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KILLING TENSOR FIELDS ON SPACES OF CONSTANT CURVATURE

By

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Introduction.

A covariant symmetric tensor field ξ on a Riemannian manifold (M, g) is called a Killing tensor field if the symmetrization of the covariant derivative of ξ vanishes identically. A Killing tensor field of order 1 is nothing but a Killing 1-form, i. e. a 1-form corresponding to a Killing vector field under the duality by means of the Riemannian metric g. The space $K(M, g)$ of all Killing tensor fields on (M, g) becomes an algebra by the symmetric product. If the algebra $K(M, g)$ is generated by Killing 1-forms, then the algebra of all linear differential operators on M which commutes with the Laplacian of (M, g) is generated by Killing vector fields (cf. Theorem 1.1).

Sumitomo-Tandai [11] proved the generation of $K(S^n, g)$ by Killing 1-forms for the unit sphere S^n with the standard metric g, by means of the notion of pseudo-connections. This was also proved by C. Tsukamoto by representation theory of compact Lie groups. Sumitomo-Tandai [11] determined moreover the spectrum of the Lichnerowicz Laplacian Δ (Lichnerowicz [8]) on $K(S^n, g)$, by giving explicitly projection operators of $K(S^n, g)$ onto eigenspaces of Δ.

In this paper, for a two-point homogeneous space of constant curvature, we compute the dimension of the space of Killing tensor fields spanned by products of p Killing 1-forms, by making use of Bott's theorem (Bott [2]) on holomorphic vector bundles over generalized flag manifolds. Together with the upper bound given by Barbance [1] for the dimension of the space $K^p(M, g)$ of Killing tensor fields of order p on a general Riemannian manifold (M, g), we prove

If (M, g) is a two-point homogeneous space of constant sectional curvature with $\dim M=n$, *then the algebra $K(M, g)$ is generated by Killing 1-forms, and*

$$\dim K^p(M, g)=\frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}, \qquad p\geqq 0.$$

We give furthemore an alternative determination of the spectrum of Δ on $K^p(S^n, g)$, applying the theory of spherical functions of E. Cartan to the manifold

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of geodesics of (S^n, g).

§ 1. Killing tensor fields.

Let V be a finite dimensional vector space over $\boldsymbol{R}$ or $\boldsymbol{C}$. A linear endomorphism S_p of the p-th tensor product $\otimes^p V$ of V, called the symmetrization, is defined by

$$S_p(v_1\otimes\cdots\otimes v_p)=\frac{1}{p\,!}\sum_{\sigma\in\mathfrak{S}_p} v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(p)} \qquad \text{for} \quad v_i\in V,$$

where $\mathfrak{S}_p$ denotes the p-th symmetric group. We put

$$S^pV=\{s\in\otimes^pV;\ S_ps=s\}, \qquad p\geqq 0.$$

Then

$$S(V)=\sum_{p\geq 0} S^pV$$

becomes a commutative associative graded algebra by the symmetric product:

$$s\cdot t=S_{p+q}(s\otimes t) \qquad \text{for} \quad s\in S^pV,\ t\in S^qV.$$

Let V^* be the dual space of V. Then S^pV^* is identified with the space of symmetric p-multilinear forms on V by

$$(\xi_1\cdot\cdots\cdot\xi_p)(v_1,\cdots,v_p)=\frac{1}{p\,!}\sum_{\sigma\in\mathfrak{S}_p}\xi_{\sigma(1)}(v_1)\cdots\xi_{\sigma(p)}(v_p)$$

for $\xi_i\in V^*$, $v_i\in V$. It is also identified with the space of homogeneous polynomials on V of degree p by

$$(\xi\cdot\cdots\cdot\xi_p)(v)=\xi_1(v)\cdots\xi_p(v) \qquad \text{for} \quad v\in V.$$

Now let M be a (connected) smooth manifold. Then $S^p(T^*M)=\bigcup_{x\in M}S^p(T^*_xM)$ where T^*_xM denotes the dual space of the tangent space T_xM of M at x, has a natural structure of smooth vector bundle over M. Let $S^p(M)$ denote the space of all smooth sections of $S^p(T^*M)$. Then

$$S(M)=\sum_{p\geq 0}S^p(M)$$

becomes a commutative associative graded algebra over $\boldsymbol{R}$ by the symmetric product $\xi\cdot\eta$. Let $\mathcal{D}_p(M)$ be the space of all linear differential operators of order p acting on the space $C^\infty(M)$ of smooth functions on M. Then

$$\mathcal{D}(M)=\bigcup_{p\geq 0}\mathcal{D}_p(M)$$

becomes an associative filtered algebra over $\boldsymbol{R}$.

In what follows we assume that (M, g) is a Riemannian manifold, ∇ the Riemannian connection for g and $\langle\ ,\ \rangle$ the inner product of tensors over M defined by g. For $\xi, \eta \in S(M)$ with compact supports, the L^2-inner product $\langle\langle \xi, \eta \rangle\rangle$ is defied by

$$\langle\langle \xi, \eta \rangle\rangle = \int_M \langle \xi, \eta \rangle dv_g ,$$

where dv_g denotes the Riemannian measure for g. We define a linear differential operator $\delta^* : S(M) \to S(M)$ of order 1 with $\delta^* : S^p(M) \to S^{p+1}(M)$, $p \geqq 0$, by

$$\delta^* \xi = S_{p+1}(\nabla \xi) \qquad \text{for} \quad \xi \in S^p(M).$$

It is known (Sumitomo-Tandai [11]) that δ^* is a derivation on $S(M)$, i. e.

$$\delta^*(\xi \cdot \eta) = (\delta^* \xi) \cdot \eta + \xi \cdot (\delta^* \eta) \qquad \text{for} \quad \xi, \eta \in S(M). \tag{1.1}$$

The kernel of $\delta^* : S^p(M) \to S^{p+1}(M)$ is denoted by $K^p(M)$. An element of $K^p(M)$ is called a *Killing p-tensor field* on (M, g). For example, $K^0(M) = \boldsymbol{R}$ (constant functions) and $K^1(M)$ is the space of all Killing 1-forms on (M, g). Killing p-tensor fields for general p are characterized as follows (Sumitomo-Tandai [11]): Let $\xi \in S^p(M)$. Then $\xi \in K^p(M)$ if and only if

$$\xi(\gamma'(t)) = \text{constant for any geodesic } \gamma \text{ of } (M, g). \tag{1.2}$$

Thus $g \in S^2(M)$ is a Killing 2-tensor field. The formula (1.1) implies that

$$K(M) = \sum_{p \geqq 0} K^p(M)$$

is a graded subalgebra of $S(M)$. We define next $\tilde{K}(M)$ to be the subalgebra of $K(M)$ generated by all Killing 1-forms, and put $\tilde{K}^p(M) = S^p(M) \cap \tilde{K}(M)$. Then

$$\tilde{K}(M) = \sum_{p \geqq 0} \tilde{K}^p(M)$$

is a graded subalgebra of $K(M)$. The following theorem was proved by Sumitomo-Tandai [11] for the standard sphere.

THEOREM 1.1. *Let $\mathcal{K}(M)$ denote the subalgebra of $\mathcal{D}(M)$ generated by all Killing vector fields on (M, g). If $\tilde{K}(M) = K(M)$, then $\mathcal{K}(M)$ coincides with the centralizer in $\mathcal{D}(M)$ of the Laplacian Δ of (M, g).*

PROOF. Since any Killing vector field $X \in \mathcal{D}_1(M)$ commutes with Δ, $\mathcal{K}(M)$ is contained in the centralizer of Δ. So we prove

$$D \in \mathcal{D}_p(M), \; D\Delta = \Delta D \Rightarrow D \in \mathcal{K}(M), \tag{1.3}$$

by the induction on p. For this purpose we define a splitting $\xi \mapsto D_\xi$ of the exact sequence:

$$0 \longrightarrow \mathcal{D}_{p-1}(M) \longrightarrow \mathcal{D}_p(M) \xrightarrow{\sigma_p} S^p(M) \longrightarrow 0,$$

where σ_p is the symbol map which is regarded as $S^p(M)$-valued by the duality by means of the metric g, as follows.

$$D_\xi=\xi^{i_1\cdots i_p}\nabla_{i_1}\cdots\nabla_{i_p} \qquad \text{for} \quad \xi\in S^p(M).$$

