL}}{\sqrt{\sigma}}\big)$ *iterations.*

Remark 3.9. Existing accelerated stochastic methods before this work (even for simpler Problem (1.3)) either do not satisfy HOOD or satisfy HOOD with an additional factor $\log(L/\sigma)$ in the number of iterations. This is why they do not yield optimal convergence rates even if Theorem 2.3 is used.

Combining Corollary 3.8 with Theorem 2.3, we have the following corollaries:

Corollary 3.10. *If each $f_i(x)$ is convex, L-smooth and $\psi(\cdot)$ is not necessarily strongly convex in Problem (1.1), then by applying* AdaptReg *on* `Katyusha` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \leq \varepsilon$ in at most*

$$O\Big(n\log\frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{\sqrt{nL}\cdot\|x_0-x^*\|}{\sqrt{\varepsilon}}\Big) \propto \frac{1}{\sqrt{\varepsilon}} \textit{ iterations. (Or equivalently } \varepsilon \propto \frac{1}{T^2}.)$$

In contrast, the best known convergence rate was

Catalyst: $O\Big(\Big(n+\frac{\sqrt{nL}\cdot\|x_0-x^*\|}{\sqrt{\varepsilon}}\Big)\log\frac{F(x_0)-F(x^*)}{\varepsilon}\log\frac{L\|x_0-x^*\|^2}{\varepsilon}\Big) \propto \frac{\log^2(1/\varepsilon)}{\sqrt{\varepsilon}}$ iterations. (Or $\varepsilon \propto \frac{\log^4 T}{T^2}$.)

Algorithm 2 $\texttt{Katyusha}^{\textsf{ns}}(x_0, S, \sigma, L)$

$m \leftarrow 2n$; ◇ *epoch length*
$\tau_2 \leftarrow \frac{1}{2}$;
$y_0 = z_0 = \widetilde{x}^0 \leftarrow x_0$; ◇ *initial vectors*
for $s \leftarrow 0$ **to** $S-1$ **do**
  $\tau_{1,s} \leftarrow \frac{2}{s+4}$, $\alpha_s \leftarrow \frac{1}{3\tau_{1,s}L}$ ◇ *different parameter choices comparing to* `Katyusha`
  $\mu^s \leftarrow \nabla f(\widetilde{x}^s)$; ◇ *compute the full gradient only once every m iterations*
  for $j \leftarrow 0$ **to** $m-1$ **do**
    $k \leftarrow (sm) + j$;
    $x_{k+1} \leftarrow \tau_{1,s} z_k + \tau_2 \widetilde{x}^s + (1 - \tau_{1,s} - \tau_2) y_k$;
    $\widetilde{\nabla}_{k+1} \leftarrow \mu^s + \nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x}^s)$ where i is randomly chosen from $\{1, 2, \dots, n\}$;
    $z_{k+1} = \arg\min_z \big\{ \frac{1}{2\alpha_s} \|z - z_k\|^2 + \langle \widetilde{\nabla}_{k+1}, z \rangle + \psi(z) \big\}$;
    Option I: $y_{k+1} \leftarrow \arg\min_y \big\{ \frac{3L}{2} \|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y \rangle + \psi(y) \big\}$;
    Option II: $y_{k+1} \leftarrow x_{k+1} + \tau_{1,s}(z_{k+1} - z_k)$ ◇ *we analyze only I but II also works*
  end for
  $\widetilde{x}^{s+1} \leftarrow \frac{1}{m} \sum_{j=1}^{m} y_{sm+j}$; ◇ *compute snapshot* $\widetilde{x}$
end for
return $\widetilde{x}^S$.

Corollary 3.11. *If each $f_i(x)$ is $\sqrt{G}$-Lipschitz continuous and $\psi(x)$ is σ-SC in Problem (1.3), then by applying* AdaptSmooth *on* `Katyusha` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \le \varepsilon$ in at most*

$$O\Big(n \log \frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{\sqrt{nG}}{\sqrt{\sigma\varepsilon}}\Big) \propto \frac{1}{\sqrt{\varepsilon}} \textit{ iterations. (Or equivalently } \varepsilon \propto \frac{1}{T^2}.)$$

In contrast, the best known convergence rate was

$$\text{APCG/SPDC:} \quad O\Big(\Big(n + \frac{\sqrt{nG}}{\sqrt{\sigma\varepsilon}}\Big) \log \frac{nG(F(x_0)-F(x^*))}{\sigma\varepsilon}\Big) \propto \frac{\log(1/\varepsilon)}{\sqrt{\varepsilon}} \text{ iterations. (Or } \varepsilon \propto \frac{\log^2 T}{T^2}.)$$

Corollary 3.12. *If each $f_i(x)$ is $\sqrt{G}$-Lipschitz continuous and $\psi(x)$ is not necessarily strongly convex in Problem (1.3), then by applying* JointAdaptRegSmooth *on* `Katyusha` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \le \varepsilon$ in at most*

$$O\Big(n \log \frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{\sqrt{nG}\|x_0 - x^*\|}{\varepsilon}\Big) \propto \frac{1}{\varepsilon} \textit{ iterations. (Or equivalently } \varepsilon \propto \frac{1}{T}.)$$

In contrast, the best known convergence rate was

$$\text{APCG/SPDC:} \quad O\Big(\Big(n + \frac{\sqrt{nG}\|x_0-x^*\|}{\varepsilon}\Big) \log \frac{nG\|x_0-x^*\|^2(F(x_0)-F(x^*))}{\varepsilon^2}\Big) \propto \frac{\log(1/\varepsilon)}{\varepsilon} \text{ iterations. (Or } \varepsilon \propto \frac{\log T}{T}.)$$

4 Katyusha in the Non-Strongly Convex Setting

Due to the increasing popularity of *non-strongly convex* minimization tasks (most notably ℓ_1-regularized problems), researchers often make additional efforts to design separate methods for minimizing the non-strongly convex variant of Problem (1.1) that are *direct*, meaning without restarting and in particular without using any reductions such as Theorem 2.3 [13, 17].

In this section, we also develop our *direct and accelerated* method for the non-strongly convex variant of Problem (1.1). We call it $\texttt{Katyusha}^{\textsf{ns}}$ and state it in Algorithm 2.

The only difference between $\texttt{Katyusha}^{\textsf{ns}}$ and $\texttt{Katyusha}$ is that we choose $\tau_1 = \tau_{1,s} = \frac{2}{s+4}$ to be a parameter that depends on the epoch index s, and accordingly $\alpha = \alpha_s = \frac{1}{3L\tau_{1,s}}$. This should not be a big surprise because in accelerated full-gradient methods, the values τ_1 and α also decrease (although with respect to k rather than s) when there is no strong convexity [11]. We note that τ_1 and τ_2 remain constant throughout an epoch, and this could simplify the implementations.

We state the following convergence theorem for $\texttt{Katyusha}^{\textsf{ns}}$ and defer its proof to Appendix B.1. The proof also relies on the one-iteration inequality in Lemma 3.7, but requires telescoping such inequalities in a different manner as compared with Theorem 3.1.

Theorem 4.1. *If each $f_i(x)$ is convex, L-smooth in Problem (1.1) and $\psi(\cdot)$ is not necessarily strongly convex, then* $\texttt{Katyusha}^{\textsf{ns}}(x_0, S, L)$ *satisfies*

$$\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \le O\Big(\frac{F(x_0) - F(x^*)}{S^2} + \frac{L\|x_0 - x^*\|^2}{mS^2}\Big)$$

In other words, choosing $m = \Theta(n)$, $\texttt{Katyusha}^{\textsf{ns}}$ *achieves an ε-additive error (i.e., $\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \le \varepsilon$) using at most $O\Big(\frac{n\sqrt{F(x_0)-F(x^*)}}{\sqrt{\varepsilon}} + \frac{\sqrt{nL}\|x_0-x^*\|}{\sqrt{\varepsilon}}\Big)$ iterations.*

Remark 4.2. $\texttt{Katyusha}^{\textsf{ns}}$ is a *direct, accelerated* solver for the non-SC case of Problem (1.1). It is illustrative to compare it with the convergence theorem of a *direct, non-accelerated* solver of the same setting. Below is the convergence theorem of SAGA after translating to our notations:

$$\text{SAGA:} \quad \mathbb{E}\big[F(x)\big] - F(x^*) \le O\Big(\frac{F(x_0) - F(x^*)}{S} + \frac{L\|x_0 - x^*\|^2}{nS}\Big) \ .$$

It is clear from this comparison that $\texttt{Katyusha}^{\textsf{ns}}$ is a factor S faster than non-accelerated methods such as SAGA, where $S = T/n$ if T is the total number of stochastic iterations. This convergence can also be written in terms of the number of iterations which is $O\big(\frac{n(F(x_0)-F(x^*))}{\varepsilon} + \frac{L\|x_0-x^*\|^2}{\varepsilon}\big)$.

Remark 4.3. Theorem 4.1 appears worse than the reduction-based complexity in Corollary 3.12. This can be fixed by setting either the parameters τ_1 or the epoch length m in a more sophisticated way. Since it complicates the proofs and the notations we refrain from doing so in this version of the paper.[16] In practice, being a direct method, $\texttt{Katyusha}^{\textsf{ns}}$ enjoys satisfactory performance.

5 Katyusha in the Mini-Batch Setting

We mentioned in earlier versions of this paper that our Katyusha method naturally generalizes to mini-batch (parallel) settings and non-uniform smoothness settings, but did not include a full proof. In this section, we carefully deal with both generalizations together.

Mini-batch. In each iteration k, instead of using a single $\nabla f_i(x_{k+1})$, one can

use the average of b stochastic gradients $\frac{1}{b}\sum_{i\in S_k} \nabla f_i(x_{k+1})$

where S_k is a random subset of $[n]$ with cardinality b. This average can be computed in a distributed manner using up to b processors. This idea is known as *mini-batch* for stochastic gradient methods.

Non-Uniform Smoothness. Suppose in Problem (1.1),

each $f_i(x)$ is L_i-smooth and $f(x) = \frac{1}{n}\sum_{i=1}^n f_i(x)$ is L-smooth.

[16] Recall that a similar issue has also happened in the non-accelerated world: the iteration complexity $O(\frac{n+L}{\varepsilon})$ in SAGA can be improved to $O(n \log\frac{1}{\varepsilon} + \frac{L}{\varepsilon})$ by doubling the epoch length across epochs [13]. Similar techniques can also be used to improve our result above.

We denote by $\overline{L} = \frac{1}{n}\sum_{i=1}^n L_i$, and assume without loss of generality $L \le \overline{L} \le nL$. [17] We note that $\overline{L}$ can sometimes be indeed much greater than L, see Remark 5.3.

Remark 5.1. L_i and L only need to be upper bounds to the minimum smoothness parameters of $f_i(\cdot)$ and $f(\cdot)$ respectively. In practice, sometimes the minimum smoothness parameters for $f_i(x)$ is efficiently computable (such as for ERM problems).

5.1 Algorithmic Changes and Theorem Restatement

To simultaneously deal with mini-batch and non-uniform smoothness, we propose the following changes to `Katyusha`:

- *Change the epoch length from $m = \Theta(n)$ to $m = \lceil \frac{n}{b} \rceil$.*

  This is standard. In each iteration we need to compute $O(b)$ stochastic gradients; therefore every $\lceil \frac{n}{b} \rceil$ iterations, we can compute the full gradient once without hurting the total performance.

- *Define distribution $\mathcal{D}$ over $[n]$ to be choosing $i \in [n]$ with probability $p_i \stackrel{\text{def}}{=} L_i/n\overline{L}$, and define gradient estimator $\widetilde{\nabla}_{k+1} \stackrel{\text{def}}{=} \nabla f(\widetilde{x}) + \frac{1}{b}\sum_{i\in S_k}\frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big)$, where $S_k \subseteq [n]$ is a multiset with b elements each i.i.d. generated from $\mathcal{D}$.*

  This is standard, see for instance Prox-SVRG [50], and it is easy to verify $\mathbb{E}[\widetilde{\nabla}_{k+1}] = \nabla f(x_{k+1})$.

- *Change τ_2 from $\frac{1}{2}$ to $\min\big\{\frac{\overline{L}}{2Lb}, \frac{1}{2}\big\}$.*

  Note that if $\overline{L} = L$ then we have $\tau_2 = \frac{1}{2b}$. In other words, the larger the mini-batch size, the smaller weight we want to give to Katyusha momentum. This should be intuitive. The reason τ_2 has a more involved form when $L \neq \overline{L}$ is explained in Remark 5.4 later.

- *Change L in gradient descent step (Line 19) to some other $L_\diamond \ge L$, and define $\alpha = \frac{1}{3\tau_1 L_\diamond}$ instead.*

  In most cases (e.g., when $\overline{L} = L$ or $L \ge \overline{L}m/b$) we choose $L_\diamond = L$. Otherwise, we let $L_\diamond = \frac{\overline{L}}{2b\tau_2} \ge L$. The reason $L_\diamond$ has a more involved form is explained in Remark 5.4 later.

- *Change τ_1 to be $\tau_1 = \min\big\{\frac{\sqrt{8bm\sigma}}{\sqrt{3\overline{L}}}\tau_2, \tau_2\big\}$ if $L \le \overline{L}m/b$ or $\tau_1 = \min\big\{\frac{\sqrt{2\sigma}}{\sqrt{3L}}, \frac{1}{2m}\big\}$ if $L > \overline{L}m/b$.*

  This corresponds to a phase-transition behavior of `Katyusha1` (see Remark 5.5 later). Intuitively, when $L \le \overline{L}m/b$ then we are in a mini-batch phase; when $L > \overline{L}m/b$ we are in a full-batch phase.

