KE Tableaux for Public Announcement Logic

Mathijs S. de Boer*
University of Luxembourg
mathijs.deboer@uni.lu

September 6, 2007

Abstract

Public announcement logic (PAL) is a simple dynamic epistemic logic extending reasoning about knowledge of agents with a modal operator for simultaneous and transparent knowledge updates. This logic is no more expressive than epistemic logic (EL) without updates, but exhibits compact representation of a number of complex epistemic situations. A labeled tableau proof system to reason with these updates directly is presented here. This system can analyse and present well-known epistemic puzzles like 'muddy children' and 'three wise men'. Using the KE tableau system as a basis, the modal and propositional characteristics of epistemic updates can be separated.

Key words: Public announcement logic; epistemic logic; KE; semantic tableaux.

1 Introduction

This paper provides a tableau proof method for PAL that is both practical and intuitive for *human* provers. Little new notation is introduced and we stay close to existing systems. Because PAL can (only) succinctly represent what would otherwise be incomprehensible in EL [9], one of the objectives of PAL must be the presentation and explanation of (formal) reasoning. Tableaux, as they are—in contrast to axiomatic proof—easy to construct and easy to read, also have their purpose in the explanation of their underlying logic and reasoning. These goals meet in the PALKE tableau system.

System PALKE extends the well-known labeled tableau scheme for modal logics from [6] in such a way that the general form of the original rules dealing with modals can be left unaltered. Using the KE tableau system from [3] allows for this elegant formulation and does not lead to a more complex system in comparison with the non-KE system. In representative cases, e.g. in solving the three wise men puzzle, PALKE gives short and informative proofs.

*This work is based on [4] as completed under the supervision of B.P. Kooi and L.C. Verbrugge at the University of Groningen, The Netherlands. An inferior (but better looking) version of this paper will appear/has appeared in the (pre-)proceedings of the Formal Approaches to Multi-Agent Systems workshop (FAMAS'007, MALLOW'007), 6 and 7 September 2007, Durham, UK.

This paper is structured as follows. After a brief formal introduction of PAL, its syntax, semantics and axiomatisation in sections 2 and 3, we meet tableau proof system PALKE in section 4. Soundness and completeness of PALKE is shown in unsurprising ways in section 5. Example tableaux are given throughout section 4 and 5, section 6 is dedicated to the example of a formalisation of the three wise men puzzle in PAL and its tableau proof. The relation of PALKE to the 'Lima' tableau proof method for PAL from [2] is discussed in section 7.

2 Syntax and Semantics

Modal logic with possible world semantics as epistemic logic was introduced in [8]. The extension of epistemic logic with public announcements or updates originates from [11]. Notations and conventions from [14] are used here.

Definition 2.1 *The language $\mathcal{L}_{pal(PA)}$ of public announcement logic is inductively defined as*

$$\phi ::= p \mid \neg\phi \mid (\phi_1 \wedge \phi_2) \mid \Box_a \phi \mid [\phi_1]\phi_2$$

where $a \in A$, a finite set of agents and $p \in P$, a countably infinite set of atoms. The language of epistemic logic $\mathcal{L}_{el}$ is the language without $[\phi_1]\phi_2$ and the language of propositional logic $\mathcal{L}_{pl}$ without $\Box_a\phi$ as well. We will use the usual abbreviations $\top$, $\bot$, $(\phi_1 \to \phi_2)$, $(\phi_1 \vee \phi_2)$, $(\phi_1 \leftrightarrow \phi_2)$, $\Diamond_a\phi$ and $\langle\phi_1\rangle\phi_2$.

Read $\Box_a\phi$ as 'agent a knows that ϕ' and $\Diamond_a\phi$ as 'agent a holds ϕ posssible. An update formula $[\phi]\psi$ reads as 'after the public announcement of ϕ, ψ is true and its dual $\langle\phi\rangle\psi$ as 'announcement ϕ can be made (that is, ϕ is true) and after it, ψ is true'.

Definition 2.2 *An epistemic model $\mathcal{M} = \langle W, R, V\rangle$ consists of a finite non-empty set of possible worlds W, an accessibility relation $R : A \to 2^{W\times W}$ and a valuation $V : W \times P \to \{1, 0\}$. We write $w \sim_a w'$ or wR_aw' if $(w, w') \in R(a)$ for 'w' is accessible from w for agent a' and $(\mathcal{M}, w)$ for an epistemic state or pointed model with $w \in W$.*

Each $R(a)$ is an equivalence relation here, as is usual for epistemic logic, which we take to be the multi-modal logic $S5_n$. So, agents only know true facts and are capable of both positive and negative introspection.

Definition 2.3 *Let $(\mathcal{M}, w)$ be an epistemic model with $\mathcal{M} = \langle W, R, V\rangle$ and $w, v \in W$, $p \in P$, $i \in A$ and ϕ, ψ, χ in $\mathcal{L}_{pal(PA)}$.*

$$\begin{array}{lll}
\mathcal{M}, w \models p & :\Leftrightarrow & V_w(p) = 1 \\
\mathcal{M}, w \models \neg\phi & :\Leftrightarrow & \mathcal{M}, w \not\models \phi \\
\mathcal{M}, w \models \phi \wedge \psi & :\Leftrightarrow & \mathcal{M}, w \models \phi \text{ and } \mathcal{M}, w \models \psi \\
\mathcal{M}, w \models \Box_i\phi & :\Leftrightarrow & \text{for all } v \in W : w \sim_i v \text{ implies } \mathcal{M}, v \models \phi \\
\mathcal{M}, w \models [\phi]\psi & :\Leftrightarrow & \mathcal{M}, w \models \phi \text{ implies } \mathcal{M}|_\phi, w \models \psi
\end{array}$$