Here $\xi^{i_1\cdots i_p}$ denotes the contravariant component of ξ, and Einstein convention is used. Then Ricci identity implies (cf. Sumitomo-Tandai [11])

$$[D_\xi, \Delta]\equiv 2D_{\delta*\xi} \mod \mathcal{D}_p(M). \tag{1.4}$$

Now let $D\in\mathcal{D}_0(M)$ with $D\Delta=\Delta D$. Then D is written as

$$Df=\phi f \qquad \text{for} \quad f\in C^\infty(M),$$

by some $\phi\in C^\infty(M)$. Applying $D\Delta=\Delta D$ to $f\in C^\infty(M)$, we get $f\Delta\phi-2\langle d\phi, df\rangle=0$, and hence $d\phi=0$. Thus ϕ=constant. Therefore (1.3) holds for $p=0$. Let next $D\in\mathcal{D}_p(M)$, $p\geqq 1$, with $D\Delta=\Delta D$, and put $\xi=\sigma_p(D)$. Then $D\equiv D_\xi \mod \mathcal{D}_{p-1}(M)$, and hence (1.4) and $D\Delta=\Delta D$ imply $\delta^*\xi=0$. Thus, from the assumption: $\tilde{K}(M)=K(M)$, we may find Killing 1-forms $\xi_1, \cdots, \xi_r$ and a homogeneous polynomial

$$F(x_1, \cdots, x_r)=\sum_{p_1+\cdots+p_r=p} a_{p_1\cdots p_r}x_1^{p_1}\cdots x_r^{p_r}$$

of degree p in r-variables such that $\xi=F(\xi_1, \cdots, \xi_r)$. Denoting by $X_1, \cdots, X_r$ the Killing vector fields corresponding to $\xi_1, \cdots, \xi_r$, we define

$$D'=D-F(X_1, \cdots, X_r).$$

Then $D'\in\mathcal{D}_{p-1}(M)$ by virtue of $\sigma_p(D)=\xi$, and $D'\Delta=\Delta D'$. Thus the induction hypothesis implies $D'\in\mathcal{K}(M)$, and hence $D\in\mathcal{K}(M)$. Therefore (1.3) holds for p. q. e. d.

The space $K^p(M)$ is always of finite dimension. Actually, Barbance [1] proved that

$$\dim K^p(M)\leqq\binom{n+p}{p}\binom{n+p-1}{p}-\binom{n+p}{p+1}\binom{n+p-1}{p-1} \tag{1.5}$$

$$=\frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$$

for any Riemannian manifold (M, g) with $\dim M=n$.

We recall next the definition of the *Lichnerowicz Laplacian* $\Delta: S(M)\to S(M)$. It is an elliptic linear differential operator of order 2 with $\Delta: S^p(M)\to S^p(M)$, $p\geqq 0$, defined by

$$(\Delta\xi)_{i_1\cdots i_p}=-\nabla^l\nabla_l\xi_{i_1\cdots i_p}+2\sum_{a<b}R^k_{i_a i_b}{}^l\xi_{i_1\cdots \overset{(a)}{k}\cdots \overset{(b)}{l}\cdots i_p}+\sum_a S_{i_a}{}^k\xi_{i_1\cdots\overset{(a)}{k}\cdots i_p}$$

$$\text{for} \quad \xi\in S^p(M),$$

where R and S are the Riemannian curvature tensor and the Ricci tensor for g, respectively. It is self-adjoint with respect to the L^2-inner product $\langle\langle\ ,\ \rangle\rangle$, and coincides on $S^0(M)=C^\infty(M)$ with the ordinary Laplacian Δ.

§ 2. Manifolds of geodesics for rank one symmetric spaces.

A Riemannian manifold (M, g) is called a *two-point homogeneous space* if for any $p, q, p', q' \in M$ with $d(p, q)=d(p', q')$, d being the Riemannian distance, there exists an isometry ϕ such that $\phi(p)=p'$ and $\phi(q)=q'$. It is known (Wang [14], Tits [13]) that if (M, g) is two-point homogeneous, (M, g) is a rank one symmetric space or a Euclidean space. If $\dim M=1$, i. e., if (M, g) is a circle or a Euclidean line, the structure of (M, g) is simple. So we assume throughout in this paper that a two-point homogenous space has always dimension $\geqq 2$.

Let (M, g) be a two-point homogeneous space. We fix an expression of M as a coset space by an almost effective symmetric pair $(G, K; \theta)$ with G locally isomorphic to the identity component $I^0(M, g)$ of the group of isometries of (M, g) (cf. Helgason [5]), i. e., (G, K) is an almost effective pair of a connected Lie group G locally isomorphic to $I^0(M, g)$ and a compact subgroup K of G such that we have an identification $G/K=M$, under which G acts on M as isometries of g. And θ is an involutive automorphism of G such that the fixed point set G_θ of θ satisfies $G_\theta^0 \subset K \subset G_\theta$, G_θ^0 being the identity component of G_θ. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebra Lie G of G and Lie K, respectively. We define

$$\mathfrak{m}=\{X \in \mathfrak{g}\,;\ \theta X=-X\},$$

where the differential of θ is also denoted by θ. Then we have the Cartan decomposition $\mathfrak{g}=\mathfrak{k}+\mathfrak{m}$, and thus $\mathfrak{m}$ is identified with the tangent space T_oM of M at the origin $o=K$. The subgroup K acts on $\mathfrak{m}$ as isometries of the Riemannian metric g_o at o. Note that (M, g) is two-point homogeneous if and only if K acts transitively on the unit sphere of $(\mathfrak{m}, g_o)$. Let $r=\dim \mathfrak{g}$.

Let γ_1, γ_2 be geodesics of (M, g) (defined on $\boldsymbol{R}$ and parametrized by arc-length). They are said to be *oriented equivalent* (resp. *equivalent*) if there exist $t_1, t_2 \in \boldsymbol{R}$ such that $\gamma_1(t_1)=\gamma_2(t_2)$ and $\gamma_1'(t_1)=\gamma_2'(t_2)$ (resp. $\gamma_1'(t_1)=\pm\gamma_2'(t_2)$). The oriented equivalence class containing a geodesic γ is denoted by $[\gamma]$. The set of all oriented equivalence classes (resp. equivalence classes) of geodesics of (M, g) is denoted by $\hat{M}_0$ (resp. by $\hat{M}$). Note that G acts on $\hat{M}_0$ and $\hat{M}$ transitively in a natural way. Moreover $\boldsymbol{Z}_2$ acts freely on $\hat{M}_0$ from the right in a natural way (reversing the orientation) in such a way that $\hat{M}$ is identified with the quotient $\hat{M}_0/\boldsymbol{Z}_2$. We study in the following the structure of $\hat{M}_0$ and $\hat{M}$.

Choose $H_0 \in \mathfrak{m}$ such that $g_o(H_0, H_0)=1$ and define a geodesic γ_0 by

$$\gamma_0(t)=(\exp tH_0)\cdot o \qquad \text{for} \quad t\in\boldsymbol{R}.$$

Let $\mathfrak{a}=\boldsymbol{R}H_0$ and A the connected (closed) subgroup of G generated by $\mathfrak{a}$. Moreover put

$$K_0=\{k\in K;\ \mathrm{Ad}(k)H_0=H_0\},\quad \mathfrak{k}_0=\mathrm{Lie}\,K_0.$$

Then $G_0=K_0A$ is a closed subgroup of G such that $\mathrm{Lie}\,G_0$ is $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{a}$. Note that G_0 is a subgroup of the centralizer $Z_G(A)$ of A.

THEOREM 2.1. *Let (M, g) be a two-point homogeneous space. Then a G-equivariant bijection $G/G_0\to\hat{M}_0$ is defined by the correspondence:*

$$aG_0\mapsto a\cdot[\gamma_0] \qquad \textit{for} \quad a\in G.$$

Thus $\hat{M}_0$ and $\hat{M}$ have natural structures of smooth G-manifolds.

PROOF. Let $\pi: UM\to M$ denote the unit tangent bundle of (M, g). Since (M, g) is two-point homogeneous, G acts transitively on UM in a natural way, and the map $G/K_0\to UM$ defined by $aK_0\mapsto a\cdot\gamma_0'(0)=a\cdot H_0$ $(a\in G)$ is a G-diffeomorphism.

We show that under this G-diffeomorphism the action of the geodesic flow ϕ_t on UM corresponds to the natural right action of $a_t=\exp tH_0\in A$ on G/K_0 defined by

$$(aK_0)\cdot a_t=aa_tK_0 \qquad \text{for} \quad a\in G.$$

For $u\in UM$, the geodesic γ of (M, g) with $\gamma(0)=\pi(u)$, $\gamma'(0)=u$ will be denoted by γ_u. Then, by definition $\phi_t u=\gamma_u'(t)$. Let $u=a\cdot H_0$ $(a\in G)$. Then $\gamma_u(t)=a\cdot\gamma_0(t)=(a\exp tH_0)\cdot o$, and hence $\gamma_u'(t)=(aa_t)\cdot H_0$. This shows the claim.