- *Due to technical reasons, we define $\widetilde{x}^s$ as a slightly different weighted average (Line 22) and output x^{out} which is a weighted combination of $\widetilde{x}^S$ and y_{Sm} as opposed to simply $\widetilde{x}^S$ (Line 24).*

We emphasize here that some of these changes are not necessary for instance in the special case of $\overline{L} = L$, but to state the strongest theorem, we have to include all such changes. It is a simple exercise to verify that, if $\overline{L} = L$ and $b = 1$, then up to only constant factors in the parameters, `Katyusha1` is exactly identical to `Katyusha`. We have the following main theorem for `Katyusha1`:

[17] It is easy to verify (using triangle inequality) that $f(x) = \frac{1}{n}\sum_{i\in[n]} f_i(x)$ must be $\overline{L}$ smooth. Also, if $f(x)$ is L-smooth then each $f_i(x)$ must be nL smooth (this can be checked via Hessian $\nabla^2 f_i(x) \preceq n\nabla^2 f(x)$ or similarly if f is not twice-differentiable).

Algorithm 3 Katyusha1$(x_0, S, \sigma, L, (L_1, \dots, L_n), b)$

$m \leftarrow \lceil n/b \rceil$ and $\overline{L} \leftarrow \frac{1}{n}(L_1 + \cdots + L_n)$; ⋄ *$m$ is epoch length*
$\tau_2 \leftarrow \min\{\frac{\overline{L}}{2Lb}, \frac{1}{2}\}$; ⋄ *if $\overline{L} = L$ then $\tau_2 = \frac{1}{2b}$ and $L_\diamond = L$*
if $L \le \frac{\overline{L}m}{b}$ **then**
  $\tau_1 \leftarrow \min\{\frac{\sqrt{8bm\sigma}}{\sqrt{3\overline{L}}}\tau_2, \tau_2\}$ and $L_\diamond \leftarrow \frac{\overline{L}}{2b\tau_2}$;
else
  $\tau_1 \leftarrow \min\{\frac{\sqrt{2\sigma}}{\sqrt{3L}}, \frac{1}{2m}\}$ and $L_\diamond \leftarrow L$;
end if
$\alpha \leftarrow \frac{1}{3\tau_1 L_\diamond}$; ⋄ *parameters*
Let distribution $\mathcal{D}$ be to output $i \in [n]$ with probability $p_i \stackrel{\text{def}}{=} L_i/(n\overline{L})$.
$y_0 = z_0 = \widetilde{x}^0 \leftarrow x_0$; ⋄ *initial vectors*
for $s \leftarrow 0$ **to** $S-1$ **do**
  $\mu^s \leftarrow \nabla f(\widetilde{x}^s)$; ⋄ *compute the full gradient once every m iterations*
  for $j \leftarrow 0$ **to** $m-1$ **do**
    $k \leftarrow (sm) + j$;
    $x_{k+1} \leftarrow \tau_1 z_k + \tau_2 \widetilde{x}^s + (1 - \tau_1 - \tau_2) y_k$;
    $S_k \leftarrow b$ independent copies of i from $\mathcal{D}$ with replacement.
    $\widetilde{\nabla}_{k+1} \leftarrow \mu^s + \frac{1}{b}\sum_{i \in S_k} \frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x}^s)\big)$;
    $z_{k+1} = \arg\min_z \{\frac{1}{2\alpha}\|z - z_k\|^2 + \langle \widetilde{\nabla}_{k+1}, z\rangle + \psi(z)\}$;
    Option I: $y_{k+1} \leftarrow \arg\min_y \{\frac{3L_\diamond}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y\rangle + \psi(y)\}$;
    Option II: $y_{k+1} \leftarrow x_{k+1} + \tau_1(z_{k+1} - z_k)$ ⋄ *we analyze only I but II also works*
  end for
  $\widetilde{x}^{s+1} \leftarrow \big(\sum_{j=0}^{m-1} \theta^j\big)^{-1} \cdot \big(\sum_{j=0}^{m-1} \theta^j \cdot y_{sm+j+1}\big)$; ⋄ *where $\theta = 1 + \min\{\alpha\sigma, \frac{1}{4m}\}$*
end for
return $x^{\mathsf{out}} \leftarrow \frac{\tau_2 m \widetilde{x}^S + (1-\tau_1-\tau_2) y_{Sm}}{\tau_2 m + (1-\tau_1-\tau_2)}$.

Theorem 5.2. *If each $f_i(x)$ is convex and L_i-smooth, $f(x)$ is L-smooth, $\psi(x)$ is σ-strongly convex in Problem (1.1), then for any $b \in [n]$, $x^{\mathsf{out}} =$ Katyusha1$(x_0, S, \sigma, L, (L_1, \dots, L_n), b)$ satisfies*

$$\mathbb{E}\big[F(x^{\mathsf{out}})\big] - F(x^*) \le \begin{cases} O\Big(\big(1 + \sqrt{b\sigma/(6\overline{L}m)}\big)^{-Sm}\Big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } \frac{m\sigma b}{L} \le \frac{3}{8} \text{ and } L \le \frac{\overline{L}m}{b}; \\ O\Big(\big(1 + \sqrt{\sigma/(6L)}\big)^{-Sm}\Big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } \frac{m^2\sigma}{L} \le \frac{3}{8} \text{ and } L > \frac{\overline{L}m}{b}; \\ O\big(1.25^{-S}\big) \cdot \big(F(x_0) - F(x^*)\big), & \text{otherwise.} \end{cases}$$

In other words, choosing $m = \lceil n/b \rceil$, Katyusha *achieves an ε-additive error (i.e., $\mathbb{E}\big[F(x^{\mathsf{out}})\big] - F(x^*) \le \varepsilon$) using at most*

$$S \cdot n = O\Big(\big(n + b\sqrt{L/\sigma} + \sqrt{n\overline{L}/\sigma}\big) \cdot \log \frac{F(x_0) - F(x^*)}{\varepsilon}\Big)$$

stochastic gradient computations.

5.2 Observations and Remarks

We explain the significance of Theorem 5.2 below.

We use *total work* to refer to the total number of stochastic gradient computations, and *iteration complexity* (also known as parallel depth) to refer to the total number of iterations.

Parallel Performance. The total work of `Katyusha1` stays the same when $b \le (n\overline{L}/L)^{1/2} \in [\sqrt{n}, n]$. This means, at least for all values $b \in \{1, 2, \dots, \lceil\sqrt{n}\rceil\}$, our `Katyusha1` achieves the same total work and thus

`Katyusha1` can be distributed to $b \le \sqrt{n}$ machines with a parallel speed-up factor b
(known as linear speed-up if ignoring communication overhead.)

In contrast, even in the special case of $\overline{L} = L$ and if no additional assumption is made, to the best of our knowledge:

- Mini-batch SVRG requires $\widetilde{O}\big(n + \frac{bL}{\sigma}\big)$ total work.

  Therefore, if SVRG is distributed to b machines, the total work is increased by a factor of b, and the parallel speed-up factor is 1 (i.e., no speed up).

- Catalyst on top of mini-batch SVRG requires $\widetilde{O}\big(n + \frac{\sqrt{bLn}}{\sqrt{\sigma}}\big)$ total work.

  Therefore, if Catalyst is distributed to b machines, the total work is increased by a factor $\sqrt{b}$, and the parallel speed-up factor is $\sqrt{b}$ only.

When preparing the journal revision (i.e., version 5), we found out at least in the case $\overline{L} = L$, some other groups of researchers very recently obtained similar results for the ERM Problem (1.3) using SPDC [48], and for the general Problem (1.1) [34].[18] These results together with Theorem 5.2 confirm the power of acceleration in the parallel regime for stochastic gradient methods.

Outperforming Full-Gradient Method. If $b = n$, the total work of `Katyusha1` becomes $\widetilde{O}\big((L/\sigma)^{1/2} n\big)$. This matches the total work of Nesterov's accelerated gradient method [11, 35, 36], and does not depend on the possibly larger parameter $\overline{L}$.

More interestingly, to achieve the *same* iteration complexity $\widetilde{O}\big((L/\sigma)^{1/2}\big)$ as Nesterov's method, our `Katyusha1` only needs to compute $b = (n\overline{L}/L)^{1/2}$ stochastic gradients $\nabla f_i(\cdot)$ per iteration (in the amortized sense). This can be much faster than computing $\nabla f(\cdot)$.

Remark 5.3. Recall $\overline{L}$ is in the range $[L, nL]$ so indeed $\overline{L}$ can be much larger than L. For instance in linear regression we have $f_i(x) = \frac{1}{2}(\langle a_i, x\rangle - b_i)^2$. Denoting by $A = [a_1, \dots, a_n] \in \mathbb{R}^{d\times n}$, we have $L = \frac{1}{n}\lambda_{\max}(A^\top A)$ and $\overline{L} = \frac{1}{n}\|A\|_F^2$. If each entry of each a_i is a random Gaussian $N(0,1)$, then $\overline{L}$ is around d and $\overline{L}$ is around only $\Theta(1 + \frac{d}{n})$ (using the Wishart random matrix theory).

Remark 5.4. The parameter specifications in `Katyusha1` look intimidating partially because we have tried to obtain the strongest statement and match the full-gradient descent performance when $b = n$. If $\overline{L}$ is equal to L, then one can simply set $\tau_2 = \frac{1}{2b}$ and $L_\diamond = L$ in `Katyusha1`.

Phase Transition between Mini-Batch and Full-Batch. Theorem 5.2 indicates a phase transition of `Katyusha1` at the point $b_0 = (n\overline{L}/L)^{1/2}$.

- If $b \le b_0$, we say `Katyusha1` is in the *mini-batch phase* and the total work is $\widetilde{O}\big(n + \sqrt{n\overline{L}/\sigma}\big)$, independent of b.
- If $b > b_0$, we say `Katyusha1` is in the *full-batch phase*, and the total work is $\widetilde{O}\big(n + b\sqrt{L/\sigma}\big)$, so essentially linearly-scales with b and matches that of Nesterov's method when $b = n$.

Remark 5.5. We set different values for τ_1 and $L_\diamond$ in the mini-batch phase and full-batch phase respectively (see Line 3). From the final complexities above, it should not be surprising that τ_1 depends on $\overline{L}$ but not L in the mini-batch phase, and depends on L but not $\overline{L}$ in the full-batch phase. In addition, one can even tune the parameters so that it suffices for `Katyusha` to output $\widetilde{x}^S$

[18] These two papers claimed that Katyusha does not enjoy linear speed-up for $b \le \sqrt{n}$, based on an earlier version of the paper (where we did not include the mini-batch theorem). As evidenced by Theorem 5.2, such claims are false.

in the mini-batch phase and y_{Sm} in the full-batch phase; we did not do so and simply choose to output x^{out} which is a convex combination of $\widetilde{x}^S$ and y_{Sm}.

Remark 5.6. In the simple case $\overline{L} = L$, Nitanda [39] obtained a total work $\widetilde{O}\big(n + \frac{n-b}{n-1}\frac{L}{\sigma} + b\sqrt{L/\sigma}\big)$, which also implies a phase transition for b. However, this result is no better than ours for all b, and in addition, in terms of total work, it is no faster than SVRG when $b \leq n/2$, and no faster than accelerated full-gradient descent when $b > n/2$.

5.3 Corollaries on Non-Smooth or Non-SC Problems

In the same way as Section 3.3, we can apply the reductions from [1] to convert the performance of Theorem 5.2 to non-smooth or non-strongly convex settings. We state the corollaries below:

Corollary 5.7. *If each $f_i(x)$ is convex and L_i-smooth, $f(x)$ is L-smooth, $\psi(\cdot)$ is not necessarily strongly convex in Problem (1.1), then for any $b \in [n]$, by applying* AdaptReg *on* `Katyusha1` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \leq \varepsilon$ in at most*

$$O\Big(n \log \frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{b\sqrt{\overline{L}}\cdot\|x_0-x^*\|}{\sqrt{\varepsilon}} + \frac{\sqrt{n\overline{L}}\cdot\|x_0-x^*\|}{\sqrt{\varepsilon}}\Big) \text{ stochastic gradient computations.}$$

Corollary 5.8. *If each $f_i(x)$ is $\sqrt{G_i}$-Lipschitz continuous and $\psi(x)$ is σ-SC in Problem (1.3), then for any $b \in [n]$, by applying* AdaptSmooth *on* `Katyusha1` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \leq \varepsilon$ in at most*

$$O\Big(n \log \frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{b\sqrt{\overline{G}}}{\sqrt{\sigma\varepsilon}} + \frac{\sqrt{n\overline{G}}}{\sqrt{\sigma\varepsilon}}\Big) \text{ stochastic gradient computations.}$$

Corollary 5.9. *If each $f_i(x)$ is $\sqrt{G_i}$-Lipschitz continuous and $\psi(x)$ is not necessarily strongly convex in Problem (1.3), then for any $b \in [n]$, by applying* JointAdaptRegSmooth *on* `Katyusha1` *with a starting vector x_0, we obtain an output x satisfying $\mathbb{E}[F(x)] - F(x^*) \leq \varepsilon$ in at most*

$$O\Big(n \log \frac{F(x_0)-F(x^*)}{\varepsilon} + \frac{b\,\overline{G}\|x_0-x^*\|}{\varepsilon} + \frac{\sqrt{n}\,\overline{G}\|x_0-x^*\|}{\varepsilon}\Big) \text{ stochastic gradient computations.}$$

6 Katyusha in the Non-Euclidean Norm Setting

In this section, we show that `Katyusha` and `Katyusha`$^{\mathsf{ns}}$ naturally extend to settings where the smoothness definition is with respect to a non-Euclidean norm.

Non-Euclidean Norm Smoothness. We consider smoothness (and strongly convexity) with respect to an arbitrary norm $\|\cdot\|$ in domain $Q \stackrel{\text{def}}{=} \{x \in \mathbb{R}^d : \psi(x) < +\infty\}$. Symbolically, we say

- f is σ-strongly convex w.r.t. $\|\cdot\|$ if $\forall x, y \in Q$, it satisfies $f(y) \geq f(x) + \langle \nabla f(x), y-x\rangle + \frac{\sigma}{2}\|x-y\|^2$;
- f is L-smooth w.r.t. $\|\cdot\|$ if $\forall x, y \in Q$, it satisfies $\|\nabla f(x) - \nabla f(y)\|_* \leq L\|x-y\|$.[19]

Above, $\|\cdot\|_* \stackrel{\text{def}}{=} \max\{\langle \xi, x\rangle : \|x\| \leq 1\}$ is the dual norm of $\|\cdot\|$. For instance, ℓ_p norm is dual to ℓ_q norm if $\frac{1}{p} + \frac{1}{q} = 1$. Some famous problems have better smoothness parameters when non-Euclidean norms are adopted, see the discussions in [11].

