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Kelvin–Helmholtz instabilities and their application to B-type variables

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Summary. A Kelvin–Holmholtz instability, formed from the differential rotation in the narrow region between the core and envelope, is proposed as a promising mechanism responsible for the excitation of pulsations in B-type variables (53 Per variables and β Cep stars), in which the unstable inertia wave resulting from this instability resonates with an eigenmode of the non-radial oscillation of the whole star. The degeneracy of the two frequencies is found to be expected at any evolutionary stage of a star. The equations for a Kelvin–Helmholtz instability have been formulated for the stellar case, and in the cylindrical configuration limit, the necessary condition for instability and characteristics of the instability have been discussed. It is shown that prograde modes with large $|m|$ for a given l are excited in almost all cases; which seems to agree with observations. The back reaction of the excited modes on the differential rotation is discussed in these stars, and it is pointed out that the differential rotation can be significantly affected by this effect in a short time.

1 Introduction

In early B-type stars, there are two small groups of oscillating variables called β Cepheid stars and 53 Persei variables. β Cep stars are restricted to between B0.5 and B2 and between luminosity classes III and IV. Lesh & Aizenman (1973) pointed out that these stars lie near the end of the core hydrogen burning stage with masses of 10–20$M_\odot$. Osaki (1971) showed that the beat phenomenon and line profile variation in some β Cep stars are most naturally explained by non-radial oscillations of a rotating star. Smith and his colleagues (Smith 1978; 1980a, 1981; Smith & McCall 1978; Smith & Buta 1979) found that 53 Per variables (53 Per, 10 Lac, ι Her, 22 Ori, υ Ori) exist around β Cep stars in the HR diagram, and show large line profile variations and comparatively small radial velocity and light variations. Smith (1980c) noted that all 53 Per variables are wholly non-radially pulsating stars.

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The physical mechanism of excitation of oscillations has been investigated but still remains unsolved. Davey (1973) has examined the pulsational instability for the stellar models corresponding to the evolutionary stage of β Cep stars by using linear non-adiabatic analysis, but he did not find any pulsational instability. However, Smith (1980b, c) recently claimed that all β Cep stars have a radial mode as an active mode. Re-examination of the stellar opacity in the relevant condition is worth studying. Also instability towards non-radial pulsations has been investigated for β Cep models by Osaki (1976), but he did not find any instability.

On the other hand, in β Cep stars a wave travelling around the equator in the same direction as the rotation with the negative azimuthal index m (i.e. $m < 0$) is, as pointed out by Osaki (1971), preferably excited. Hansen, Cox & Carroll (1978) have claimed that the opacity-driven mechanism has a tendency for the negative m-modes in a rotating star to be the least stable ones against pulsation. However, their physical reason was not given. Therefore, it is suggested that the excitation mechanism may be closely related to the rotation or some similar effect.

Osaki (1974) proposed a new physical mechanism of resonant excitation of non-radial oscillations by oscillatory convection in the stellar core, and showed that if a star has a rapidly spinning core at the zero-age main sequence (ZAMS), the degeneracy of the oscillatory convective mode and the non-radial f-mode is possible only in the late stage of the main sequence band, since the f-mode is found to have a significant amplitude only in the convective core and near the surface. On the other hand, higher harmonic g-modes (radial index $k \sim 10$–20) which are prograde ones (though only in 22 Ori have retrograde modes been identified by Smith 1980a) are found in 53 Per variables.

In this paper we investigate the possibility of a Kelvin–Helmholtz instability as another alternative mechanism, which is expected in the differentially rotating region. In Section 2 we will formulate the equations of Kelvin–Helmholtz instabilities in stars and derive their characteristics. The realization of this instability including its possible resonant excitation with non-radial oscillations will be discussed in Section 3, and Section 4 contains a discussion.

2 Kelvin–Helmholtz instability in stars

Kelvin–Helmholtz instabilities have been studied, mainly in applied mathematics and geophysical fluid dynamics, as instabilities of plane parallel flow (see the excellent review article of Drazin & Howard 1966). This is overstability of inertia waves and was explained by Gill (1965) to be formed as a result of an unstable distribution of vorticity in the flow. We may therefore expect the possibility of this instability in a star where there is vorticity distribution (i.e. shear flow).

Throughout the lifetime of a star, it can, more or less, have differential rotation. As Kippenhahn, Meyer-Hofmeister & Thomas (1970) have discussed under certain assumptions, a rapidly spinning core in a star may be developed due to core contraction during its evolution. In addition to this fact, Bodenheimer & Ostriker (1970) suggested that even in the ZAMS stage a central core may rotate rapidly. Thus it is expected that generally there is a shear flow due to differential rotation in the narrow region between core and envelope. If a Kelvin–Helmholtz instability in this narrow region results in the occurrence of inertia waves and if the frequencies of these waves happen to coincide with those of non-radial oscillations of the whole star, the non-radial modes can be excited. As these modes have a significant amplitude at the surface (even for g-modes: see Unno *et al.* 1979), they may be observed at the stellar surface; this is a mechanism we propose here. We will now derive the basic equations for Kelvin–Helmholtz instabilities.

