QUARTERLY OF APPLIED MATHEMATICS
VOLUME LXI, NUMBER 4
DECEMBER 2003, PAGES 731-757

KERNEL SECTIONS FOR DAMPED NON-AUTONOMOUS WAVE EQUATIONS WITH LINEAR MEMORY AND CRITICAL EXPONENT

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Abstract. We prove the existence of kernel sections for the process generated by a non-autonomous wave equation with linear memory when there is nonlinear damping and the nonlinearity has a critically growing exponent; we also obtain a more precise estimate of upper bound of the Hausdorff dimension of the kernel sections. And we point out that in the case of autonomous systems with linear damping, the obtained upper bound of the Hausdorff dimension decreases as the damping grows for suitable large damping.

1. Introduction and Main Results.

In this paper, we consider the existence of the compact kernel sections and estimate the Hausdorff dimension of sections for non-autonomous wave equations with linear memory when there is nonlinear damping and the nonlinearity satisfies the critical growth condition.

Let Ω be an open bounded set of R^3 with a smooth boundary $\partial\Omega$. We consider the following non-autonomous wave equation with linear memory term:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}+h(\dfrac{\partial u}{\partial t}) - k(0)\Delta u - \displaystyle\int_0^\infty k'(s) \triangle u(t-s)ds+f(u,t)=g(x,t), x\in\Omega, t>\tau, \tau\in R, \\ u(x,t)|_{x\in\partial\Omega} = 0, \quad t \in R, \\ u(x,t) = u_0(x,t), \quad x \in \Omega, \quad t \leq \tau, \end{cases} \tag{1}$$

with $k(0)$, $k(\infty) > 0$ and $k'(s) \leq 0$ for every $s \in R^+$, where $u = u(x,t)$ is a real-valued function on $\Omega \times [\tau, +\infty)$, $\tau \in R$, $u(t-s) = u(x,t-s)$, $h(v) \in C^1(R;R)$, $f(u,t) \in C^1(R \times R; R)$, $g(\cdot,t)$, $g'_t(\cdot,t) = \frac{\partial}{\partial t}g(\cdot,t) \in C_b(R, L^2(\Omega))$, and $C_b(R, L^2(\Omega))$ denotes the set of continuous bounded functions from R into $L^2(\Omega)$.

Received November 27, 2001.
2000 *Mathematics Subject Classification.* Primary 34B40, 35L70.
Key words and phrases. Wave equation, process, kernel section, Hausdorff dimension.
This research is supported by the National Natural Science Foundation of China under Grant 10171072, and the Developing Foundation of Science and Technology of Institutions of Higher Learning of Shanghai under Grant 03AK32.
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Equations of this kind (1) occur in the description of viscoelastic solids with fading memory and dissipation due to the viscous resistance of the surrounding medium, in the presence of a nonlinear external force, of which u represents the displacement vector. Particularly, Eq. (1) can be regarded as a model of a viscoelastic membrane, where u is the vertical displacement. If $k' \equiv 0$, (1) reduces to a damped non-autonomous wave equation (cf. [1]).

Following the idea of Dafermos [3], we introduce a new variable

$$\eta(x,t,s) = u(x,t) - u(x,t-s). \tag{2}$$

For simplicity, we set $\mu(s) = -k'(s)$ and $k(\infty) = 1$. Setting $v(x,t) = u_t(x,t)$, Eq. (1) can then be transformed into the following three-dimensional system:

$$\begin{cases} u_t = v, \\ v_t = \triangle u + \int_0^\infty \mu(s) \triangle \eta(s) ds - h(v) - f(u,t) + g(x,t), \\ \eta_t = v - \eta_s \end{cases} \tag{3}$$

with initial-boundary conditions

$$\begin{cases} u(x,t)|_{x\in\partial\Omega} = 0, \quad t \geq \tau, \\ \eta(x,t,s)|_{x\in\partial\Omega} = 0, \quad s \in R^+, \quad t \geq \tau, \\ \eta(x,t,0) = 0, \quad x \in \Omega, \quad t \geq \tau, \\ u(x,\tau) = u_{0\tau}(x), \quad x \in \Omega, \\ v(x,\tau) = v_{0\tau}(x), \quad x \in \Omega, \\ \eta(x,\tau,s) = \eta_{0\tau}(x,s), \quad (x,s) \in \Omega \times R^+, \end{cases} \tag{4}$$

where we set

$$\begin{cases} u_{0\tau}(x) = u_0(x,\tau), \\ v_{0\tau}(x) = \frac{\partial}{\partial t} u_0(x,t)|_{t=\tau}, \\ \eta_{0\tau}(x,s) = u_0(x,\tau) - u_0(x,\tau - s). \end{cases}$$

We assume the memory kernel μ satisfies:
(F_1): $\mu \in C^1(R^+) \cap L^2(R^+)$, $\mu(s) \geq 0$, $\mu'(s) \leq 0$, $\forall s \in R^+$.
(F_2): $\mu'(s) + \delta\mu(s) \leq 0$, $\forall s \in R^+$ and some $\delta > 0$.

We denote by $H^m(\Omega)$ the Sobolev space consisting of all functions for which, up to mth-order, generalized derivatives are all in $L^2(\Omega)$ and $H_0^1(\Omega) = \{g \in H^1(\Omega) : g(x)|_{x\in\partial\Omega} = 0\}$. Let $A = -\triangle$ with Dirichlet boundary conditions, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. We can define the powers A^r of A for $r \in R$. The space $V_{2r} = D(A^r)$ is a Hilbert space with the inner product and norm:

$$(u,v)_{2r} = (A^r u, A^r v), \quad ||u||_{2r}^2 = (A^r u, A^r u).$$

In particular, $V_{-1} = H^{-1}(\Omega)$, $V_0 = L^2(\Omega)$, $V_1 = H_0^1(\Omega)$ and

$$(A^{1/2}u, A^{1/2}v) = (\nabla u, \nabla v), \quad \forall u, v \in H_0^1(\Omega).$$

The injection $V_{r_1} \hookrightarrow V_{r_2}$ is compact if $r_1 > r_2$.

In view of (F_1), let the "history space" $\Re = L^2_\mu(R^+, H_0^1(\Omega))$ be the Hilbert space of $H_0^1(\Omega)$-valued functions on R^+, endowed with the inner product and norm

$$(\eta, \eta_1)_\Re = \int_0^\infty \mu(s)(\nabla\eta(s), \nabla\eta_1(s))ds, \ ||\eta||^2_\Re = (\eta, \eta)_\Re = \int_0^\infty \mu(s)(\nabla\eta(s), \nabla\eta(s))ds, \quad \forall \eta, \eta_1 \in H.$$

The linear operator $-\partial_s$ on $\Re$ is of domain

$$D(-\partial_s) = \{\eta \in H^1_\mu(R^+, H_0^1) : \eta(0) = 0\} \quad \text{where} \quad H^1_\mu(R^+, H_0^1) = \{\eta : \eta(s), \ \partial_s\eta \in L^2_\mu(R^+, H_0^1)\}$$

which generates a right-translation semigroup (cf. [2]).

Introduce the Hilbert space

$$E = H_0^1(\Omega) \times L^2(\Omega) \times \Re$$

with the inner product:

$$(Z_1, Z_2)_E = (u_1, u_2)_{H_0^1} + (v_1, v_2)_{L^2} + (\eta_1, \eta_2)_\Re, \ \forall Z_i = (u_i, v_i, \eta_i) \in E, \ i = 1, 2.$$

Setting the triplet $Z = (u, v, \eta)^T$, then the system (3)–(4) is equivalent to the following initial value problem in the Hilbert space E :

$$\begin{cases} Z_t = L(Z) + N(Z, t), \quad (x, s) \in \Omega \times R^+, \quad t \geq \tau, \\ Z(\tau) = Z_{0\tau} = (u_{0\tau}(x), v_{0\tau}(x), \eta_{0\tau}(x, s))^T, \quad (x, s) \in \Omega \times R^+, \end{cases} \tag{5}$$

where

$$L(Z) = \begin{pmatrix} v \\ \Delta u + \int_0^\infty \mu(s) \Delta \eta(s)ds \\ v - \eta_s \end{pmatrix}, \quad N(Z, t) = \begin{pmatrix} 0 \\ -h(v) - f(u, t) + g(x, t) \\ 0 \end{pmatrix}, \tag{6}$$

$$D(L) = \left\{ Z \in E \,\middle|\, \begin{array}{c} u + \int_0^\infty \mu(s)\eta(s)ds \in H^2(\Omega) \cap H_0^1(\Omega), \\ v \in H_0^1(\Omega), \ \eta(s) \in H^1_\mu(R^+, H_0^1(\Omega)), \ \eta(0) = 0 \end{array} \right\}.$$

Let $f(u, t) = f_1(u, t) + f_2(u, t)$, $G_i(u, t) = \int_0^u f_i(r, t)dr$, $i = 1, 2$. We make the following assumptions on functions $G_i(u, t)$, $f_i(u, t)$, $i = 1, 2$, $h(v)$:

(F_3):

$$f_1(u, t)u \geq 0, \quad \lim_{|u| \to +\infty} \inf \frac{G_2(u, t)}{u^2} \geq 0, \quad \forall u, t \in R.$$

(F_4): There exist positive constants $c_{0i} > 0,\ i = 1, 2$, such that

$$\lim_{|u|\to+\infty} \inf \frac{uf_i(u,t) - c_{0i}G_i(u,t)}{u^2} \geq 0, \quad \forall u, t \in R.$$

(F_5): There are constants $c_0 > 0$ and sufficient small $\gamma \geq 0$ such that

$$G'_{1,t}(u,t) \leq \gamma G_1(u,t), \quad G'_{2,t}(u,t) \leq \gamma G_2(u,t) + c_0, \quad \forall u, t \in R.$$

(F_6): $f_1(u,t) \in C^2(R \times R, R)$, $f'_{1,u}(0,t) = 0$ and there exists a constant $c_1 > 0$ such that

$$|f''_{1,u}(u,t)| \leq c_1(1 + |u|), \quad |f'_{1,t}(u,t)| \leq c_1(1 + |u|^3), \quad \forall u, t \in R.$$

(F_7): There exists a constant $c_2 > 0$ such that

$$|f'_{2,u}(u,t)| \leq c_2(1 + |u|^p), \quad |f'_{2,t}(u,t)| \leq c_2(1 + |u|^{p+1}), \quad 0 \leq p < 2, \quad \forall u, t \in R.$$

(F_8): There exist two positive constants α, β such that

$$h(0) = 0, \quad 0 < \alpha \leq h'(v) \leq \beta < +\infty, \quad \forall v \in R.$$

(F_9): The partial derivatives of $G'_{i,t}(u,t) = \frac{\partial}{\partial t} G_i(u,t)$, $i = 1,\ 2$ and $g'_t(x,t)$ satisfy

$$G'_{1,t}(u,t) + G'_{2,t}(u,t) - g'_t(x,t)u \leq 0, \forall u, t \in R, x \in \Omega.$$

(F_{10}): For $M > 0$, there exist $c_3 = c_3(M)$ and $\delta_1 > 0$ such that for any $\forall u_1,\ u_2 \in H_0^1(\Omega)$, $||u_1||_1$, $||u_2||_1 \leq M$,

$$||f'_u(u_1,t) - f'_u(u_2,t)||_{L(H_0^1(\Omega), L^2(\Omega))} \leq c_3(M)||u_1 - u_2||_1^{\delta_1}, t \in R.$$

(F_{11}): For $M' > 0$, there exist $c_4 = c_4(M')$ and $\delta_2 > 0$ such that for any $v_1, v_2 \in L^2(\Omega)$, $||v_1||_0$, $||v_2||_0 \leq M'$,

$$||h'(v_1) - h'(v_2)||_{L(L^2(\Omega), L^2(\Omega))} \leq c_4(M')||v_1 - v_2||_0^{\delta_2}.$$

Where $||\cdot||_0$, $||\cdot||_1$ denote the norms of $L^2(\Omega)$ and $H_0^1(\Omega)$, respectively, $||\cdot||_{L(X,Y)}$ denotes the norm of operator of $L(X, Y)$ (the space of linear continuous operators from X into Y), $f'_{i,q}$ is partial derivative with respect to q, and the inner product in $L^2(\Omega)$ is denoted by $(\cdot, \cdot)$.