Bregman Divergence. Following the traditions in the non-Euclidean norm setting [11], we

[19]This definition has another equivalent form: $\forall x, y \in Q$, it satisfies $f(y) \leq f(x) + \langle \nabla f(x), y-x\rangle + \frac{L}{2}\|y-x\|^2$.

- select a *distance generating function* $w(\cdot)$ that is 1-strongly convex w.r.t. $\|\cdot\|$, and[20]
- define the *Bregman divergence function* $V_x(y) \stackrel{\text{def}}{=} w(y) - w(x) - \langle \nabla w(x), y - x\rangle$.

The final algorithms and proofs will be described using $V_x(y)$ and $w(x)$.

Generalized Strong Convexity of $\psi(\cdot)$. We require $\psi(\cdot)$ to be σ-strongly convexity with respect to function $V_x(y)$ rather than the $\|\cdot\|$ norm; or symbolically,

$$\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y - x\rangle + \sigma V_x(y) \ .$$

(For instance, this is satisfied if $\omega(y) \stackrel{\text{def}}{=} \frac{1}{\sigma}\psi(y)$.) This is known as the "generalized strong convexity" [43] and is necessary for any linear-convergence result in the SC setting. Of course, in the non-SC setting, we do not require any (general or not) strong convexity for $\psi(\cdot)$.

6.1 Algorithm Changes and Theorem Restatements

Suppose each $f_i(x)$ is L_i-smooth with respect to norm $\|\cdot\|$, and a Bregman divergence function $V_x(y)$ is given. We perform the following changes to the algorithms:

- In Line 9 of `Katyusha` (resp. Line 10 of `Katyusha`$^{\mathsf{ns}}$), we choose i with probability proportional to L_i instead of uniformly at random.
- In Line 10 of `Katyusha` (resp. Line 11 of `Katyusha`$^{\mathsf{ns}}$), we change the arg min to be its non-Euclidean norm variant [11]: $z_{k+1} = \arg\min_z \big\{\frac{1}{\alpha}V_{z_k}(z) + \langle \widetilde{\nabla}_{k+1}, z\rangle + \psi(z)\big\}$
- We forbidden Option II and use Option I only (but *without* replacing $\|y - x_{k+1}\|^2$ with $V_{x_{k+1}}(y)$).

Interested readers can find discussions regarding why such changes are natural in [11]. We call the resulting algorithms `Katyusha2` and `Katyusha2`$^{\mathsf{ns}}$, and include them in Appendix D for completeness' sake. We state our final theorems below (recall $\overline{L} = \frac{1}{n}\sum_{i=1}^n L_i$).

Theorem 6.1 (ext. of Theorem 3.1)**.** *If each $f_i(x)$ is convex and L_i-smooth with respect to some norm $\|\cdot\|$, $V_x(y)$ is a Bregman divergence function for $\|\cdot\|$, and $\psi(x)$ is σ-strongly convex with respect to $V_x(y)$, then* `Katyusha2`$(x_0, S, \sigma, (L_1, \ldots, L_n))$ *satisfies*

$$\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq \begin{cases} O\Big(\big(1 + \sqrt{\sigma/(9\overline{L}m)}\big)^{-Sm}\Big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } m\sigma/\overline{L} \leq \frac{9}{4}; \\ O\big(1.5^{-S}\big) \cdot \big(F(x_0) - F(x^*)\big), & \text{if } m\sigma/\overline{L} > \frac{9}{4}. \end{cases}$$

In other words, choosing $m = \Theta(n)$, `Katyusha2` *achieves an ε-additive error (i.e., $\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq \varepsilon$) using at most $O\big((n + \sqrt{n\overline{L}/\sigma}) \cdot \log\frac{F(x_0) - F(x^*)}{\varepsilon}\big)$ iterations.*

Theorem 6.2 (ext. of Theorem 4.1)**.** *If each $f_i(x)$ is convex and L_i-smooth with respect to some norm $\|\cdot\|$, $V_x(y)$ is a Bregman divergence function for $\|\cdot\|$, and $\psi(\cdot)$ is not necessarily strongly convex, then* `Katyusha2`$^{\mathsf{ns}}(x_0, S, (L_1, \ldots, L_n))$ *satisfies*

$$\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq O\Big(\frac{F(x_0) - F(x^*)}{S^2} + \frac{\overline{L}V_{x_0}(x^*)}{nS^2}\Big) \ .$$

In other words, `Katyusha2`$^{\mathsf{ns}}$ *achieves an ε-additive error (i.e., $\mathbb{E}\big[F(\widetilde{x}^S)\big] - F(x^*) \leq \varepsilon$) using at most $O\Big(\frac{n\sqrt{F(x_0) - F(x^*)}}{\sqrt{\varepsilon}} + \frac{\sqrt{nLV_{x_0}(x^*)}}{\sqrt{\varepsilon}}\Big)$ iterations.*

[20] For instance, if $Q = \mathbb{R}^d$ and $\|\cdot\|_p$ is the ℓ_p norm for some $p \in (1, 2]$, one can choose $w(x) = \frac{1}{2(p-1)}\|x\|_p^2$; if $Q = \{x \in \mathbb{R}^d : \sum_i x_i = 1\}$ is the probability space and $\|\cdot\|_1$ is the ℓ_1 norm, one can choose $w(x) = \sum_i x_i \log x_i$.

The proofs of Theorem 6.1 and Theorem 6.2 follow exactly the same proof structures of Theorem 3.1 and Theorem 4.1, so we include them only in Appendix D.

6.2 Remarks

We highlight one main difference between the proof of `Katyusha2` and that of `Katyusha`: if ξ is a random vector and $\|\cdot\|$ is an arbitrary norm, we do not necessarily have $\mathbb{E}[\|\xi - \mathbb{E}[\xi]\|_*^2] \leq \mathbb{E}[\|\xi\|_*^2]$. Therefore, we only used $\mathbb{E}[\|\xi - \mathbb{E}[\xi]\|_*^2] \leq 2\mathbb{E}[\|\xi\|_*^2] + 2\|\mathbb{E}[\xi]\|_*^2$ (see Lemma D.2) and this loses a constant factor in some parameters. (For instance, α now becomes $\frac{1}{9\tau_1 \overline{L}}$ as opposed to $\frac{1}{3\tau_1 L}$).

More interestingly, one may ask how our revised algorithms `Katyusha2` or `Katyusha2`$^{\mathsf{ns}}$ perform in the mini-batch setting (just like we have studied in Section 5 for the Euclidean case). We are optimistic here, but unfortunately do not have a clean worst-case statement for how much speed-up we can get. The underlying reason is that, if $\mathcal{D}$ is a distribution for vectors, $\mu = \mathbb{E}_{\xi\sim\mathcal{D}}[\xi]$ is its expectation, and $\xi_1, \dots, \xi_b$ are b i.i.d. samples from $\mathcal{D}$, then letting $\overline{\xi} = \frac{1}{b}(\xi_1 + \cdots + \xi_b)$, we do not necessarily have $\mathbb{E}[\|\overline{\xi} - \mu\|_*^2] \leq \frac{1}{b}\mathbb{E}_{\xi\sim\mathcal{D}}[\|\xi - \mu\|_*^2]$. In other words, using a mini-batch version of the gradient estimator, the "variance" with respect to an arbitrary norm may not necessarily go down by a factor of b. For such reason, in the mini-batch setting, the best total work we can cleanly state, say for `Katyusha2` in the SC setting, is only $O\Big(\big(n + \sqrt{bn\overline{L}/\sigma}\big) \cdot \log \frac{F(x_0) - F(x^*)}{\varepsilon}\Big)$.

7 Empirical Evaluations

We conclude this paper with empirical evaluations to our theoretical speed-ups. We work on Lasso and ridge regressions (with regularizer $\frac{\lambda}{2}\|x\|^2$ for ridge and regularizer $\lambda\|x\|_1$ for Lasso) on the following six datasets: adult, web, mnist, rcv1, covtype, sensit. We defer dataset and implementation details to Appendix A.

Algorithms and Parameter Tuning. We have implemented the following algorithms, all with mini-batch size 1 for this version of the paper:

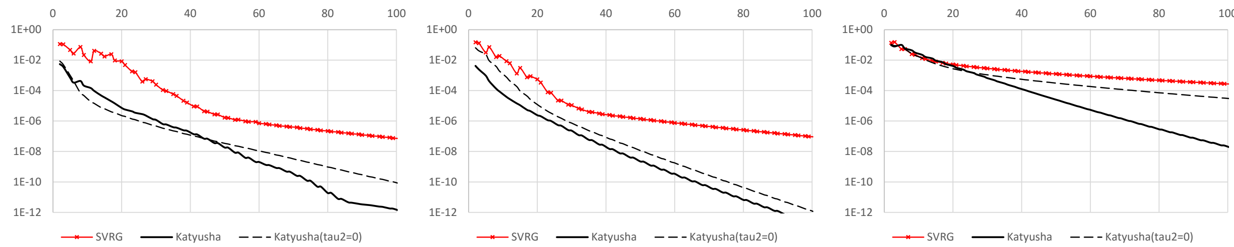
- SVRG [23] with default epoch length $m = 2n$. We tune only *one parameter*: the learning rate.
- `Katyusha` for ridge and `Katyusha`$^{\mathsf{ns}}$ for Lasso. We tune only *one parameter*: the learning rate.
- SAGA [17]. We tune only *one parameter*: the learning rate.
- Catalyst [29] on top of SVRG. We tune *three parameters*: SVRG's learning rate, Catalyst's learning rate, as well as the regularizer weight in the Catalyst reduction.
- APCG [30]. We tune the learning rate. For Lasso, we also tune the ℓ_2 regularizer weight.
- APCG+AdaptReg (Lasso only). Since APCG intrinsically require an ℓ_2 regularizer to be added on Lasso, we apply AdaptReg from [1] to adaptively learn this regularizer and improve APCG's performance. Two parameters to be tuned: APCG's learning rate and σ_0 in AdaptReg.

All of the parameters were equally, fairly, and automatically tuned by our code base. For interested readers, we discuss more details in Appendix A.

We emphasize that `Katyusha` is *as simple as SAGA or SVRG in terms of parameter tuning.* In contrast, APCG for Lasso requires two parameters to be tuned, and Catalyst requires three. [28]

Performance Plots. Following the tradition of ERM experiments, we use the number of "passes" of the dataset as the x-axis. Letting n be the number of feature vectors, each new stochastic gradient computation $\nabla f_i(\cdot)$ counts as $1/n$ pass, and a full gradient computation $\nabla f(\cdot)$ counts as 1 pass.

The y-axis in all of our plots represent the objective distance to the minimum. We emphasize that it is practically also crucial to study high-accuracy regimes (such as objective distance $\leq 10^{-7}$).



(a) web, ridge $\lambda = 10^{-6}$ (b) mnist, ridge $\lambda = 10^{-6}$ (c) rcv1, ridge $\lambda = 10^{-6}$

Figure 1: Comparing SVRG vs. `Katyusha` vs. `Katyusha` with $\tau_2 = 0$.

This is because nowadays there is an increasing number of methods that reduce large-scale machine learning tasks to multiple black-box calls to ERM solvers [4, 5]. In all such applications, due to error blowups between oracle calls, the ERM solver is required to be *very accurate in training error.*

7.1 Effectiveness of Katyusha Momentum

In our `Katyusha` method, τ_1 controls to the classical Nesterov's momentum and τ_2 controls our newly introduced Katyusha momentum. We find in our theory that setting $\tau_2 = 1/2$ is a good choice so we universally set it to be $1/2$ without tuning in all our experiments. (Of course, if time permits, tuning τ_2 could only help in performance.)

Before this paper, researchers have tried heuristics that is to add Nesterov's momentums directly to stochastic gradient methods [39], and this corresponds to setting $\tau_2 = 0$ in `Katyusha`. In Figure 1, we compare `Katyusha` with $\tau_2 = 1/2$ and $\tau_2 = 0$ in order to illustrate the importance and effectiveness of our Katyusha momentum.

We conclude that the old heuristics (i.e., $\tau_2 = 0$) sometimes indeed make the method faster after careful parameter tuning. However, for certain tasks such as Figure 1(c), without Katyusha momentum the algorithm does not even enjoy an accelerated convergence rate.

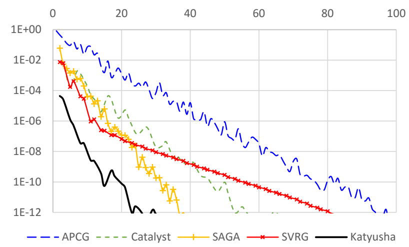
7.2 Performance Comparison Across Algorithms

For each of the six datasets and each objective (ridge or lasso), we experiment on three different magnitudes of regularizer weights.[21] This totals 36 performance charts, and we include them in full at the end of this paper. For the sake of cleanness, in Figure 2 we select 6 representative charts for ridge regression and make the following observations.

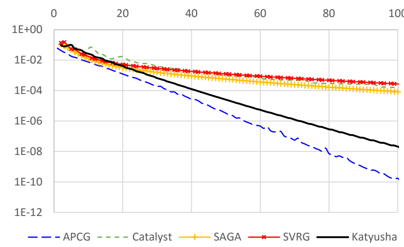
- Accelerated methods are more powerful when the regularizer weights are small (cf. [12, 30, 47]). For instance, Figure 2(c) and 2(f) are for large values of λ and `Katyusha` performs relatively the same as compared with SVRG / SAGA; however, `Katyusha` significantly outperforms SVRG / SAGA for small values of λ, see for instance Figure 2(b) and 2(e).
- `Katyusha` almost always either outperform or equal-perform its competitors. The only notable place it gets outperformed is by SVRG (see Figure 2(f)); however, this performance gap cannot be large because `Katyusha` is capable of recovering SVRG if $\tau_1 = \tau_2 = 0$.[22]

[21] We choose three values λ that are powers of 10 and around $10/n, 1/n, 1/10n$. This range can be verified to contain the best regularization weights using cross validation.