Now the assertion follows from the fact that for $u, u'\in UM$, γ_u is oriented equivalent to $\gamma_{u'}$ if and only if $\phi_t u=u'$ for some $t\in\boldsymbol{R}$. q. e. d.

In what follows in this section, we assume that (M, g) is a rank one symmetric space. In this case G is semisimple, and so there exists uniquely a non-degenerate G-invariant symmetric bilinear from B on $\mathfrak{g}$ such that $B(\mathfrak{k}, \mathfrak{m})=0$ and $B|\mathfrak{m}\times\mathfrak{m}=g_o$. Note that B is positive-definite if and only if (M, g) is of compact type. Let

$$S_B^{r-1}=\{X\in\mathfrak{g};\ B(X, X)=1\},$$

and $P_{r-1}(\boldsymbol{R})$ the real projective space associated to $\mathfrak{g}$. We denote by $\pi:\mathfrak{g}-\{0\}\to P_{r-1}(\boldsymbol{R})$ the natural projection. It is G-equivariant with respect to natural actions of G. For a geodesic γ of (M, g), let τ_t denote the transvection associated to the geodesic segment $\gamma|[0, t]$. Let X_γ be the Killing vector field generated by the 1-parameter group of isometries $\{\tau_t\}$. Then it depends only on the class

$[\gamma]$. So it will be denoted by $X_{[\gamma]}$. Identifying $\mathfrak{g}$ with the space of all Killing vector fields on (M, g), we define a map $\iota_0 : \hat{M}_0 \to \mathfrak{g}$ by

$$\iota_0[\gamma] = X_{[\gamma]} \quad \text{for} \quad [\gamma] \in \hat{M}_0.$$

Under these notations we have the following theorem.

THEOREM 2.2. *Let (M, g) be a rank one symmetric space. Then*

1) *The map $\iota_0 : \hat{M}_0 \to \mathfrak{g}$ is a G-equivariant imbedding such that $\iota_0(\hat{M}_0) \subset S_B^{r-1}$ and $\iota_0[\gamma_0] = H_0$;*

2) *The composite $\pi \circ \iota_0 : \hat{M}_0 \to P_{r-1}(\boldsymbol{R})$ induces a G-equivariant imbedding $\iota : \hat{M} \to P_{r-1}(\boldsymbol{R})$.*

PROOF. 1) We show first that under the identification $G/G_0 = \hat{M}_0$ our map ι_0 corresponds to the map $aG_0 \mapsto \mathrm{Ad}(a) H_0$ $(a \in G)$. Let $\gamma = \gamma_u$ with $u = a \cdot H_0$ $(a \in G)$. Then $\gamma(t) = (a \exp tH_0) \cdot o = \exp t(\mathrm{Ad}(a) H_0) a \cdot o$ and hence τ_t for γ is the left translation by $\exp t(\mathrm{Ad}(a) H_0)$. Therefore $X_\gamma = \mathrm{Ad}(a) H_0$, which shows the claim.

It remains therefore to show $G_0 = Z_G(A)$. Assume first that (M, g) is of compact type. In this case G is compact and hence $Z_G(A)$ is connected by Hopf's theorem (cf. Helgason [5]). Since $G_0 \subset Z_G(A)$ and $\mathrm{Lie}\, Z_G(A) = \mathfrak{g}_0$, we get $G_0 = Z_G(A)$. Assume next that (M, g) is of non-compact type. Let $\mathfrak{g}^u = \mathfrak{k} + \sqrt{-1}\,\mathfrak{m}$, which is a compact real form of the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$, and G^u the compact simply connected Lie group with $\mathrm{Lie}\, G^u = \mathfrak{g}^u$. Denoting by σ the complex conjugation of $\mathfrak{g}^C$ with respect to $\mathfrak{g}$, we extend σ to a smooth automorphism σ of the complexification G^C of G^u such that $\sigma(G^u) = G^u$. We define a compact subgroup K^u of G^u with $\mathrm{Lie}\, K^u = \mathfrak{k}$ by

$$K^u = \{a \in G^u ; \sigma(a) = a\}, \tag{2.1}$$

which is known to be connected (E. Cartan [4]). Let G' be the connected subgroup of G^C generated by $\mathfrak{g}$. We have then an identification $M = G'/K^u$ since M is simply connected, and therefore we have an identification $\hat{M}_0 = G'/G'_0$ with $G'_0 = K_0^u A'$ by the previous construction for the pair (G', K^u). We show $G'_0 = Z_{G'}(A')$; this will imply that ι_0 is an imbedding, which means $G_0 = Z_G(A)$. We define a subgroup G_0^C of G^C with $\sigma(G_0^C) = G_0^C$ by

$$G_0^C = \{a \in G^C ; \mathrm{Ad}(a) H_0 = H_0\}.$$

Then G_0^C contains $Z_{G'}(A')$ and has the polar decomposition:

$$G_0^C = G_0^u \exp \sqrt{-1}\, \mathfrak{g}_0^u, \tag{2.2}$$

with $G_0^u = G_0^C \cap G^u$ and $\mathfrak{g}_0^u = \mathrm{Lie}\, G_0^u = \mathfrak{k}_0 + \sqrt{-1}\,\mathfrak{a}$, which are stable under σ. Let

$a\in Z_{G'}(A')$ be arbitrary. Decompose it by (2.2) as

$$a=a_0\exp\sqrt{-1}X_0,\quad a_0\in G_0^u,\quad X_0\in\mathfrak{g}_0^u.$$

Since $\sigma(a)=a$, we have $\sigma(a_0)=a_0$ and $\sigma X_0=-X_0$. Therefore $a_0\in K^u$ by (2.1) and $X_0\in\sqrt{-1}\mathfrak{a}$. Thus $a_0\in K_0^u$ and $\exp\sqrt{-1}X_0\in A'$, which implies $a\in K_0^uA'=G_0'$. This proves $G_0'=Z_{G'}(A')$.

2) This follows from that $\boldsymbol{Z}_2$ acts on $\mathfrak{g}-\{0\}$ from the right in a natural way and the map $\iota_0:\hat{M}_0\to\mathfrak{g}-\{0\}$ is $\boldsymbol{Z}_2$-equivariant. q.e.d.

We define

$$\hat{\mathfrak{m}}=\{X\in\mathfrak{g}\,;\,B(X,\mathfrak{g}_0)=0\}. \tag{2.3}$$

Then it is stable under G_0, $\mathfrak{g}=\mathfrak{g}_0+\hat{\mathfrak{m}}$ (direct sum as vector space) and $B|\hat{\mathfrak{m}}\times\hat{\mathfrak{m}}$ is a G_0-invariant non-degenerate symmetric bilinear form. In fact, since $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{a}$ with $B(\mathfrak{k}_0,\mathfrak{a})=0$ and both $B|\mathfrak{k}_0\times\mathfrak{k}_0$ and $B|\mathfrak{a}\times\mathfrak{a}$ are definite, $B|\mathfrak{g}_0\times\mathfrak{g}_0$ is non-degenerate. Thus the assertions follow.

Therefore B defines a normal homogeneous pseudo-Riemannian metric on $\hat{M}_0=G/G_0$, which will be denoted by $\hat{g}$. Note that $\hat{g}$ is Riemannian if and only if (M, g) is of compact type.

In the following we assume further that (M, g) is a compact rank one symmetric space, and identify as $\hat{M}_0\subset\mathfrak{g}$ and $\hat{M}\subset P_{r-1}(\boldsymbol{R})$ through the imbeddings ι_0 and ι, respectively. Let $(S\mathfrak{g}^*)^G$ denote the algebra of all G-invariant polynomials on $\mathfrak{g}$.

LEMMA 2.3. *There exist homogeneous elements $I_1,\cdots,I_{l-1}$ of $(S\mathfrak{g}^*)^G$, where $l=\operatorname{rank}\mathfrak{g}$, such that*

$$\hat{M}_0=\{X\in\mathfrak{g}\,;\,B(X)=1,\ I_i(X)=0 \textit{ for each } i,\ 1\leqq i\leqq l-1\}.$$

Here B is regarded as a homogeneous element of $(S\mathfrak{g}^*)^G$ of degree 2.

PROOF. If homogeneous elements $I_1,\cdots,I_{l-1}$ of $(S\mathfrak{g}^*)^G$ satisfy

$$B,\ I_1,\cdots,I_{l-1}\ \text{generate}\ (S\mathfrak{g}^*)^G, \tag{2.4}$$

$$B(H_0)=1,\quad I_i(H_0)=0\quad(1\leqq i\leqq l-1), \tag{2.5}$$

then they have the required property, since the correspondence:

$$X\mapsto{}^t(B(X),\ I_1(X),\cdots,I_{l-1}(X))\in\boldsymbol{R}^l$$

induces an injection from the orbit space $G\backslash\mathfrak{g}$ into $\boldsymbol{R}^l$ (cf. Helgason [5]). So we shall find $I_1,\cdots,I_{l-1}$ with (2.4) and (2.5).