2.1 BASIC EQUATIONS

Here we are interested in the characteristics of Kelvin–Helmholtz instabilities and in their back reaction on the differential rotation (mean flow). We therefore assume, for simplicity, adiabatic motion throughout the region considered here, though this assumption is not always valid. We neglect the effect of molecular viscosity, because it is so small that it takes more than the lifetime of a star to transport the angular momentum and to smear out the stellar differential rotation. The basic equations are given by

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p - \rho\nabla\Phi, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

$$c^2\left[\frac{\partial \rho}{\partial t} + \mathbf{u}\cdot\nabla\rho\right] = \frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p \tag{3}$$

and

$$\nabla^2\Phi = 4\pi G\rho, \tag{4}$$

where $c^2 = \gamma p/\rho$, and the other symbols have the usual meanings. We assume the angular velocity Ω is axially symmetric, and that all perturbed quantities behave as $\exp\{i(\sigma t + m\phi)\}$, where ϕ is the azimuthal coordinate.

The governing equations of mean flow (i.e. differential rotation) can be obtained by averaging equations (1)–(4) along the azimuthal direction, and are

$$\rho \frac{\partial(\varpi\Omega)}{\partial t} + \frac{1}{\varpi}\nabla_1\cdot(\rho\varpi\overline{\mathbf{V}v_\phi}) = 0, \tag{5}$$

$$\rho(\Omega\times\Omega\times\mathbf{r}) = -\nabla_1 p - \rho\nabla_1\Phi, \tag{6}$$

$$\frac{\partial \rho}{\partial t} = 0, \tag{7}$$

$$c^2\frac{\partial \rho}{\partial t} = \frac{\partial p}{\partial t} \tag{8}$$

and

$$\nabla_1^2\Phi = 4\pi G\rho, \tag{9}$$

where ϖ, $\mathbf{V}$, v_ϕ, and ∇_1 are cylindrical polar radius, poloidal component, azimuthal component of perturbed velocity, and poloidal component of the differential operator ∇, respectively, and other physical quantities represent those of the mean flow, and a bar above quantities means the average along the azimuthal direction. Among non-linear terms, turbulent mass transport or conduction $\rho'\mathbf{v}$ is very small, and has been dropped from the basic equations. The turbulent Reynolds stress $\overline{\nabla\cdot(\rho\mathbf{vv})}$ is retained in equation (5), but can be neglected in equation (6) governing hydrostatic equilibrium, since its effect on the mean structure may be considered to be negligibly small.

By subtracting equations (5)–(8) for the mean flow from the basic equations (1)–(3), we can get the equations for the perturbed motion. After some manipulations, we get the following equations

$$\rho\left[\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\phi}\right)v_i\right]\mathbf{e}_i + 2\Omega\times\mathbf{v} + (\mathbf{v}\cdot\nabla\Omega)\varpi\mathbf{e}_\phi = -\nabla p' - \rho'\nabla\Phi + \rho\mathbf{f}', \tag{10}$$

$$\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right]\rho'+\ \nabla\cdot(\rho\mathbf{v})=0 \tag{11}$$

and

$$c^2\left[\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)\rho'+\mathbf{v}\cdot\nabla\rho\right]=\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right)p'+\mathbf{v}\cdot\nabla p, \tag{12}$$

where quantities denoted by a prime are the Eulerian perturbations, and $\mathbf{v}$, $\mathbf{e}_\phi$, and $\mathbf{f}'$ are perturbed velocity, unit vector of azimuthal direction, and turbulent Reynolds stress (non-linear term), respectively. It should be noted that the centrifugal potential of rotation appearing in equation (6) is already involved in the expression Φ in equation (10). We have also adopted the Cowling approximation ($\Phi'=0$) for simplicity, which is good approximation for almost all cases.

It is worthwhile to be reminded that equation (5) is the general equation describing the angular momentum transport for any periodic motion along the azimuthal direction. The set of equations (10)–(12) describe the motion due to Kelvin–Helmholtz instabilities, and its expression has already been given by Papaloizou & Pringle (1978) in terms of the orthogonal coordinates (ψ, χ, ϕ), where ψ is the distance of equipotential surface from the centre along the rotation axis, and ϕ is the azimuthal coordinate.

Next we derive the energy equation to see the energy balance of this instability and the back reaction of this instability on the differential rotation, whose actual effect will be later discussed. From equation (5) we get

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho\varpi^2\Omega^2\right]+\Omega\nabla_1\cdot(\rho\varpi\overline{\mathbf{v}v_\phi})=0, \tag{13}$$

and by further integrating this equation over the whole volume with the condition $\mathbf{v}=0$ at the boundary surface, we obtain

$$\frac{\partial E_1}{\partial t}+M=0, \tag{14}$$

where

$$E_1=\int \tfrac{1}{2}\rho\varpi^2\Omega^2\,dV \tag{15}$$

and

$$M=-\int\rho\varpi\overline{\mathbf{v}v_\phi}\cdot\nabla_1\Omega\,dV \tag{16}$$

are the total rotation energy and the total energy supply from rotation to the motion induced by Kelvin–Helmholtz instability through Reynolds stress, respectively. Following a similar derivation to that of Unno *et al.* (1979), who gave energy equation for non-radial oscillations of non-rotating star, we can give the energy equation for a rotating star from equations (10)–(12), that is