Model $\mathcal{M}|_\phi := \langle W', R', V'\rangle$ is defined as

$$\begin{array}{lll}
W' & := & \{w' \in W \mid \mathcal{M}, w' \models \phi\} \\
R'(i) & := & R(i) \cap (W' \times W') \\
V'_w(p) & := & V_w(p) \restriction W'
\end{array}$$

all propositional tautologies	
$\vdash \Box_a(\phi \to \psi) \to (\Box_a\phi \to \Box_a\psi)$	distribution
$\vdash \Box_a\phi \to \phi$	truth
$\vdash \Box_a\phi \to \Box_a\Box_a\phi$	positive introspection
$\vdash \Diamond_a\phi \to \Box_a\Diamond_a\phi$	negative introspection
$\vdash [\phi]p \leftrightarrow (\phi \to p)$	atoms
$\vdash [\phi]\neg\psi \leftrightarrow (\phi \to \neg[\phi]\psi)$	partial function
$\vdash [\phi](\psi \wedge \chi) \leftrightarrow ([\phi]\psi \wedge [\phi]\chi)$	distribution
$\vdash [\phi]\Box_i\psi \leftrightarrow (\phi \to \Box_i[\phi]\psi)$	knowledge update
$\vdash [\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$	combined updates
from $\vdash \phi$ and $\vdash \phi \to \psi$ infer $\vdash \psi$	modus ponens
from $\vdash \phi$ and $\vdash \Box_i\phi$	necessity for $\Box_i$
from $\vdash \phi$ and $\vdash [\psi]\phi$	necessity for $[\psi]$

Table 1: Proof system PAL_{PA} with ϕ, ψ, χ formulas in $\mathcal{L}_{\mathsf{PAL}(PA)}$, a some agent in A and p some propositional atom in P.

We can read $\mathcal{M}|_\phi$ as a submodel of $\mathcal{M}$, containing exactly the possible worlds of $\mathcal{M}$ where ϕ was true. So, semantically, public announcement logic is a world elimination game, illustrated in [14, 11, 12, 13].

3 Axiomatisation

The axiomatisation for PAL in table 1 from [14] is only briefly discussed here and included for use in the next section. Because PAL is no more expressive than EL its Hilbert-style proof system sports so-called reduction axioms. Systematic application of the reduction axioms translates any formula in PAL into a formula in EL without updates, preserving truth value. As is shown in [9], the length of these translations might be exponential in the length of the original formula in PAL.

4 Tableau Method

A tableau for a formula ϕ is a tree with root the labeled formula $1\ \neg\phi$ and is constructed by the application of tableau rules. A label is a sequence of integers with agent subscripts but starting with '1', for instance label $1.1_a.2_e$, possibly preceded by a sequence of formulas in brackets: $(\Diamond_a p)(\Box_e(q \vee p))1.1_i$. Labels represent paths through (updated) epistemic models, e.g. $(p)(q)1.1_a\ p$ is a picture of $\mathcal{M}|_q|_p, w' \models p$ with $w \sim_a w' \in R'$.

A *closed* tableau with root $1\ \neg\phi$ is a proof for ϕ. A tableau is closed if all its branches are closed. Closed branches have two nodes of the form $\sigma\ \phi$ and $\sigma\ \neg\phi$ and are leafed '='. A tableau branch that cannot be closed by the application of tableau rules is a counter example of ϕ and is leafed '↑'. If a branch $\mathcal{B}$ of a tableau is a counter example for ϕ then a countermodel $\mathcal{M}$ can be constructed from the formulas on $\mathcal{B}$ with $\mathcal{M}, w \models \neg\phi$.

The tableau rules in table 2 form the proof system PALKE. Omitting the rules concerning updates results in the system KEEL. Leaving out the rules for

$$\frac{\sigma\ \ \phi\wedge\psi}{\begin{array}{ll}\sigma & \phi\\ \sigma & \psi\end{array}}\ (\alpha) \qquad \frac{\begin{array}{ll}\sigma & \neg(\phi\wedge\psi)\\ \sigma & \phi\end{array}}{\sigma\ \ \neg\psi}\ (\beta_1) \qquad \frac{\begin{array}{ll}\sigma & \neg(\phi\wedge\psi)\\ \sigma & \psi\end{array}}{\sigma\ \ \neg\phi}\ (\beta_2) \qquad \frac{}{\underline{\sigma}\ \ \phi \mid \underline{\sigma}\ \ \neg\phi}\ (\text{PB})$$

$$\frac{\sigma\ \ \Box_a\phi}{\underline{\sigma.h_a}\ \ \phi}\ (K^a) \qquad \frac{\sigma\ \ \neg\Box_a\phi}{\overline{\sigma.n_a}\ \ \neg\phi}\ (M^a) \qquad \frac{\sigma.m_a\ \ \Box_a\phi}{\underline{\sigma.h_a}\ \ \phi}\ (K^a_=) \qquad \frac{\sigma.m_a\ \ \neg\Box_a\phi}{\overline{\sigma.n_a}\ \ \neg\phi}\ (M^a_=) \qquad \frac{\sigma\ \ \Box_a\phi}{\sigma\ \ \phi}\ (T^a) \qquad \frac{\sigma.m_a\ \ \Box_a\phi}{\underline{\sigma}\ \ \phi}\ (B^a)$$

$$\text{updates: } \frac{\sigma\ \ [\chi]\psi}{\underline{(\chi)\sigma}\ \ \psi} \quad \frac{\sigma\ \ \neg[\phi]\psi}{(\phi)\sigma\ \ \neg\psi} \qquad \text{persistence: } \frac{(\phi)\sigma\ \ p}{\sigma\ \ p} \quad \frac{(\phi)\sigma\ \ \neg p}{\sigma\ \ \neg p} \qquad \text{correspondence: } \frac{(\phi)\sigma}{\sigma\ \ \phi} \quad \frac{\sigma\ \ \phi}{(\phi)\sigma}$$

Table 2: The PALKE tableau system. Underlined labels must already exist on the branch (label *selection*) and over-lined labels must be new (label *introduction*). In rule M^a, σ must be a non-a-label.

agent knowledge as well, we have unsigned KE for propositional logic.