The exponential decay of the semigroup associated to the system (5) with $N(Z, t) \equiv 0$ has been investigated by Z. Liu and S. Zheng and others; see [4–5]. When the damping is linear ($h(v) = \alpha v$) and the nonlinearity satisfies the noncritical growth conditions ($f_1 = 0$), for the non-autonomous (or autonomous) system (5) with f independent of t, V. Pata and A. Zucchi *et al.* proved the existence of its (uniform) global attractor and obtained an estimate of the Hausdorff dimension of attractor; see [1, 2] and references therein. For the non-autonomous system (5) where $k' \equiv 0$, i.e., (5) reduces to a non-autonomous semilinear wave equation, V. Chepyzhov and M. Vishik in [6] proved the existence of its kernel sections and obtained an upper bound of the Hausdorff dimension of section. We observed from their estimates of [1, 6] that the upper bound of the Hausdorff dimension increases as the damping α grows and tends to infinity as α tends to positive infinity. From the physical intuition of Eq. (1), the Hausdorff dimension of attractor (or section) should be smaller when the damping grows. If $k' \equiv 0$, $h(v) = \alpha v$ and $f(u,t) = f(u)$, $g(x,t) = g(x)$ are independent of t, then the system (5) reduces to an autonomous semilinear wave equation with linear damping for which the existence and estimate of the Hausdorff dimension of the global attractor have been widely studied; see [7-10]. Here it is worth mentioning that the author in [9] made a mistake in the proof of the uniform boundedness of the compact component of the semigroup. Later, Y. Huang *et al.* in [10] gave a correct proof.

In this paper, we generalize the existence and estimate of upper bound of the Hausdorff dimension of the kernel sections (or global attractor) in the previous works to the process generated by system (5) with $k' \neq 0$ when there is a nonlinear damping and the nonlinearity has a critical growth exponent. A more precise upper bound of the Hausdorff dimension for the kernel sections is obtained by carefully estimating the positivity of operator in the corresponding evolution equation of the first order in time. According to our estimate of dimension, in the case of autonomous systems with linear damping, the kernel of process is just the global attractor, and the upper bound of the Hausdorff dimension decreases as the damping grows for suitable large damping. The main results are the following theorems.

THEOREM I. If the functions $\mu(s)$, $f(u,t)$, and $h(v)$ satisfy conditions (F_1)-(F_9), then the (mild) solutions of problem (5) exist globally and define a process

$$U(t,\tau) : (u_{0\tau}, v_{0\tau}, \eta_{0\tau})^T \to (u(t), v(t), \eta(t))^T, \ \ E \to E, \quad t \geq \tau \tag{7}$$

which possesses a non-empty kernel

$$K = \{Z(\cdot) : Z(t),\ t \in R,\ \text{is a solution of (5)},\quad ||Z(t)||_E \leq M_Z,\ \forall t \in R\}$$

consisting of all bounded complete trajectories of the process, and the kernel sections at times $s \in R$:

$$K(s) = \{Z(s) : Z(\cdot) \in K\} \tag{8}$$

are all compact. Moreover, let

$$\varepsilon = \frac{2\alpha}{3 + \kappa\alpha + \beta^2/\lambda_1 + \sqrt{(3 + \kappa\alpha + \beta^2/\lambda_1)^2 - 12\kappa\alpha}}, \tag{9}$$

$$\kappa = \frac{2||\mu||_{L^1(R^+)}}{\delta} > 0, \quad \sigma = \min\{\frac{\varepsilon}{2}, \frac{\delta}{4}\}.$$

If (F_{10})–(F_{11}) hold, then the Hausdorff dimension d_H of $K(\tau)$, $\tau \in R$ satisfies:

$$d_H \leq \min\left\{ m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-4\nu_0} < \frac{2\alpha\sigma}{k} \right\} \leq \left[\left(\frac{c'\lambda_1 k}{2\alpha\sigma} \right)^{\frac{3}{8\nu_0}} \right] + 1, \tag{10}$$

where $\{\lambda_j\}_{j\in N} : 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$, are the eigenvalues of operator $-\Delta$ with the Dirichlet boundary condition on Ω, $0 < \nu_0 < \min\{\frac{5}{4} - \frac{3}{2\iota}, \frac{1}{4}\}$, $\iota \in (\frac{6}{5}, \frac{6}{3+p})$, p is as in (F_7), k is a positive constant independent of τ, $c' > 0$ is a constant only depending on the shape of Ω and ν_0, and $[\ell]$ denotes the largest integer which is less than or equal to ℓ.

THEOREM II. If $h(v) = \alpha v$ is linear, the functions f, g are independent of t, and (F_1)–F_{11} hold, then the process $U(t, 0)$ associated to the autonomous system (5) with $\tau = 0$ is a semigroup

$$U(t,0) = S(t) : (u_0, v_0, \eta_0)^T \to (u(t), v(t), \eta(t))^T, \quad E \to E, \quad t \geq 0.$$

The kernel section $K(0) = \Theta$ is just the global attractor of semigroup $\{S(t), t \geq 0\}$, and for any fixed $\alpha_0 > 0$, if $\alpha \geq \alpha_0$, then there exists a constant $\alpha_1 > 0$ such that for any $\alpha \geq \alpha_1$, the Hausdorff dimension $d_H(\Theta)$ of attractor Θ satisfies:

$$\begin{aligned} d_H(\Theta) &\leq \min\left\{ m \,\middle|\, m \in N, \quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-4\nu_0} < \frac{\alpha\varepsilon_0}{k_0} \right\} \\ &\leq \left[\left(\frac{c'\lambda_1 k_0}{\alpha\varepsilon_0} \right)^{\frac{3}{8\nu_0}} \right] + 1 \\ &\leq \left[\left(c'\lambda_1 k_0 \left(\frac{3}{\alpha^2} + \frac{\kappa}{\alpha} + \frac{1}{\lambda_1} \right) \right)^{\frac{3}{8\nu_0}} \right] + 1, \end{aligned} \tag{11}$$

where k_0 is a positive constant which is independent of α, and

$$\varepsilon_0 = \frac{2\alpha}{3 + \kappa\alpha + \alpha^2/\lambda_1 + \sqrt{(3 + \kappa\alpha + \alpha^2/\lambda_1)^2 - 12\kappa\alpha}}. \tag{12}$$

Particularly, if

$$\alpha \geq \max\left\{ \alpha_1, \quad \frac{k_0\kappa\lambda_1}{2(\lambda_1^{4\nu_0+1} - k_0)} \right\}, \tag{13}$$

then $d_H(\Theta) = 0$.

It is easy to see that d_H in (9) is uniformly bounded. The upper bound in the right side of (11) is a decreasing function in α for large damping α. Therefore, the asymptotical behavior of (mild) solutions of system (1) ((5)) can be described by a finite number of parameters.

2. Existence and Uniqueness of Solutions.

In this section, we present the existence, uniqueness, and continuous dependence of (mild) solutions of the initial problem (5) in E.

Assume that conditions (F_1)–(F_8) hold. We know from [4] that the operator L in (6) is the infinitesimal generator of a C_0-process e^{Lt} of contractions on the Hilbert space E under assumptions (F_1)–(F_2).

By the embedding relation $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, for any $\forall Z_1,\ Z_2 \in E$, $||Z_1||_E \leq b$, $||Z_2||_E \leq b$, and $t \in R$,

$$\begin{aligned} ||N(Z_1,t) - N(Z_2,t)||_E^2 \leq & ||h(v_1) - h(v_2)||_0^2 + ||f(u_1,t) - f(u_2,t)||_0^2 \\ \leq & \beta^2 ||v_1 - v_2||_0^2 + c_5(b)||u_1 - u_2||_1^2 \\ \leq & L_0(b)||Z_1 - Z_2||_E^2, \end{aligned}$$

that is, the function $N(Z,t) : Z = (u,v,\eta)^T \to (0, -h(v) - f(u,t) + g(x,t), 0)^T$ is locally Lipschitz continuous with respect to Z from E into E and it is easy to see that $N(Z,t)$ is continuously differentiable from $E \times R \to E$. By the standard theory of semigroup of operators concerning the existence and uniqueness of solutions of evolution equations in Chapter 6 of [11], we have the following Lemma.

LEMMA 1. Consider the initial value problem (5) on the Hilbert space E.

(i) For any $Z_{0\tau} \in E$, there exists a unique function $Z(\cdot) = Z(\cdot, Z_{0\tau}) \in C([\tau, +\infty); E)$ such that $Z(\tau, Z_{0\tau}) = Z_{0\tau}$ and $Z(t)$ satisfies the integral equation

$$Z(t) = e^{L(t-\tau)} Z_{0\tau} + \int_\tau^t e^{L(t-r)} N(Z(r), r) dr, \quad \forall t \geq \tau. \tag{14}$$

In this case, $Z(t)$ is called a mild solution of (5).

(ii) If $Z_{0\tau} \in D(L)$, there exists $Z(\cdot) \in C([\tau, +\infty); D(L)) \cap C^1([\tau, +\infty); E)$ which satisfies (5).

(iii) $Z(t, Z_{0\tau})$ is jointly continuous in t and $Z_{0\tau}$.

The local existence of mild solutions of (5) in E is obtained from Theorem VI. 1.4 and Theorem VI. 1.5 of [11], and the global existence of solutions can be obtained by the boundedness of solutions in Lemma 3 below.

For any $t \geq \tau$, we introduce a map $U(t,\tau) : Z_{0\tau} \mapsto Z(t, Z_{0\tau})$, where $Z(t, Z_{0\tau})$ is the mild solution (or solution) of (5), then $\{U(t,\tau), t \geq \tau\}$ define a strongly continuous process:

$$U(t,\tau) : (u_{0\tau}, v_{0\tau}, \eta_{0\tau})^T \to (u(t), v(t), \eta(t))^T, \quad t \geq \tau \tag{15}$$

on E (or $D(L)$), which fulfills the following properties: (i) $U(t,\tau) : E \to E$ (or $D(L) \to D(L)$) for all $t \geq \tau$, $\tau \in R$; (ii) $U(\tau,\tau)$ is the identity on E (or $D(L)$) for all $t \geq \tau$, $\tau \in R$; (iii) $U(t,s)U(s,\tau) = U(t,\tau)$ for all $t \geq s \geq \tau$, $\tau \in R$; (iv) $U(t,\tau)Z \to Z$ as $t \searrow \tau$ for all $Z \in E$ (or $D(L)$), $\tau \in R$; and (v) $U(t,\tau) \in C(E,E)$ (or $C(D(L), D(L))$ for all $t \geq \tau$, $\tau \in R$.

In this article, we will prove the existence of non-empty compact kernel sections at times $s \in R$:

$$K(s) = \{Z(s) : Z(t) \text{ is a solution of (5)}, \quad ||Z(t)||_E \leq M_Z, \ \forall t \in R\} \tag{16}$$

for the process $\{U(t,\tau), t \geq \tau\}$ in E and give an upper bound of the Hausdorff dimension of the kernel section $K(\tau)$.

3. Uniform Boundedness of Solutions.

Let

$$\varphi = \begin{pmatrix} u \\ w \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} u \\ v+\varepsilon u \\ \eta \end{pmatrix}, \tag{17}$$

where ε is chosen as in (10), here $\mu(s) \neq 0$. The initial problem (5) is equivalent to the following system in Hilbert space E :

$$\dot{\varphi} + H(\varphi) = F(\varphi, t), \quad \varphi(\tau) = (u_{0\tau}, v_{0\tau} + \varepsilon u_{0\tau}, \eta_{0\tau})^T, \quad t \geq \tau, \quad \tau \in R, \tag{18}$$

where

$$H(\varphi) = \begin{pmatrix} \varepsilon u - w \\ Au + \varepsilon^2 u - \varepsilon w + h(w - \varepsilon u) + \int_0^\infty \mu(s) A\eta(s) ds \\ \varepsilon u - w + \eta_s \end{pmatrix}, \tag{19}$$

$$F(\varphi, t) = \begin{pmatrix} 0 \\ -f(u,t) + g(x,t) \\ 0 \end{pmatrix}.$$

In this section, we suppose the assumptions (F_1)–(F_8) hold. Firstly, we present a positivity property of the operator $H(\varphi)$ in (19) which plays an important role in this article.