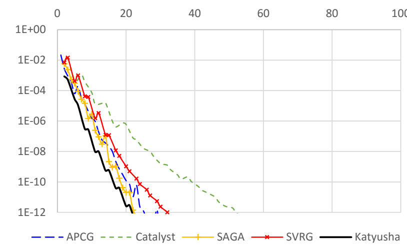
[22] The only reason `Katyusha` does not match the performance of SVRG in Figure 2(f) is because we have not tuned parameter τ_2. If we also tune τ_2 for the best performance, `Katyusha` shall no longer be outperformed by SVRG. In any case, it is not really necessary to tune τ_2 because the performance of `Katyusha` is already superb.



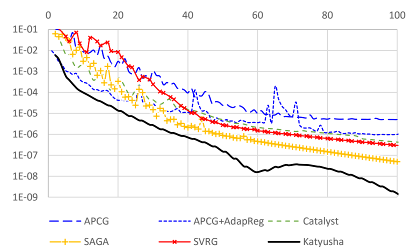
(a) covtype, ridge $\lambda = 10^{-6}$ (small)



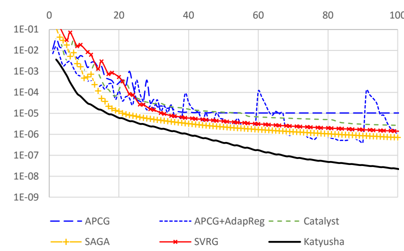
(b) rcv1, ridge $\lambda = 10^{-7}$ (small)



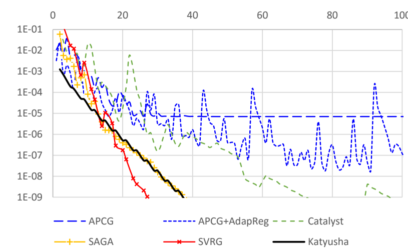
(c) adult, ridge $\lambda = 10^{-4}$ (large)



(d) web, ridge $\lambda = 10^{-6}$ (small)



(e) mnist, ridge $\lambda = 10^{-6}$ (small)



(f) sensit, ridge $\lambda = 10^{-4}$ (large)

Figure 2: Some representative performance charts where λ is the regularizer weight.

- Catalyst does not work as beautiful as its theory in high-accuracy regimes, even though we have carefully tuned parameters α_0 and κ in Catalyst in addition to its learning rate. Indeed, in Figure 2(a), 2(c) and 2(f) Catalyst (which is a reduction on SVRG) is outperformed by SVRG.
- APCG performs poorly on all Lasso tasks (cf. Figure 2(d), 2(e), 2(f)) because it is not designed for non-SC objectives. The reduction in [1] helps to fix this issue, but not by a lot.
- APCG can sometimes be largely dominated by SVRG or SAGA (cf. Figure 2(f)): this is because for datasets such as sensit, dual-based methods (such as APCG) cannot make use of the implicity local strong convexity in the objective. In such cases, `Katyusha` is not lost to SVRG or SAGA.

8 Conclusion

The novel Katyusha momentum technique introduced in this paper gives rise to accelerated convergence rates even in the stochastic setting. For many classes of the problems, such convergence rates are the first to match the theoretical lower bounds [49]. The algorithms generated by Katyusha momentum are simple yet highly practical and parallelizable.

More importantly, this new technique has the potential to enrich our understanding of accelerated methods in a broader sense. Currently, although acceleration methods are becoming more and more important to the field of computer science, they are still often regarded as "analytical tricks" [15, 24] and lacking complete theoretical understanding. The Katyusha momentum presented in this paper, however, adds a new level of decoration on top of the classical Nesterov momentum. This decoration is shown valuable for stochastic problems in this paper, but may also lead to future applications as well. In general, the author hopes that the technique and analysis in this paper could facilitate more studies in this field and thus become a stepping stone towards the ultimate goal of unveiling the mystery of acceleration.

Appendix

A Experiment Details

The datasets we used in this paper are downloaded from the LibSVM website [18]:

- the adult (a9a) dataset (32,561 samples and 123 features).
- the web (w8a) dataset (49,749 samples and 300 features).
- the covtype (binary.scale) dataset (581,012 samples and 54 features).
- the mnist (class 1) dataset (60,000 samples and 780 features).
- the rcv1 (train.binary) dataset (20,242 samples and 47,236 features).
- the sensit (combined) dataset (78,823 samples and 100 features).

To make easier comparison across datasets, we scale every vector by the average Euclidean norm of all the vectors in the dataset. In other words, we ensure that the data vectors have an average Euclidean norm 1. This step is for comparison only and not necessary in practice.

Parameter-tuning details. We select learning rates from the set $\{10^{-k}, 2\times 10^{-k}, 5\times 10^{-k} : k\in\mathbb{Z}\}$, and select regularizer weights (for APCG) from the set $\{10^{-k} : k\in\mathbb{Z}\}$. We have fully automated the parameter tuning procedure to ensure a fair and strong comparison.

While the learning rates were explicitly defined for SVRG and SAGA, there were implicit for all accelerated methods. For Catalyst, the learning rate is in fact their α_0 in the paper [28]. Instead of choosing it to be the theory-predicted value, we multiply it with an extra factor to be tuned and call this factor the "learning rate". Similarly, for `Katyusha` and $\texttt{Katyusha}^{\texttt{ns}}$, we multiply the theory-predicted τ_1 with an extra factor and this serves as a learning rate. For APCG, we use their Algorithm 1 in the paper and multiply their theory-predicted μ with an extra factor.

For Catalyst, in principle one also has to tune the stopping criterion. After communicating with an author of Catalyst, we learned that one can terminate the inner loop whenever the duality gap becomes no more than, say one fourth, of the last duality gap from the previous epoch [28]. This stopping criterion was also found by the authors of [1] to be a good choice for reduction-based methods.

B Appendix for Section 4

B.1 Proof of Theorem 4.1

Proof of Theorem 4.1. First of all, the parameter choices satisfy the presumptions in Lemma 3.6, so again by defining $D_k \stackrel{\text{def}}{=} F(y_k) - F(x^*)$ and $\widetilde{D}^s \stackrel{\text{def}}{=} F(\widetilde{x}^s) - F(x^*)$, we can rewrite Lemma 3.7 as follows:

$$0 \le \frac{\alpha_s(1-\tau_{1,s}-\tau_2)}{\tau_{1,s}} D_k - \frac{\alpha_s}{\tau_{1,s}}\mathbb{E}\big[D_{k+1}\big] + \frac{\alpha_s\tau_2}{\tau_{1,s}}\widetilde{D}^s + \frac{1}{2}\|z_k - x^*\|^2 - \frac{1}{2}\mathbb{E}\big[\|z_{k+1}-x^*\|^2\big] \enspace .$$

Summing up the above inequality for all the iterations $k = sm, sm+1, \dots, sm+m-1$, we have

$$\begin{aligned}&\mathbb{E}\Big[\alpha_s\frac{1-\tau_{1,s}-\tau_2}{\tau_{1,s}}D_{(s+1)m} + \alpha_s\frac{\tau_{1,s}+\tau_2}{\tau_{1,s}}\sum_{j=1}^{m} D_{sm+j}\Big] \\ &\le \alpha_s\frac{1-\tau_{1,s}-\tau_2}{\tau_{1,s}}D_{sm} + \alpha_s\frac{\tau_2}{\tau_{1,s}}m\widetilde{D}^s + \frac{1}{2}\|z_{sm}-x^*\|^2 - \frac{1}{2}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \enspace . \end{aligned} \qquad \text{(B.1)}$$

Note that in the above inequality we have assumed all the randomness in the first $s-1$ epochs are fixed and the only source of randomness comes from epoch s.

If we define $\widetilde{x}^s = \frac{1}{m}\sum_{j=1}^m y_{(s-1)m+j}$, then by the convexity of function $F(\cdot)$ we have $m\widetilde{D}^s \leq \sum_{j=1}^n D_{(s-1)m+j}$. Therefore, using the parameter choice $\alpha_s = \frac{1}{3\tau_{1,s}L}$, for every $s \geq 1$ we can derive from (B.1) that

$$\mathbb{E}\Big[\frac{1}{\tau_{1,s}^2} D_{(s+1)m} + \frac{\tau_{1,s}+\tau_2}{\tau_{1,s}^2}\sum_{j=1}^{m-1} D_{sm+j}\Big]$$
$$\leq \frac{1-\tau_{1,s}}{\tau_{1,s}^2} D_{sm} + \frac{\tau_2}{\tau_{1,s}^2}\sum_{j=1}^{m-1} D_{(s-1)m+j} + \frac{3L}{2}\|z_{sm}-x^*\|^2 - \frac{3L}{2}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \ . \tag{B.2}$$

For the base case $s=0$, we can also rewrite (B.1) as

$$\mathbb{E}\Big[\frac{1}{\tau_{1,0}^2} D_m + \frac{\tau_{1,0}+\tau_2}{\tau_{1,0}^2}\sum_{j=1}^{m-1} D_j\Big]$$
$$\leq \frac{1-\tau_{1,0}-\tau_2}{\tau_{1,0}^2} D_0 + \frac{\tau_2 m}{\tau_{1,0}^2}\widetilde{D}^0 + \frac{3L}{2}\|z_0-x^*\|^2 - \frac{3L}{2}\mathbb{E}\big[\|z_m-x^*\|^2\big] \ . \tag{B.3}$$

At this point, if we choose $\tau_{1,s} = \frac{2}{s+4} \leq \frac{1}{2}$, it satisfies

$$\frac{1}{\tau_{1,s}^2} \geq \frac{1-\tau_{1,s+1}}{\tau_{1,s+1}^2} \quad \text{and} \quad \frac{\tau_{1,s}+\tau_2}{\tau_{1,s}^2} \geq \frac{\tau_2}{\tau_{1,s+1}^2} \ .$$

Using these two inequalities, we can telescope (B.3) and (B.2) for all $s = 0, 1, \ldots, S-1$. We obtain in the end that

$$\mathbb{E}\Big[\frac{1}{\tau_{1,S-1}^2} D_{Sm} + \frac{\tau_{1,S-1}+\tau_2}{\tau_{1,S-1}^2}\sum_{j=1}^{m-1} D_{(S-1)m+j} + \frac{3L}{2}\|z_{Sm}-z^*\|^2\Big]$$
$$\leq \frac{1-\tau_{1,0}-\tau_2}{\tau_{1,0}^2} D_0 + \frac{\tau_2 m}{\tau_{1,0}^2}\widetilde{D}^0 + \frac{3L}{2}\|z_0-x^*\|^2 \tag{B.4}$$

Since we have $\widetilde{D}^S \leq \frac{1}{m}\sum_{j=1}^m D_{(S-1)m+j}$ which is no greater than $\frac{2\tau_{1,S-1}^2}{m}$ times the left hand side of (B.4), we conclude that

$$\mathbb{E}\big[F(\widetilde{x}^S) - F(x^*)\big] = \mathbb{E}\big[\widetilde{D}^S\big] \leq O\Big(\frac{\tau_{1,S}^2}{m}\Big)\cdot\Big(\frac{1-\tau_{1,0}-\tau_2}{\tau_{1,0}^2} D_0 + \frac{\tau_2 m}{\tau_{1,0}^2}\widetilde{D}^0 + \frac{3L}{2}\|z_0-x^*\|^2\Big)$$
$$= O\Big(\frac{1}{mS^2}\Big)\cdot\Big(m\big(F(x_0)-F(x^*)\big) + L\|x_0-x^*\|^2\Big) \ . \qquad \square$$

C Appendix for Section 5

C.1 One-Iteration Analysis

Similar as Section 3.1, we first analyze the behavior of `Katyusha1` in a single iteration (i.e., for a fixed k). We view y_k, z_k and x_{k+1} as fixed in this section so the only randomness comes from the choice of i in iteration k. We abbreviate in this subsection by $\widetilde{x} = \widetilde{x}^s$ where s is the epoch that iteration k belongs to, and denote by $\sigma_{k+1}^2 \stackrel{\text{def}}{=} \|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|^2$.