Unlike the usual Smullyan tableaux TAB, system KE has a *principle of bivalence* (PB) as the only forking rule and non-forking β rules. The PB can be seen as the tableau counterpart of the sequent *cut* rule. Rule PB allows the introduction of arbitrary formulas on tableau branches and therefore KE is not *analytic*. Limiting the use of PB to (labeled, negated) subformulas of formulas already on the branch gives access to a whole class of analytic applications of (PAL)KE. A possible analytic application of KE in the propositional case (and for EL too) could be the limitation of PB to introducing $\sigma\ \phi \mid \sigma\ \neg\phi$ and $\sigma\ \psi \mid \sigma\ \neg\psi$ on a branch $\mathcal{B}$ with formula $\sigma\ \neg(\phi\wedge\psi)$ already on $\mathcal{B}$ and $\sigma\ \phi$, $\sigma\ \psi$ not. Note that KE does not need a double negation rule ($\sigma\ \neg\neg\phi/\sigma\ \phi$), but it would be acceptable to use it as an abbreviation.

System KEEL for epistemic logic is a variation on the more straightforward proof system $KMT44r$ from [5]. System $KMT44r$ applies rules K^a, M^a and T^a from table 2 and rules 4^a and $4r^a$ as shown in table 3 on page 4. System KEEL is more well-behaved than $KMT44r$ and forms a finite, loop-checking free decision procedure for proving formulas in epistemic logic almost in itself. In using KEEL, for any given formula $\sigma\ \neg\Box_a\phi$ we have, as usual, only one choice: if σ is an a-label we use rule $M^a_=$, rule M^a otherwise. Analysing formulas $\sigma\ \Box_a\phi$ we can select a label $\sigma'.h_a$ or σ', where $\sigma = \sigma'.m_a$ or $\sigma = \sigma'$. When we consider $\sigma' = \sigma$ as selection of an 'empty' prefix, label $\sigma.\varepsilon_a$, implementation of KEEL with free variables in labels can prevent duplication of formulas and moves all choice points to the moment of branch closure, see [4]. Implementation of a $KMT44r$ style system is more difficult because endless repetition of rules 4^a and $4r^a$ must be prevented. It must be admitted however, that this implementational trickery seems to works less well when updates come into play.

1	$\langle p \wedge \neg\Box_a p\rangle(p \wedge \neg\Box_a p)$		1.
$(p \wedge \neg\Box_a p)1$	$p \wedge \neg\Box_a p$	$(upd.\ 1)$	2.
$(p \wedge \neg\Box_a p)1$	p	$(\alpha\ 2)$	3.
$(p \wedge \neg\Box_a p)1$	$\neg\Box_a p$	$(\alpha\ 2)$	4.
$(p \wedge \neg\Box_a p)1.1_a$	$\neg p$	$(M^a\ 4)$	5.
1.1_a	$\neg p$	$(pers.\ 5)$	6.
1.1_a	$p \wedge \neg\Box_a p$	$(corr.\ 5)$	7.
1.1_a	p	$(\alpha\ 7)$	8.
	$=$	(8×6)	

Figure 1: A tableau proof for an *unsuccessful update* where a formula is in fact rendered false by it's own announcement. For reference, formulas are numbered and applied rules are given on the right. Note that abbreviation $\sigma\ \langle\phi\rangle\psi$ is here directly analyzed to $(\phi)\sigma\ \psi$.

The tableau rules for epistemic logic do not interact with PB and we could use the Smullyan system TAB to give a sound and complete tableau proof system for epistemic logic. Unlike KEEL, PALKE explicitly needs PB for completeness; observe the following cases. Analysis of formulas $\sigma\ [\phi]\psi$ when $(\phi)\sigma$ is not a label on the branch and $\sigma\ \phi$ is not a formula on the branch, requires a fork $\sigma\ \phi \mid \sigma\ \neg\phi$. Bringing a label $\sigma.n_a$ in reach for selection by a formula $(\phi)\sigma\ \Box_a\phi$ might require a fork with $\sigma.n_i\ \phi \mid \sigma.n_i\ \neg\phi$ to see whether $\sigma.n_i$ belongs in $\mathcal{M}'|_\phi$. For example, if we want to expand a branch $\mathcal{B}$ by selection of a label 1.1_i from $(\phi)(\psi)(\chi)1\ \Box_i\xi$ and even $(\chi)1.1_i$ is not on $\mathcal{B}$ we could apply PB three times, creating four branches, eventually only expanding one of them with $(\phi)(\psi)(\chi)1.1_i\ \xi$. To begin with a fork $1.1_i\ \neg[\chi][\psi][\phi]\bot \mid 1.1_i\ [\chi][\psi][\phi]\bot$ would of course also be acceptable. Note that we have to restrict PB to existing labels or we could introduce $(\bot)1\ \phi \mid (\bot)1\ \neg\phi$, closing both branches by correspondence.