LEMMA 2. For any $\varphi = (u, w, \eta)^T \in E$,

$$(H(\varphi), \varphi)_E \geq \frac{\varepsilon}{2}(||u||_1^2 + ||w||_0^2) + \frac{\delta}{4}||\eta||_{\Re}^2 + \frac{\alpha}{2}||w||_0^2. \tag{20}$$

Proof. It is sufficient to prove (20) for $\varphi = (u, w, \eta)^T \in D(L)$ since $D(L)$ is dense in E. Let $\varphi = (u, w, \eta)^T \in D(L)$, from ($F_8$), (19), and the Poincaré inequality:

$$\lambda_1 ||w||_0^2 \leq ||w||_1^2, \quad \forall w \in H_0^1(\Omega), \tag{21}$$

we have that

$$\begin{aligned} &(H(\varphi), \varphi)_E - \frac{\varepsilon}{2}(||u||_1^2 + ||w||_0^2) - \frac{\alpha}{2}||w||_0^2 \\ &\geq \frac{\varepsilon}{2}||u||_1^2 + (\frac{\alpha}{2} - \frac{3\varepsilon}{2})||w||_0^2 - \frac{\varepsilon\beta}{\sqrt{\lambda_1}}||u||_1 \cdot ||w||_0 + \frac{1}{2}\int_0^\infty \mu(s) d||\nabla\eta(s)||_0^2 + \varepsilon(u, \eta)_{\Re}. \end{aligned} \tag{22}$$

Integration by parts and (F_2) yields

$$\int_0^\infty \mu(s) d||\nabla\eta(s)||_0^2 = -\int_0^\infty \mu'(s)||\nabla\eta(s)||_0^2 ds \geq \delta||\eta||_{\Re}^2,$$

$$\varepsilon(u, \eta)_{\Re} \geq -\frac{\kappa}{2}\varepsilon^2||u||_1^2 - \frac{\delta}{4}||\eta||_{\Re}^2.$$

Simple computation shows that

$$(\frac{\varepsilon}{2}-\frac{\kappa}{2}\varepsilon^2)(\frac{\alpha}{2}-\frac{3\varepsilon}{2})=\frac{\varepsilon^2\beta^2}{4\lambda_1}. \tag{23}$$

Thus by (22)–(23), we have that

$$\begin{aligned}&(H(\varphi),\varphi)_E-\frac{\varepsilon}{2}(||u||_1^2+||w||_0^2)-\frac{\alpha}{2}||w||_0^2\\ &\geq(\frac{\varepsilon}{2}-\frac{\kappa}{2}\varepsilon^2)||u||_1^2+(\frac{\alpha}{2}-\frac{3\varepsilon}{2})||w||_0^2-\frac{\varepsilon\beta}{\sqrt{\lambda_1}}\|u\|_1\cdot||w||_0+\frac{\delta}{4}||\eta||_{\Re}^2\\ &\geq\frac{\delta}{4}||\eta||_{\Re}^2.\end{aligned}$$

The proof is completed. □

It is easy to check from (17) and Lemma 1 that for any initial data $\varphi(\tau)\in E$, there exists a unique continuous mild solution $\varphi(t)\in C((\tau,+\infty),E)$ of system (18) which defines a process

$$U_\varepsilon(t,\tau):(u_{0\tau},v_{0\tau}+\varepsilon u_{0\tau},\eta_{0\tau})^T\to(u(t),v(t)+\varepsilon u(t),\eta(t))^T,\quad t\geq\tau \tag{24}$$

from E into E, and $U_\varepsilon(t,\tau)=R_\varepsilon U(t,\tau)R_{-\varepsilon}$, where

$$R_\varepsilon=\begin{pmatrix}1&0&0\\ \varepsilon&1&0\\ 0&0&1\end{pmatrix}:(a,b,c)^T\to(a,b+\varepsilon a,c)^T$$

is an isomorphism of E. So we need to consider the equivalent system (18) only.

LEMMA 3. If $2\gamma\leq\varepsilon\min\{c_{01},c_{02}\}$, then there exists a ball B_0 of E, $B_0=B_E(0,r_0)$, centered at 0 of radius $r_0>0$ such that for any bounded set B of E, there exists $T_0(B)\geq 0$ such that the mild solution $\varphi(t)=(u(t),w(t),\eta(t))^T\in E$ of (18) with $\varphi(\tau)\in B$ satisfies

$$||\varphi(t)||_E=(||u(t)||_1^2+||w(t)||_0^2+||\eta(t)||_{\Re}^2)^{\frac{1}{2}}\leq r_0,\quad\forall t\geq T_0(B)\geq\tau, \tag{25}$$

in which $w=u_t+\varepsilon u$ and r_0 is independent of $\tau\in R$; that is, the ball $B_0=B_E(0,r_0)$ of E is a uniformly bounded absorbing set of the process $\{U_\varepsilon(t,\tau),\,t\geq\tau\}$ in E with respect to $\tau\in R$.

Proof. Let

$$G(u,t)=\int_0^u f(r,t)dr=G_1(u,t)+G_2(u,t),\quad\overline{G}(u,t)=\int_\Omega G(u,t)dx.$$

Let $\varphi=(u(t),w(t),\eta(t))^T\in E$, $t\geq\tau$, be a mild solution of (18) with initial value $\varphi(\tau)=(u_{0\tau},v_{0\tau}+\varepsilon u_{0\tau},\eta_{0\tau})^T\in E$. Taking the inner product $(\cdot,\cdot)_E$ of (18) with $\varphi=(u(t),w(t),\eta(t))^T$ in which $w=u_t+\varepsilon u$, we find

$$\frac{1}{2}\frac{d}{dt}[||\varphi||_E^2+2\overline{G}(u,t)]+(H(\varphi),\varphi)_E-\overline{G}_t'(u,t)+\varepsilon(f(u,t),u)=(g(x,t),w),\quad\forall t\geq\tau. \tag{26}$$

By (F_3)–(F_5), we have

$$G_1(u,t)\geq 0, \quad G_t'(u,t)\leq \gamma G(u,t)+c_0,$$

$$\lim_{|u|\to+\infty}\inf\frac{G(u,t)}{u^2}\geq 0, \quad \lim_{|u|\to+\infty}\inf\frac{uf(u,t)-c_6G(u,t)}{u^2}\geq 0, \quad \forall u\in H_0^1(\Omega), \quad \forall t\geq\tau,$$

where $c_6=\min\{c_{01},c_{02}\}$. Thus, there exist two positive constants k_1, $k_2\geq 0$ such that

$$\overline{G}(u,t)+\frac{1}{4+4c_6}||u||_1^2+k_1\geq 0, \quad \forall u\in H_0^1(\Omega), \quad \forall t\geq\tau,$$

$$(u,f(u,t))-c_6\overline{G}(u,t)+\frac{1}{4}||u||_1^2+k_2\geq 0, \quad \forall u\in H_0^1(\Omega), \quad \forall t\geq\tau. \tag{27}$$

Thus, by $\gamma\leq\frac{1}{2}c_6$ and (20),

$$(H(\varphi),\varphi)_E-\overline{G}_t'(u,t)+\varepsilon(f(u,t),u)\geq\frac{1}{2}\rho y-(\frac{1}{2}\varepsilon k_1c_6+k_2\varepsilon+c_0|\Omega|)+\frac{\alpha}{2}||w||_0^2, \quad \forall t\geq\tau,$$

where

$$y=||\varphi||_E^2+2\overline{G}(u,t)+2k_1\geq\frac{1}{2}||\varphi||_E^2\geq 0,$$

$\rho=\min(\frac{\varepsilon}{4},\frac{\delta}{2},\frac{\varepsilon}{2}c_6)$, $|\Omega|=\int_\Omega dx$. Hence,

$$\frac{d}{dt}y+\rho y\leq\frac{1}{\alpha}||g||_0^2+2(\frac{1}{2}\varepsilon k_1c_6+k_2\varepsilon+c_0|\Omega|), \quad \forall t\geq\tau, \tag{28}$$

where $||g||_0=\sup_{t\in R}||g(x,t)||_0$. By Gronwall's inequality, we have the following absorbing property:

$$\begin{aligned}||\varphi(t)||_E^2\leq&2y(t)\\ \leq&2y(\tau)e^{-\rho(t-\tau)}+2\left(\frac{1}{\alpha\rho}||g||_0^2+\frac{2(\frac{1}{2}\varepsilon k_1c_6+k_2\varepsilon+c_0|\Omega|)}{\rho}\right), \quad \forall t\geq\tau.\end{aligned} \tag{29}$$

Choosing $r_0^2=4\left(\frac{1}{\alpha\rho}||g||_0^2+\frac{2(\frac{1}{2}\varepsilon k_1c_6+k_2\varepsilon+c_0|\Omega|)}{\rho}\right)$ (independent of $\tau\in R$), by (29), the proof is completed. □

COROLLARY 4. For any initial value $\varphi(\tau)\in B_0$, i.e.,

$$||\varphi(\tau)||_E^2=||u_{0\tau}||_1^2+||v_{0\tau}+\varepsilon u_{0\tau}||_0^2+||\eta_{0\tau}||_{\Re}^2\leq r_0^2, \tag{30}$$

there exists a constant $r_1=r_1(r_0)$ such that the mild solution of (18) $\varphi(t)=(u(t),w(t),\eta(t))^T$ satisfies $||\varphi(t)||_E\leq r_1$, $\forall t\geq\tau$.

4. Existence of Compact Kernel Sections.

To prove the existence of non-empty compact kernel sections for the process $\{U(t,\tau), t \geq \tau\}$ in E, we first present some definitions and results from [6].

DEFINITION 5. The kernel K of a process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$K = \{Z(\cdot) : Z(t),\ \ t \in R,\ \ \text{is a solution of (5)},\ \ ||Z(t)||_E \leq M_Z,\ \ \forall t \in R\} \tag{31}$$

and the section $K(s) \subset E$ of the kernel K at time $s \in R$ is

$$K(s) = \{Z(s) : Z(\cdot) \in K\}. \tag{32}$$

DEFINITION 6. A set $\Lambda \subset E$ is said to be a uniformly attracting set of a process $\{U(t,\tau)\}$ if for any bounded set $B \subset E$,

$$\sup_{\tau \in R} \operatorname{dist}_H(U(t+\tau,\tau)B, \Lambda) \to 0 \ \text{ as } \quad t \to +\infty,$$

where dist_H denotes the Hausdorff semidistance in E, defined as

$$\operatorname{dist}_H(B_1, B_2) = \sup_{Z_1 \in B_1} \inf_{Z_2 \in B_2} ||Z_1 - Z_2||_E.$$

DEFINITION 7. A process $\{U(t,\tau)\}$ possessing a compact uniformly attracting set is said to be uniformly asymptotically compact.

LEMMA 8. (cf. [6, Thm 2.2].) Let $\{U(t,\tau)\}$ be a uniformly asymptotically compact process acting in a space E, with a compact uniformly attracting set $\Lambda \subset E$. Each mapping $U(t,\tau) : E \to E$ is assumed continuous. Then the kernel K of the process $\{U(t,\tau)\}$ is non-empty, the kernel sections $K(s)$ are all compact, and $K(s) \subseteq \Lambda$.

In this section, we will prove that the process $\{U_\varepsilon(t,\tau), t \geq \tau\}$ is uniformly asymptotically compact in E, that is, $\{U_\varepsilon(t,\tau), t \geq \tau\}$ possesses a uniformly attracting compact set in E with respect to $\tau \in R$.

Assume conditions (F_1)–(F_9) hold. Let $\varphi(t)$ (or $u(t)$) be a (mild) solution of system (18) (or (1)) with the initial value $\varphi(\tau) = (u_{0\tau}, v_{0\tau} + \varepsilon u_{0\tau}, \eta_{0\tau})^T \in B_0$, i.e., $||\varphi(\tau)||_E^2 = ||u_{0\tau}||_1^2 + ||v_{0\tau} + \varepsilon u_{0\tau}||_0^2 + ||\eta_{0\tau}||_{\Re}^2 \leq r_0^2$. We decompose $u(t)$ into $u(t) = u_L(t) + u_N(t)$, where $u_L(t)$ and $u_N(t)$ satisfy, respectively,

$$\begin{cases} u_{L,tt} + h(u_t) - h(u_{N,t}) - k(0)\Delta u_L - \displaystyle\int_0^\infty k'(s) \triangle u_L(t-s)ds + f_1(u_L,t) = 0, \\ u_L(x,t)|_{x\in\partial\Omega} = 0, \quad t \in R, \\ u_L(x,\tau) = u_{0\tau}(x), \quad u_{L,t}(x,\tau) = v_{0\tau}(x), \quad x \in \Omega, \\ u_L(x,t) = u_0(x,t), \quad x \in \Omega, \quad t \leq \tau, \end{cases} \tag{33}$$

and

$$\begin{cases} u_{N,tt} + h(u_{N,t}) - k(0)\Delta u_N - \int_0^\infty k'(s) \triangle u_N(t-s)ds + f(u,t) - f_1(u_L,t) = g(x,t), \\ u_N(x,t)|_{x\in\partial\Omega} = 0, \quad t \in R, \\ u_N(x,\tau) = u_{N,t}(x,\tau) = 0, \quad x \in \Omega, \\ u_N(x,t) = 0, \quad x \in \Omega, \quad t \le \tau, \end{cases} \tag{34}$$

where $u_{L,tt} = \frac{\partial^2 u_L}{\partial t^2}$, $u_{L,t} = \frac{\partial u_L}{\partial t}$.