$$\left[\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\phi}\right](\rho e_{\mathrm{K}})+\ \nabla\cdot\mathbf{F}=\mathbf{v}\cdot[\rho\mathbf{f}'-\rho\varpi v_\phi\nabla_1\Omega], \tag{17}$$

where

$$e_{\mathrm{K}}=\frac{1}{2}\left[\mathbf{v}^2+\frac{(\nabla\Phi)^2}{A}\left(\frac{p'}{\gamma p}-\frac{\rho'}{\rho}\right)^2+\left(\frac{p'}{\rho c}\right)^2\right], \tag{18}$$

with the Brunt–Väisälä frequency

$$A = \nabla\Phi \cdot \nabla \ \ln\,(p^{1/\gamma}/\rho)$$

and (19)

$$\mathbf{F} = p'\mathbf{v}.$$

After integration of this equation over the whole volume, we obtain

$$\frac{\partial E_2}{\partial t} - M + N = 0, \tag{20}$$

where

$$E_2 = \int \rho e_{\mathrm{K}}\, dV \tag{21}$$

and

$$N = -\int \rho \mathbf{v} \cdot \mathbf{f}'\, dV \tag{22}$$

are the total wave energy and the dissipative energy by turbulent viscosity, respectively. It is interesting that equations (14) and (20) lead to the following physical progression of the Kelvin–Helmholtz instability through the energy flow. If this instability takes place under proper differential rotation, the wave energy of the inertia wave induced by this instability grows through energy supply accompanying angular momentum transport from the mean flow (differential rotation) according to equation (20). This instability continues to grow until the non-linear effect (turbulent viscosity) becomes significant and balances Reynolds stress from the mean flow, and after that the steady state is realized. On the other hand, mean flow, that is differential rotation, is significantly modified according to equation (14). This modification time-scale is usually long compared with the growth time of this instability, and thus we can estimate how long this instability is sustained. This problem is very important in the realization of this instability in stars, which will be discussed later.

2.2 KELVIN–HELMHOLTZ INSTABILITY IN STARS

For the sake of theoretical clarification, we assume here the applicability of the Bousinesq approximation, in which density variation is only taken into account in the buoyancy term, and the pressure perturbation P' in equation (12) is neglected. This approximation is valid for the inertia wave considered in the narrow region, where the differential rotation is usually confined, as noted before. Keeping in mind that all perturbed quantities behave as $\exp\{i(\sigma t + m\phi)\}$, the equation for the Kelvin–Helmholtz instability from equations (10)–(12) is

$$\nabla_1(\varpi^2\ \nabla_1 \cdot \mathbf{V}) + \frac{m[\ \nabla_1 h\ \nabla_1 \cdot \mathbf{V} - \nabla_1(\mathbf{V} \cdot \nabla_1 h)]}{(\sigma + m\Omega)} - m^2 \mathscr{A}\mathbf{V} = 0, \tag{23}$$

where

$$\mathscr{A} = \begin{pmatrix} 1 - \dfrac{A}{(\sigma + m\Omega)^2} & 0 \\ & \\ 0 & 1 \end{pmatrix}, \tag{24}$$

$$h = \varpi^2 \Omega,$$

and also turbulent viscosity has been neglected in order to investigate linear stability.

It is difficult to discuss this problem using equations (23) in a general way, since these equations are second-order partial differential equations of motion. No one has so far attacked this problem directly, except by using the general properties, like the energy and vorticity integral. Papaloizou & Pringle (1978) reduced these equations to the ordinary differential equation for the 'r-mode' with the assumption of $v_\psi = p' = 0$. However, this mode is a toroidal mode, and thus scarcely propagates along the ψ-direction. Therefore, we cannot apply the equation of motion for the 'r-mode' to discuss the characteristics of the Kelvin–Helmholtz instability propagating along the ψ-direction. For this purpose, we should use equation (23) directly, which is very difficult for the reason mentioned above. We reduce equation (23) to an ordinary differential equation valid near the equatorial region (i.e. $\chi \sim 90°$). The resultant equation is quite similar to that for the cylindrical configuration, and thus the characteristics obtained may be, as shown later, applied to those for the cylindrical one (see Howard & Gupta 1962). This approximation is valid only for the higher tesseral harmonic waves and/or for the waves trapped in the equatorial region, which is realized for g-modes in a rapidly rotating star (see Papaloizou & Pringle 1978). However, the results obtained from our following discussion can be considered not to be completely inapplicable to the general discussion, since the cylindrical configuration is sometimes a good approximation in the general one (ellipsoidal body). After some manipulation the corresponding equation becomes

$$D^2(r^2 v_r) - l^2 \left[1 - \frac{A}{(\sigma + m\Omega)^2} + \frac{mr^2 D(Dh/r^2)}{l^2(\sigma + m\Omega)} - \frac{k^2 D(h^2)}{l^2(\sigma + m\Omega)^2 r^3}\right] v_r = 0, \tag{25}$$

where $D = d/dr$, and we replace the coordinates (ψ, χ, ϕ) with the spherical ones (r, θ, ϕ), and k is a wavenumber for the θ-direction, and $l^2 = m^2 + k^2$. We also assume Ω depends only upon r.