5 Soundness and Completeness

Soundness and completeness of KEEL is easily shown by comparison with the $S5_n$ tableau system $KMT44r$. Rules $M^{\underline{a}}_{=}$ and $K^{\underline{a}}_{=}$ are multi-modal variants of the simple mono-modal $S5$ rules from [5, page 113], [6] and are sound directly by semantics: the $S5$ accessability relation for single agents holds between any two connected worlds. Rule B^a is $S5$-sound by symmetry, we could also see it as the application of rules $4r^a + T^a$. For completeness we show that rules 4^a and $4r^a$ from table 3 have in fact become redundant.

Proposition 5.1 *Proofs in KEEL simulate proofs in $KMT44r$.*

Proof If we can close a branch $\mathcal{B}$ by the application of rule 4^a on $\sigma\ \Box_a\phi$, meaning we have a formula $\sigma.h_a\neg\Box_a\phi$ already on $\mathcal{B}$, we can also close $\mathcal{B}$ by application of $M^{\underline{a}}_{=}$ on $\sigma.h_a\neg\Box_a\phi$ and K^a on $\sigma\ \Box_a\phi$. The same goes for the other variations of $\Box_a\phi$ and $\Diamond_a\phi$ formulas and for closure by application of rule $4r^a$, by way of rules M^a and B^a.

Non-closing applications of rules 4^a and $4r^a$, meaning combinations of applications of rules $4^a + K^a$, $4r^a + T^a$ etc. (the second rule must always be a rule

$$4^a \quad \frac{\sigma \quad \Box_a\phi}{\sigma.h_a \quad \Box_a\phi} \qquad\qquad 4r^a \quad \frac{\sigma.n_a \quad \Box_a\phi}{\sigma \quad \Box_a\phi}$$

Table 3: Tableau rules 4^a and $4r^a$ of system $KMT44r$

on formulas of the form $\sigma' \ \Box_a\phi$, the result of both 4^a and $4r^a$), is equivalent to application of a rule in KEEL in the following cases:

- $4^a + K^a_= = K^a$
- $4^a + T^a = K^a$
- $4^a + B^a = T^a$
- $4^a + 4r^a = 4^a_r + 4^a =$ do nothing
- $4r^a + K^a = K^a$
- $4r^a + K^a_= = K^a$
- $4r^a + T^a = B^a$.

Applications $4^a + K^a$, $4^a + 4^a$, $4r^a + 4r^a$ and $4r^a + B^a$ deal with labels of the form $\sigma.m_a.n_a$ which are not required in proofs: KEEL explicitly disallows labels of this form. Any branch closure with two labels $1 \ldots .m_a.n_a \ldots$ could be closed on these labels with m_a stripped from the label by either rule $M^a_=$, introducing n_a in place of m_a or $K^a_=$, selecting n_a in place of m_a. ∎

Soundness of PALKE is by the usual arguments, from [6]. A set S of labeled formulas is PAL satisfiable in an epistemic model $\mathcal{M} = \langle W, R, V \rangle$ if there is a mapping $\mathcal{N}(\sigma) = (\mathcal{M}', w)$ from every $\sigma \ \phi \in S$ so that $w \in W$ and $\mathcal{M}'$ is $\mathcal{M}$ or the submodel corresponding to σ, e.g. $\mathcal{N}((\Box_a p)(p \to q)1) = (\mathcal{M}|_{p\to q}|_{\Box_a p}, w)$ and $\mathcal{N}(\sigma) \models \phi$. A tableau branch is satisfiable if its set of formulas is satisfiable and a tableau is satisfiable if at least one of its branches is.

Proposition 5.2 *A satisfiable tableau cannot be closed by application of the PALKE tableau rules.*

Proof We only show this for the rules not in KEEL. We take tableau branch $\mathcal{B}$ satisfying model $\mathcal{M} = \langle W, R, V \rangle$.

- Update$_1$: $\sigma \ [\phi]\psi$ and label $(\phi)\sigma$ are on $\mathcal{B}$ and we add $\{(\phi)\sigma \ \psi\}$ to $\mathcal{B}$. We now have $\mathcal{N}((\phi)\sigma) \models \psi$ which follows from $\mathcal{N}(\sigma) \models [\phi]\psi$, $\mathcal{N}(\sigma) = (\mathcal{M}', w)$, $\mathcal{N}((\phi)\sigma) = (\mathcal{M}'|_\phi, w)$, and so $(\mathcal{M}', w) \models \phi$.

- Update$_2$: $\sigma \ \neg[\phi]\psi$ is on $\mathcal{B}$ and we add $(\phi)\sigma \ \neg\psi$. Now we must have $\mathcal{N}((\phi)\sigma) \models \neg\psi$ which follows from $\mathcal{N}(\sigma) \models \neg[\phi]\psi$ and $\mathcal{N}(\sigma) = (\mathcal{M}', w)$ and $\mathcal{N}((\phi)\sigma) = (\mathcal{M}'|_\phi, w)$.

- Persistence$_1$: $(\phi)\sigma \ p$ is on $\mathcal{B}$. Take $\mathcal{N}(\sigma) = (\mathcal{M}', w)$ with $\mathcal{M}' = \langle W', R', V' \rangle$ and $\mathcal{N}((\phi)\sigma) = (\mathcal{M}'|_\phi, w)$ with $\mathcal{M}'|_\phi = \langle W'', R'', V'' \rangle$. We have by definition $V''_w(p) := V'_w(p)$ (obviously $W'' \subseteq W'$).