LEMMA 9. There exist two positive constants $M_1(r_0)$ and $\sigma_1(r_0)$ such that

$$||u_L(t)||_1^2 + ||u_{L,t}(t)||_0^2 + ||\eta_L(t)||_{\Re}^2 \le M_1(r_0)\exp(-\sigma_1(r_0)(t-\tau)), \quad \forall t \ge \tau, \tag{35}$$

provided that γ in (F_5) is small enough (see below), where $\eta_L(x,t,s) = u_L(x,t) - u_L(x,t-s)$.

Proof. Let $\eta_L(x,t,s) = u_L(x,t) - u_L(x,t-s)$, $w_L = u_{L,t} + \varepsilon u_L$, $Z_L = (u_L, w_L, \eta_L)^T$. Then (33) can be written as

$$Z_{L,t} + H_L(Z_L) + F_L(Z_L,t) = 0, \quad Z_L(\tau) = (u_{0\tau}, v_{0\tau} + \varepsilon u_{0\tau}, \eta_{0\tau})^T \in B_0, \quad t \ge \tau, \tag{36}$$

where

$$H_L(Z_L) = \begin{pmatrix} \varepsilon u_L - w_L \\ Au_L + \varepsilon^2 u_L - \varepsilon w_L + h(u_t) - h(u_{N,t}) - \int_0^\infty \mu(s) \triangle \eta_L(s)ds \\ \varepsilon u_L - w_L + \eta_{L,s} \end{pmatrix}, \tag{37}$$

$$F_L(Z_L,t) = \begin{pmatrix} 0 \\ f_1(u_L,t) \\ 0 \end{pmatrix}.$$

Write $\overline{G}_1(u_L,t) = \int_\Omega G_1(u_L,t)dx$, similar to Lemma 2, for any $Z_L = (u_L, w_L, \eta_L)^T \in E$,

$$(H_L(Z_L), Z_L)_E \ge \frac{\varepsilon}{2}(||u_L||_1^2 + ||w_L||_0^2) + \frac{\delta}{4}||\eta_L||_{\Re}^2 + \frac{\alpha}{2}||w_L||_0^2,$$

and similar to Corollary 4, there exists a constant $c_7 = c_7(r_0) > 0$ such that

$$||Z_L(t)||_E^2 = ||u_L(t)||_1^2 + ||w_{L,t}(t)||_0^2 + ||\eta_L||_{\Re}^2 \le c_7^2, \quad \forall t \ge \tau.$$

From (F_3) and (F_6), we deduce that $f_1(0,t) = 0$ and $|f_1(u,t)| \le c_8(|u|^3 + |u|)$ $(\forall t \in R)$; hence for every $u_L \in H_0^1(\Omega)$, by the Sobolev embedding $H_0^1(\Omega) \subset L^4(\Omega)$, we have

$$0 \le \overline{G}_1(u_L,t) \le c_9(||u_L||_{L^4}^4 + ||u_L||_0^2) \le c_{10}(r_0)||u_L||_1^2, \tag{37}$$

i.e.,

$$\frac{\varepsilon}{2}||u_L||_1^2 \ge \frac{\varepsilon}{2c_{10}(r_0)}\overline{G}_1(u_L,t), \quad \forall t \in R.$$

By (F_3),

$$\varepsilon(f_1(u_L,t),u_L)\geq 0.$$

Choosing $\gamma<\frac{\varepsilon}{2c_{10}(r_0)}$ in (F_5), and taking the inner product $(\cdot,\cdot)_E$ of (36) with $Z_L=(u_L,w_L,\eta_L)^T$, we obtain that $\forall t\geq\tau$,

$$\frac{d}{dt}[||Z_L||_E^2+2\overline{G}_1(u_L,t)]+\frac{\varepsilon}{2}(||u_L||_1^2+||w_L||_0^2)+\frac{\delta}{2}||\eta_L||_{\Re}^2+[\frac{\varepsilon}{2c_{10}(r_0)}-\gamma]\overline{G}_1(u_L,t)\leq 0.$$

Thus,

$$\frac{d}{dt}[||Z_L||_E^2+2\overline{G}_1(u_L,t)]+\sigma_1(r_0)[||Z_L||_E^2+2\overline{G}_1(u_L,t)]\leq 0,\quad \forall t\geq\tau,$$

where

$$\sigma_1(r_0)=\frac{1}{2}\min\{\varepsilon,\ \delta,\ [\frac{\varepsilon}{2c_{10}(r_0)}-\gamma]\}. \tag{38}$$

By Gronwall's inequality, we have

$$\begin{aligned}||Z_L||_E^2\leq&[||Z_L(\tau)||_E^2+2\overline{G}_1(u_L(\tau),\tau)]\exp(-\sigma_1(r_0)(t-\tau))\\ \leq&M_1(r_0)\exp(-\sigma_1(r_0)(t-\tau)),\quad \forall t\geq\tau,\end{aligned}\tag{39}$$

where $M_1(r_0)=r_0^2(1+2c_{10}(r_0))$. The proof is completed. □

LEMMA 10. There exist constants $M_2(r_0)>0$ (independent of τ) and $\nu_0\in(0,\min\{\frac{5}{4}-\frac{3}{2\iota},\frac{1}{4}\})$, $\iota\in(\frac{6}{5},\frac{6}{3+p})$ such that $u_N(t)$ satisfies

$$||A^{\nu_0+\frac{1}{2}}u_N(t)||_0^2+||A^{\nu_0}u_{N,t}(t)||_0^2+||\eta_N||_{\mu,2\nu_0+1}^2\leq M_2(r_0),\quad \forall t\geq\tau, \tag{40}$$

where p is as in (F_7), $\eta_N(x,t,s)=u_N(x,t)-u_N(x,t-s)$.

Proof. u_N and η_N satisfy the following equation with zero initial value conditions at time $t=\tau\in R$:

$$u_{N,tt}+h(u_{N,t})+Au_N+\int_0^\infty\mu(s)A\eta_N(s)ds+f(u,t)-f_1(u_L,t)=g(x,t),\quad t\geq\tau, \tag{41}$$

where $u=u_L+u_N$ is the solution of (1).

Set $w_N=u_{N,t}+\varepsilon u_N=\eta_{N,t}+\eta_{N,s}+\varepsilon u_N$, where ε is as in (10). Taking the inner product of (41) in $L^2(\Omega)$ with $A^{2\nu}w_N=A^{2\nu}(u_{N,t}+\varepsilon u_N)=A^{2\nu}(\eta_{N,t}+\eta_{N,s}+\varepsilon u_N)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(||A^{\nu+1/2}u_N||_0^2+||A^\nu w_N||_0^2+||\eta_N||_{\mu,2\nu+1}^2+2\int_\Omega[f(u,t)-f_1(u_L,t)-g(x,t)]A^{2\nu}u_Ndx\right)\\ &+\varepsilon||A^{\nu+1/2}u_N||_0^2-\varepsilon||A^\nu w_N||_0^2+\varepsilon^2(A^\nu u_N,A^\nu w_N)+(A^\nu h(w_N-\varepsilon u_N),A^\nu w_N)\\ &+\frac{1}{2}\int_0^\infty\mu(s)\cdot d||A^{1/2+\nu}\eta(s)||_0^2\\ &+\varepsilon(\int_0^\infty\mu(s)A\eta_N(s)ds,A^{2\nu}u_N)+\varepsilon\int_\Omega[f(u,t)-f_1(u_L,t)-g(x,t)]\cdot A^{2\nu}u_Ndx\\ &+\int_\Omega[g_t'(x,t)-f_{1,t}'(u(t),t)-f_{2,t}'(u(t),t)+f_{1,t}'(u_L(t),t)]\cdot A^{2\nu}u_Ndx\\ &-\int_\Omega\left([f_{1,u}'(u,t)-f_{1,u}'(u_L,t)]u_t+f_{1,u}'(u_L,t)u_{N,t}+f_{2,u}'(u,t)u_t\right)\cdot A^{2\nu}u_Ndx\\ &=0,\quad t\geq\tau.\end{aligned}\tag{42}$$

Following [7, 8], we introduce the intermediate Sobolev spaces $H^{\nu}(\Omega)$, $\nu \in (0,1)$ with the standard scalar product:

$$< z_1,\ z_2 >_{\nu} = (z_1,\ z_2) + \int_{\Omega}\int_{\Omega} \frac{(z_1(x)-z_1(y))(z_2(x)-z_2(y))}{|x-y|^{3+2\nu}} dxdy. \quad (43)$$

Setting $0 \le \nu \le \frac{1}{4}$, we have $H_0^{\nu}(\Omega) = D(A^{\frac{\nu}{2}}) = H^{\nu}(\Omega)$[7, 8]. Thus, by ($F_8$) and the mean value theorem, we obtain that

$$\begin{aligned}
&(A^{\nu} h(w_N - \varepsilon u_N), A^{\nu} w_N) \\
&= (h(w_N - \varepsilon u_N), w_N) \\
&\quad + \int_{\Omega}\int_{\Omega} \frac{(h(w_N(x,t)-\varepsilon u_N(x,t)) - h(w_N(y,t)-\varepsilon u_N(y,t)))(w_N(x,t)-(w_N(y,t)))}{|x-y|^{3+4\nu}} dxdy \\
&= (h'(\xi_1) w_N, w_N) - \varepsilon (h'(\xi_1) u_N, w_N) \\
&\quad + \int_{\Omega}\int_{\Omega} \frac{h'(\xi_2)(w_N(x,t)-\varepsilon u_N(x,t) - w_N(y,t)+\varepsilon u_N(y,t))(w_N(x,t)-(w_N(y,t)))}{|x-y|^{3+4\nu}} dxdy \\
&\ge \alpha \left((w_N, w_N) + \int_{\Omega}\int_{\Omega} \frac{(w_N(x,t)-w_N(y,t))(w_N(x,t)-(w_N(y,t)))}{|x-y|^{3+4\nu}} dxdy \right) \\
&\quad - \varepsilon\beta \left((u_N, w_N) + \int_{\Omega}\int_{\Omega} \frac{(u_N(x,t)-u_N(y,t))(w_N(x,t)-(w_N(y,t)))}{|x-y|^{3+4\nu}} dxdy \right) \\
&= \alpha ||A^{\nu} w_N||_0^2 - \varepsilon\beta (A^{\nu} u_N, A^{\nu} w_N).
\end{aligned}$$

Thus, similarly to the proof of Lemma 3, we have

$$\begin{aligned}
&\varepsilon ||A^{\nu+1/2} u_N||_0^2 - \varepsilon ||A^{\nu} w_N||_0^2 + \varepsilon^2 (A^{\nu} u_N, A^{\nu} w_N) + (A^{\nu} h(w_N - \varepsilon u_N), A^{\nu} w_N) \\
&+ \frac{1}{2}\int_0^{\infty} \mu(s) d||A^{1/2+\nu}\eta_N(s)||_0^2 + \varepsilon (\int_0^{\infty} \mu(s) A\eta_N(s) ds, A^{2\nu} u_N) \\
&\ge \frac{\varepsilon}{2} ||A^{\nu+1/2} u_N||_0^2 + (\frac{\varepsilon}{2} + \frac{\alpha}{2}) ||A^{\nu} w_N||_0^2 + \frac{\delta}{4} ||A^{1/2+\nu}\eta_N||_0^2.
\end{aligned} \quad (44)$$

From Corollary 4 and Lemma 9, we know that $||u_N(t)||_1$ and $||u_{N,t}(t)||_0$ are uniformly bounded, i.e.,

$$||u_N(t)||_1 \le c_{11}(r_0), \quad ||u_{N,t}(t)||_0 \le c_{12}(r_0), \quad \forall t \ge \tau. \quad (45)$$

We recall the embedding relations:

$$H^{\nu_1}(\Omega) \subset H^{\nu_2}(\Omega) \quad \text{if} \quad \nu_1 \ge \nu_2 \quad \text{and} \quad H^{\nu}(\Omega) \subset L^q(\Omega), \quad \text{where} \quad \frac{1}{q} = \frac{1}{2} - \frac{\nu}{3}. \quad (46)$$