Let us derive the characteristics of Kelvin–Helmholtz instability. We consider that the motion is unstable if equation (25) possesses a non-trivial solution with boundary condition $v_r(r_1) = v_r(r_2) = 0$ and $\operatorname{Im} \sigma < 0$. Supposing we have an unstable case, set $v_r = H(\sigma + m\Omega)^{1/2}$, some definite branch being selected. In terms of H, equation (25) becomes

$$\begin{aligned} D[(\sigma + m\Omega)D(r^2 H)] - l^2(\sigma + m\Omega) \Bigg[1 &+ \frac{1}{l^2(\sigma + m\Omega)} \left\{2mr^2 D\left(\frac{\Omega}{r}\right) + \frac{1}{2} mr^2 D^2\Omega\right\} \\ &+ \frac{1}{l^2(\sigma + m\Omega)^2}\left\{\frac{1}{4} m^2 r^2 (D\Omega)^2 - \frac{k^2 D(r^4\Omega^2)}{r^3} - l^2 A\right\}\Bigg] H = 0. \end{aligned} \tag{26}$$

Multiplying this equation by $r^2 H^*$ and integrating over range (r_1, r_2), we get

$$\begin{aligned} &\int_{r_1}^{r_2} (\sigma + m\Omega)\,[|D(r^2 H)|^2 + l^2 r^2 |H|^2]\, dr + \int_{r_1}^{r_2} \left[2mr^2 D\left(\frac{\Omega}{r}\right) + \frac{1}{2} m^2 r^2 D^2\Omega\right] r^2 |H|^2\, dr \\ &+ \int_{r_1}^{r_2} \frac{1}{(\sigma + m\Omega)}\left[\frac{1}{4} m^2 r^2 (D\Omega)^2 - \frac{k^2 D(r^4\Omega^2)}{r^3} - l^2 A\right] r^2 |H|^2\, dr = 0. \end{aligned} \tag{27}$$

The imaginary part of this equation gives

$$\int_{r_1}^{r_2} [|D(r^2 H)|^2 + l^2 r^2 |H|^2]\, dr = -\int_{r_1}^{r_2} \left[l^2 A + \frac{k^2 D(r^4\Omega^2)}{r^3} - \frac{1}{4} m^2 r^2 (D\Omega)^2\right] \times \frac{r^2 |H|^2}{|\sigma + m\Omega|^2}\, dr. \tag{28}$$

Equation (28) gives as a sufficient condition for stability

$$J \geqslant 1/4 \qquad \text{everywhere,} \tag{29}$$

where

$$J \equiv \frac{l^2 A + k^2 D(r^4 \Omega^2)/r^3}{m^2 r^2 (D\Omega)^2}. \tag{30}$$

Therefore, the effect of stratification through A usually stabilizes the motion. Using the rotation scale height $H_\Omega = (dr/d \ln \Omega)$, we now discuss the necessary condition for instability which violates condition (29). This condition is obviously given by

$$J < 1/4 \qquad \text{somewhere,} \tag{31}$$

where

$$J = \left(\frac{l^2 A}{m^2 \Omega^2} + 4\frac{k^2}{m^2}\right)\left(\frac{H_\Omega}{r}\right)^2 + 2\frac{k^2}{m^2}\left(\frac{H_\Omega}{r}\right). \tag{32}$$

From equation (31), we obtain

$$\alpha < \frac{H_\Omega}{r} < \beta, \tag{33}$$

where

$$0 < \beta < -\alpha.$$

As shown later in equation (49), $\sigma \sim |m|\Omega$, and in B-type stars discussed here the (A/σ^2) values for the evolved stage and for the near ZAMS stage are usually 100 and 1, respectively (see Unno *et al.* 1979). So if we assume $l = 2$, $m = 1$, the strengths of differential rotation necessary for instability for each case are

$$\frac{|H_\Omega|}{r} \lesssim \frac{1}{20} \quad \text{for the evolved stage}$$

and (34)

$$\frac{|H_\Omega|}{r} \lesssim \frac{1}{3} \quad \text{for the near ZAMS stage.}$$

This implies that instability usually requires the rotation scale height H_Ω to be smaller compared with the radius at the relevant region, and that stronger differential rotation is demanded for instability with an increase in the effect of stratification.

Here we should give one comment on the k^2-term in equation (30) to avoid confusion. This term is the so called 'Rayleigh criterion' for stability, when the condition $D(r^2\Omega) \geqslant 0$ is satisfied. Some people have confused this condition with a necessary and sufficient condition for stability, which is, as pointed out with physical argument by Kippenhahn & Thomas (1981), not true. In fact even if this Rayleigh condition $D(r^2\Omega) \geqslant 0$ contributes to stabilizing, instability can be, as indicated by equation (33), conceivable as far as the differential rotation is strong enough. In equation (30), the m^2-term in the denominator is thus so large that the J-value may become smaller than 1/4. In any case, to understand Kelvin–Helmholtz instabilities more, it is necessary to discuss them from the viewpoint of interactions between rotation and waves than to do so with each term in equation (30). This discussion will be given elsewhere.