Our first lemma is analogous to Lemma 3.3, where note that we have replaced the use of L in Lemma 3.3 with $L_\diamond \geq L$:

Lemma C.1 (proximal gradient descent)**.** *If* $L_\diamond \geq L$ *and*

$$y_{k+1} = \arg\min_y \Big\{ \frac{3L_\diamond}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \enspace, \qquad \textit{and}$$

$$\mathsf{Prog}(x_{k+1}) \stackrel{\mathrm{def}}{=} -\min_y \Big\{ \frac{3L_\diamond}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \geq 0 \enspace,$$

we have (where the expectation is only over the randomness of $\widetilde{\nabla}_{k+1}$*)*

$$F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] \geq \mathbb{E}\big[\mathsf{Prog}(x_{k+1})\big] - \frac{1}{4L_\diamond}\mathbb{E}\big[\sigma_{k+1}^2\big] \enspace.$$

Proof.

$$\begin{aligned}
\mathsf{Prog}(x_{k+1}) &= -\min_y \Big\{ \frac{3L_\diamond}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \\
&\stackrel{①}{=} -\Big(\frac{3L_\diamond}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&= -\Big(\frac{L_\diamond}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \nabla f(x_{k+1}), y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&\quad + \Big(\langle \nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle - L_\diamond\|y_{k+1} - x_{k+1}\|^2 \Big) \\
&\stackrel{②}{\leq} -\Big(f(y_{k+1}) - f(x_{k+1}) + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) + \frac{1}{4L_\diamond}\|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|^2 \enspace.
\end{aligned}$$

Above, ① is by the definition of y_{k+1}, and ② uses the smoothness of function $f(\cdot)$, as well as Young's inequality $\langle a, b\rangle - \frac{1}{2}\|b\|^2 \leq \frac{1}{2}\|a\|^2$. Taking expectation on both sides we arrive at the desired result. □

The following lemma is analogous to Lemma 3.4. The main difference is that since we have not chosen a mini-batch of size b, one should expect the variance to decrease by a factor of b. Also, since we are in the non-uniform case one should expect the use of L in Lemma 3.4 to be replaced with $\overline{L}$:

Lemma C.2 (variance upper bound)**.**

$$\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|^2\big] \leq \frac{2\overline{L}}{b} \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big) \enspace.$$

Proof. Each $f_i(x)$, being convex and L_i-smooth, implies the following inequality which is classical in convex optimization and can be found for instance in Theorem 2.1.5 of the textbook of Nesterov [36].

$$\|\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\|^2 \leq 2L_i \cdot \big(f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle \nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big)$$

Therefore, taking expectation over the random choice of i, we have

$$\begin{aligned}
&\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|^2\big] \\
&= \mathbb{E}_{S_k}\Big[\Big\|\Big(\frac{1}{b}\sum_{i\in S_k}\Big(\nabla f(\widetilde{x}) + \frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big)\Big)\Big) - \nabla f(x_{k+1})\Big\|^2\Big] \\
&= \frac{1}{b}\mathbb{E}_{i\sim\mathcal{D}}\Big[\Big\|\Big(\nabla f(\widetilde{x}) + \frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big)\Big) - \nabla f(x_{k+1})\Big\|^2\Big] \\
&= \frac{1}{b}\mathbb{E}_{i\sim\mathcal{D}}\Big[\Big\|\frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big) - \big(\nabla f(x_{k+1}) - f(\widetilde{x})\big)\Big\|^2\Big] \\
&\overset{①}{\leq} \frac{1}{b}\mathbb{E}_{i\sim\mathcal{D}}\Big[\Big\|\frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big)\Big\|_*^2\Big] \\
&\overset{②}{\leq} \frac{1}{b}\cdot\sum_{i\in[n]}\frac{2L_i}{n^2p_i}\Big(f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle\nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\Big) \\
&= \frac{2\overline{L}}{b}\cdot\big(f(\widetilde{x}) - f(x_{k+1}) - \langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big) \ .
\end{aligned}$$

Above, ① is because for any random vector $\zeta \in \mathbb{R}^d$, it holds that $\mathbb{E}\|\zeta - \mathbb{E}\zeta\|^2 = \mathbb{E}\|\zeta\|^2 - \|\mathbb{E}\zeta\|^2$; ② follows from the first inequality in this proof. □

The next lemma is completely identical to Lemma 3.5 so we skip the proof.

Lemma C.3 (proximal mirror descent)**.** *Suppose $\psi(\cdot)$ is σ-SC. Then, fixing $\widetilde{\nabla}_{k+1}$ and letting*

$$z_{k+1} = \arg\min_z\Big\{\frac{1}{2}\|z - z_k\|^2 + \alpha\langle\widetilde{\nabla}_{k+1}, z - z_k\rangle + \alpha\psi(z) - \alpha\psi(z_k)\Big\} \ ,$$

it satisfies for all $u \in \mathbb{R}^d$,

$$\alpha\langle\widetilde{\nabla}_{k+1}, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \leq -\frac{1}{2}\|z_k - z_{k+1}\|^2 + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\|z_{k+1} - u\|^2 \ .$$

The following lemma combines Lemma C.1, Lemma C.2 and Lemma C.3 all together, using the special choice of x_{k+1} which is a convex combination of y_k, z_k and $\widetilde{x}$:

Lemma C.4 (coupling step 1)**.** *If $x_{k+1} = \tau_1 z_k + \tau_2\widetilde{x} + (1 - \tau_1 - \tau_2)y_k$, where $\tau_1 \leq \frac{1}{3\alpha L_\diamond}$ and $\tau_2 = \frac{\overline{L}}{2L_\diamond b}$,*

$$\begin{aligned}
&\alpha\langle\nabla f(x_{k+1}), z_k - u\rangle - \alpha\psi(u) \\
&\leq \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 f(x_{k+1}) - \tau_2\langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\Big) \\
&\quad + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - u\|^2\big] + \frac{\alpha(1 - \tau_1 - \tau_2)}{\tau_1}\psi(y_k) - \frac{\alpha}{\tau_1}\psi(x_{k+1}) \ .
\end{aligned}$$

Proof. We first apply Lemma C.3 and get

$$\begin{aligned}
&\alpha\langle\widetilde{\nabla}_{k+1}, z_k - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&= \alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle + \alpha\langle\widetilde{\nabla}_{k+1}, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&\leq \alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2 + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\|z_{k+1} - u\|^2 \ . \qquad \text{(C.1)}
\end{aligned}$$

By defining $v \stackrel{\text{def}}{=} \tau_1 z_{k+1} + \tau_2 \widetilde{x} + (1-\tau_1-\tau_2) y_k$, we have $x_{k+1} - v = \tau_1(z_k - z_{k+1})$ and therefore

$$\begin{aligned}
&\mathbb{E}\Big[\alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2\Big] = \mathbb{E}\Big[\frac{\alpha}{\tau_1}\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{1}{2\tau_1^2}\|x_{k+1} - v\|^2\Big] \\
&= \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{1}{2\alpha\tau_1}\|x_{k+1} - v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{①}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{3L_\diamond}{2}\|x_{k+1} - v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{②}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{1}{4L_\diamond}\sigma_{k+1}^2\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{③}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{\overline{L}}{2L_\diamond b}\big(f(\widetilde{x}) - f(x_{k+1}) - \langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big)\Big) \\
&\qquad + \frac{\alpha}{\tau_1}\Big(\tau_1\psi(z_{k+1}) + \tau_2\psi(\widetilde{x}) + (1-\tau_1-\tau_2)\psi(y_k) - \psi(x_{k+1})\Big)\Big] \ . \qquad \text{(C.2)}
\end{aligned}$$

Above, ① uses our choice $\tau_1 \leq \frac{1}{3\alpha L}$, ② uses Lemma C.1, ③ uses Lemma C.2 together with the convexity of $\psi(\cdot)$ and the definition of v. Finally, noticing that $\mathbb{E}[\langle\widetilde{\nabla}_{k+1}, z_k - u\rangle] = \langle\nabla f(x_{k+1}), z_k - u\rangle$ and $\tau_2 = \frac{1}{2}$, we obtain the desired inequality by combining (C.1) and (C.2). □

The next lemma simplifies the left hand side of Lemma C.4 using the convexity of $f(\cdot)$, and gives an inequality that relates the objective-distance-to-minimizer quantities $F(y_k) - F(x^*)$, $F(y_{k+1}) - F(x^*)$, and $F(\widetilde{x}) - F(x^*)$ to the point-distance-to-minimizer quantities $\|z_k - x^*\|^2$ and $\|z_{k+1} - x^*\|^2$.

Lemma C.5 (coupling step 2)**.** *Under the same choices of τ_1, τ_2 as in Lemma C.4, we have*

$$\begin{aligned}
0 \leq \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - F(x^*)) - \frac{\alpha}{\tau_1}\big(\mathbb{E}\big[F(y_{k+1})\big] - F(x^*)\big) + \frac{\alpha\tau_2}{\tau_1}\big(F(\widetilde{x}) - F(x^*)\big) \\
+ \frac{1}{2}\|z_k - x^*\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - x^*\|^2\big] \ .
\end{aligned}$$

Proof. We first compute that

$$\begin{aligned}
&\alpha\big(f(x_{k+1}) - f(u)\big) \overset{①}{\leq} \alpha\langle\nabla f(x_{k+1}), x_{k+1} - u\rangle \\
&= \alpha\langle\nabla f(x_{k+1}), x_{k+1} - z_k\rangle + \alpha\langle\nabla f(x_{k+1}), z_k - u\rangle \\
&\overset{②}{=} \frac{\alpha\tau_2}{\tau_1}\langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\langle\nabla f(x_{k+1}), y_k - x_{k+1}\rangle + \alpha\langle\nabla f(x_{k+1}), z_k - u\rangle \\
&\overset{③}{\leq} \frac{\alpha\tau_2}{\tau_1}\langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(f(y_k) - f(x_{k+1})) + \alpha\langle\nabla f(x_{k+1}), z_k - u\rangle \ .
\end{aligned}$$

Above, ① uses the convexity of $f(\cdot)$, ② uses the choice that $x_{k+1} = \tau_1 z_k + \tau_2\widetilde{x} + (1-\tau_1-\tau_2)y_k$, and ③ uses the convexity of $f(\cdot)$ again. By applying Lemma C.4 to the above inequality, we have

$$\begin{aligned}
&\alpha\big(f(x_{k+1}) - F(u)\big) \leq \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - f(x_{k+1})) \\
&+ \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 f(x_{k+1})\Big) + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - u\|^2\big] - \frac{\alpha}{\tau_1}\psi(x_{k+1})
\end{aligned}$$

which implies

$$\begin{aligned}
&\alpha\big(F(x_{k+1}) - F(u)\big) \leq \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}(F(y_k) - F(x_{k+1})) \\
&+ \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2 F(x_{k+1})\Big) + \frac{1}{2}\|z_k - u\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1} - u\|^2\big] \ .
\end{aligned}$$

After rearranging and setting $u = x^*$, the above inequality yields

$$0 \le \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\big(F(y_k)-F(x^*)\big) - \frac{\alpha}{\tau_1}\big(\mathbb{E}\big[F(y_{k+1})-F(x^*)\big]\big) + \frac{\alpha\tau_2}{\tau_1}\big(F(\widetilde{x})-F(x^*)\big) + \frac{1}{2}\|z_k-x^*\|^2 - \frac{1+\alpha\sigma}{2}\mathbb{E}\big[\|z_{k+1}-x^*\|^2\big] \ . \qquad \square$$

C.2 Proof of Theorem 5.2

We are now ready to combine the analyses across iterations, and derive our final Theorem 5.2. Our proof next requires a careful telescoping of Lemma C.5 together with our specific parameter choices.

Proof of Theorem 5.2. Define $D_k \stackrel{\text{def}}{=} F(y_k) - F(x^*)$, $\widetilde{D}^s \stackrel{\text{def}}{=} F(\widetilde{x}^s) - F(x^*)$, and rewrite Lemma C.5:

$$0 \le \frac{(1-\tau_1-\tau_2)}{\tau_1}D_k - \frac{1}{\tau_1}D_{k+1} + \frac{\tau_2}{\tau_1}\mathbb{E}\big[\widetilde{D}^s\big] + \frac{1}{2\alpha}\|z_k-x^*\|^2 - \frac{1+\alpha\sigma}{2\alpha}\mathbb{E}\big[\|z_{k+1}-x^*\|^2\big] \ .$$

At this point, let us θ be an arbitrary value in $\big[1, 1+\alpha\sigma\big]$ and multiply the above inequality by θ^j for each $k = sm + j$. Then, we sum up the resulting m inequalities for all $j = 0, 1, \ldots, m-1$:

$$0 \le \mathbb{E}\Big[\frac{(1-\tau_1-\tau_2)}{\tau_1}\sum_{j=0}^{m-1} D_{sm+j}\cdot\theta^j - \frac{1}{\tau_1}\sum_{j=0}^{m-1} D_{sm+j+1}\cdot\theta^j\Big] + \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\big[\|z_{(s+1)m}-x^*\|^2\big] \ .$$

Note that in the above inequality we have assumed all the randomness in the first $s-1$ epochs are fixed and the only source of randomness comes from epoch s. We can rearrange the terms in the above inequality and get

$$\mathbb{E}\Big[\frac{\tau_1+\tau_2-(1-1/\theta)}{\tau_1}\sum_{j=1}^{m} D_{sm+j}\cdot\theta^j\Big] \le \frac{(1-\tau_1-\tau_2)}{\tau_1}\Big(D_{sm} - \theta^m\mathbb{E}\big[D_{(s+1)m}\big]\Big) + \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \ .$$

Using the special choice that $\widetilde{x}^{s+1} = \big(\sum_{j=0}^{m-1}\theta^j\big)^{-1}\cdot\sum_{j=0}^{m-1} y_{sm+j+1}\cdot\theta^j$ and the convexity of $F(\cdot)$, we derive that $\widetilde{D}^{s+1} \le \big(\sum_{j=0}^{m-1}\theta^j\big)^{-1}\cdot\sum_{j=0}^{m-1} D_{sm+j+1}\cdot\theta^j$. Substituting this into the above inequality, we get

$$\frac{\tau_1+\tau_2-(1-1/\theta)}{\tau_1}\theta\mathbb{E}\big[\widetilde{D}^{s+1}\big]\cdot\sum_{j=0}^{m-1}\theta^j \le \frac{(1-\tau_1-\tau_2)}{\tau_1}\Big(D_{sm} - \theta^m\mathbb{E}\big[D_{(s+1)m}\big]\Big) + \frac{\tau_2}{\tau_1}\widetilde{D}^s\cdot\sum_{j=0}^{m-1}\theta^j + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2 - \frac{\theta^m}{2\alpha}\mathbb{E}\big[\|z_{(s+1)m}-x^*\|^2\big] \ . \tag{C.3}$$

We consider two cases (and four subcases) next.