From (F_6)–(F_7), we obtain that

$$\begin{aligned}
&\int_{\Omega} |[f(u,t) - f_1(u_L,t) - g(x,t)] A^{2\nu} u_N| dx \\
&\quad \le ||f(u(t),t) - f_1(u_L(t),t) - g(x,t)||_0 \cdot ||A^{2\nu} u_N(t)||_0 \\
&\quad \le c_{13}(r_0), \quad \forall t \ge \tau
\end{aligned} \quad (47)$$

and by $g'_t(x,t) \in C_b(R, L^2(\Omega))$,

$$\int_\Omega [|g'_t(x,t)|+|f'_{1,t}(u(t),t)|+|f'_{2,t}(u(t),t)|+|f'_{1,t}(u_L(t),t)|]|A^{2\nu}u_N(s)|dx \leq c_{14}(r_0), \ \forall t \geq \tau. \tag{48}$$

Now we estimate the last term of the left-hand side of (42) which is similar to the proof of *Proposition 2* in [10]. Choosing two positive numbers $\tilde{\delta}$ and κ such that

$$\frac{\tilde{\delta}}{6}+\frac{\tilde{\delta}}{6}+\frac{1}{\kappa}=1, \quad \frac{1}{\tilde{\delta}\kappa}=\frac{1}{2}-\frac{2\nu}{3}, \quad \frac{1}{\tilde{\delta}}=\frac{5}{6}-\frac{2\nu}{3}, \tag{49}$$

by Hölder's inequality and $(\mathbf{F}_6)$, we have

$$\begin{aligned} ||f'_{1,u}(u_L(t),t)u_{N,t}(t)||_{L^{\tilde{\delta}}} \leq & c_{15}||(|u_L(t)|(1+|u_L(t)|)|u_{N,t}(t)|)||_{L^{\tilde{\delta}}} \\ \leq & c_{16}||u_L(t)||_{L^6}\cdot||(1+|u_L(t)|)||_{L^6}\cdot||u_{N,t}(t)||_{L^{\tilde{\delta}\kappa}} \\ \leq & c_{17}(r_0)||u_N(t)||_1\cdot(||A^\nu w_N(s)||_0+\varepsilon), \quad \forall t\geq\tau. \end{aligned} \tag{50}$$

Similarly,

$$||(f'_{1,u}(u(t),t)-f'_{1,u}(u_L(t),t))u_t(t)||_{L^{\tilde{\delta}}} \leq c_{18}(r_0)||u_t(t)||_0\cdot||A^{\nu+\frac{1}{2}}u_N(t)||_0, \quad \forall t\geq\tau. \tag{51}$$

On the other hand, we have

$$||f'_{2,u}(u(t),t)u_t(t)||_{L^\iota} \leq c_{19}(r_0)||u_t(t)||_0, \quad \forall\iota\in(\frac{6}{5},\frac{6}{3+p}). \tag{52}$$

Let $\tilde{\delta}<\iota$, then

$$||f'_{2,u}(u(t),t)u_t(t)||_{L^{\tilde{\delta}}} \leq c_{20}||f'_{2,u}(u(t),t)u_t(t)||_{L^\iota} \leq c_{21}(r_0)||u_t(t)||_0, \quad \forall t\geq\tau. \tag{53}$$

Again,

$$||A^{2\nu}u_N(t)||_{L^r} = ||A^{\nu-\frac{1}{2}}A^{\nu+\frac{1}{2}}u_N(t)||_{L^r} \leq c_{22}||A^{\nu+\frac{1}{2}}u_N(t)||_0, \quad \forall t\geq\tau, \tag{54}$$

where $\frac{1}{r}=1-\frac{1}{\tilde{\delta}}$.

Set $\nu=\min\{\frac{5}{4}-\frac{3}{2\iota},\ \frac{1}{4}\}$, here $\iota\in(\frac{6}{5},\ \frac{6}{3+p})$. By the above inequalities and (42), for all $t\geq\tau$,

$$\begin{aligned} &\frac{d}{dt}\tilde{y}(t)+\tilde{\rho}\tilde{y}(t)+\frac{\alpha}{2}||A^\nu w_N||_0^2 \\ &\leq c_{23}(r_0)\{||u_t||_0\cdot||A^{\nu+\frac{1}{2}}u_N||_0^2+||u_L||_1\cdot||A^\nu w_N||_0\cdot||A^{\nu+\frac{1}{2}}u_N||_0 \\ &\quad +||u_t(t)||_0\cdot||A^{\nu+\frac{1}{2}}u_N||_0+\varepsilon\}+\varepsilon c_{13}(r_0)+c_{14}(r_0). \end{aligned}$$

By the Young inequality, we have

$$||u_L||_1\cdot||A^\nu w_N||_0\cdot||A^{\nu+\frac{1}{2}}u_N||_0 \leq \frac{3}{2\alpha}||u_L||_1^2\cdot||A^{\nu+\frac{1}{2}}u_N||_0^2+\frac{\alpha}{2}||A^\nu w_N||_0^2$$

and

$$||u_t(t)||_0 \cdot ||A^{\nu+\frac{1}{2}}u_N||_0 = \quad ||u_t(t)||_0 \cdot \frac{1}{2}(||A^{\nu+\frac{1}{2}}u_N||_0^2 + 1),$$

thus,

$$||u_t||_0 \cdot ||A^{\nu+\frac{1}{2}}u_N||_0^2 + ||u_t(t)||_0 \cdot ||A^{\nu+\frac{1}{2}}u_N||_0 = \frac{3}{2}||u_t(t)||_0||A^{\nu+\frac{1}{2}}u_N||_0^2 + \frac{1}{2}\ ||u_t(t)||_0$$

hence,

$$\begin{aligned}\frac{d}{dt}\tilde{y}(t) + \frac{\tilde{\rho}}{2}\tilde{y}(t) \leq& c_{24}(r_0)[||u_t||_0 + \frac{1}{\alpha}||u_L||_1^2] \cdot ||A^{\nu+\frac{1}{2}}u_N||_0^2 + c_{25}(r_0)||u_t||_0 + c_{26}(r_0)\\ \leq& c_{27}(r_0)[||u_t||_0 + \frac{1}{\alpha}||u_L||_1^2]\tilde{y}(t) + c_{25}(r_0)||u_t||_0 + c_{26}(r_0), \quad \forall t \geq \tau,\end{aligned} \tag{55}$$

where $\tilde{\rho} = \min\{\varepsilon,\ \frac{\delta}{2}\}$ and

$$\begin{aligned}\tilde{y}(t) =& \frac{1}{2}||A^{\nu+1/2}u_N||_0^2 + \frac{1}{2}||A^{\nu}w_N||_0^2 + \frac{1}{2}||\eta_N||_{\mu,2\nu+1}^2\\ &+ \int_\Omega [f(u,t) - f_1(u_L,t) - g]A^{2\nu}u_N dx + c_{13}(r_0)\\ \geq& 0, \quad \forall t \geq \tau.\end{aligned} \tag{56}$$

By applying Gronwall's inequality to (55), we have that for any $t \geq s \geq \tau$,

$$\tilde{y}(t) \leq \tilde{y}(s)e^{-\int_s^t m(r)dr} + c_{25}(r_0)\int_s^t ||u_t(\xi)||_0 e^{-\int_\xi^t m(r)dr}d\xi + c_{26}(r_0)\int_s^t e^{-\int_\xi^t m(r)dr}d\xi, \tag{57}$$

where

$$m(r) = \frac{\tilde{\rho}}{2} - c_{27}(r_0)[||u_t(r)||_0 + \frac{1}{\alpha}||u_L(r)||_1^2]. \tag{58}$$

Taking the inner product of (5) with $Z = (u, u_t, \eta)^T \in E$ in E, by (F_2) and (F_8), we obtain that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||\nabla u||_0^2 + ||u_t||_0^2 + ||\eta||_{\Re}^2 + 2G(u,t) - (g(x,t),u))\\ &+ \frac{\delta}{2}||\eta||_{\Re}^2 + \alpha||u_t||_0^2 - \int_\Omega [G_t'(u,t) - g_t'(x,t)u]dx\\ &\leq 0, \quad \forall t \geq \tau.\end{aligned}$$

From Corollary 4 and (F_9), it is easy to obtain that

$$\alpha\int_\tau^{+\infty} ||u_t(r)||_0^2 dr \leq c_{28}(r_0). \tag{59}$$

We know from Lemma 9 that

$$||u_L(t)||_1^2 \leq M_1(r_0)e^{-\sigma_1(r_0)(t-\tau)}, \quad \forall t \geq \tau. \tag{60}$$

Thus, for any $t \geq s \geq \tau$,

$$\begin{aligned}\int_s^t m(r)dr =& \int_s^t \left(\frac{\tilde{\rho}}{2} - c_{27}(r_0)[||u_t(r)||_0 + \frac{1}{\alpha}||u_N(r)||_1^2]\right) dr \\ \geq& \frac{\tilde{\rho}}{2}(t-s) - \frac{c_{28}(r_0)}{\sqrt{\alpha}}\sqrt{t-s} - \frac{c_{29}(r_0)}{\alpha}\int_s^t e^{-\sigma_1(r_0)(r-\tau)}dr \\ \geq& \frac{\tilde{\rho}}{2}(t-s) - \frac{c_{28}(r_0)}{\sqrt{\alpha}}\sqrt{t-s} - \frac{c_{30}(r_0)}{\alpha\sigma_1(r_0)}. \end{aligned} \tag{61}$$

So,

$$\begin{aligned}\int_\tau^t e^{-\int_\xi^t m(r)dr}d\xi \leq& e^{c_{31}(r_0)/\alpha}\int_\tau^t e^{-[\frac{\tilde{\rho}}{2}(t-\xi) - \frac{c_{28}(r_0)}{\sqrt{\alpha}}\sqrt{t-\xi}]}d\xi \\ =& e^{c_{31}(r_0)/\alpha}\int_\tau^t e^{-\frac{1}{2}[(\sqrt{\tilde{\rho}(t-\xi)} - \frac{c_{28}(r_0)}{\sqrt{\tilde{\rho}\alpha}})^2 + \frac{c_{28}^2(r_0)}{2\alpha\tilde{\rho}}]}d\xi \\ \leq& \frac{4}{\tilde{\rho}}e^{c_{32}(r_0)/\alpha}[1 + \frac{c_{33}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}], \quad \forall t \geq \tau. \end{aligned} \tag{62}$$

Similarly,

$$\int_\tau^t e^{-2\int_\xi^t m(r)dr}d\xi \leq \frac{4}{\tilde{\rho}}e^{c_{34}(r_0)/\alpha}[1 + \frac{c_{35}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}], \quad \forall t \geq \tau. \tag{63}$$

By (47), (56), and the zero value conditions $u_N(x,\tau) = u_{N,t}(x,\tau) = \eta_N(x,\tau) = 0$, $x \in \Omega$ at $t = \tau$,

$$\tilde{y}(\tau) = \int_\Omega [f(u,t) - f_1(u_L,t) - g]A^{2\nu}u_N dx + c_{13}(r_0) \leq 2c_{13}(r_0).$$

From (57)–(63), for $t \geq \tau$, we have

$$\begin{aligned}\tilde{y}(t) \leq& \tilde{y}(\tau)e^{-\frac{\tilde{\rho}}{2}(t-\tau) + \frac{c_{28}(r_0)}{\sqrt{\alpha}}\sqrt{t-\tau} + \frac{c_{30}(r_0)}{\alpha\sigma_1(r_0)}} \\ &+ c_{25}(r_0)\left(\int_\tau^t ||u_t(\xi)||_0^2 d\xi\right)^{\frac{1}{2}}\left(\int_\tau^t e^{-2\int_\xi^t m(r)dr}d\xi\right)^{\frac{1}{2}} + c_{26}(r_0)\int_\tau^t e^{-\int_\xi^t m(r)dr}d\xi \\ \leq& 2c_{13}(r_0)e^{-\frac{\tilde{\rho}}{2}(t-\tau) + \frac{c_{28}(r_0)}{\sqrt{\alpha}}\sqrt{t-\tau} + \frac{c_{30}(r_0)}{\alpha\sigma_1(r_0)}} + c_{36}(r_0)\left(\frac{4}{\alpha\tilde{\rho}}e^{c_{84}(r_0)/\alpha}[1 + \frac{c_{35}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}]\right)^{\frac{1}{2}} \\ &+ \frac{4}{\tilde{\rho}}c_{26}(r_0)e^{c_{32}(r_0)/\alpha}[1 + \frac{c_{33}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}] \\ \leq& 2c_{13}(r_0)e^{\frac{c_{30}(r_0)}{\alpha\sigma_1(r_0)}}\left(1 + e^{\frac{2c_{28}^2(r_0)}{\alpha\tilde{\rho}}}\right) + c_{36}(r_0)\left(\frac{4}{\alpha\tilde{\rho}}e^{c_{84}(r_0)/\alpha}[1 + \frac{c_{35}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}]\right)^{\frac{1}{2}} \\ &+ \frac{4}{\tilde{\rho}}c_{26}(r_0)e^{c_{32}(r_0)/\alpha}[1 + \frac{c_{33}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}]. \end{aligned} \tag{64}$$

Hence for any $t \geq \tau$, we have

$$
\begin{aligned}
&||A^{\nu+1/2}u_N||_0^2 + ||A^\nu w_N||_0^2 + ||\eta_N||_{\mu,2\nu+1}^2 \\
&\leq 2\tilde{y}(t) \\
&\leq 4c_{13}(r_0)e^{\frac{c_{30}(r_0)}{\alpha\sigma_1(r_0)}}\left(1+e^{\frac{2c_{28}^2(r_0)}{\alpha\tilde{\rho}}}\right) + 2c_{36}(r_0)\left(\frac{4}{\alpha\tilde{\rho}}e^{c_{84}(r_0)/\alpha}[1+\frac{c_{35}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}]\right)^{\frac{1}{2}} \\
&\quad + \frac{8}{\tilde{\rho}}c_{26}(r_0)e^{c_{32}(r_0)/\alpha}[1+\frac{c_{33}(r_0)}{\sqrt{\tilde{\rho}\alpha}}\sqrt{2\pi}] \\
&= M_2(r_0).
\end{aligned}
$$

The proof is completed. □

COROLLARY 11. Let $\eta_N(x,t,s)$ and ν_0 are as in Lemma 11, then $||\partial_s\eta_N||_{\mu,2\nu_0}^2 \leq M_2(r_0)$, $\forall t \geq \tau$.