- Persistence$_2$: $(\phi)\sigma \ \neg p$ is on $\mathcal{B}$. As persistence$_1$.

- Correspondence$_1$: $(\phi)\sigma$ is a label on $\mathcal{B}$ and we add $\sigma \ \phi$. $\mathcal{N}(\sigma)$ maps to some $(\mathcal{M}', w)$, $w \in W'$ and now $\mathcal{N}((\phi)\sigma)$ maps to $(\mathcal{M}'|_\phi, w)$, exactly following semantics: $W'' := \{w \in W' | \mathcal{M}', w \models \phi\}$.

$$
\begin{array}{llr}
1 & \neg((\phi \to \Box_i[\phi]\psi) \to [\phi]\Box_i\psi) & 1. \\
1 & \quad (\phi \to \Box_i[\phi]\psi) & (\alpha_1)\ 2. \\
1 & \neg[\phi]\Box_i\psi & (\alpha_2)\ 3. \\
(\phi)1 & \neg\Box_i\psi & (upd.)\ 4. \\
(\phi)1.1_i & \neg\psi & (M^i)\ 5. \\
1 & \quad \phi & (corr.)\ 6. \\
1.1_i & \quad \phi & (corr.)\ 7. \\
1 & \quad \Box_i[\phi]\psi & (\beta)\ 8. \\
1.1_i & \quad [\phi]\psi & (K^i)\ 9. \\
(\phi)1.1_i & \quad \psi & (upd.)\ 10. \\
 & = &
\end{array}
$$

Figure 2: A tableau for a ‘knowledge update’. This is one half of the knowledge update axiom of table 1. With some effort we can prove all PAL reduction axioms in PALKE.

- Correspondence$_2$: $\sigma\ \phi$ is on $\mathcal{B}$ and we add label $(\phi)\sigma$. As correspondence$_1$.

∎

Proposition 5.3 (Soundness) *If there is a PALKE tableau proof for ϕ then for any pointed model we have $(\mathcal{M}, w) \models \phi$.*

Proof By the application of tableau rules we can close $1\ \neg\phi$. If there is a model $(\mathcal{M}, w) \models \neg\phi$ then $\mathcal{N}(1) \models \neg\phi$ must be satisfiable and closable at the same time, which is impossible as then we must have $(\mathcal{M}', w') \models \phi \wedge \neg\phi$. ∎

For completeness it suffices to prove that the PAL reduction axioms from table 1 can be proven in tableaux. For the most part this can be done directly, only the combined updates axiom requires some more work. We have already seen a proof for (half of) the knowledge update axiom in figure 2, the other axiom tableaux are similar and not given here.

Proposition 5.4 *PALKE is complete for the PAL reduction axioms*

Proof Only the axiom $\vdash [\phi][\psi]\chi \leftrightarrow [\phi \wedge [\phi]\psi]\chi$ does not appear to have a direct proof. Figure 3 shows an attempted proof for half of the combined updates axiom. We prove closure on $\sigma_1\ \chi$ and $\sigma_2\ \neg\chi$ where σ_1 stands for $(\phi \wedge [\phi]\psi)\sigma$ and σ_2 for $(\psi)(\phi)\sigma$.

- base case: χ is p or χ is $\neg p$, closure on persistence.
- χ is of the form $\chi'_1 \wedge \chi'_2$. We fork on $\sigma_2\ \chi'_2 \mid \sigma_2\ \neg\chi'_2$ and will have to close the two branches on $\sigma_1\ \chi'_1$, $\sigma_2\ \neg\chi'_1$ and $\sigma_1\ \chi'_2$, $\sigma_2\ \neg\chi'_2$. The same for the other propositional connectives.
- χ is of the form $\Box_a\chi'$. We analyse to $\sigma_2.1_a\ \neg\chi'$ (label introduction) and we fork with $\sigma.1_i\ \phi \wedge [\phi]\psi \mid \sigma.1_i\ \neg(\phi \wedge [\phi]\psi)$ and we are in the same situation as the tableau in figure 3, formulas 8 and 9, but now in label $\sigma.1_i$ with χ' instead of σ with χ and the analysis goes the same. The same for $\Diamond_a\chi'$ etc.

$$
\begin{array}{rll}
1 & \neg([\phi \wedge [\phi]\psi]\chi \rightarrow [\phi][\psi]\chi) & 1. \\
1 & [\phi \wedge [\phi]\psi]\chi & 2. \\
1 & \neg[\phi][\psi]\chi & 3. \\
(\phi)1 & \neg[\psi]\chi & 4. \\
(\psi)(\phi)1 & \neg\chi & 5. \\
(\phi)1 & \psi & 6. \\
1 & \phi & 7.
\end{array}
$$

$$
\begin{array}{rll}
1 & \phi \wedge [\phi]\psi & 8. \\
(\phi \wedge [\phi]\psi)1 & & 10. \\
(\phi \wedge [\phi]\psi)1 & \chi & 11. \\
 & \vdots &
\end{array}
\qquad\qquad
\begin{array}{rll}
1 & \neg(\phi \wedge [\phi]\psi) & 9. \\
1 & \neg[\phi]\psi & 12. \\
(\phi)1 & \neg\psi & 13. \\
 & = &
\end{array}
$$

Figure 3: A tableau for combined updates. For any explicitly formulated χ we can close the tableau.