Equation (31) can bound on the ratio m/l possible for unstable waves, that is

$$S < \left(\frac{m}{l}\right)^2 \leqslant 1, \tag{35}$$

where

$$S = \min \left[\frac{A + D(r^4\Omega^2)/r^3}{\tfrac{1}{4} r^2 (D\Omega)^2 + D(r^4\Omega^2)/r^3} \right]. \tag{36}$$

Equation (28) may also give a limitation on eigenvalues $\sigma = \sigma_r + i\sigma_i$ possible for any instability and it gives

$$\int_{r_1}^{r_2} \left[|\sigma + m\Omega|^2 - \frac{m^2 r^2 (D\Omega)^2}{l^2} \left(\frac{1}{4} - J \right) \right] \frac{l^2 r^2 |H|^2}{|\sigma + m\Omega|^2} \, dr = - \int_{r_1}^{r_2} |D(r^2 H)|^2 \, dr \leqslant 0. \tag{37}$$

Therefore, we find

$$\min \left[|\sigma + m\Omega|^2 \right] \leqslant \max \left[\frac{m^2 r^2 (D\Omega)^2}{l^2} \left(\frac{1}{4} - J \right) \right] \equiv R^2,$$

i.e.

$$(\sigma_r + m\Omega^+)^2 + \sigma_i^2 \leqslant R^2, \tag{38}$$

where Ω^+ is a proper value of Ω at a certain point, and chosen for each eigenvalue in such a way that $|\sigma + m\Omega^+| = \min [|\sigma + m\Omega|]$. Equation (38) is similar to the 'semicircle theorem' for plane parallel flow derived by Howard (1961). Equation (38) gives

$$-m\Omega^+ - R \leqslant \sigma_r \leqslant -m\Omega^+ + R. \tag{39}$$

This leads to the important conclusion that since $\sigma_r \geqslant 0$, unstable waves are prograde modes for the majority of cases, but in some preferable conditions retrograde unstable modes are also possible. Equation (38) also indicates that the growth rate of instability is of the same order of magnitude as the rotational one.

While the several properties above have been obtained from the simplified equation valid for the equatorial region, they are also valid for a cylindrical rotating body, as shown by Howard & Gupta (1962). As supposed from equation (23), we can also expect a bound on possible l- values for unstable waves.

As mentioned before, the strong differential rotation may usually exist in the narrow region between core and envelope, which is also a preferable situaton for instability. This sometimes allows us to assume that $(r_2 - r_1)$, while small compared with r_1, is large compared with the scale of variation of Ω. In this case, we can take over as examples any solutions to the stability problem for unbounded parallel stratified flow with gravity, which has been discussed in detail by Drazin & Howard (1966). In fact, we get

$$\frac{d^2 v}{dy^2} - l^2 \left[1 - \frac{A_0}{(\omega + m\Omega_0)^2} + \frac{m(d^2\Omega_0/dy^2)}{l^2(\omega + m\Omega_0)} + \frac{k^2(d\Omega_0^2/dy)}{l^2(\omega + m\Omega_0)^2} \right] v = 0, \tag{40}$$

where we have adopted non-dimensional values according to

$$\begin{aligned} &\omega^2 = \sigma^2 (r_1/\nabla\Phi), \quad y = r/r_1, \quad v = v_r/(r_1 \nabla\Phi)^{1/2}, \\ &\Omega_0 = \Omega/(\nabla\Phi/r_1)^{1/2}, \quad \text{and} \quad A_0 = A/(\nabla\Phi/r_1). \end{aligned} \tag{41}$$

In this approximation, general characteristics obtained previously, of course, result. However, relation (38) is given in a more strict way. We set $v = G(\omega + m\Omega_0)$. In terms of G, equation (40) becomes

$$\frac{d}{dy} \left[(\omega + m\Omega_0) \frac{dG}{dy} \right] - l^2 (\omega + m\Omega_0)^2 \left[1 - \frac{l^2 A + k^2 (d\Omega_0^2/dy)}{l^2 (\omega + m\Omega_0)^2} \right] G = 0. \tag{42}$$

We multiply equation (42) by G^* and integrate over $(-\infty, \infty)$, which gives

$$\int_{-\infty}^{\infty} (\omega + m\Omega_0)^2 \left[\left| \frac{dG}{dy} \right|^2 + l^2 |G|^2 \right] dy = \int_{-\infty}^{\infty} \left[l^2 A + k^2 \frac{d\Omega_0^2}{dy} \right] |G|^2 \, dy. \tag{43}$$

Supposing $[l^2 A + k^2(d\Omega_0^2/dy)] \geqslant 0$, and setting $\omega = \omega_r + i\omega_i$ gives

$$\left[\omega_r + \frac{m}{2}(a+b) \right]^2 + \omega_i^2 \leqslant \frac{m^2}{4}(a-b)^2, \tag{44}$$

where $a = \max(\Omega_0)$ and $b = \min(\Omega_0)$. Equation (44) is called 'semicircle theorem' which was discovered by Howard (1961). This relation allows only prograde modes for unstable waves, in which the pattern of an unstable wave travels with rotation rate at a certain point. Therefore, generally speaking, retrograde modes can be scarcely excited in a real star.