Proof. It is directly obtained from (40) and

$$
\eta_N(x,t,s) = \begin{cases} u_N(x,t) - u_N(x,t-s), & t-s \geq \tau, \\ u_N(x,t), & t-s \leq \tau, \end{cases} \tag{65}
$$

$$
\eta_{N,s}(x,t,s) = \begin{cases} u_{N,t}(x,t-s), & t-s \geq \tau, \\ 0, & t-s \leq \tau. \end{cases}
$$

To obtain the existence of a uniformly attracting compact set for the process $U_\varepsilon(t,\tau)$ defined by (18), we need to use a Lemma from [1]. □

LEMMA 12 (cf. [1, Lemma 5.5]). Let X_0, X, X_1 be three Banach spaces such that

$$
X_0 \hookrightarrow X \hookrightarrow X_1,
$$

the first injection being compact. Let $Y \subset L_\mu^2(R^+, X)$ satisfy the following hypotheses:

(i). Y is bounded in $L_\mu^2(R^+, X_0) \cap H_\mu^1(R^+, X_1)$.

(ii). $\sup_{\eta\in Y} ||\eta(s)||_X^2 \leq l(s)$, $\forall s \in R^+$ for some $l \in L_\mu^1(R^+)$.

Then Y is relatively compact in $L_\mu^2(R^+, X)$.

Let $B_0 \subset E$ be the bounded absorbing set for the process $U_\varepsilon(t,\tau)$ in E as in Lemma 4. Define a set $\tilde{B}$ as

$$
\tilde{B} = \cup_{\varphi\in B_0} \cup_{t\geq\tau} \eta_N(x,t,s), \quad \varphi = (u,w,\eta)^T \text{ is solution of (18)}, \quad \tau \in R. \tag{66}
$$

From Lemma 9 and Lemma 10, $\tilde{B}$ is uniformly bounded in $L_\mu^2(R^+, V_{1+2\nu_0}) \cap H_\mu^1(R^+, V_{2\nu_0})$ with respect to $\tau \in R$, where ν_0 is defined by Lemma 10. It is easy to see from (65) that $\sup_{\eta\in\tilde{B}, s\in R^+} ||\nabla\eta(s)||_0^2$ is bounded. Thus we have that the set $\tilde{B}$ in (66) is relatively compact in $L_\mu^2(R^+, H_0^1(\Omega))$.

LEMMA 13. Suppose the conditions (F_1)–(F_9) hold; then the process $\{U_\varepsilon(t,\tau), t \geq \tau\}$ associated with (18) possesses a uniformly attracting compact set $\Lambda \subset E$ with respect to $\tau \in R$, i.e., the process $\{U_\varepsilon(t,\tau), t \geq \tau\}$ is uniformly asymptotically compact in E.

Proof. In view of Lemma 10, let B_{ν_0} be the ball of $V_{1+2\nu_0} \times V_{2\nu_0}$ of radius $M_2(r_0)$ and set

$$\Lambda = B_{\nu_0} \times \tilde{B} \subset E. \tag{67}$$

From the compact embedding $V_{1+2\nu_0} \times V_{2\nu_0} \hookrightarrow H_0^1(\Omega) \times L^2(\Omega)$ and the relative compactness of $\tilde{B}$ in $L^2_\mu(R^+, H_0^1(\Omega))$, Λ is compact in E. Now we show the attraction property of Λ. Let $B \subset E$ be a bounded set, with $r = \sup_{\varphi \in B} ||\varphi||_E$ and let $t^* = t^*(B)$ such that $U_\varepsilon(t,\tau)B \subset B_0$, $\forall t > t^*$. Let $t > t^*$ and $t_0 = t - t^* > 0$. Using the process property (iii), we have that

$$U_\varepsilon(t_0 + t^*, \tau)B = U_\varepsilon(t_0 + t^*, t^*)U_\varepsilon(t^*, \tau)B \subset U_\varepsilon(t_0 + t^*, t^*)B_0. \tag{68}$$

Pick any $\varphi(t) = (u(t), w(t), \eta(t))^T \in U_\varepsilon(t,\tau)B$ for $t > t^*$. From (68) and Lemma 10, we have $\varphi_N(t) = \varphi(t) - \varphi_L(t) \in \Lambda$ where $\varphi_N(t) = (u_N(t), w_N(t), \eta_N(t))^T$ is given by (34). Therefore, by Lemma 9,

$$\inf_{\psi \in \Lambda} ||\varphi(t) - \psi||_E^2 \leq ||\varphi_L(t)||_E^2 \leq M_1(r_0)\exp(-\sigma_1(r_0)(t-\tau)), \quad \forall t > t^* \geq \tau.$$

So,

$$\mathrm{dist}_H(U_\varepsilon(t,\tau)B, \Lambda) \leq \sqrt{M_1(r_0)}\exp(-\frac{\sigma_1(r_0)}{2}(t-\tau)), \quad \forall t > t^* \geq \tau.$$

The proof is completed. □

LEMMA 14. The process $\{U_\varepsilon(t,\tau), t \geq \tau\}$ possesses a non-empty kernel

$$K = \{\varphi(\cdot) : \varphi(t),\ \ t \in R,\ \text{ is a solution of (18)},\ ||\varphi(t)||_E \leq M_\varphi,\ \forall t \in R\} \tag{69}$$

such that the kernel section $K(s)$ at the time s:

$$K(s) = \{\varphi(s) : \varphi(t)\ \text{ is a solution of (18)},\ ||\varphi(t)||_E \leq M_u,\ \forall t \in R\} \tag{70}$$

is compact and $K(s) \subseteq \Lambda$, $\forall s \in R$.

Proof. It is immediately proved from Lemma 8 and Lemma 13. □

5. Hausdorff Dimension of Kernel Sections.

In this section, we assume the assumptions (F_1)–(F_{11}) hold. Firstly we prove the differentiability of the process $U(t,\tau)$ defined by (3) (or (5)).

LEMMA 15. Consider the linearized equation of (3) (or (5)) with initial-boundary conditions:

$$\begin{cases} U_t = V, \\ V_t = \triangle U - h'(v)V + \int_0^\infty \mu(s) \triangle \Upsilon(s)ds - f'_u(u,t)U, \\ \Upsilon_t = V - \Upsilon_s, \\ U(x,t) = V(x,t) = \Upsilon(x,t,s) = 0,\ x \in \partial\Omega \text{ or } s = 0,\ t \geq \tau, \\ U(x,\tau) = U_{0\tau},\ V(x,\tau) = V_{0\tau},\ \Upsilon(x,\tau,s) = \Upsilon_{0\tau},\ (x,s) \in \Omega \times R^+ \end{cases} \tag{71}$$

where $Z(x,t,s) = (u(x,t), v(x,t), \eta(x,t,s))^T$ is a solution of (5) (or (3)). Then (71) is a well posed problem in E, the process $U(t,\tau)$ defined by (3) (or (5)) is uniformly quasidifferentiable on the kernel section $\{K(\tau), \tau \in R\}$ for $t \geq \tau$, the quasidifferentiability of $U(t,\tau)$ at $Z(\tau) = (u_{0\tau}, v_{0\tau}, \eta_{0\tau})^T$ is the linear operator $U'(t,\tau,Z)$ on E :

$$U'(t,\tau,Z) : \widetilde{\Psi}_{0\tau} = (U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T \mapsto \widetilde{\Psi}(t) = (U(t), V(t), \Upsilon(t))^T \in E,$$

where $\widetilde{\Psi} = (U, V, \Upsilon)^T$ is the solution of (71).

Proof. It is clear from the assumptions (F_1)–(F_{11}) that the linear system (71) is well posed in E. We first consider the Lipschitz property of $U(t,\tau)$ on the bounded sets of E. Let

$$Z_{0\tau} = (u_{0\tau}, v_{0\tau}, \eta_{0\tau})^T \in E, \ \ \widetilde{Z}_{0\tau} = (u_{0\tau} + U_{0\tau}, v_{0\tau} + V_{0\tau}, \eta_{0\tau} + \Upsilon_{0\tau})^T \in E$$

with

$$||Z_{0\tau}||_E \leq R_0, \ \ ||\widetilde{Z}_{0\tau}||_E \leq R_0 \tag{72}$$

and

$$U(t,\tau)Z_{0\tau} = Z(t) = (u(t), v(t), \eta(t,s))^T \in E,$$
$$U(t,\tau)\widetilde{Z}_{0\tau} = \widetilde{Z}(t) = (\widetilde{u}(t), \widetilde{v}(t), \widetilde{\eta}(t,s))^T \in E.$$

Similarly to the proof of Corollary 5, there exists a constant $r' = r'(R_0)$ such that

$$||Z(t)||_E \leq r', \ \ ||\widetilde{Z}(t)||_E \leq r', \ \ \forall t \geq \tau. \tag{73}$$

The difference $\psi = \widetilde{Z} - Z = (\psi_1, \psi_2, \psi_3)^T$ satisfies

$$\begin{cases} \partial_t \psi_1 = \psi_2, \\ \partial_t \psi_2 = \triangle \psi_1 - h(\widetilde{v}(t)) + h(v(t)) + \int_0^\infty \mu(s) \triangle \psi_3(s) ds - f(\widetilde{u}, t) + f(u,t), \\ \partial_t \psi_3 = \psi_2 - \partial_s \psi_3, \\ \psi_1(x,\tau) = U_{0\tau}, \ \ \psi_2(x,\tau) = V_{0\tau}, \ \ \psi_3(x,\tau,s) = \Upsilon_{0\tau}, \ \ (x,s) \in \Omega \times R^+, \end{cases} \tag{74}$$

where $\partial_t = \frac{\partial}{\partial t}$. By (F_6)–(F_8), Young inequality, Poincaré inequality, embedding theorem, and (73),

$$||f(\widetilde{u}(t),t) - f(u(t),t)||_0 = ||f'(u + \vartheta_1(\widetilde{u} - u)) \cdot (\widetilde{u} - u)||_0 \leq c_{37}(r')||\psi_1||_1, \ \ \forall t \geq \tau$$

$$||h(\widetilde{v}) - h(v)||_0 = ||h'(v + \vartheta_2(\widetilde{v} - v))(\widetilde{v} - v)||_0 \leq \beta ||\psi_2||_0, \ \ \forall t \geq \tau$$

where $\vartheta_i \in (0,1)$, $i = 1, 2$. Taking the inner product of (74) with ψ in E, we have

$$\frac{d}{dt}(||\psi_1||_1^2 + ||\psi_2||_0^2 + ||\psi_3||_{\Re}^2) \leq c_{38}(r')(||\psi_1||_1^2 + ||\psi_2||_0^2 + ||\psi_3||_{\Re}^2), \ \ \forall t \geq \tau.$$

So, we have the Lipschitz property

$$||\widetilde{Z}(t) - Z(t)||_E^2 \leq ||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^2 e^{c_{38}(r')(t-\tau)}, \ \ \forall t \geq \tau. \tag{75}$$