Case (a): M is the n-sphere, real projective n-space with n even, quaternion projective n-space with $n\geqq 2$ or Cayley projective plane.

In this case the degrees of homogeneous generators of $(S\mathfrak{g}^*)^G$ are all even (cf. Bourbaki [3]). Choose $I_1', \cdots, I_{l-1}'$ such that $B, I_1', \cdots, I_{l-1}'$ generate $(S\mathfrak{g}^*)^G$, and suppose that $\deg I_i'=2n_i$ and $I_i'(H_0)=a_i$ $(1\leqq i\leqq l-1)$. Put

$$I_i=I_i'-a_iB^{n_i}\quad(1\leqq i\leqq l-1).$$

Then $I_1, \cdots, I_{l-1}$ have the properties (2.4) and (2.5).

Case (b): M is the n-sphere or real projective n-space with n odd.

We may assume (cf. § 3) that $\mathfrak{g}=\mathfrak{o}(n+1)$, $\mathfrak{k}=\mathfrak{o}(n)$ and

$$H_0=\left(\begin{array}{cc|c} 0 & -1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right).$$

We define a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ with $H_0\in\mathfrak{t}$ by

$$\mathfrak{t}=\left\{\begin{pmatrix} \begin{matrix} 0 & -\lambda_1 \\ \lambda_1 & 0 \end{matrix} & & & 0 \\ & \begin{matrix} 0 & -\lambda_2 \\ \lambda_2 & 0 \end{matrix} & & \\ & & \ddots & \\ 0 & & & \begin{matrix} 0 & -\lambda_l \\ \lambda_l & 0 \end{matrix} \end{pmatrix};\ \lambda_i\in\boldsymbol{R}\right\},\quad l=\frac{n+1}{2},$$

and regard each λ_i as an element of $\mathfrak{t}^*$. It is known that $(S\mathfrak{g}^*)^G$ is isomorphic to the algebra of W-invariant polynomials on $\mathfrak{t}$ by the restriction, where W is the Weyl group of $\mathfrak{g}$. Therefore there exist $I_1, \cdots, I_{l-1}\in(S\mathfrak{g}^*)^G$ such that $I_i|\mathfrak{t}=(i+1)$-th elementary symmetric polynomial of $\lambda_1^2, \cdots, \lambda_l^2$ $(1\leqq i\leqq l-2)$ and $I_{l-1}|\mathfrak{t}=\lambda_1\cdots\lambda_l$. They have then the properties (2.4) and (2.5).

Case (c): M is the complex projective n-space with $n\geqq 2$.

We may assume that $\mathfrak{g}=\mathfrak{su}(n+1)$, $\mathfrak{k}=\mathfrak{s}(\mathfrak{u}(1)\oplus\mathfrak{u}(n))$ and

$$H_0=\sqrt{-1}\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline 0 & & 0 \end{array}\right).$$

In the same way as in Case (b), we define

$$\mathfrak{t}=\left\{\sqrt{-1}\begin{pmatrix}\begin{matrix} y_0 & x_0 \\ x_0 & y_0\end{matrix} & 0 \\ 0 & \begin{matrix} x_2 & & 0 \\ & \ddots & \\ 0 & & x_n\end{matrix}\end{pmatrix};\ \begin{matrix} x_0, y_0, x_2, \cdots, x_n\in \boldsymbol{R} \\ 2y_0+x_2+\cdots+x_n=0\end{matrix}\right\},$$

and put $\lambda_1=x_0+y_0$, $\lambda_2=x_2, \cdots, \lambda_n=x_n$, $\lambda_{n+1}=y_0-x_0$. Then there exist $I_1, \cdots, I_{l-1} \in (S\mathfrak{g}^*)^G$ with $l=n$ such that $I_i|\mathfrak{t}=(i+2)$-th elementary symmetric polynomial of $\lambda_1, \cdots, \lambda_{n+1}$ $(1\leqq i\leqq l-1)$. They have the required properties. q. e. d.

LEMMA 2.4. *We define*

$$C(\hat{M})=\{tY\ ;\ t\in \boldsymbol{R}-\{0\},\ Y\in\hat{M}_0\}.$$

Then we have

$$C(\hat{M})=\{X\in\mathfrak{g}-\{0\}\ ;\ I_i(X)=0 \quad \textit{for each } i,\ 1\leqq i\leqq l-1\}.$$

PROOF. Let $X\in\mathfrak{g}-\{0\}$ with $I_i(X)=0$ $(1\leqq i\leqq l-1)$. Then $B(X)>0$ since $X\neq 0$. Putting $t=\sqrt{B(X)}$, we define $Y=\frac{1}{t}X$. Then $B(Y)=1$ and $I_i(Y)=I_i(X)/t^{m_i}=0$ $(1\leqq i\leqq l-1)$, where $m_i=\deg I_i$. Therefore $X=tY$ with $Y\in\hat{M}_0$ by Lemma 2.3, and thus $X\in C(\hat{M})$.

Conversely, for $X=tY$ with $t\in\boldsymbol{R}-\{0\}$, $Y\in\hat{M}_0$, we have $I_i(X)=t^{m_i}I_i(Y)=0$ $(1\leqq i\leqq l-1)$. q. e. d.

Now Lemma 2.4 implies the following

THEOREM 2.5. *If (M, g) is a compact rank one symmetric space, then $\hat{M}$ is a real projective algebraic manifold defined by*

$$\hat{M}=\{(x)\in P_{r-1}(\boldsymbol{R})\ ;\ I_i(x)=0 \ \textit{for each } i,\ 1\leqq i\leqq l-1\},$$

where (x) denotes the 1-dimensional subspace of $\mathfrak{g}$ spanned by $x\in\mathfrak{g}-\{0\}$. If we denote by

$$J=\sum_{p\geqq 0} J^p\subset S(\mathfrak{g}^*)$$

the homogeneous ideal for $\hat{M}\subset P_{r-1}(\boldsymbol{R})$, then J^p coincides with the kernel of the restriction map $\iota_0^*: S^p\mathfrak{g}^*\to C^\infty(\hat{M}_0)$.

Let $P_{r-1}(\boldsymbol{C})$ be the complex projective space associated to the complexification $\mathfrak{g}^C$ of $\mathfrak{g}$, and $P_{r-1}(\boldsymbol{R})$ be regarded as a submanifold of $P_{r-1}(\boldsymbol{C})$. We identify $S^p\mathfrak{g}^*$ with a real form of $S^p(\mathfrak{g}^C)^*$, and define a complex projective algebraic set $\hat{M}^C$ of $P_{r-1}(\boldsymbol{C})$ with $\hat{M}=\hat{M}^C\cap P_{r-1}(\boldsymbol{R})$ by

$$\hat{M}^C=\{(z)\in P_{r-1}(\boldsymbol{C})\,;\ F(z)=0 \text{ for each } F\in J^p,\ p\geqq 0\}.$$

We denote by

$$J^C=\sum_{p\geqq 0}(J^C)^p\subset S((\mathfrak{g}^C)^*)$$

the homogeneous ideal for $\hat{M}^C\subset P_{r-1}(\boldsymbol{C})$. Each $(J^C)^p$ is stable under the complex conjugation $F\mapsto\bar{F}$ of $S^p(\mathfrak{g}^C)^*$ with respect to $S^p\mathfrak{g}^*$. We call $\hat{M}^C$ the *smooth complexification* of $\hat{M}$ if $\hat{M}^C$ is a connected complex submanifold of $P_{r-1}(\boldsymbol{C})$. Note that then for each $x\in\hat{M}$ there exists a holomorphic coordinate $\{z^i\}$ of $\hat{M}^C$ around x such that $\hat{M}$ is given by $\bar{z}^i=z^i$ around x.

LEMMA 2.6. *Suppose that $\hat{M}^C$ is the smooth complexification of $\hat{M}$. Then*

1) *We have*

$$J^p=\{F\in(J^C)^p\,;\ \bar{F}=F\}\,;$$

2) *Let L denote the holomorphic line bundle over $\hat{M}^C$ associated to a hyperplane section, and $\Gamma(\hat{M}^C, L^p)$ the space of all holomorphic sections of the p-th tensor product L^p of L. Suppose that the canonical map $\psi: S^p(\mathfrak{g}^C)^*\to\Gamma(\hat{M}^C, L^p)$ is surjective. Then we have*

$$\dim S^p\mathfrak{g}^*/J^p=\dim_C\Gamma(\hat{M}^C, L^p)\,.$$

PROOF. 1) It is obvious that J^p contains the right hand side. Let F be an arbitrary element of J^p. Then the holomorphic section $\psi(F)$ of L^p vanishes on $\hat{M}$. Since $\hat{M}^C$ is the smooth complexification of $\hat{M}$, $\psi(F)$ vanishes around a point of $\hat{M}$, and hence it vanishes on $\hat{M}^C$ by the maximum principle, which means $F\in(J^C)^p$. This shows that J^p is contained in the right hand side.