We will give a solution of equation (40) as an example for the following rotation profile

$$\Omega_0 = \begin{cases} \Omega_1 & (y < 0), \\ \Omega_2 & (y > 0). \end{cases} \qquad (\Omega_1 > \Omega_2 > 0) \tag{45}$$

With boundary condition $v(-\infty) = v(\infty) = 0$, and the continuity of pressure and normal velocity of fluid at $y = 0$, we can get the eigenvalue relation and eigenfunction for the case of small A

$$\omega = -\frac{m(\Omega_1 + \Omega_2)}{2} \pm \frac{\sqrt{Q - 2l^2 A}}{2l} i \tag{46}$$

and

$$v = \begin{cases} (\omega + m\Omega_1)\exp(\nu y) & (y < 0) \\ (\omega + m\Omega_2)\exp(-\mu y) & (y > 0), \end{cases} \tag{47}$$

where

$$Q = l^2 m^2 (\Omega_1 - \Omega_2)^2 + 2lk^2(\Omega_1^2 - \Omega_2^2) > 0,$$

$$\left.\begin{aligned} \nu &= l\left[1 - \frac{A}{2(\omega + m\Omega_1)^2}\right], \\ \mu &= l\left[1 - \frac{A}{2(\omega + m\Omega_2)^2}\right] \end{aligned}\right. \tag{48}$$

The above results indicate that the eigenvalues satisfy the semicircle theorem and that gravity has a stabilizing effect. Also, the amplitude of perturbation is shown to concentrate on the differential rotation region and to die out rapidly in the outside. In this example, the travelling speed of the wave pattern does not depend on the wavenumbers. However, in general, it depends on wavenumbers, as shown by Drazin & Howard (1966).

Finally we will summarize several characteristics of Kelvin–Helmholtz instability in stars as follows.

(1) In almost all cases, the wave pattern of the unstable mode travels around the rotation axis with the rotation speed at a certain point within the differentially rotating region, whose position depends on the wavenumbers, that is:

$$\sigma_r \sim -m\Omega(r_0). \tag{49}$$

Thus prograde modes can usually be excited. However, there still remains the possibility of excitation of retrograde modes in some circumstances.

(2) For a given total wavenumber l, the modes with larger azimuthal wavenumber $|m|$ can be preferably excited.

(3) In general, gravity has a stabilizing effect on this instability.

(4) The growth rate of the unstable wave is equal in order of magnitude to rotation rate.

(5) The amplitude of the eigenfunction for the unstable wave concentrates on the differentially rotating region and dies out quickly in the outside.

2.3 BACK REACTION OF KELVIN–HELMHOLTZ INSTABILITIES ON DIFFERENTIAL ROTATION

On the basis of energy equations obtained in Section 2.1, we will estimate the time-scale of modification of the differential radiation. From equation (5), the time-scale τ of angular momentum transport becomes

$$\tau = \left(\frac{r_1 \Omega_1}{u_1}\right)^2 \frac{1}{\Omega_1 \epsilon}, \tag{50}$$

where ϵ stands for measure of orthogonality of Reynolds stress $\overline{\mathbf{V} v_\phi}$ defined by

$$|\overline{\mathbf{V} v_\phi}| = \epsilon u^2. \tag{51}$$

The value of ϵ indicates the degree of the progressive nature of the unstable waves along the r- and/or θ-direction (i.e. energy transport efficiency). Using equation (25), we obtain

$$\epsilon \sim \frac{\sigma_i}{|\sigma|} \simeq \frac{\Delta\Omega}{\Omega_1}, \tag{52}$$

where $\Delta\Omega$ is a representative difference of rotation rate throughout the stellar layers. Thus the time-scale τ becomes

$$\tau = \left(\frac{r_1 \Omega_1}{u_1}\right)^2 \frac{1}{\Delta\Omega}. \tag{53}$$

3 Possible excitation of non-radial oscillations by Kelvin–Helmholtz instabilities

In this section we will restrict the problem to the excitation of non-radial oscillations in B-type variables (53 Per variables and β Cep stars), while Smith (1980b, c) claimed that all β Cep stars have a radial mode as an active mode.

Obviously from the results obtained in Section 2 we cannot observe the unstable inertia waves at the surface. Therefore, it is important to examine the possibility of resonant excitation of non-radial oscillations, which are, even if they are g-modes, observed at the surface. As mentioned before, a rapidly spinning core may be developed with the stellar evolution, whether or not a star has uniform rotation at the ZAMS stage. In the early-type stars (massive star), the differential rotation can be, as Kippenhahn *et al.* (1970) has shown, expected in the narrow region between convective core and envelope, which corresponds to the μ-gradient zone. Therefore, in these stars Kelvin–Helmholtz instabilities are possible in the μ-gradient zone if the condition (34) obtained before is satisfied. In such a case the frequencies of unstable waves are, as shown in the previous section, roughly equal to rotational ones. On the other hand, in the early-type stars, the rotation rate usually ranges from 10^{-5} to $10^{-4}\,\mathrm{s}^{-1}$, which corresponds to that of f- and g-modes (*cf.* Unno *et al.* 1979), and also in the μ-gradient zone these modes can propagate and thus have significant amplitudes. Accordingly, if the frequency of the unstable wave driven by differential rotation happens to coincide with one of the *f*- and *g*-modes, non-radial oscillations may be excited, which is observable at the stellar surface since even *g*-modes have a significant amplitude at the surface (see Unno *et al.* 1979). The kinetic energy in *f*- and/or *g*-modes is usually much greater than that in unstable inertia waves excited in the narrow region. Therefore, in order to realize the resonant excitation of non-radial oscillations by unstable inertia waves, it is necessary that the damping time of non-radial oscillations is longer than the growth time of unstable inertia waves, at least in the case of small amplitude oscillations, and the higher degeneracy in the frequency is also required to transport enough energy to the motion of non-radial oscillations. In fact, the damping time of *f*- and *g*-modes in these stars is about 10^2 yr; on the other hand, the growth time of unstable inertia waves is about the revolution time of a star. We can thus expect this resonant excitation of non-radial oscillations in these stars if higher phase coherency is fulfilled. It should be noted that though the resultant amplitude of the non-radial oscillation may be determined by a non-linear effect, its amplitude in the coupling region cannot be larger than that of the unstable inertia wave. In this case the results obtained in the previous section about the unstable inertia waves can explain the following observational facts in 53 Per variables and β Cep stars.