Case 1. Suppose $L \le \frac{\overline{L}m}{b}$. In this case, we choose

$$\tau_2 = \min\Big\{\frac{\overline{L}}{2Lb}, \frac{1}{2}\Big\} \in \Big[\frac{1}{2m}, \frac{1}{2}\Big] \quad \text{and} \quad L_\diamond = \frac{\overline{L}}{2b\tau_2} \ge L$$

Case 1.1. Suppose $\frac{m\sigma b}{\overline{L}} \leq \frac{3}{8}$. In this subcase, we choose

$$\alpha = \frac{\sqrt{b}}{\sqrt{6m\sigma\overline{L}}} \quad , \qquad \tau_1 = \frac{1}{3\alpha L_\diamond} = 4m\alpha\sigma\tau_2 = \frac{\sqrt{8\tau_2^2 bm\sigma}}{\sqrt{3\overline{L}}} \in \big[0, \tau_2\big] \subseteq \big[0, \frac{1}{2}\big] \quad , \quad \text{and} \quad \theta = 1 + \alpha\sigma$$

We have

$$\alpha\sigma = \frac{1}{\sqrt{6m^2}} \frac{\sqrt{b\sigma m}}{\sqrt{\overline{L}}} \leq \frac{1}{4m}$$

and therefore the following inequality holds:

$$\tau_2(\theta^{m-1} - 1) + (1 - 1/\theta) = \tau_2((1 + \alpha\sigma)^{m-1} - 1) + (1 - \frac{1}{1 + \alpha\sigma}) \leq 2\tau_2 m\alpha\sigma + \alpha\sigma \leq 4\tau_2 m\alpha\sigma = \tau_1 \enspace .$$

In other words, we have $\tau_1 + \tau_2 - (1 - 1/\theta) \geq \tau_2\theta^{m-1}$ and thus (C.3) implies that

$$\mathbb{E}\Big[\frac{\tau_2}{\tau_1}\widetilde{D}^{s+1} \cdot \sum_{j=0}^{m-1} \theta^j + \frac{1 - \tau_1 - \tau_2}{\tau_1} D_{(s+1)m} + \frac{1}{2\alpha}\|z_{(s+1)m} - x^*\|^2\Big]$$

$$\leq \theta^{-m} \cdot \Big(\frac{\tau_2}{\tau_1}\widetilde{D}^{s} \cdot \sum_{j=0}^{m-1} \theta^j + \frac{1 - \tau_1 - \tau_2}{\tau_1} D_{sm} + \frac{1}{2\alpha}\|z_{sm} - x^*\|^2\Big) \enspace .$$

If we telescope the above inequality over all epochs $s = 0, 1, \ldots, S-1$, we obtain

$$\begin{aligned}
\mathbb{E}\big[F(x^{\mathsf{out}}) - F(x^*)\big] &\overset{①}{\leq} \frac{1}{\tau_2 m + (1 - \tau_1 - \tau_2)} \mathbb{E}\big[\tau_2 \widetilde{D}^S \cdot \sum_{j=0}^{m-1} \theta^j + \big(1 - \tau_1 - \tau_2\big) D_{Sm}\big] \\
&\overset{②}{\leq} \theta^{-Sm} \cdot O\Big(\widetilde{D}^0 + D_0 + \frac{\tau_1}{\alpha\tau_2 m}\|x_0 - x^*\|^2\Big) \\
&\overset{③}{\leq} \theta^{-Sm} \cdot O\Big(1 + \frac{\tau_1}{\alpha\tau_2 m\sigma}\Big) \cdot (F(x_0) - F(x^*)) \\
&\overset{④}{=} O((1 + \alpha\sigma)^{-Sm}) \cdot \big(F(x_0) - F(x^*)\big) \enspace . \qquad \text{(C.4)}
\end{aligned}$$

Above, inequality ① uses the choice $x^{\mathsf{out}} = \frac{\tau_2 m \widetilde{x}^S + (1 - \tau_1 - \tau_2) y_{Sm}}{\tau_2 m + (1 - \tau_1 - \tau_2)}$, the convexity of $F(\cdot)$, and the fact $\sum_{j=0}^{m-1} \theta^j \geq m$; inequality ② uses the fact that $\sum_{j=0}^{m-1} \theta^j \leq O(m)$ (because $\alpha\sigma \leq \frac{1}{4m}$), and the fact that $\tau_2 \geq \frac{1}{2m}$; inequality ③ uses the strong convexity of $F(\cdot)$ which implies $F(x_0) - F(x^*) \geq \frac{\sigma}{2}\|x_0 - x^*\|^2$; and inequality ④ uses our choice of τ_1.

Case 1.2. Suppose $\frac{m\sigma b}{\overline{L}} > \frac{3}{8}$. In this case, we choose

$$\tau_1 = \tau_2 \quad \text{and} \quad \alpha = \frac{1}{3\tau_1 L_\diamond} = \frac{2b}{3\overline{L}} \geq \frac{1}{4\sigma m} \quad , \quad \theta = 1 + \frac{1}{4m}$$

(Note that we can choose $\theta = 1 + \frac{1}{4m}$ because $\frac{1}{4m} \leq \alpha\sigma$.)

Under these parameter choices, we can calculate that

$$\frac{(\tau_1 + \tau_2 - (1 - 1/\theta))\theta}{\tau_2} = 2 - \frac{1 - 2\tau_2}{4m\tau_2} \geq \frac{3}{2} > \frac{5}{4} \quad \text{and} \quad \theta^m \geq \frac{5}{4}$$

thus (C.3) implies that

$$\mathbb{E}\Big[\frac{\tau_2}{\tau_1}\widetilde{D}^{s+1} \cdot \sum_{j=0}^{m-1} \theta^j + \frac{1 - \tau_1 - \tau_2}{\tau_1} D_{(s+1)m} + \frac{1}{2\alpha}\|z_{(s+1)m} - x^*\|^2\Big]$$

$$\leq \frac{4}{5} \cdot \Big(\frac{\tau_2}{\tau_1}\widetilde{D}^{s} \cdot \sum_{j=0}^{m-1} \theta^j + \frac{1 - \tau_1 - \tau_2}{\tau_1} D_{sm} + \frac{1}{2\alpha}\|z_{sm} - x^*\|^2\Big) \enspace .$$

If we telescope the above inequality over all epochs $s = 0, 1, \dots, S-1$, we obtain

$$\mathbb{E}\big[F(x^{\mathsf{out}}) - F(x^*)\big] \overset{①}{\leq} \frac{1}{\tau_2 m + (1-\tau_1-\tau_2)}\mathbb{E}\big[\tau_2 \widetilde{D}^S \cdot \sum_{j=0}^{m-1}\theta^j + \big(1-\tau_1-\tau_2\big)D_{Sm}\big]$$
$$\overset{②}{\leq} \big(\frac{5}{4}\big)^{-S} \cdot O\Big(\widetilde{D}^0 + D_0 + \frac{\tau_1}{\alpha\tau_2 m}\|x_0 - x^*\|^2\Big)$$
$$\overset{③}{\leq} \big(\frac{5}{4}\big)^{-S} \cdot O\Big(1 + \frac{\tau_1}{\alpha\tau_2 m\sigma}\Big) \cdot (F(x_0) - F(x^*))$$
$$\overset{④}{=} O((5/4)^{-S}) \cdot \big(F(x_0) - F(x^*)\big) \ . \tag{C.5}$$

Above, inequality ① uses the choice $x^{\mathsf{out}} = \frac{\tau_2 m \widetilde{x}^S + (1-\tau_1-\tau_2)y_{Sm}}{\tau_2 m + (1-\tau_1-\tau_2)}$, the convexity of $F(\cdot)$, and the fact $\sum_{j=0}^{m-1}\theta^j \geq m$; inequality ② uses the fact that $\sum_{j=0}^{m-1}\theta^j \leq O(m)$, and the fact that $\tau_2 \geq \frac{1}{2m}$; inequality ③ uses the strong convexity of $F(\cdot)$ which implies $F(x_0) - F(x^*) \geq \frac{\sigma}{2}\|x_0 - x^*\|^2$; and inequality ④ uses our choice of τ_1 and α.

Case 2. Suppose $L > \frac{\overline{L}m}{b}$. In this case, we choose

$$L_\diamond = L \quad \text{and} \quad \tau_2 = \frac{\overline{L}}{2L_\diamond b} = \frac{\overline{L}}{2Lb} \in \big[0, \frac{1}{2m}\big]$$

Case 2.1. Suppose $\frac{m^2\sigma}{L} \leq \frac{3}{8}$. In this subcase, we choose

$$\alpha = \frac{1}{\sqrt{6\sigma L}} \ , \qquad \tau_1 = \frac{1}{3\alpha L} = 2\alpha\sigma = \frac{\sqrt{2\sigma}}{\sqrt{3L}} \in \big[0, \frac{1}{2m}\big] \ , \quad \theta = 1 + \alpha\sigma$$

We have $\alpha\sigma \leq \frac{1}{4m}$ and therefore the following inequality holds:

$$\tau_2(\theta^{m-1} - 1) + (1 - 1/\theta) = \tau_2((1+\alpha\sigma)^{m-1} - 1) + (1 - \frac{1}{1+\alpha\sigma}) \leq 2\tau_2 m\alpha\sigma + \alpha\sigma \leq 2\alpha\sigma = \tau_1 \ .$$

In other words, we have $\tau_1 + \tau_2 - (1 - 1/\theta) \geq \tau_2\theta^{m-1}$ and thus (C.3) implies that

$$\mathbb{E}\Big[\frac{\tau_2}{\tau_1}\widetilde{D}^{s+1} \cdot \sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{(s+1)m} + \frac{1}{2\alpha}\|z_{(s+1)m} - x^*\|^2\Big]$$
$$\leq \theta^{-m} \cdot \Big(\frac{\tau_2}{\tau_1}\widetilde{D}^s \cdot \sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{sm} + \frac{1}{2\alpha}\|z_{sm} - x^*\|^2\Big) \ .$$

If we telescope the above inequality over all epochs $s = 0, 1, \dots, S-1$, we obtain

$$\mathbb{E}\big[F(x^{\mathsf{out}}) - F(x^*)\big] \overset{①}{\leq} \frac{1}{\tau_2 m + (1-\tau_1-\tau_2)}\mathbb{E}\big[\tau_2 \widetilde{D}^S \cdot \sum_{j=0}^{m-1}\theta^j + \big(1-\tau_1-\tau_2\big)D_{Sm}\big]$$
$$\overset{②}{\leq} \theta^{-Sm} \cdot O\Big(\widetilde{D}^0 + D_0 + \frac{\tau_1}{\alpha}\|x_0 - x^*\|^2\Big)$$
$$\overset{③}{\leq} \theta^{-Sm} \cdot O\Big(1 + \frac{\tau_1}{\alpha\sigma}\Big) \cdot (F(x_0) - F(x^*))$$
$$\overset{④}{=} O((1+\alpha\sigma)^{-Sm}) \cdot \big(F(x_0) - F(x^*)\big) \ . \tag{C.6}$$

Above, inequality ① uses the choice $x^{\mathsf{out}} = \frac{\tau_2 m \widetilde{x}^S + (1-\tau_1-\tau_2)y_{Sm}}{\tau_2 m + (1-\tau_1-\tau_2)}$, the convexity of $F(\cdot)$, and the fact $\sum_{j=0}^{m-1}\theta^j \geq m$; inequality ② uses the fact that $\sum_{j=0}^{m-1}\theta^j \leq O(m)$ (because $\alpha\sigma \leq \frac{1}{4m}$), and the fact that $\tau_2 m + (1-\tau_1-\tau_2) \geq 1-\tau_1+(m-1)\tau_2 \geq 1/2$; inequality ③ uses the strong convexity of $F(\cdot)$ which implies $F(x_0)-F(x^*) \geq \frac{\sigma}{2}\|x_0-x^*\|^2$; and inequality ④ uses our choice of τ_1.

Case 2.2. Suppose $\frac{m^2\sigma}{L} > \frac{3}{8}$. In this case, we choose

$$\tau_1 = \frac{1}{2m} \quad \text{and} \quad \alpha = \frac{1}{3\tau_1 L} = \frac{2m}{3L} > \frac{1}{4\sigma m} \ , \quad \theta = 1 + \frac{1}{4m}$$

(Note that we can choose $\theta = 1 + \frac{1}{4m}$ because $\frac{1}{4m} \leq \alpha\sigma$.)

Under these parameter choices, we can calculate that

$$\frac{(\tau_1+\tau_2-(1-1/\theta))\theta}{\tau_2} = \frac{\tau_1+\tau_2}{\tau_2} - \frac{1-2\tau_2}{4m\tau_2} \geq 1 + \frac{\tau_1 - 1/4m}{\tau_2} \geq \frac{3}{2} > \frac{5}{4} \quad \text{and} \quad \theta^m \geq \frac{5}{4}$$

thus (C.3) implies that

$$\mathbb{E}\Big[\frac{\tau_2}{\tau_1}\widetilde{D}^{s+1} \cdot \sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{(s+1)m} + \frac{1}{2\alpha}\|z_{(s+1)m}-x^*\|^2\Big]$$
$$\leq \frac{4}{5} \cdot \Big(\frac{\tau_2}{\tau_1}\widetilde{D}^{s} \cdot \sum_{j=0}^{m-1}\theta^j + \frac{1-\tau_1-\tau_2}{\tau_1}D_{sm} + \frac{1}{2\alpha}\|z_{sm}-x^*\|^2\Big) \ .$$

If we telescope the above inequality over all epochs $s = 0, 1, \dots, S-1$, we obtain

$$\begin{aligned}
\mathbb{E}\big[F(x^{\mathsf{out}}) - F(x^*)\big] &\overset{①}{\leq} \frac{1}{\tau_2 m + (1-\tau_1-\tau_2)}\mathbb{E}\big[\tau_2 \widetilde{D}^S \cdot \sum_{j=0}^{m-1}\theta^j + \big(1-\tau_1-\tau_2\big)D_{Sm}\big] \\
&\overset{②}{\leq} \big(\frac{5}{4}\big)^{-S} \cdot O\Big(\widetilde{D}^0 + D_0 + \frac{\tau_1}{\alpha}\|x_0-x^*\|^2\Big) \\
&\overset{③}{\leq} \big(\frac{5}{4}\big)^{-S} \cdot O\Big(1 + \frac{\tau_1}{\alpha\sigma}\Big) \cdot (F(x_0) - F(x^*)) \\
&\overset{④}{=} O((5/4)^{-S}) \cdot \big(F(x_0)-F(x^*)\big) \ .
\end{aligned} \tag{C.7}$$

Above, inequality ① uses the choice $x^{\mathsf{out}} = \frac{\tau_2 m \widetilde{x}^S + (1-\tau_1-\tau_2)y_{Sm}}{\tau_2 m + (1-\tau_1-\tau_2)}$, the convexity of $F(\cdot)$, and the fact $\sum_{j=0}^{m-1}\theta^j \geq m$; inequality ② uses the fact that $\sum_{j=0}^{m-1}\theta^j \leq O(m)$, and that $\tau_2 m + (1-\tau_1-\tau_2) \geq 1-\tau_1+(m-1)\tau_2 \geq 1/2$; inequality ③ uses the strong convexity of $F(\cdot)$ which implies $F(x_0)-F(x^*) \geq \frac{\sigma}{2}\|x_0-x^*\|^2$; and inequality ④ uses our choice of τ_1 and α. □

D Appendix for Section 6

In this section, we first include the complete pseudo-codes for `Katyusha2` and `Katyusha2`$^{\mathsf{ns}}$. Then, we provide a one-iteration analysis for both algorithms, in the same spirit as Section 3.1.