Consider the difference $\theta = \widetilde{Z} - Z - \widetilde{\Psi} = \psi - \widetilde{\Psi} = (\theta_1, \theta_2, \theta_3)^T \in E$, with $\widetilde{\Psi}$ the solution of the linearized system (71). Obviously,

$$\begin{cases} \partial_t \theta_1 = \theta_2, \\ \partial_t \theta_2 = \triangle \theta_1 + \int_0^\infty \mu(s) \triangle \theta_3(s) ds + d, \\ \partial_t \theta_3 = \theta_2 - \partial_s \theta_3, \\ \theta_1(x,\tau) = \theta_2(x,\tau) = \theta_3(x,\tau,s) = 0, \ \ (x,s) \in \Omega \times R^+ \end{cases} \tag{76}$$

where

$$\begin{aligned} d &= -[f(\widetilde{u},t) - f(u,t) - f'_u(u,t)(\widetilde{u}-u) + f'_u(u,t)\theta_1 + h(\widetilde{v}) - h(v) - h'_v(v)(\widetilde{v}-v) + h'_v(v)\theta_2] \\ &= -[f'_u(u + \vartheta_3(\widetilde{u}-u), t) - f'_u(u,t)](\widetilde{u}-u) - f'_u(u,t)\theta_1 \\ &\quad - [h'(v + \vartheta_4(\widetilde{v}-v)) - h'(v)](\widetilde{v}-v) - h'(v)\theta_2, \end{aligned}$$

where $\vartheta_i \in (0,1)$, $i = 3, 4$. By the assumptions (F_9) and (F_{10}),

$$||f'_u(u + \vartheta_3(\widetilde{u}-u), t) - f'_u(u,t)||_{L(H_0^1(\Omega), L^2(\Omega))} \le c_{39} \vartheta_3^{\delta_1} ||\widetilde{u} - u||_1^{\delta_1}, \tag{77}$$

and

$$||h'(v + \vartheta_4(\widetilde{v}-v)) - h'(v)||_{L(L^2(\Omega), L^2(\Omega))} \le c_{40} \vartheta_4^{\delta_2} ||\widetilde{v} - v||_0^{\delta_2}. \tag{78}$$

Taking the inner product of each side of (76) with θ in E, we obtain that

$$\frac{d}{dt}||\theta||_E^2 \le c_{38}(r')||\theta||_E^2 + c_{39}(r')(||\widetilde{u}(t) - u(t)||_1^{2+2\delta_1} + ||\widetilde{v}(t) - v(t)||_0^{2+2\delta_2}), \ \ t \ge \tau.$$

By the Gronwall inequality, we obtain

$$||\theta(t)||_E^2 \le c_{41} e^{c_{42}(t-\tau)} \cdot [||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^{2+2\delta_1} + ||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^{2+2\delta_2}], \ \ \forall t \ge \tau.$$

Therefore,

$$\begin{aligned} \frac{||\widetilde{Z}(t) - Z(t) - \widetilde{\Psi}(t)||_E^2}{||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^2} \le & c_{41} e^{c_{42}(t-\tau)} \cdot [||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^{2\delta_1} + ||(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T||_E^{2\delta_2}] \\ \to & 0, \quad \forall t \ge \tau, \end{aligned}$$

as $(U_{0\tau}, V_{0\tau}, \Upsilon_{0\tau})^T \to 0$ in E. The proof is completed. □

Now, we estimate the Hausdorff dimension of the kernel sections $K(\tau)$ in E, $\tau \in R$. For this purpose, we consider the first variation equation of the equivalent system (18) with initial condition

$$\Psi_t + H'(\varphi)\Psi = F'_\varphi(\varphi, t)\Psi, \quad \Psi(\tau) = (U_\tau, W_\tau, \Upsilon_\tau)^T \in E, \tag{79}$$

where $\Psi = (U, W, \Upsilon)^T \in E$ and $\varphi(t) = (u, w, \eta)^T \in E$, $t \ge \tau$ is the solution of (18),

$$H'(\varphi) = \begin{pmatrix} \varepsilon I & -I & 0 \\ A + \varepsilon^2 I - \varepsilon h'(v - \varepsilon u) I & h'(v - \varepsilon u) I - \varepsilon I & \int_0^\infty \mu(s) A \cdot ds \\ \varepsilon I & I & \partial_s \end{pmatrix}, \tag{80}$$

$$F'_{\varphi}(\varphi, t) = \begin{pmatrix} 0 & 0 & 0 \\ -f'_u(u,t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see from Lemma 15 that (79) is a well-posed problem in E, the mapping $U_\varepsilon(t,\tau)$ is uniformly quasidifferentiable on the kernel sections $\{K(\tau), \tau \in R\}$ for $t \geq \tau$, that is, there exists a family of bounded linear operators (quasidifferentials) $\{U'_\varepsilon(t,\tau,\varphi) : \varphi \in K(\tau), t \geq \tau, \tau \in R\}$ such that $U'_\varepsilon(t,\tau,\varphi) : E \to E$ and

$$||U_\varepsilon(t,\tau)\varphi_1 - U_\varepsilon(t,\tau)\varphi - U'_\varepsilon(t,\tau,\varphi)(\varphi_1 - \varphi)||_E \leq \ell(t-\tau, ||\varphi_1 - \varphi||_E)||\varphi_1 - \varphi||_E,$$

where $\varphi_1, \varphi \in K(\tau)$, $\ell(t-\tau, ||\varphi_1 - \varphi||_E) \to 0$ as $\varphi_1 \to \varphi$ for all $t \geq \tau$. Here the quasidifferential of $U_\varepsilon(t,\tau)$ at $\varphi(\tau) = (u_{0\tau}, v_{0\tau} + \varepsilon u_{0\tau}, \eta_{0\tau})^T$ is the linear operator $U'_\varepsilon(t,\tau,\varphi)$ on $E : (U_\tau, W_\tau, \Upsilon_\tau)^T \mapsto (U, W, \Upsilon)^T \in E$, where $(U, W, \Upsilon)^T$ is the solution of (79).

LEMMA 16 (cf. [6, Thm. 4.1]). Consider the system (18). Let Φ denote a set of m vectors $\{\Phi_1, \Phi_2, \cdots, \Phi_m\}$ which are orthonormal in E. If

$$q_m = \lim_{t\to+\infty} \inf \sup_{\tau\in R} \sup_{\Phi\subset E} \sup_{\varphi(\tau)\in K(\tau)} \frac{1}{t}\int_\tau^{\tau+t} \sum_{j=1}^m ((-H'(\varphi(s)) + F'_\varphi(\varphi(s), s))\Phi_j(s), \Phi_j(s))_E ds < 0, \tag{81}$$

and there exists a continuous function of $(t-\tau)$, $\forall t \geq \tau$ such that

$$\sup_{\varphi_\tau\in K(\tau)} ||U'_\varepsilon(t,\tau,\varphi_\tau)||_{L(E,E)} \leq C(t-\tau), \quad \forall t \geq \tau, \tag{82}$$

then the Hausdorff dimension of the kernel section $K(\tau)$ is less than or equal to m, $\forall \tau \in R$.

LEMMA 17. For any orthonormal family of elements of E, $\{(\xi_j, \hat{\eta}_j, \zeta_j)^T\}_{j=1}^m$, we have

$$\sum_{j=1}^m ||A^{\frac{1}{2}\nu}\xi_j||_0^2 \leq \sum_{j=1}^m \lambda_j^{\nu-1}, \quad \forall \nu \in [0,1). \tag{83}$$

Proof. Similar to the proof of Lemma VI. 6.3 of [12]. □

LEMMA 18. For any $\varphi = (u, w, \eta)^T$, $\varphi_1 = (\xi, \hat{\eta}, \zeta)^T \in E$,

$$(H'(\varphi)\varphi_1, \varphi_1)_E \geq \frac{\varepsilon}{2}(||\xi||_1^2 + ||\hat{\eta}||_0^2) + \frac{\delta}{4}||\zeta||_{\Re}^2 + \frac{\alpha}{2}||\hat{\eta}||_0^2. \tag{84}$$

Proof. Similar to the proof of Lemma 2. □

LEMMA 19. If the assumptions (F_1)-(F_{11}) hold, then the Hausdorff dimension $d_H(K(\tau))$, $\tau \in R$ of the kernel section $K(\tau)$ of the process $\{U_\varepsilon(t,\tau),\ t \geq \tau\}$ generated by system (18) in E satisfies

$$d_H(K(\tau)) \leq \min\left\{ m \,\middle|\, m \in N,\quad \frac{1}{m}\sum_{j=1}^{m} \lambda_j^{-4\nu_0} < \frac{2\alpha\sigma}{k} \right\} \leq \left[\left(\frac{c'\lambda_1 k}{2\alpha\sigma} \right)^{\frac{3}{8\nu_0}} \right] + 1, \tag{85}$$

where $k = k(r_0)$ is a positive constant, σ, ε, c', ν_0, are defined in Theorem I.

Proof. It is easy to obtain (82) by taking the inner product of (79) with Ψ in E, and the function $C(t-\tau)$ is an exponential function of $(t-\tau)$.

Let $m \in N$ be fixed. Consider m solutions Ψ_1, Ψ_2, $\cdots$, Ψ_m of (79). At a given time $q \geq \tau$, let $Q_m(q)$ denote the orthogonal projection in E onto the space span $\{\Psi_1(q), \Psi_2(q), \cdots, \Psi_m(q)\}$. Let $\Phi_j(q) = (\xi_j, \hat{\eta}_j, \zeta_j)^T \in E$, $\quad j = 1, 2, \cdots, m$, be an orthonormal basis of $Q_m(q)E$.

Suppose $\varphi(\tau) = (u(\tau), w(\tau), \eta(\tau))^T \in K(\tau) \subset B_0$; then $||\varphi(t)||_E \leq r_1$, $t \geq \tau$ (r_1 is defined by Corollary 5). By Lemma 18 and $||\Phi_j||_E = 1$, we have

$$\begin{aligned} -(H'_\varphi(\varphi(q))\Phi_j(q), \Phi_j(q))_E \leq &-\frac{\varepsilon}{2}(||\xi||_1^2 + ||\hat{\eta}||_0^2) - \frac{\delta}{4}||\zeta||_{\Re}^2 - \frac{\alpha}{2}||\hat{\eta}_j||_0^2. \\ \leq &-\sigma - \frac{\alpha}{2}||\hat{\eta}_j||_0^2, \qquad q \geq \tau. \end{aligned} \tag{86}$$

On the other hand,

$$(F'_\varphi(\varphi(q), q)\Phi_j(q), \Phi_j(q))_E = (-f'_u(u(q), q)\xi_j(q), \eta_j(q))_{L^2} \leq ||f'_u(u(q), q)\xi_j(q)||_0 \cdot ||\eta_j||_0.$$

By (F_6)–(F_7), Lemma 9, Lemma 10, Young's inequality, Holder's inequality, and the embedding theorem,

$$\begin{aligned} ||f'_u(u(q), q)\xi_j(q)||_0^2 \leq & c_{43} \int_\Omega [1 + (u_L(q) + u_N(q))^2]^2 \xi_j^2(q) dx \\ \leq & c_{44} \int_\Omega [1 + u_L^4(q) + u_N^4(t)] \xi_j^2(q) dx \\ \leq & c_{45} [||u_L(q)||_{L^6}^4 ||\xi_j(q)||_{L^6}^2 + ||\xi_j(q)||_0^2 + ||u_N(q)||^4_{L^{\frac{6}{1-4\nu_0}}} ||\xi_j(q)||^2_{L^{\frac{6}{1+8\nu_0}}}] \\ \leq & c_{46}(r_0)[e^{-4\sigma_1(r_0)(q-\tau)}||\xi_j(q)||_1^2 + ||A^{\nu_0+\frac{1}{2}} u_N(q)||_0^4 ||A^{\frac{1-4\nu_0}{2}} \xi_j(q)||_0^2] \\ \leq & c_{47}(r_0)[e^{-4\sigma_1(r_0)(q-\tau)}||\xi_j(q)||_1^2 + ||A^{\frac{1-4\nu_0}{2}} \xi_j(q)||_0^2], \quad \forall q \geq \tau, \end{aligned}$$

where ν_0 is as in Lemma 10. Thus, there exists a constant $k = k(r_0) > 0$ such that

$$\begin{aligned} &(F'_\varphi(\varphi(q), q)\Phi_j(q), \Phi_j(q))_E \\ &\leq \frac{k(r_0)}{2\alpha}[e^{-4\sigma_1(r_0)(q-\tau)}||\xi_j(q)||_1^2 + ||A^{\frac{1-4\nu_0}{2}} \xi_j(q)||_0^2] + \frac{\alpha}{2}||\eta_j(\tau)||_0^2, \ \forall q \geq \tau. \end{aligned} \tag{87}$$