(i) Prograde modes have been identified in the majority of cases, though for the case of 22 Ori (one of 53 Per variables) a retrograde mode has been observed.

(ii) The observed non-radial modes prefer to sectorial modes ($|m| = l$) or modes with larger $|m|$.

(iii) There often exist multiple periods, some of which show commensurable relations between them.

If Kelvin–Helmholtz instabilities are responsible for the variability of these stars, we can guess the angular velocity of the stellar core Ω_c for an active mode of these stars, since the pattern phase velocity of non-radial oscillations σ/m is roughly equal to Ω_c. We have given these values in Table 1 with other quantities for 53 Per variables and β Cep stars, where we have adopted mean quantities for β Cep stars from Table 7.1 of Unno *et al.* (1979), and data for 53 Per variables are based on the recent works by Smith (1978, 1979, 1980a, 1981). From this table, we notice that the values of Ω_c for 53 Per variables are roughly equal to mean angular velocity at the surface for early B-type main sequence stars, and on the other hand that the surface angular velocity Ω_e's are less by about a factor of 10 than Ω_c's, except for 53 Per. Further for β Cep stars, Ω_c's are larger roughly by a factor of 10 than Ω_e's, and Ω_c's may be equal to the value expected from angular momentum evolution (Kippenhahn's α evolution) if the core has a mean angular velocity for an early B-type main sequence star at the ZAMS stage. In any case, we should postulate that in these stars (53 Per variables and β Cep stars) the core rotates at the rate expected from the mean

Table 1. Data for 53 Per variables and β Cep stars.

	Spectral type	$M/M_\odot$	$R/R_\odot$	$V_{e,rot}$ (km s^{-1})	Period (hr)	m	Ω_e (s^{-1})	Ω_c (s^{-1})	v_{osc} (km s^{-1})
53 Per	B4.5 V	7	4.0	20	41.7	−3	7.1×10^{-6}	1.4×10^{-5}	12
22 Ori	B3 IV	10	4.0	>11	22.9	−3	$>3.9 \times 10^{-6}$	2.5×10^{-5}	4–8
υ Ori	O9 V	20	7.2	>20	12.0	−2	$>4.0 \times 10^{-6}$	7.3×10^{-5}	4–5
10 Lac	O9 V	22	7.5	25	4.9	−3	4.8×10^{-6}	1.2×10^{-4}	4–10
ι Her	B3 V	8	4.0	8	9.9	−2	2.9×10^{-6}	8.8×10^{-5}	3–6
β Cep stars	B0.5 – B2	10	8.0	50	4.5	−2	1.0×10^{-5}	2.0×10^{-4}	10

rotation velocity at the surface of early B-type stars, while the envelope does at a slower rate (by about a factor of 10), and is a slow rotator. However, differential rotation for a massive star at the ZAMS, may not be inconceivable as Bodenheimer & Ostriker (1970) discussed.

Next we shall investigate the durability of this instability. To do so, it is necessary to estimate the time-scale of the angular momentum transport by a non-axisymmetric wave using equation (53). We can now modify the expression $(r_1 \Omega_1/u_1)$ into the following convenient form

$$\frac{r_1\Omega_1}{u_1} = \gamma\left(\frac{r_0\Omega_0}{v_{\mathrm{osc},0}}\right)\left(\frac{\Omega_1}{\Omega_0}\right)\left(\frac{v_{\mathrm{osc}}}{u_1}\right), \tag{54}$$

where the quantities with suffix (0) correspond to those at the surface, and v_{osc} and γ are the amplitude of a non-radial mode and the relative displacement $(\delta\xi_r/r)$ of a non-radial mode at the surface with a corresponding value at the core as unity, respectively. Since $(r_0\Omega_0/v_{\mathrm{osc},0}) \sim 5$, $(\Omega_1/\Omega_0) \sim 10$, $\Delta\Omega \sim 5\times10^{-5}$ from Table 1 and $\gamma \sim 5$ for $10M_\odot$ star model from Unno *et al.* (1979), we get from equation (53)

$$\tau \sim 40\left(\frac{v_{\mathrm{osc}}}{u_1}\right)^2 \text{ yr.} \tag{55}$$

The value of (v_{osc}/u_1) is not certain, since no non-linear coupling calculation exists, but it is reasonable to assume that $u_1 \sim v_{\mathrm{osc}}$, as mentioned before. Therefore, the time-scale of angular momentum transport becomes 40 yr at the differential rotation region of B-type stars. This means that the differential rotation profile at that region can be significantly affected in a short time, and thus that the instability cannot continue for the lifetime of the main sequence stage. However, we should notice that the back reaction may contribute to stopping the instability, but that this cannot lead to the complete uniform rotation in a short time, since it was pointed out by Gill (1965) that for plane parallel flow the instability is significantly affected even if the mean flow is modified infinitesimally. We can expect that the non-radial oscillations excited by this mechanism may be maintained not continuously, but intermittently.