2) From the assumption we have

$$\dim_C S^p(\mathfrak{g}^C)^*/(J^C)^p=\dim_C\Gamma(\hat{M}^C, L^p)\,.$$

On the other hand, by 1) we have

$$\dim S^p\mathfrak{g}^*/J^p=\dim_C S^p(\mathfrak{g}^C)^*/(J^C)^p\,.$$

Thus we get the required equality. q. e. d.

§ 3. Manifolds of geodesics of spheres.

In this section we give explicitly $\hat{M}_0$ and $\hat{M}$ for the standard n-sphere M. In this case, $r=\dim\mathfrak{g}$ is given by $r=\frac{1}{2}n(n+1)$.

We recall first the Plücker imbedding of a Grassmann manifold. Let $\wedge^2\boldsymbol{R}^{n+1}$

denote the second exterior product of the Euclidean $(n+1)$-space $\boldsymbol{R}^{n+1}$. Note that $\dim \wedge^2 \boldsymbol{R}^{n+1}=r$. Making use of the standard inner product $(\ ,\ )$ of $\boldsymbol{R}^{n+1}$, we define an inner product $(\ ,\ )$ on $\wedge^2 \boldsymbol{R}^{n+1}$ by

$$(u\wedge v,\ x\wedge y)=(u,\ x)(v,\ y)-(v,\ x)(u,\ y).$$

We identify $\wedge^2 \boldsymbol{R}^{n+1}$ with the space $A_{n+1}(\boldsymbol{R})$ of all real alternating $(n+1)\times(n+1)$ matrices by the correspondence:

$$u\wedge v \mapsto v\,{}^t u - u\,{}^t v \qquad \text{for} \quad u,\ v\in \boldsymbol{R}^{n+1}.$$

The inner product $(\ ,\ )$ on $A_{n+1}(\boldsymbol{R})$ corresponding to $(\ ,\ )$ on $\wedge^2 \boldsymbol{R}^{n+1}$ is given by

$$(A,\ B)=-\frac{1}{2}\mathrm{Tr}(AB) \qquad \text{for} \quad A,\ B\in A_{n+1}(\boldsymbol{R}).$$

The unit sphere in $A_{n+1}(\boldsymbol{R})$ and the real projective space associated to $A_{n+1}(\boldsymbol{R})$ are denoted by S^{r-1} and $P_{r-1}(\boldsymbol{R})$, respectively.

Let $\tilde{G}_{2,n-1}(\boldsymbol{R})$ (resp. $G_{2,n-1}(\boldsymbol{R})$) denote the Grassmann manifold of all oriented 2-dimensional subspaces (resp. all 2-dimensional subspaces) of $\boldsymbol{R}^{n+1}$. We define an imbedding $\tilde{p}:\tilde{G}_{2,n-1}(\boldsymbol{R})\to A_{n+1}(\boldsymbol{R})$ as follows: For $P\in\tilde{G}_{2,n-1}(\boldsymbol{R})$, choose a positively oriented orthonormal basis $\{u,\ v\}$ of P. Then $u\wedge v\in A_{n+1}(\boldsymbol{R})$ depends only on P. We define

$$\tilde{p}(P)=u\wedge v.$$

The image $\tilde{p}(G_{2,n-1}(\boldsymbol{R}))$ is a compact smooth submanifold of S^{r-1}. The imbedding $\tilde{p}$ induces an imbedding $p: G_{2,n-1}(\boldsymbol{R})\to P_{r-1}(\boldsymbol{R})$, whose image $p(G_{2,n-1}(\boldsymbol{R}))$ is a real projective algebraic submanifold of $\boldsymbol{P}_{r-1}(\boldsymbol{R})$. In the following $\tilde{G}_{2,n-1}(\boldsymbol{R})$ and $G_{2,n-1}(\boldsymbol{R})$ will be identified with submanifolds of S^{r-1} and $P_{r-1}(\boldsymbol{R})$, respectively, through these imbeddings $\tilde{p}$ and p.

Now let M be the unit sphere:

$$M=\{x\in \boldsymbol{R}^{n+1};\ \sum_i x_i^2=1\},$$

with the metric g induced from the standard Riemannian metric $(\ ,\ )$ on $\boldsymbol{R}^{n+1}$. We take $G=SO(n+1)$ and

$$K=\left\{\begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix};\ \alpha\in SO(n)\right\}\cong SO(n).$$

We have then an identification $M=G/K$ such that the point ${}^t(1,0,\cdots,0)$ corresponds to the origin o. We have $\mathfrak{g}=\mathfrak{o}(n+1)=A_{n+1}(\boldsymbol{R})$, $\mathfrak{k}=\mathfrak{o}(n)$ and

$$\mathfrak{m}=\left\{\begin{bmatrix} 0 & -{}^t x \\ x & 0 \end{bmatrix};\ x\in \boldsymbol{R}^n\right\}.$$

Moreover $B(X, Y)=-\frac{1}{2}\mathrm{Tr}(XY)=(X, Y)$ for $X, Y\in\mathfrak{g}=A_{n+1}(\boldsymbol{R})$. We choose $H_0\in\mathfrak{m}$ as in Lemma 2.3, Case (b). Then

$$G_0=\left\{\begin{bmatrix}\alpha & 0\\ 0 & \beta\end{bmatrix};\ \begin{matrix}\alpha\in SO(2),\\ \beta\in SO(n-1)\end{matrix}\right\}\cong SO(2)\times SO(n-1).$$

The imbedding $\iota_0:\hat{M}_0\to\mathfrak{g}=A_{n+1}(\boldsymbol{R})$ is given by

$$\iota_0[\gamma]=\gamma(0)\wedge\gamma'(0) \qquad \text{for} \quad [\gamma]\in\hat{M}_0.$$

The image $\iota_0(\hat{M}_0)$ coincides with $\tilde{G}_{2,n-1}(\boldsymbol{R})$, and hence we have $\iota(\hat{M})=G_{2,n-1}(\boldsymbol{R})$.

§ 4. Killing tensor fields on spaces of constant curvature.

Let (M, g) be a two-point homogeneous space. As is seen in the proof of Theorem 2.1, the subgroup A of G acts on the unit tangent bundle UM from the right in such a way that $\hat{M}_0$ is diffeomorphic to the quotient UM/A. So we identify $C^\infty(\hat{M}_0)$ with the space $C^\infty(UM)^A$ of all smooth functions on UM which is invariant under A. We define the evaluation map $\varepsilon: S(M)\to C^\infty(UM)$ by regarding ξ_x as a polynomial on T_xM for each $\xi\in S(M)$ and $x\in M$. Note that the map ε is a G-homomorphism with respect to natural actions of G and that $K(M)$ is stable under G. By (1.2) the map ε induces a G-homomorphism $\varepsilon: K(M)\to C^\infty(\hat{M}_0)$. The map ε is injective on $S^p(M)$ or on $K^p(M)$. But it is not injective on $S(M)$ nor on $K(M)$. Actually we have the following lemma.

LEMMA 4.1. *The kernel of* $\varepsilon: K(M)\to C^\infty(\hat{M}_0)$ *coincides with* $(1-g)\cdot K(M)$.

PROOF. Suppose $\xi\in K(M)$ with $\varepsilon(\xi)=0$. At each $x\in M$, $\xi_x\in S(T_x^*M)$ vanishes on the unit sphere of T_xM. Therefore there exists uniquely $\eta_x\in S(T_x^*M)$ such that $(1-g_x)\cdot\eta_x=\xi_x$. Now $\{\eta_x\}_{x\in M}$ defines a section $\eta\in S(M)$ such that $(1-g)\cdot\eta=\xi$. By (1.1) we have

$$0=\delta^*\xi=\delta^*(1-g)\cdot\eta+(1-g)\cdot\delta^*\eta=(1-g)\cdot\delta^*\eta,$$

and hence $\eta\in K(M)$. Thus we have proved that the kernel of $\varepsilon: K(M)\to C^\infty(\hat{M}_0)$ is contained in $(1-g)\cdot K(M)$. The converse inclusion is obvious from the above argument. q. e. d.