- χ is of the form $[\chi'_1]\chi'_2$. We get $(\chi'_1)\sigma_2\ \neg\chi'_2$ and fork with $\sigma_1\ \chi'_1 \mid \sigma_1\ \neg\chi'_1$. The left branch is to be closed on $(\chi'_1)\sigma_2\ \neg\chi'_2$ and $(\chi'_1)\sigma_1\ \chi'_2$ and the right branch on $\sigma_2\ \chi'_1$ and $\sigma_1\ \neg\chi'_1$. The same for $\langle\chi'_1\rangle\chi'_2$ etc.

In the last three cases we have produced branches with formulas $\sigma'_1\ \chi'$ and $\sigma'_2\ \neg\chi'$ with χ' a subformula of χ. By induction on the structure of χ, all instances of χ' reduce to the base case and the entire tableau closes. ∎

Proposition 5.5 *PALKE is complete for update reduction $\phi \leftrightarrow \phi^t$ for every PAL formula ϕ and its translation to EL ϕ^t.*

Proof (Sketch) From completeness of the reduction axioms [14] we know there is a finite sequence of formulas $\phi^{t_0} \leftrightarrow \phi^{t_1}, ..., \phi^{t_{n-1}} \leftrightarrow \phi^{t_n}$ with $\phi = \phi^{t_0}$, $\phi^{t_n} = \phi^t$ and $\phi^{t_{k+1}}$ a formula where exactly one instance of an update is reduced from ϕ^{t_k}. From completeness for EL and the reduction axioms of PALKE it follows we can prove each $\phi^{t_{k-1}} \leftrightarrow \phi^{t_k}$ so we can can also prove $\phi^{t_0} \leftrightarrow \phi^{t_1} \wedge ... \wedge \phi^{t_{n-1}} \leftrightarrow \phi^{t_n}$ and by propositional logic it follows that $\phi \leftrightarrow \phi^t$. ∎

Proposition 5.6 (Completeness) *If $\models_{PAL} \phi$ then there is a PALKE tableau proof for ϕ.*

Proof We can prove $\phi \leftrightarrow \phi^t$ in a tableau. By completeness for EL, we can also prove ϕ^t in a tableau. If we cannot prove ϕ in a tableau we also cannot prove $\neg(\phi \leftrightarrow \phi^t) \vee \neg\phi^t \vee \phi$, because all three formulas are unprovable. However, we can prove $\neg(\phi \leftrightarrow \phi^t) \vee \neg\phi^t \vee \phi$ by propositional logic, rendering the unprovability of ϕ inconsistent. ∎

6 An Example

As an example of a more elaborate proof in PALKE we formalise the following variation of the well-known three wise men puzzle from [10]:

> A certain king wishes to test his three wise men. He arranges them in a circle so that they can see and hear each other and tells them that he will put a white or black spot on each of their foreheads but that at least one spot will be white. In fact all three spots are white. He then asks them, "Do you know the color of your spot?" The first answers "I don't know the color of my spot". The second answers "I don't know either". The third then answers "Now I know!".

Variations of this puzzle and some very similar accounts of the muddy children puzzle are formalized in, just to name a few, [10, 7, 12, 5]. The puzzle is also an example in the Logics Workbench manuals[1]. The formulation given here is closest to an alternative formalisation for the LWB by Van Ditmarsch[2].

To solve the three wise men puzzle we don't have to find a single correct answer ("all spots are white"), but an *explanation* showing there is some actual sound reasoning going on. PAL and PALKE are well suited for this task and capture the puzzle's full dynamics. We give an informative, direct and short proof showing that the puzzle indeed captures an eternal truth.

The three wise men puzzle is formulated in PAL as follows. Let agents 1, 2, 3 be our three wise men and a, b, c the facts that respectively 1, 2, 3 have a white spot. The men observing each other (but not themselves) is represented by $(\neg a \rightarrow (\Box_2 \neg a \wedge \Box_3 \neg a)) \wedge (\neg b \rightarrow (\Box_1 \neg b \wedge \Box_3 \neg b)) \wedge (\neg c \rightarrow (\Box_1 \neg c \wedge \Box_2 \neg c))$. When 1 has a black spot (that is, not a white spot) 2 and 3 know this etc. The fact that at least one of the spots is white is represented by $(a \vee b \vee c)$. The utterances of 1 not knowing, 2 not knowing and 3 knowing are then $\Diamond_1 a$, $\Diamond_2 b$ and $\Box_3 c$. The tableau in figure 4 proves that, after the announcement of the fact that at least one spot is white and the men observing each other and the announcements that 1 does not know and 2 does not know, 3 *does* know his spot is white. So, the given formulation is true in any epistemic model.

From the proof in figure 4, we can observe some well-known facts about the three wise men puzzle. First, the three wise men puzzle is a K theorem as only modal rules K^a and M^a are used in the proof. Second, the observation of the spot of 1 $(\neg a \rightarrow (\Box_2 \neg a \wedge \Box_3 \neg a))$ plays no part in the proof. This is the case where 1 would hide his spot. Then, from the announcements of 1 and 2 and the observation of the spots of 2 and 3 alone, 3 would still know he has a white spot. When 1 says "I don't know", 2 and 3 must conclude that 1 sees either one or two white spots. When 2 now says "I don't know", 3 must conclude that 2 sees a white spot as well or 2 would have know to have a white spot and 3 still concludes to have a white spot. So now, for agents with $S5$ knowledge we could also prove $[(\neg a \vee \neg b \vee \neg c) \wedge (b \rightarrow \Box_1 \Box_3 b) \wedge (c \rightarrow \Box_1 \Box_2 c)][\Diamond_1 a][\Diamond_2 b]\Box_3 \neg c$ (here a, b, c would represent black spots).