The final proofs of Theorem 6.1 and Theorem 6.2 are direct corollaries of such one-iteration analysis, where the details we have already given in Section 3.2 and in Section B.1 respectively.

D.1 Pseudo-Codes

Algorithm 4 `Katyusha2`$(x_0, S, \sigma, (L_1, \ldots, L_n))$

$m \leftarrow n$; $\overline{L} = (L_1 + \cdots + L_n)/n$;
$\tau_2 \leftarrow \frac{1}{2}$, $\tau_1 \leftarrow \min\big\{\sqrt{m\sigma/9\overline{L}}, \frac{1}{2}\big\}$, $\alpha \leftarrow \frac{1}{9\tau_1 \overline{L}}$;
$y_0 = z_0 = \widetilde{x}^0 \leftarrow x_0$;
for $s \leftarrow 0$ **to** $S-1$ **do**
  $\mu^s \leftarrow \nabla f(\widetilde{x}^s)$;
  for $j \leftarrow 0$ **to** $m-1$ **do**
    $k \leftarrow (sm) + j$;
    $x_{k+1} \leftarrow \tau_1 z_k + \tau_2 \widetilde{x}^s + (1 - \tau_1 - \tau_2) y_k$;
    Pick i randomly from $\{1, 2, \ldots, n\}$, each with probability $L_i / n\overline{L}$;
    $\widetilde{\nabla}_{k+1} \leftarrow \mu^s + \nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x}^s)$;
    $z_{k+1} = \arg\min_z \big\{\frac{1}{\alpha} V_{z_k}(z) + \langle \widetilde{\nabla}_{k+1}, z \rangle + \psi(z)\big\}$;
      ⋄ *$V_x(y)$ is the Bregman divergence function, see Section 6*
    $y_{k+1} \leftarrow \arg\min_y \big\{\frac{9\overline{L}}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y \rangle + \psi(y)\big\}$;
  end for
  $\widetilde{x}^{s+1} \leftarrow \big(\sum_{j=0}^{m-1} (1+\alpha\sigma)^j\big)^{-1} \cdot \big(\sum_{j=0}^{m-1} (1+\alpha\sigma)^j \cdot y_{sm+j+1}\big)$;
end for
return $\widetilde{x}^S$.

Algorithm 5 `Katyusha2`$^{\mathsf{ns}}(x_0, S, \sigma, (L_1, \ldots, L_n))$

$m \leftarrow n$; $\overline{L} = (L_1 + \cdots + L_n)/n$;
$\tau_2 \leftarrow \frac{1}{2}$;
$y_0 = z_0 = \widetilde{x}^0 \leftarrow x_0$;
for $s \leftarrow 0$ **to** $S-1$ **do**
  $\tau_{1,s} \leftarrow \frac{2}{s+4}$, $\alpha_s \leftarrow \frac{1}{9\tau_{1,s}\overline{L}}$
  $\mu^s \leftarrow \nabla f(\widetilde{x}^s)$;
  for $j \leftarrow 0$ **to** $m-1$ **do**
    $k \leftarrow (sm) + j$;
    $x_{k+1} \leftarrow \tau_{1,s} z_k + \tau_2 \widetilde{x}^s + (1 - \tau_{1,s} - \tau_2) y_k$;
    Pick i randomly from $\{1, 2, \ldots, n\}$, each with probability $L_i / n\overline{L}$;
    $\widetilde{\nabla}_{k+1} \leftarrow \mu^s + \nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x}^s)$;
    $z_{k+1} = \arg\min_z \big\{\frac{1}{\alpha_s} V_{z_k}(z) + \langle \widetilde{\nabla}_{k+1}, z \rangle + \psi(z)\big\}$;
      ⋄ *$V_x(y)$ is the Bregman divergence function, see Section 6*
    $y_{k+1} \leftarrow \arg\min_y \big\{\frac{9\overline{L}}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y \rangle + \psi(y)\big\}$;
  end for
  $\widetilde{x}^{s+1} \leftarrow \frac{1}{m}\sum_{j=1}^{m} y_{sm+j}$;
end for
return $\widetilde{x}^S$.

D.2 One-Iteration Analysis

Similar as Section 3.1, we first analyze the behavior of `Katyusha2` in a single iteration (i.e., for a fixed k). We view y_k, z_k and x_{k+1} as fixed in this section so the only randomness comes from the choice of i in iteration k. We abbreviate in this subsection by $\widetilde{x} = \widetilde{x}^s$ where s is the epoch that iteration k belongs to, and denote by $\sigma_{k+1}^2 \stackrel{\text{def}}{=} \|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|_*^2$.

Our first lemma is analogous to Lemma D.1 except the change of the parameter and the norm.

Lemma D.1 (proximal gradient descent)**.** *If*

$$y_{k+1} = \arg\min_y \Big\{ \frac{9\overline{L}}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \ , \qquad \textit{and}$$

$$\mathsf{Prog}(x_{k+1}) \stackrel{\text{def}}{=} -\min_y \Big\{ \frac{9\overline{L}}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \geq 0 \ ,$$

we have (where the expectation is only over the randomness of $\widetilde{\nabla}_{k+1}$*)*

$$F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] \geq \mathbb{E}\big[\mathsf{Prog}(x_{k+1})\big] - \frac{1}{16\overline{L}}\mathbb{E}\big[\sigma_{k+1}^2\big] \ .$$

Proof.

$$\begin{aligned}
\mathsf{Prog}(x_{k+1}) &= -\min_y \Big\{ \frac{9\overline{L}}{2}\|y - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y - x_{k+1}\rangle + \psi(y) - \psi(x_{k+1}) \Big\} \\
&\stackrel{①}{=} -\Big(\frac{9\overline{L}}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&= -\Big(\frac{\overline{L}}{2}\|y_{k+1} - x_{k+1}\|^2 + \langle \nabla f(x_{k+1}), y_{k+1} - x_{k+1}\rangle + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) \\
&\quad + \Big(\langle \nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}, y_{k+1} - x_{k+1}\rangle - 4\overline{L}\|y_{k+1} - x_{k+1}\|^2 \Big) \\
&\stackrel{②}{\leq} -\Big(f(y_{k+1}) - f(x_{k+1}) + \psi(y_{k+1}) - \psi(x_{k+1}) \Big) + \frac{1}{16\overline{L}}\|\nabla f(x_{k+1}) - \widetilde{\nabla}_{k+1}\|_*^2 \ .
\end{aligned}$$

Above, ① is by the definition of y_{k+1}, and ② uses the smoothness of function $f(\cdot)$, as well as Young's inequality $\langle a, b\rangle - \frac{1}{2}\|b\|^2 \leq \frac{1}{2}\|a\|_*^2$. Taking expectation on both sides we arrive at the desired result. □

The next lemma is analogous to Lemma 3.4 but with slightly different proof.

Lemma D.2 (variance upper bound)**.**

$$\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|_*^2\big] \leq 8\overline{L} \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big) \ .$$

Proof. Each $f_i(x)$, being convex and L_i-smooth, implies the following inequality which is classical in convex optimization and can be found for instance in Theorem 2.1.5 of the textbook of Nesterov [36].

$$\|\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\|_*^2 \leq 2L_i \cdot \big(f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle \nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \big) \qquad \text{(D.1)}$$

Therefore, taking expectation over the random choice of i, we have

$$\begin{aligned}
&\mathbb{E}\big[\|\widetilde{\nabla}_{k+1} - \nabla f(x_{k+1})\|_*^2\big] \\
&= \mathbb{E}\Big[\Big\|\frac{1}{np_i}\big(\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big) - \big(\nabla f(x_{k+1}) - \nabla f(\widetilde{x})\big)\Big\|_*^2\Big] \\
&\stackrel{①}{\leq} 2\mathbb{E}\Big[\frac{1}{n^2p_i^2}\big\|\nabla f_i(x_{k+1}) - \nabla f_i(\widetilde{x})\big\|_*^2\Big] + 2\big\|\nabla f(x_{k+1}) - \nabla f(\widetilde{x})\big\|_*^2 \\
&\stackrel{②}{\leq} 4 \cdot \mathbb{E}\Big[\frac{L_i}{n^2p_i^2}\big(f_i(\widetilde{x}) - f_i(x_{k+1}) - \langle \nabla f_i(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big)\Big] + 2\big\|\nabla f(x_{k+1}) - \nabla f(\widetilde{x})\big\|_*^2 \\
&\stackrel{③}{=} 4\overline{L} \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big) + 2\big\|\nabla f(x_{k+1}) - \nabla f(\widetilde{x})\big\|_*^2 \\
&\leq 8\overline{L} \cdot \big(f(\widetilde{x}) - f(x_{k+1}) - \langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big) \ .
\end{aligned}$$

Above, inequality ① is because $\|a+b\|_*^2 \le (\|a\|_* + \|b\|_*)^2 \le 2\|a\|_*^2 + 2\|b\|_*^2$; inequality ② follows from (D.1); equality ③ follows from the probability distribution that we select i with probability $p_i = L_i/(n\overline{L})$; inequality ④ uses (D.1) again but replacing $f_i(\cdot)$ with $f(\cdot)$. $\square$

The next lemma is classical for mirror descent with respect to a general Bregman divergence.

Lemma D.3 (proximal mirror descent)**.** *Suppose $\psi(\cdot)$ is σ-SC with respect to $V_x(y)$. Then, fixing $\widetilde{\nabla}_{k+1}$ and letting*

$$z_{k+1} = \arg\min_z \left\{ V_{z_k}(z) + \alpha\langle \widetilde{\nabla}_{k+1}, z - z_k\rangle + \alpha\psi(z) - \alpha\psi(z_k) \right\} ,$$

it satisfies for all $u \in \mathbb{R}^d$,

$$\alpha\langle \widetilde{\nabla}_{k+1}, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \le -\frac{1}{2}\|z_k - z_{k+1}\|^2 + V_{z_k}(u) - (1+\alpha\sigma)V_{z_{k+1}}(u) .$$

Proof. By the minimality definition of z_{k+1}, we have that

$$\nabla V_{z_k}(z_{k+1}) + \alpha\widetilde{\nabla}_{k+1} + \alpha g = 0$$

where g is *some* subgradient of $\psi(z)$ at point $z = z_{k+1}$. This implies that for every u it satisfies

$$0 = \left\langle \nabla V_{z_k}(z_{k+1}) + \alpha\widetilde{\nabla}_{k+1} + \alpha g, z_{k+1} - u \right\rangle .$$

At this point, using the equality $\langle \nabla V_{z_k}(z_{k+1}), z_{k+1} - u\rangle = V_{z_k}(z_{k+1}) - V_{z_k}(u) + V_{z_{k+1}}(u)$ (known as the "three-point equality of Bregman divergence", see [40]), as well as the inequality $\langle g, z_{k+1} - u\rangle \ge \psi(z_{k+1}) - \psi(u) + \sigma V_{z_{k+1}}(u)$ which comes from the strong convexity of $\psi(\cdot)$, we can write

$$\begin{aligned}
&\alpha\langle \widetilde{\nabla}_{k+1}, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&= -\langle z_{k+1} - z_k, z_{k+1} - u\rangle - \langle \alpha g, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&\le -V_{z_k}(z_{k+1}) + V_{z_k}(u) - (1+\alpha\sigma)V_{z_{k+1}}(u) .
\end{aligned}$$

Finally, using $V_{z_k}(z_{k+1}) \ge \frac{1}{2}\|z_k - z_{k+1}\|^2$ which comes from the strong convexity of $w(x)$ with respect to $\|\cdot\|$, we complete the proof. $\square$

The following lemma combines Lemma D.1, Lemma D.2 and Lemma D.3 all together, using the special choice of x_{k+1} which is a convex combination of y_k, z_k and $\widetilde{x}$:

Lemma D.4 (coupling step 1)**.** *If $x_{k+1} = \tau_1 z_k + \tau_2\widetilde{x} + (1-\tau_1-\tau_2)y_k$, where $\tau_1 \le \frac{1}{9\alpha\overline{L}}$ and $\tau_2 = \frac{1}{2}$,*

$$\begin{aligned}
&\alpha\langle \nabla f(x_{k+1}), z_k - u\rangle - \alpha\psi(u) \\
&\le \frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - \mathbb{E}\big[F(y_{k+1})\big] + \tau_2 F(\widetilde{x}) - \tau_2\mathbb{E}\big[F(x_{k+1})\big] - \tau_2\langle \nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle \Big) \\
&\quad + V_{z_k}(u) - (1+\alpha\sigma)\mathbb{E}\big[V_{z_{k+1}}(u)\big] + \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\psi(y_k) - \frac{\alpha}{\tau_1}\psi(x_{k+1}) .
\end{aligned}$$

Proof. We first apply Lemma D.3 and get

$$\begin{aligned}
&\alpha\langle \widetilde{\nabla}_{k+1}, z_k - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&= \alpha\langle \widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle + \alpha\langle \widetilde{\nabla}_{k+1}, z_{k+1} - u\rangle + \alpha\psi(z_{k+1}) - \alpha\psi(u) \\
&\le \alpha\langle \widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2 + V_{z_k}(u) - (1+\alpha\sigma)V_{z_{k+1}}(u) .
\end{aligned} \tag{D.2}$$