LEMMA 4.2. *Let* (M, g) *be a rank one symmetric space. Then the G-homomorphism*:

$$S^p\mathfrak{g}^* \underset{B}{\cong} S^p\mathfrak{g} \underset{g}{\cong} S^p(K^1(M)) \underset{\mu}{\longrightarrow} \tilde{K}^p(M) \underset{\varepsilon}{\longrightarrow} C^\infty(\hat{M}_0)$$

coincides with $\iota_0^*: S^p\mathfrak{g}^*\to C^\infty(\hat{M}_0)$. *Here the first map (resp. the second map) is the* *duality by means of* B *(resp. by means of* g*) and* μ *is the multiplication. Therefore we have*

$$\iota_0^* S^p \mathfrak{g}^* = \varepsilon \tilde{K}^p(M).$$

PROOF. Let $\lambda \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$ correspond to λ by B, $\xi \in K^1(M)$ correspond to X by g. Let γ be a geodesic of (M, g). Choose $a \in G$ such that $\gamma(0) = a \cdot o$ and $\gamma'(0) = a \cdot H_0$. Then

$$\begin{aligned}\varepsilon(\xi)[\gamma] &= \xi(\gamma'(0)) = \langle X_{a\cdot o}, a \cdot H_0 \rangle \\ &= \langle a \cdot (\mathrm{Ad}(a^{-1})X)_o, a \cdot H_0 \rangle = B(\mathrm{Ad}(a^{-1})X, H_0) \\ &= B(X, \mathrm{Ad}(a) H_0) = \lambda(\mathrm{Ad}(a) H_0) \\ &= \lambda(\iota_0[\gamma]) = (\iota_0^* \lambda)[\gamma].\end{aligned}$$

Therefore we have $\varepsilon(\xi) = \iota_0^* \lambda$, which implies the assertion. q. e. d.

By Theorem 2.5 we have the following

COROLLARY. *If* (M, g) *is a compact rank one symmetric space,* $\tilde{K}(M)$ *is isomorphic to* $S(\mathfrak{g}^*)/J$. *In particular, we have*

$$\dim \tilde{K}^p(M) = \dim S^p \mathfrak{g}^* / J^p, \qquad p \geqq 0.$$

LEMMA 4.3. *Let* $M = S^n$ *be the unit sphere with the standard metric* g. *Then we have*

$$\dim \tilde{K}^p(M) = \frac{1}{n} \binom{n+p}{p+1} \binom{n+p-1}{p}, \qquad p \geqq 0.$$

PROOF. By § 3, $\hat{M} = G_{2, n-1}(\boldsymbol{R}) \subset P_{r-1}(\boldsymbol{R})$. Thus $\hat{M}^C$ is the complex Grassmann manifold $G_{2, n-1}(\boldsymbol{C})$ of all 2-dimensional subspaces of $\boldsymbol{C}^{n+1}$ imbedded in the complex projective space $P_{r-1}(\boldsymbol{C})$ associated to the space $A_{n+1}(\boldsymbol{C})$ of complex alternating $(n+1) \times (n+1)$ matrices. Therefore $\hat{M}^C$ is the smooth complexification of $\hat{M}$. In thie case the canonical map $\psi: S^p(\mathfrak{g}^C)^* \to \Gamma(\hat{M}^C, L^p)$ is surjective for each $p \geqq 0$ (cf. Sakane-Takeuchi [10]), and so we may apply Lemma 2.6 to get $\dim S^p \mathfrak{g}^* / J^p = \dim_C \Gamma(\hat{M}^C, L^p)$. Therefore, by the above Corollary we have

$$\dim \tilde{K}^p(M) = \dim_C \Gamma(\hat{M}^C, L^p).$$

Now $\dim_C \Gamma(\hat{M}^C, L^p)$ is computed as follows. We take the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(n+1, \boldsymbol{C})$ consisting of all diagonal matrices in $\mathfrak{sl}(n+1, \boldsymbol{C})$. Then the real part $\mathfrak{h}_{\boldsymbol{R}}$ of $\mathfrak{h}$ is given by

$$\mathfrak{h}_{\boldsymbol{R}} = \left\{ \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_{n+1} \end{bmatrix} ;\ \lambda_i \in \boldsymbol{R},\ \sum \lambda_i = 0 \right\}.$$

We introduce a lexicographic order $>$ on $\mathfrak{h}_R^*$ by $\lambda_1 > \cdots > \lambda_n$. Then by Bott's theorem (Bott [2]), $\dim_C \Gamma(\hat{M}^C, L^p)$ is the degree of irreducible representation of $\mathfrak{sl}(n+1, \boldsymbol{C})$ with the highest weight $p(\lambda_1+\lambda_2)$, which is equal to

$$\frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}$$

by Weyl's degree formula. q. e. d.

THEOREM 4.4. *Let (M, g) be a two-point homogeneous space of constant sectional curvature with $\dim M=n$. Then the algebra $K(M)$ of Killing tensor fields on (M, g) is generated by Killing 1-forms, and*

$$\dim K^p(M) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p}, \qquad p \geqq 0.$$

Therefore (by Theorem 1.1) the centralizer in $\mathcal{D}(M)$ of the Laplacian is generated by Killing vector fields.

PROOF. We show first that for any open set U of M the restriction map $r: \tilde{K}^p(M) \to \tilde{K}^p(U)$, $p \geqq 0$, is an isomorphism. It is known (Barbance [1]) that the restriction $r: K^p(M) \to K^p(U)$ is injective for a general Riemannian manifold. In our case, $\dim K^1(M) = \dim \mathfrak{g} = \frac{1}{2}n(n+1)$ and $\dim K^1(U) \leqq \frac{1}{2}n(n+1)$ by (1.5), and hence $r: K^1(M) \to K^1(U)$ is an isomorphism. It follows that $r: \tilde{K}^p(M) \to \tilde{K}^p(U)$ is surjective, which implies the assertion.

Now, since our (M, g) is of constant curvature, it is locally projectively equivalent to the standard sphere S^n, i.e., there are open sets U of M and V of S^n and a diffeomorphism $\varphi: U \to V$ which maps a geodesic of U to a geodesic of V (up to parametrization). Now it is not hard to see that the correspondence $\xi \mapsto (\varphi^{-1})^*[(\varphi^* v_V / v_U)^{2/n+1}\xi]$, $v_\cdot$ being the volume element, gives an isomorphism $K^1(U) \to K^1(V)$. Thus $\tilde{K}^p(U)$ is isomorphic to $\tilde{K}^p(V)$ for each $p \geqq 0$. Therefore by the above fact we get $\dim \tilde{K}^p(M) = \dim \tilde{K}^p(S^n)$. Thus by Lemma 4.3 we obtain

$$\dim \tilde{K}^p(M) = \frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p},$$

and hence $\tilde{K}^p(M) = K^p(M)$ by (1.5). This implies the assertions of the theorem. q. e. d.

§ 5. Lichnerowicz Laplacian on symmetric spaces.

Let (M, g) be a symmetric space. Take a coset space expression $M=G/K$ as in the beginning of § 2. We decompose the pair $(\mathfrak{g}, \mathfrak{k})$ as the direct sum:

$$(\mathfrak{g}, \mathfrak{k})=(\mathfrak{g}_0, \mathfrak{k}_0)\oplus(\mathfrak{g}_1, \mathfrak{k}_1)$$

of the Euclidean part $(\mathfrak{g}_0, \mathfrak{k}_0)$ and the semisimple part $(\mathfrak{g}_1, \mathfrak{k}_1)$, with Cartan decompositions $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{m}_0$ and $\mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{m}_1$, respectively (cf. Helgason [5]). We further decompose $\mathfrak{m}_0$ as the direct sum:

$$\mathfrak{m}_0=\mathfrak{m}_0'\oplus\mathfrak{m}_0''$$

of the trivial part $\mathfrak{m}_0'$ and the non-trivial part $\mathfrak{m}_0''$ with respect to the action of $\mathfrak{k}_0$. We put

$$\mathfrak{g}'=\mathfrak{m}_0'\oplus\mathfrak{g}_1, \quad \mathfrak{k}'=\mathfrak{k}_1, \quad \mathfrak{m}'=\mathfrak{m}_0'\oplus\mathfrak{m}_1, \quad \mathfrak{g}''=\mathfrak{k}_0+\mathfrak{m}_0''.$$

We have then another decomposition:

$$(\mathfrak{g}, \mathfrak{k})=(\mathfrak{g}', \mathfrak{k}')\oplus(\mathfrak{g}'', \mathfrak{k}_0),$$

with Cartan decompositions $\mathfrak{g}'=\mathfrak{k}'+\mathfrak{m}'$ and $\mathfrak{g}''=\mathfrak{k}_0+\mathfrak{m}_0''$. Now there exists uniquely a $\mathfrak{g}'$-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}'$ such that $B(\mathfrak{k}', \mathfrak{m}')=0$ and $B|\mathfrak{m}'\times\mathfrak{m}'=g_o|\mathfrak{m}'\times\mathfrak{m}'$. Choosing basis $\{X_i\}$, $\{Y_i\}$ for $\mathfrak{g}'$ and a basis $\{Z_k\}$ for $\mathfrak{m}_0''$ such that $B(X_i, Y_j)=\delta_{ij}$ and $g_o(Z_k, Z_l)=\delta_{kl}$, we define an element C of the universal enveloping algebra of $\mathfrak{g}$ by

$$C=-\sum_i X_iY_i-\sum_k Z_k^2,$$

which is independent of the choice of basis. Then C acts on $C^\infty(G)$ as a two-sided invariant linear differential operator.