7 Related Work, the Lima System

A different and independently conceived tableau method for (an extension of) public announcement logic is presented in [1]. Apart from notational differences,

[1] The LWB is a popular theorem prover developed at the University of Bern, Switzerland; http://www.lwb.unibe.ch.

[2] http://tcw2.ppsw.rug.nl/mas/LOK/lwb

$\mathsf{WM} \equiv (a \vee b \vee c) \wedge (\neg a \to (\Box_2 \neg a \wedge \Box_3 \neg a)) \wedge (\neg b \to (\Box_1 \neg b \wedge \Box_3 \neg b)) \wedge (\neg c \to (\Box_1 \neg c \wedge \Box_2 \neg c))$

1	$\neg[\mathsf{WM}][\Diamond_1 \neg a][\Diamond_2 \neg b]\Box_3 c$		1.
$(\mathsf{WM})1$	$\neg[\Diamond_1 \neg a][\Diamond_2 \neg b]\Box_3 c$	$(upd.\ 1)$	2.
$(\Diamond_1 \neg a)(\mathsf{WM})1$	$\neg[\Diamond_2 \neg b]\Box_3 c$	$(upd.\ 2)$	3.
$(\Diamond_2 \neg b)(\Diamond_1 \neg a)(\mathsf{WM})1$	$\neg\Box_3 c$	$(upd.\ 3)$	4.
$(\Diamond_2 \neg b)(\Diamond_1 \neg a)(\mathsf{WM})1.1_3$	$\neg c$	$(M^1\ 4)$	5.
$(\Diamond_1 \neg a)(\mathsf{WM})1.1_3$	$\Diamond_2 \neg b$	$(corr.\ 5)$	6.
$(\Diamond_1 \neg a)(\mathsf{WM})1.1_3.1_2$	$\neg b$	$(M^2\ 6)$	7.
$(\mathsf{WM})1.1_3.1_2$	$\Diamond_1 \neg a$	$(corr.\ 7)$	8.
$(\mathsf{WM})1.1_3.1_2.1_1$	$\neg a$	$(M^1\ 8)$	9.
1.1_3	WM	$(corr.\ abbr.\ 6)$	10.
1.1_3	$\neg c$	$(pers.\ abbr.\ 5)$	11.
1.1_3	$\Box_2 \neg c$	$(\beta\ abbr.\ 10, 11)$	12.
$1.1_3.1_2$	$\neg c$	$(K^2\ 12)$	13.
$1.1_3.1_2$	WM	$(corr.\ 8)$	14.
$1.1_3.1_2$	$\neg b$	$(pers.\ abbr.\ 7)$	15.
$1.1_3.1_2$	$\Box_1 \neg c$	$(\beta\ abbr.\ 13, 14)$	16.
$1.1_3.1_2$	$\Box_1 \neg b$	$(\beta\ abbr.\ 15, 14)$	17.
$1.1_3.1_2.1_1$	$\neg c$	$(K^1\ 16)$	18.
$1.1_3.1_2.1_1$	$\neg b$	$(K^1\ 17)$	19.
$1.1_3.1_2.1_1$	$\neg a$	$(pers.\ 9)$	20.
$1.1_3.1_2.1_1$	WM	$(corr.\ 9)$	21.
$1.1_3.1_2.1_1$	$(a \vee b \vee c)$	$(\alpha\ 21)$	22.
	$=$	$(abbr.\ [18, 19, 20] \times 22)$	

Figure 4: A tableau for the three wise men puzzle. Some intermittent steps are omitted, marked by the most relevant rule used and *abbr*.

this system, here referred to as the 'Lima' system (as I believe the PALKE is referred to as the 'Boer' system) does not use a bivalence rule and the propositional and modal aspects of updates are not separated as was done here. To compare these systems a rendering of the Lima system in PALKE notation is given in table 4. These rules, as is necessary for a cut-free proof system, exactly represent the worst case scenario of the application of the bivalence rule of (PAL)KE. Lima tableaus close only on propositional atoms: $(\phi_n)\ldots(\phi_1)\sigma\ p \times (\psi_n)\ldots(\psi_1)\sigma\ \neg p$, so Lima is again not *substitution-closed*, that is, not all rules apply to all formulas alike [13].

Clearly, PALKE can linearly simulate the Lima system. For instance, the Lima rules for $(\psi_n)\ldots(\psi_1)1.\sigma\ \Box_a\phi$ give $n{+}1$ branches with $1\ldots n{+}1$ conclusions corresponding with n times PB and $0\ldots n$ times correspondence on each branch and a PALKE $\sigma\ \Box_a\phi$ rule once on the rightmost branch. Formulas already on the branch should be omitted as is intended in the Lima system.