How about the angular momentum transport by non-axisymmetric non-radial mode itself (i.e. *f*- and/or *g*-modes)? In this case, from equation (52) we can take $\epsilon \sim \sigma_i/\sigma_r$ $(10^{-6}$–$10^{-8})$ and thus $\tau \sim 4\times10^7$–10^9 (yr). This effect is therefore inefficient during the lifetime in the main sequence stage of a massive star. We conclude that if a star has a core with the same rotation rate as the mean rotation one in B-type main sequence stars at the ZAMS stage, inertia waves can be excited by Kelvin–Helmholtz instabilities in the narrow region between core and envelope and a resonance between unstable inertia waves and non-radial modes of a whole star is possible.

Finally, a Kelvin–Helmholtz instability is very sensitive to the second derivative of Ω with r (see Gill 1965). We therefore expect that unstable behaviour in the period and amplitude and mode switching on a short time-scale observed in 53 Per variables, could arise from the above fact.

4 Discussion

We have proposed a mechanism which could be responsible for non-radial oscillations of 53 Per variables and β Cep stars, in which the non-radial oscillations resonate with the inertia waves driven by a Kelvin–Helmholtz instability in the μ-gradient zone. If these stars have a core with the same rotation rate as the mean rotation one in B-type main sequence stars at the ZAMS stage (while the envelope rotates at the slower rate by about a factor of 10 than that of the core), inertia waves can be excited by Kelvin–Helmholtz instabilities and these waves may resonate in the μ-gradient zone with f- and/or g-modes, even higher g-modes. It is necessary for this instability that the differential rotation scale height H_Ω is smaller than the radius at the relevant region. It is shown that the differential rotation can be affected in a short time by the back reaction. The characteristics of unstable inertia waves may explain the main aspects of the observations in these stars.

So far we have mentioned the favourable side of this mechanism, but there exist several difficulties in it. The first one concerns the excitation of the radial mode in β Cep stars, which was claimed by Smith (1980b, c) to be an active mode in these stars. Smith (1980b) also proposed a possible resonant excitation of radial mode by axisymmetric non-radial mode ($m = 0$). However, the theory of Kelvin–Helmholtz instabilities in the differential rotation predicts zero frequency ($\sigma_r = 0$) for axisymmetric modes, which is easily derived from equation (28). It we are to get the non-zero frequency for this mode, the poloidal current should be postulated in these stars. We suppose the meridional circulation as a reasonable candidate. But the derived frequencies are quite long compared with observational ones, since the frequency of axisymmetric modes in this case is equal in order of magnitude to the circulation time-scale (Helmholtz–Kelvin time-scale: 10^6–10^7 yr). Therefore it is very difficult to excite a radial mode in these stars by this mechanism.

The second one relates to the existence of several closer periodicities, which are usually interpreted by rotation splitting. They cannot be excited directly by this mechanism, and we may investigate mode interaction as one of many possibilities.

The third one concerns the angular momentum transport from the core to the envelope. We have pointed out a possibility that a Kelvin–Helmholtz instability can take place intermittently with the evolution of a star and mentioned that this instability does not always mean the effective transport of angular momentum from the core to the envelope. However, there remains a possibility of the effective transport of the angular momentum by this instability. We cannot give an exact answer at present, since this is a non-linear problem related to the stellar evolution. We would also like to point out that there remains a problem about the origin of a slowly rotating envelope in these stars, which is beyond the scope of this paper.

We should note two main differences between the mechanism proposed by Osaki (1974) and ours. First, the former requires a spinning convective core, while the latter a differential rotation. Second, the former prefers f-modes and at most lower g-modes, but the latter may select any f-and/or g-modes as long as the degeneracy is fulfilled. Therefore, Kelvin–Helmholtz instabilities can be expected for a larger number of cases. In fact Papaloizou & Pringle (1978) applied this mechanism to dwarf novae. Here we apply it to the solar 160-min oscillation (see Brookes, Isaak & van der Raay 1976; Severny, Kotov & Tsap 1976) as a possible example. If we assume this oscillaton to be one of the g-modes and to be excited by this mechanism, the solar core rotation rate becomes $(6.5 \times 10^{-4}/m)\,\mathrm{s}^{-1}$, where m is the azimuthal wavenumber. This value is larger than that estimated by Dicke (1970) ($\sim 6 \times 10^{-5}\,\mathrm{s}^{-1}$) from his observations of solar oblateness, though the value m is not specified. However, the above discussion is interesting as one of several lines of attack in this problem.

Despite several difficulties involved in this mechanism, it still seems promising for 53 Per variables and β Cep stars, and the applicability of this mechanism to many problems is attractive and will stimulate further investigation.

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