By defining $v \stackrel{\text{def}}{=} \tau_1 z_{k+1} + \tau_2 \widetilde{x} + (1-\tau_1-\tau_2) y_k$, we have $x_{k+1} - v = \tau_1(z_k - z_{k+1})$ and therefore

$$
\begin{aligned}
&\mathbb{E}\Big[\alpha\langle\widetilde{\nabla}_{k+1}, z_k - z_{k+1}\rangle - \frac{1}{2}\|z_k - z_{k+1}\|^2\Big] = \mathbb{E}\Big[\frac{\alpha}{\tau_1}\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{1}{2\tau_1^2}\|x_{k+1} - v\|^2\Big] \\
&= \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{1}{2\alpha\tau_1}\|x_{k+1} - v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{①}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(\langle\widetilde{\nabla}_{k+1}, x_{k+1} - v\rangle - \frac{9\overline{L}}{2}\|x_{k+1} - v\|^2 - \psi(v) + \psi(x_{k+1})\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{②}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{1}{16\overline{L}}\sigma_{k+1}^2\Big) + \frac{\alpha}{\tau_1}\Big(\psi(v) - \psi(x_{k+1})\Big)\Big] \\
&\overset{③}{\leq} \mathbb{E}\Big[\frac{\alpha}{\tau_1}\Big(F(x_{k+1}) - F(y_{k+1}) + \frac{1}{2}\big(f(\widetilde{x}) - f(x_{k+1}) - \langle\nabla f(x_{k+1}), \widetilde{x} - x_{k+1}\rangle\big)\Big) \\
&\quad + \frac{\alpha}{\tau_1}\Big(\tau_1\psi(z_{k+1}) + \tau_2\psi(\widetilde{x}) + (1-\tau_1-\tau_2)\psi(y_k) - \psi(x_{k+1})\Big)\Big] \ . \qquad \text{(D.3)}
\end{aligned}
$$

Above, ① uses our choice $\tau_1 \leq \frac{1}{9\alpha\overline{L}}$, ② uses Lemma D.1, ③ uses Lemma D.2 together with the convexity of $\psi(\cdot)$ and the definition of v. Finally, noticing that $\mathbb{E}[\langle\widetilde{\nabla}_{k+1}, z_k - u\rangle] = \langle\nabla f(x_{k+1}), z_k - u\rangle$ and $\tau_2 = \frac{1}{2}$, we obtain the desired inequality by combining (D.2) and (D.3). □

The next lemma is completely analogous to Lemma 3.7 except that we use Lemma D.4 rather than Lemma 3.6. We ignore the proof since it is a simple copy-and-paste.

Lemma D.5 (coupling step 2). *Under the same choices of τ_1, τ_2 as in Lemma D.4, we have*

$$
\begin{aligned}
0 \leq \frac{\alpha(1-\tau_1-\tau_2)}{\tau_1}\big(F(y_k) - F(x^*)\big) - \frac{\alpha}{\tau_1}\big(\mathbb{E}\big[F(y_{k+1})\big] - F(x^*)\big) + \frac{\alpha\tau_2}{\tau_1}\big(F(\widetilde{x}) - \tau_2 F(x^*)\big) \\
+ V_{z_k}(x^*) - (1+\alpha\sigma)\mathbb{E}\big[V_{z_{k+1}}(x^*)\big] \ .
\end{aligned}
$$

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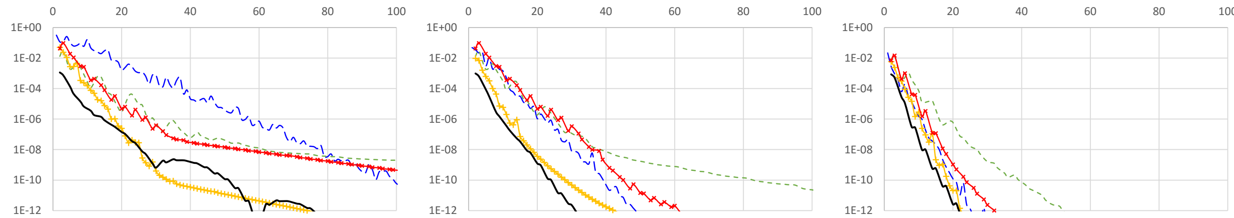

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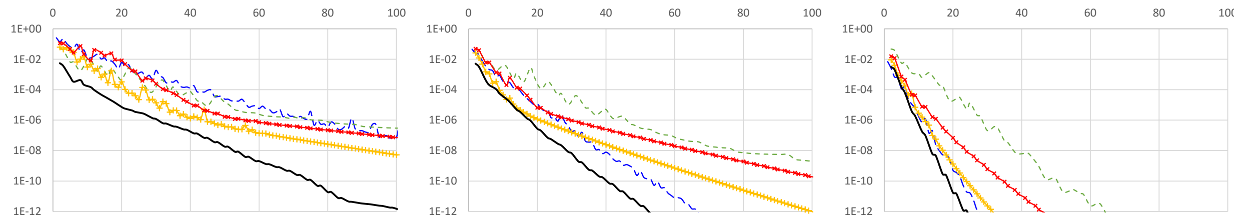
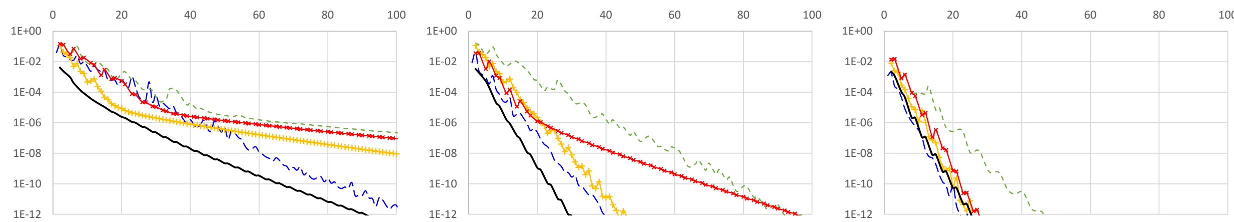
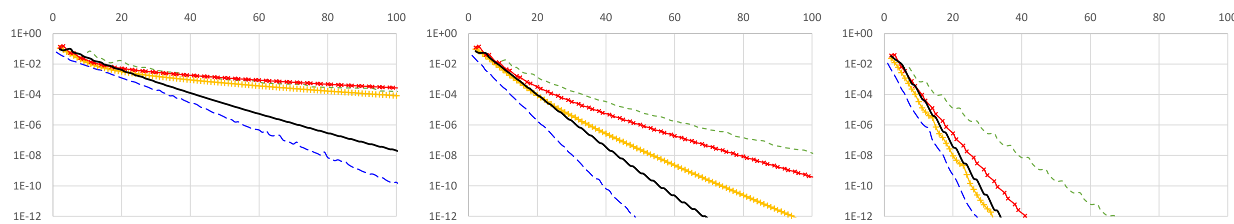


(d) web, $\lambda = 10^{-6}$ (e) web, $\lambda = 10^{-5}$ (f) web, $\lambda = 10^{-4}$



(g) mnist, $\lambda = 10^{-6}$ (h) mnist, $\lambda = 10^{-5}$ (i) mnist, $\lambda = 10^{-4}$



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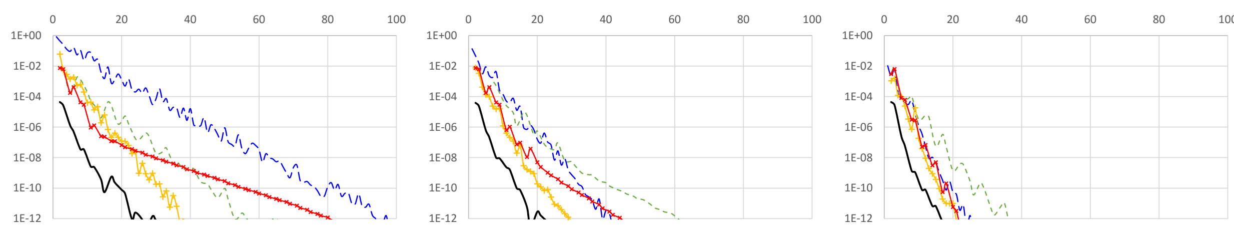

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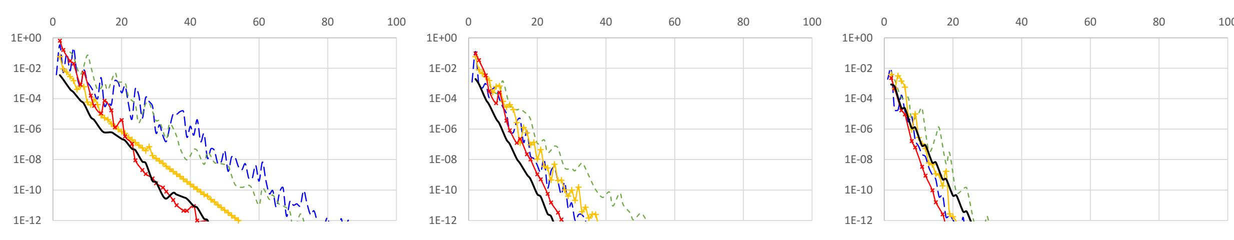
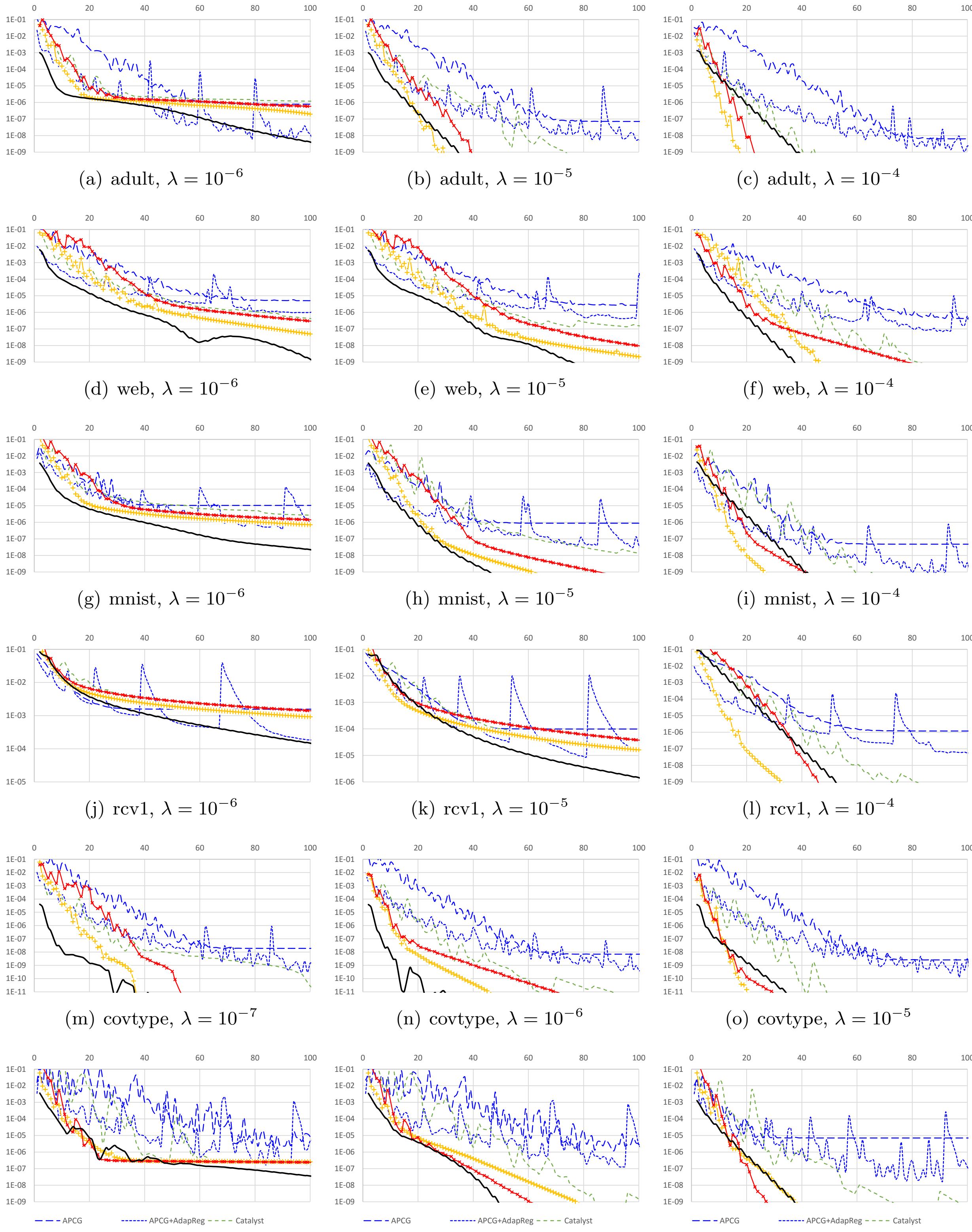


(p) sensit, $\lambda = 10^{-6}$ (q) sensit, $\lambda = 10^{-5}$ (r) sensit, $\lambda = 10^{-4}$

Figure 3: Experiments on ridge regression with ℓ_2 regularizer weight λ.



(a) adult, $\lambda = 10^{-6}$ (b) adult, $\lambda = 10^{-5}$ (c) adult, $\lambda = 10^{-4}$

(d) web, $\lambda = 10^{-6}$ (e) web, $\lambda = 10^{-5}$ (f) web, $\lambda = 10^{-4}$

(g) mnist, $\lambda = 10^{-6}$ (h) mnist, $\lambda = 10^{-5}$ (i) mnist, $\lambda = 10^{-4}$

(j) rcv1, $\lambda = 10^{-6}$ (k) rcv1, $\lambda = 10^{-5}$ (l) rcv1, $\lambda = 10^{-4}$

(m) covtype, $\lambda = 10^{-7}$ (n) covtype, $\lambda = 10^{-6}$ (o) covtype, $\lambda = 10^{-5}$

(p) sensit, $\lambda = 10^{-6}$ (q) sensit, $\lambda = 10^{-5}$ (r) sensit, $\lambda = 10^{-4}$

Figure 4: Experiments on Lasso with ℓ_1 regularizer weight λ.