From the fact that PALKE can linearly simulate the Lima system and the Lima system cannot simulate PALKE at all (directly from PB), it follows that the complexity of PALKE is *at most* equal to that of the Lima system. The Lima system can be seen as an analytic application of the PALKE system and

$$
\begin{array}{c|c|c|c|c}
\multicolumn{5}{c}{(\psi_n)\ldots(\psi_1)1.\sigma\ \Box_a\phi} \\
\hline
1.\sigma'\ \ \neg\psi_1 & 1.\sigma'\ \ \psi_1 & \ldots & 1.\sigma'\ \ \psi_1 & 1.\sigma'\ \ \psi_1 \\
 & (\psi_1)1.\sigma'\ \ \neg\psi_2 & \ldots & (\psi_1)1.\sigma'\ \ \psi_2 & (\psi_1)1.\sigma'\ \ \psi_2 \\
 & & \ddots & \vdots & \vdots \\
 & & & (\psi_{n-1})\ldots 1.\sigma'\ \ \neg\psi_n & (\psi_{n-1})\ldots 1.\sigma'\ \ \psi_n \\
 & & & & (\psi_n)(\psi_{n-1})\ldots 1.\sigma'\ \ \phi'
\end{array}
$$

$$
\frac{\sigma\ \ \Box_a\phi}{\sigma\ \ \phi}
\qquad
\begin{array}{r}
(\psi_n)\ldots(\psi_1)1.\sigma\ \ \neg\Box_a\phi \\
\hline
1.\sigma.n_a\ \ \ \psi_1 \\
\vdots \\
(\psi_{n-1})\ldots(\psi_1)1.\sigma.n_a\ \ \ \psi_n \\
(\psi_n)(\psi_{n-1})\ldots(\psi_1)1.\sigma.n_a\ \ \neg\phi
\end{array}
\qquad
\begin{array}{c}
\sigma\ \ \neg[\chi]\phi \\
\hline
\sigma\ \ \chi \\
(\chi)\sigma\ \ \neg\phi
\end{array}
\qquad
\begin{array}{c|c}
\multicolumn{2}{c}{\sigma\ \ [\chi]\phi} \\
\hline
\sigma\ \ \neg\chi & \sigma\ \ \chi \\
 & (\chi)\sigma\ \ \phi
\end{array}
$$

Table 4: The Lima system as rendered in PALKE notation. Formula $\ldots\sigma'\ \phi'$ is selected according to rules K^a,4^a or $4r^a$ as before and $1.\sigma.n_a$ is a new label. The TAB rules for propositional connectives are omitted.

it may be the "right" analytic application. However, there might be analytic applications of PALKE that are better at least for many cases.

In [2], proof procedures are given for the Lima system that stay within the PSPACE bound for multi- K, KT and $S4$ modals + PAL. It seems likely that there also exist such procedures for full $S5_n$-PAL but none have been found that stay within polynomial space. These results equally apply to the PALKE system.

8 Conclusion

The simple proof system PALKE for public announcement logic was introduced. This system is capable of analysing formalisations in PAL in an appealing way. Using KE as the underlying propositional formalism retains the pleasant features of labeled tableaux for modal logic. The KEEL tableau rules also improve on labeled tableaux for epistemic logic. The PALKE system's analytical applications allow for further and more general research in decision procedures for dynamic epistemic logics.

References

[1] P. Balbiani, A. Herzig, T. De Lima, and H.P. van Ditmarsch. What becomes true after arbitrary announcements. Technical report, Department of Computer Science University of Otago, 2006.

[2] P. Balbiani, H.P. van Ditmarsch, A. Herzig, and T. de Lima. A tableau method for public announcement logics. In N. Olivetti, editor, *Proceedings of the International Conference on Automated Reasoning with Ana-*

lytic Tableaux and Related Methods (TABLEAUX), pages 43–59. Springer-Verlag, 2007.

[3] M. D'Agostino and M. Mondadori. The taming of the cut. *Journal of Logic and Computation*, 4(3):285–319, 1994.

[4] M.S. de Boer. Praktische bewijzen in public announcement logica (practical proofs in public announcement logic). Master's thesis, Department of Artificial Intelligence, University of Groningen, 2006. in Dutch.

[5] M. Fitting. Modal proof theory. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of Modal Logic*, volume 3 of *Studies in Logic and Practical Reasoning*, chapter 2, pages 85–138. Elsevier, 2006.

[6] M. Fitting and R.L. Mendelsohn. *First-Order Modal Logic*, volume 277 of *Synthese Library*. Kluwer Academic Publishers, 1999.

[7] J. Gerbrandy and W. Groeneveld. Reasoning about information change. *Journal of Logic, Language and Information*, 6(2):147–169, 1997.

[8] J. Hintikka. *Knowledge and belief, An Introduction of the Logic of the Two notions*. Cornell University Press, 1962.

[9] C. Lutz. Complexity and succinctness of public announcement logic. In *Proceedings of the fifth international joint conference on Autonomous agents and multiagent systems*, pages 137–143. ACM Press, 2005.

[10] J. McCarthy. Formalisation of two puzzles involving knowledge. In V. Lifschitz, editor, *Formalizing Common Sense: Papers by John McCarthy*, Ablex series in artificial intelligence. Northwood, New Jersey: Ablex Publishing Corporation, 1990.

[11] J.A. Plaza. Logics of public communication. In M.S. Pfeifer, M. Hadzikadic, and Z.W. Ras, editors, *Proceedings of the* 4^{th} *International Symposium on Methodologies for Intelligent Systems*, pages 201–216. Oak Ridge National Laboratory, 1989.

[12] J.F.A.K. van Benthem. One is a lonely number: on the logic of communication. Technical Report PP–2002–27, ILLC, Amsterdam, 2002. to appear in P. Köpke, editor, *Colloquium Logicum*, Münster, AMS Publications, Providence.

[13] J.F.A.K. van Benthem. Open problems in logical dynamics. In M. Zakharysashev D. Gabbay, S. Goncharov, editor, *Mathematical Problems from Applied Logic I*, International Mathematical Series , Vol. 4, pages 137–192. Springer, New York & Novosibirsk, 2006.

[14] H.P. van Ditmarsch, W. van der Hoek, and B.P. Kooi. *Dynamic epistemic logic*, volume 337 of *Synthese Library*. Springer, 2007.