Thus, by $||\xi_j(q)||_1^2 \le ||\Phi_j(q)||_E^2 = 1$, (86)–(87), we have

$$
\begin{aligned}
q_m &= \lim_{t\to+\infty} \inf \sup_{\tau\in R} \sup_{\Phi\subset E} \sup_{\varphi(\tau)\in K(\tau)} \frac{1}{t}\int_\tau^{\tau+t} \sum_{j=1}^m ((-H'(\varphi(q))+F'_\varphi(\varphi(q),q))\Phi_j(q),\Phi_j(q))_E dq \\
&\le \lim_{t\to+\infty} \left(-m\sigma + \frac{k(r_0)m}{8\sigma_1(r_0)\alpha t}(1-e^{-4\sigma_1(r_0)t}) + \frac{k(r_0)}{2\alpha}\sum_{j=1}^m \lambda_j^{-4\nu_0} \right) \\
&\le -\frac{mk(r_0)}{2\alpha}\left(\frac{2\alpha\sigma}{k(r_0)} - \frac{1}{m}\sum_{j=1}^m \lambda_j^{-4\nu_0} \right).
\end{aligned}
$$

If

$$
\frac{1}{m}\sum_{j=1}^m \lambda_j^{-4\nu_0} < \frac{2\alpha\sigma}{k(r_0)},
$$

then $q_m < 0$. By Lemma 16, the first inequality in (85) is proved. By Remark VI. 6.1 of [12],

$$
\frac{1}{m}\sum_{j=1}^m \lambda_j^{-4\nu_0} \le c'\lambda_1 m^{-\frac{8\nu_0}{3}},
$$

where the constant c' is as in Theorem I. The second inequality in (85) is obtained provided that

$$
c'\lambda_1 m^{-\frac{8\nu_0}{3}} < \frac{2\alpha\sigma}{k(r_0)}, \quad \text{i.e.,} \quad m > \left(\frac{c'\lambda_1 k}{2\alpha\sigma}\right)^{\frac{3}{8\nu_0}}.
$$

Combining with Lemma 1, Lemma 14, and Lemma 19, the proof of Theorem I is completed. □

It is easy to see that Theorem II is a simple corollary of Theorem I. In the following, we state that the upper bound of the Hausdorff dimension decreases as the damping grows for suitable large damping.

If the damping $h(v) = \alpha v$ is linear and the functions f, g are independent of t, then the problem (5) reduces to the following autonomous system with the initial value at $\tau = 0$ in Hilbert space E :

$$
\begin{cases} Z_t = L_0(Z) + N_0(Z), & (x,s)\in\Omega\times R^+, \ t\ge 0, \\ Z(0) = Z_0 = (u_0(x), v_0(x), \eta_0(x,s))^T, & (x,s)\in\Omega\times R^+, \end{cases} \tag{88}
$$

where $Z = (u, v, \eta)^T$,

$$
L_0(Z) = \begin{pmatrix} v \\ \triangle u - \alpha v + \int_0^\infty \mu(s)\triangle\eta(s)ds \\ v - \eta_s \end{pmatrix}, \quad N_0(Z) = \begin{pmatrix} 0 \\ -f(u) + g(x) \\ 0 \end{pmatrix}. \tag{89}
$$

Obviously, the solutions of autonomous system (88) define a semigroup on E :

$$
S(t) : (u_0, v_0, \eta_0)^T \to (u(t), v(t), \eta(t))^T, \quad E\to E, \quad t\ge 0. \tag{90}
$$

For fixed $\alpha_0 > 0$, setting $\alpha_2 = \max\{\alpha_0, \frac{\lambda_1}{\delta}[d_0 - ||\mu||_{L^1(\Omega)}]\}$, where $d_0 = \min(\frac{1}{2}, c_{01}.c_{02})$. It is possible to check from the proof of Lemma 3 that if $\alpha \geq \alpha_2$, then the radius number $\tilde{r}_0$ of the bounded absorbing set $\tilde{B}_0$ for semigroup $\{S(t)\}_{t\geq 0}$ can be chosen as

$$\tilde{r}_0 = 2\sqrt{\frac{1}{d_0}[b_0||g||_0^2 + 4(k_1 c_6 + 2k_2)]}, \; c_6, \; k_1, \; k_2 \text{ are in (29)}, \tag{91}$$

$$b_0 = \frac{3}{\alpha_2^2} + \frac{\kappa}{\alpha_2} + \frac{1}{\lambda_1} + \sqrt{\left(\frac{3}{\alpha_2^2} + \frac{\kappa}{\alpha_2} + \frac{1}{\lambda_1}\right)^2 - \frac{12\kappa}{\alpha_2^3}} \tag{92}$$

which is independent of α.

Set $\alpha_3 = \max\{\frac{\lambda_1}{\delta}[2 - ||\mu||_{L^1(\Omega)}], \frac{\lambda_1}{\delta}[d_1(\tilde{r}_0) - ||\mu||_{L^1(\Omega)}]\}$, where $d_1(\tilde{r}_0) = \min\{1, \frac{1}{2c_{10}(\tilde{r}_0)}\}$, $c_{10}(\tilde{r}_0) = c_{48}(\tilde{r}_0^2 + 1)$ is defined by (38) in which c_{47} is constant depending on c_1, embedding constant of $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ and λ_1. Set $\alpha_1 = \max\{\alpha_2, \alpha_3\}$. From the proof of Lemma 9 and Lemma 10, if $\alpha \geq \alpha_1$, we can carefully choose the constant $M_2(\tilde{r}_0)$ in Lemma 10 such that it is also independent of α, say,

$$\begin{aligned} M_2(\tilde{r}_0) =& 4c_{49}(\tilde{r}_0)\left(4b_0 e^{c_{49}(\tilde{r}_0)b_0}[1 + c_{51}(\tilde{r}_0)\sqrt{2b_0\pi}]\right)^{\frac{1}{2}} \\ &+ 16c_{52}(\tilde{r}_0)e^{c_{53}(\tilde{r}_0)b_0}[1 + c_{54}(\tilde{r}_0)\sqrt{2b_0\pi}]. \end{aligned} \tag{93}$$

Hence, the constant k_0 in the right side of inequality (11) is independent of α and $\varepsilon_0 \leq \frac{\delta}{2}$. Obviously, the upper bound on the right side of (11) is decreasing in α for $\alpha \geq \alpha_1$. Particularly, if

$$\alpha \geq \max\left\{\alpha_1, \; \frac{k_0\kappa\lambda_1}{2(\lambda_1^{4\nu_0+1} - k_0)}\right\},$$

then $\frac{2\alpha\varepsilon_0}{k(r_0)} - \lambda_1^{-4\nu_0} > 0$. Thus, for any unit element $\Phi = (\xi, \hat{\eta}, \zeta)^T \in E$,

$$q_1 \leq -\frac{k(r_0)}{2\alpha}\left(\frac{2\alpha\sigma}{k(r_0)} - \lambda_1^{-4\nu_0}\right) < 0,$$

hence, the first largest Lyapunov exponent μ_1 of semigroup $\{S(t), t \geq 0\}$ on attractor Θ is negative: $\mu_1 < 0$, which implies that the Hausdorff dimension of Θ is zero.

REMARK. In the autonomous case, if f, f' are uniformly bounded, i.e., $|f(s)| \leq a_0$ (*const*), $|f'(s)| \leq a_0$, $\forall s \in R$ and $g(x) \in H_0^1(\Omega)$, then we can prove the existence of a global attractor for the semigroup defined by (88) when the damping term $h(v)$ vanishes. Let's state this fact. In this case, the linear operator L_0 in (88) is $L_0 = \begin{pmatrix} 0 & I & 0 \\ \triangle & 0 & \int_0^\infty \mu(s) \triangle \cdot ds \\ 0 & I & -\partial_s \end{pmatrix}$. From [4], we know that the semigroup $e^{L_0 t}$, $t \geq 0$ generated by L_0 decays exponentially, that is, there exist positive constants ϖ, $\omega > 0$ such that $|||e^{L_0 t}||| \leq \varpi e^{-\omega t}$, $\forall t \geq 0$, where $||| \cdot |||$ denotes the norm of operator. By Lemma 1, the solution $Z(t) = S(t)Z_0$ of (88) can be expressed as $S(t)Z_0 = e^{L_0 t}Z_0 + \int_0^t e^{L_0(t-r)}N_0(S(r)Z_0)dr$, $\forall t \geq 0$. Then there exists a bounded set $\overline{B}_0$ attracting any bounded set B of E. Since $|f(s)| \leq a_0$, $\forall s \in R$, $g(x) \in L^2(\Omega)$, and $N_0(Z) = (0, g(x) -$

$f(u), 0)^T$, there exists a constant $a_1 > 0$ such that $||N_0(Z(r))||_E \leq a_1$. For any $Z_0 \in B$, we have

$$||S(t)Z_0||_E \leq \varpi e^{-\omega t}||Z_0||_E + \frac{\varpi a_1}{\omega}(1 - e^{-\omega t}), \quad \forall t \geq 0. \tag{94}$$

Thus we can choose $\overline{B}_0$ being a ball with radius $\bar{r} = \frac{\varpi a_1}{\omega}$. For any $Z_0 \in \overline{B}_0$, let $Z(t) = S(t)Z_0$ be a solution of (88) and let $Z_L = S_L(t)Z_0 = e^{L_0 t}Z_0 = (u_L, v_L, \eta_L)^T$ and $Z_N = S_N(t)Z_0 = \int_0^t e^{L_0(t-r)}N_0(S(r)Z_0)dr = (u_N, v_N, \eta_N)^T, \forall t \geq 0$; then $S(t) = S_L(t) + S_N(t)$, where $|||S_L(t)||| \leq \varpi e^{-\omega t}$, $\forall t \geq 0$ and $S_2(0)Z_0 = 0$. It is easy to see that $Z_N = S_N(t)Z_0$ satisfies:

$$\partial_t Z_N(t) = L_0(Z_N) + N_0(S(t)Z_0), \quad Z_N(0) = (0,0,0)^T, \quad \forall t \geq 0. \tag{95}$$

Let $\zeta = \partial_t Z_N(t) = (\partial_t u_N, \partial_t v_N, \partial_t \eta_N)^T$. By differentiating (95), $\partial_t \zeta = L_0(\zeta) + (0, -f'(u)v, 0)^T$, $\zeta(0) = (0, g(x) - f(u_0), 0)^T$, $\forall t \geq 0$. Thus $\zeta(t)$ can be expressed as

$$\zeta(t) = e^{L_0 t}\zeta(0) + \int_0^t e^{L_0(t-r)}(0, -f'(u(r))v(r), 0)^T dr, \quad \forall t \geq 0.$$

Here $||\zeta(0)||_E \leq a_1$, $|f'(s)| \leq a_0$, and $||v(t)||_0 \leq (\varpi + 1)\bar{r}$ imply that there exists a constant $a_2 > 0$ such that $||(0, -f'(u(t))v(t), 0)^T||_E \leq a_2$ for all $t \geq 0$. Similarly to (94), we have $||\zeta(t)||_E \leq \varpi a_1 + \frac{\varpi a_2}{\omega}$, $\forall t \geq 0$; hence, $\zeta(t)$, $v_N = \partial_t u_N$, $\partial_s \eta_N$ are uniformly bounded in E, $H_0^1(\Omega)$, $L^2_\mu(R_+; H_0^1(\Omega))$, respectively. Let $\xi = A^{1/2}Z_N$, by (95), $\partial_t \xi = L_0(\xi) + (0, A^{1/2}[g(x) - f(u)], 0)^T$, $\zeta(0) = (0,0,0)^T$, $\forall t \geq 0$. Thus, $\xi(t) = \int_0^t e^{L_0(t-r)}(0, A^{1/2}[g(x) - f(u)], 0)^T dr$, $\forall t \geq 0$. And $||(0, A^{1/2}[g(x) - f(u)], 0)^T||_E^2 = || - f'(u)\nabla u + \nabla g(x)]||_0^2 \leq 8[a_0^2\bar{r}^2 + ||g||_1^2] = a_3^2$, $t \geq 0$. Thus we obtain that $||\zeta(t)||_E = \left(||Au_N||^2 + ||A^{1/2}v_N||^2 + ||\eta_N||^2_{\mu,2}\right)^{1/2} \leq \frac{\varpi a_3}{\omega}$, $\forall t \geq 0$ which implies the corresponding results in Lemma 10, and finally we obtain the existence of the global attractor for the semigroup $\{S(t)\}_{t\geq 0}$.

Acknowledgments.
The author thanks Professor V. Pata very much for his useful discussion and suggestions. The author thanks the reviewer very much for his reading carefully this paper and his giving useful suggestions.

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