Let $\sigma: K\to GL(S^p\mathfrak{m}^*)$ denote the natural action of K on $S^p\mathfrak{m}^*$, and $R(k)$: $C^\infty(G)\to C^\infty(G)$ the right translation by $k\in K$, i.e., $(R(k)f)(a)=f(ak)$ for $f\in C^\infty(G)$, $a\in G$. Now K acts on $C^\infty(G)\otimes S^p\mathfrak{m}^*$ by the tensor product $R\otimes\sigma$, and the space $(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K$ of all K-invariants in $C^\infty(G)\otimes S^p\mathfrak{m}^*$ is canonically identified with $S^p(M)$. It is seen that $C\otimes 1$ leaves $(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K$ invariant. Under these definitions we have

THEOREM 5.1 (Koiso [7]). *For a symmetric space (M, g), the Lichnerowicz Laplacian Δ on $S^p(M)$ corresponds to $C\otimes 1$ on $(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K$ under the canonical identification $S^p(M)=(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K$.*

For $\xi\in S(\mathfrak{m}^*)$ (resp. $\xi\in S(M)$) the multiplication by ξ is denoted by $\mu(\xi)$, i.e., $\mu(\xi)\eta=\xi\cdot\eta$ for $\eta\in S(\mathfrak{m}^*)$ (resp. $\eta\in S(M)$). The action of $X\in\mathfrak{g}$ on $C^\infty(G)$ as a left invariant vector field is denoted by $\nu(X)$.

LEMMA 5.2. *Let $\{X_i\}$ be a basis for $\mathfrak{m}$ and $\{\xi_i\}$ the basis for $\mathfrak{m}^*$ dual to $\{X_i\}$, i.e., $\xi_i(X_j)=\delta_{ij}$. Then the operator $\delta^*: S^p(M)\to S^{p+1}(M)$ identified with the map $(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K\to(C^\infty(G)\otimes S^{p+1}\mathfrak{m}^*)^K$ is given by*

$$\delta^*=\sum_i \nu(X_i)\otimes\mu(\xi_i)\,.$$

PROOF. If we write $e(\xi)\tau=\xi\otimes\tau$ for $\xi\in\mathfrak{m}^*$ and $\tau\in\otimes^p\mathfrak{m}^*$, the covariant derivation $\nabla: C^\infty(\otimes^p T^*M)\to C^\infty(\otimes^{p+1}T^*M)$ identified with the map

$$(C^\infty(G)\otimes(\otimes^p\mathfrak{m}^*))^K\to(C^\infty(G)\otimes(\otimes^{p+1}\mathfrak{m}^*))^K$$

is given (cf. Koiso [7]) by

$$\nabla=\sum_i \nu(X_i)\otimes e(\xi_i)\,.$$

Thus $\delta^*=S_{p+1}\nabla$ is given by the above formula. q. e. d.

THEOREM 5.3. (Sumitomo-Tandai [11]). *Let (M, g) be a locally symmetric space. Then*

1) *$\Delta\delta^*=\delta^*\Delta$. Therefore $\Delta K^p(M)\subset K^p(M)$ for each $p\geqq 0$;*
2) *$\Delta\mu(g)=\mu(g)\Delta$.*

PROOF. We may assume that (M, g) is a symmetric space.

1) Since $\nu(X)C=\nu(X)C$ on $C^\infty(G)$ for each $X\in\mathfrak{g}$, by Theorem 5.1 and Lemma 5.2 we get $\Delta\delta^*=\delta^*\Delta$.

2) Since the operator $\mu(g): S^p(M)\to S^{p+2}(M)$ identified with the map $(C^\infty(G)\otimes S^p\mathfrak{m}^*)^K\to(C^\infty(G)\otimes S^{p+2}\mathfrak{m}^*)^K$ is given by $\mu(g)=1\otimes\mu(g_o)$, by Theorem 5.1 we get $\Delta\mu(g)=\mu(g)\Delta$. q. e. d.

Now let (M, g) be two-point homogeneous. We use the notation in the previous sections. The space $C^\infty(G)^{K_0}$ (resp. $C^\infty(G)^{G_0}$) of all smooth functions on G which is invariant under the right translation by K_0 (resp. by G_0) will be identified with $C^\infty(UM)$ (resp. with $C^\infty(\hat{M}_0)$). Then the map $\varepsilon_{H_0}: C^\infty(G)\otimes S^p\mathfrak{m}^*\to C^\infty(G)$ defined by $\varepsilon_{H_0}(f\otimes\xi)=\xi(H_0)f$ $(f\in C^\infty(G),\ \xi\in S^p\mathfrak{m}^*)$ induces the map $\varepsilon_{H_0}: (C^\infty(G)\otimes S^p\mathfrak{m}^*)^K\to C^\infty(G)^{K_0}$, which corresponds to the evaluation map $\varepsilon: S^p(M)\to C^\infty(UM)$.

We assume further that (M, g) is a compact rank one symmetric space. We denote by $\varpi: C^\infty(UM)\to C^\infty(\hat{M}_0)$ the orthogonal projection with respect to the L^2-inner product $\langle\!\langle\,,\rangle\!\rangle$. If it is identified with the map $C^\infty(G)^{K_0}\to C^\infty(G)^{G_0}$, then

$$(\varpi f)(b)=\int_A f(ba)da \qquad \text{for}\quad f\in C^\infty(G)^{K_0},\ b\in G,$$

where da is the normalized Haar measure of the total subgroup A. Both $C^\infty(G)^{K_0}$ and $C^\infty(G)^{G_0}$ are stable under C, and we have $C\varpi=\varpi C$ on $C^\infty(G)^{K_0}$, which follows from the above expression for ϖ.

Now it is known that any geodesic of a compact rank one symmetric space

(M, g) is periodic and has the same period, say l. We define a linear map $\wedge: S(M)\rightarrow C^{\infty}(\hat{M}_0)$, called the *Randon transform*, by

$$\hat{\xi}([\gamma])=\frac{1}{l}\int_0^l \xi(\gamma'(t))dt \qquad \text{for} \quad \xi\in S^p(M),\ [\gamma]\in\hat{M}_0.$$

The following lemma is an immediate consequence of definitions.

LEMMA 5.4. *Let (M, g) be a compact rank one symmetric space. Then the composite $S^p(M) \xrightarrow{\varepsilon} C^{\infty}(UM) \xrightarrow{\varpi} C^{\infty}(\hat{M}_0)$ coincides with the Radon transform on $S^p(M)$. In particular, the evaluation $\varepsilon: K^p(M)\rightarrow C^{\infty}(\hat{M}_0)$ coincides with the Radon transform on $K^p(M)$.*

The following theorem was proved by Sumitomo-Tandai [11] for standard spheres, and by Michel [9] for $p=2$.

THEOREM 5.5. *Let (M, g) be a compact rank one symmetric space, $\hat{\Delta}$ the Laplacian of $(\hat{M}_0, \hat{g})$, where $\hat{g}$ is the Riemannian metric on $\hat{M}_0$ defined in §2. Then*

$$\widehat{\Delta\xi}=\hat{\Delta}\hat{\xi} \qquad \textit{for each} \quad \xi\in S^p(M).$$

PROOF. Since $C\varpi=\varpi C$ on $C^{\infty}(G)^{K_0}$, the following diagram is commutative.

$$\begin{array}{ccccc}
(C^{\infty}(G)\otimes S^p\mathfrak{m}^*)^K & \xrightarrow{\varepsilon_{H_0}} & C^{\infty}(G)^{K_0} & \xrightarrow{\varpi} & C^{\infty}(G)^{G_0} \\
\downarrow{\scriptstyle C\otimes 1} & & & & \downarrow{\scriptstyle C} \\
(C^{\infty}(G)\otimes S^p\mathfrak{m}^*)^K & \xrightarrow{\varepsilon_{H_0}} & C^{\infty}(G)^{K_0} & \xrightarrow{\varpi} & C^{\infty}(G)^{G_0}.
\end{array}$$

On the other hand, since $\hat{g}$ is a normal homogeneous Riemannian metric on $\hat{M}_0$, the operator C on $C^{\infty}(G)^{G_0}$ corresponds to the Laplacian $\hat{\Delta}$ on $C^{\infty}(\hat{M}_0)$. Therefore, by Theorem 5.1 and Lemma 5.4 we get the required equality. q. e. d.

We define

$$P^p(M)=\{\xi\in K^p(M);\ \langle\langle\xi,\ g\cdot K^{p-2}(M)\rangle\rangle=0\}, \qquad p\geqq 0,$$

under the convention: $K^p(M)=0$ for $p<0$. An element of $P^p(M)$ is called a *primitive Killing p-tensor field* on (M, g). From Theorem 5.3 and the self-adjointness of Δ, we have $\Delta P^p(M)\subset P^p(M)$. Recall that $K^p(M)$ is stable under G, and hence $P^p(M)$ is also stable under G. We put

$$P(M)=\sum_{p\geqq 0} P^p(M).$$

It is seen by the induction on p that

$$K^{p}(M)=\sum_{0\leq k\leq[p/2]}g^{k}\cdot P^{p-2k}(M)\,. \tag{5.1}$$

Therefore, if we denote the spectrum of Δ by Spec Δ, by Theorem 5.3 we have

$$\mathrm{Spec}\,\Delta\ \text{ on }\ K^{p}(M)=\bigcup_{0\leq k\leq[p/2]}(\mathrm{Spec}\,\Delta\ \text{ on }\ P^{p-2k}(M))\,. \tag{5.2}$$

LEMMA 5.6. *The evaluation map* $\varepsilon: P(M)\rightarrow C^{\infty}(\hat{M}_{0})$ *on* $P(M)$ *is injective.*

PROOF. Suppose $\xi\in P(M)$, $\varepsilon(\xi)=0$. Assuming that

$$\xi=\xi_{0}+\xi_{1}+\cdots+\xi_{p}\,,\quad \xi_{i}\in P^{i}(M)\,,\quad \xi_{p}\neq 0\,,$$

we shall lead to a contradiction. By Lemma 4.1 there exists $\eta\in K(M)$ such that $\xi=(1-g)\cdot\eta$. Therefore we have $p\geqq 2$. Now η can be written as

$$\eta=\eta_{0}+\eta_{1}+\cdots+\eta_{p-2}\,,\quad \eta_{i}\in K^{i}(M)\,.$$

Then we have $\xi_{p}=g\cdot\eta_{p-2}$. From $\langle\langle P^{p}(M), g\cdot K^{p-2}(M)\rangle\rangle=0$ we get $\xi_{p}=0$. This is a contradiction. q. e. d.

In the following, for various real vector spaces V defined previously, the complexification of V will be denoted by V^{C}, and the $\boldsymbol{C}$-linear extensions of various real linear maps will be denoted by the same notation.

We denote by $\mathcal{S}(\hat{M}_{0})$ the space of all functions $f\in C^{\infty}(\hat{M}_{0})^{C}$ such that the $\boldsymbol{C}$-linear span of $\{a\cdot f\,;\,a\in G\}$ is of finite dimension. Note that $\mathcal{S}(\hat{M}_{0})$ is a G-submodule of $C^{\infty}(\hat{M}_{0})^{C}$. An element of $\mathcal{S}(\hat{M}_{0})$ is called a *spherical function* on $\hat{M}_{0}=G/G_{0}$.

THEOREM 5.7. *Let* (M, g) *be a compact rank one symmetric space. Then the evaluation* $\varepsilon: K(M)\rightarrow C^{\infty}(\hat{M}_{0})$ *induces a* G*-isomorphism* $\varepsilon: P(M)^{C}\rightarrow\mathcal{S}(\hat{M}_{0})$ *such that* $\varepsilon\Delta=\hat{\Delta}\varepsilon$.

PROOF. Note first that $\varepsilon K(M)^{C}\subset\mathcal{S}(\hat{M}_{0})$. This follows from that $K^{p}(M)$ is a finite dimensional G-module and ε is a G-homomorphism. Now, by Lemma 2.3 $\hat{M}_{0}$ is affine algebraic in $\mathfrak{g}$, and so by Iwahori-Sugiura [6] we have $\iota_{0}^{*}S((\mathfrak{g}^{C})^{*})=\mathcal{S}(\hat{M}_{0})$. On the other hand, by Lemma 4.2 we have $\iota_{0}^{*}S((\mathfrak{g}^{C})^{*})=\varepsilon\tilde{K}(M)^{C}$. Therefore we get $\varepsilon K(M)^{C}=\mathcal{S}(\hat{M}_{0})$. Now (5.1) implies the surjectivity of $\varepsilon: P(M)^{C}\rightarrow\mathcal{S}(\hat{M}_{0})$. The injectivity follows from Lemma 5.6. The commutativity $\varepsilon\Delta=\hat{\Delta}\varepsilon$ follows from Lemma 5.4 and Theorem 5.5. q. e. d.

§ 6. Spectrum of Lichnerowicz Laplacian on $K^{p}(S^{n})$.

We recall first the definition of a weight of a compact connected Lie group G. Let $\mathfrak{g}=\mathrm{Lie}\,G$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Let V be a finite dimensional

G-module (over $\boldsymbol{C}$). It becomes a $\mathfrak{g}$-module by differentiation. We mean by a *weight* of V relative to $\mathfrak{t}$ an element λ of $\mathfrak{t}^*$ such that there exists $v \in V - \{0\}$ with $H \cdot v = \sqrt{-1}\lambda(H)v$ for each $H \in \mathfrak{t}$.

In what follows, let M be the unit sphere S^n of dimension $n \geqq 2$ with the standard metric g. We use the notation in § 3, and let $\hat{\mathfrak{m}}$ be the subspace of $\mathfrak{g}$ defined by (2.3).

In this case, the pair $(G, G_0) = (SO(n+1), SO(2) \times SO(n-1))$ is a compact symmetric pair with the associated Cartan decomposition $\mathfrak{g} = \mathfrak{g}_0 + \hat{\mathfrak{m}}$. The G-module structure of $\mathcal{S}(M_0)$ for such a pair is determined in the following way by the theory of E. Cartan on spherical functions (cf. Takeuchi [12]). Let $\hat{\mathfrak{a}}$ be a maximal abelian subalgebra in $\hat{\mathfrak{m}}$, and put

$$\hat{\Gamma} = \{H \in \hat{\mathfrak{a}}\ ;\ \exp H \in G_0\}.$$

Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\hat{\mathfrak{a}}$ and put $\hat{\mathfrak{b}} = \mathfrak{t} \cap \mathfrak{g}_0$. Let $(\ ,\ )$ denote the inner product on $\mathfrak{t}^*$ defined by $B|\mathfrak{t} \times \mathfrak{t}$. Take a compatible lexicographic order $>$ on $\mathfrak{t}^*$. Let $\hat{D}$ be the set of all $\lambda \in \mathfrak{t}^*$ such that $(\lambda, \alpha) \geqq 0$ for each positive root α of $\mathfrak{g}$, $\lambda|\hat{\mathfrak{b}} = 0$ and $\lambda(\hat{\Gamma}) \in 2\pi\boldsymbol{Z}$. Let finally denote by ρ_λ the irreducible representation of G with the highest weight $\lambda \in \hat{D}$. Then the decomposition of $\mathcal{S}(\hat{M}_0)$ as G-module is given by

$$\mathcal{S}(\hat{M}_0) = \sum_{\lambda \in \hat{D}} \rho_\lambda . \tag{6.1}$$

We assume first $n \geqq 3$. We take

$$\hat{\mathfrak{a}} = \left\{ \left(\begin{array}{cc|cc|c} 0 & 0 & -\lambda_1 & 0 & \\ 0 & 0 & 0 & -\lambda_2 & 0 \\ \hline \lambda_1 & 0 & 0 & 0 & \\ 0 & \lambda_2 & 0 & 0 & \\ \hline & 0 & & & 0 \end{array} \right) ;\ \lambda_1, \lambda_2 \in \boldsymbol{R} \right\},$$

$$\mathfrak{t} = \left\{ \left(\begin{array}{cccccccccc} 0 & 0 & -\lambda_1 & 0 & & & & & & \\ 0 & 0 & 0 & -\lambda_2 & & & & 0 & & \\ \lambda_1 & 0 & 0 & 0 & & & & & & \\ 0 & \lambda_2 & 0 & 0 & & & & & & \\ & & & & 0 & -\lambda_3 & & & & \\ & & & & \lambda_3 & 0 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & 0 & -\lambda_l & \\ & & 0 & & & & & \lambda_l & 0 & \\ & & & & & & & & & (0) \end{array} \right) ;\ \lambda_i \in \boldsymbol{R} \right\},\quad l = \left[\frac{n+1}{2}\right].$$

Define $\lambda_1 > \cdots > \lambda_l > 0$. Then we have

(6.2) $$\hat{D} = \{m_1\lambda_1 + m_2\lambda_2\,;\ m_1,\ m_2 \in \boldsymbol{Z},\ m_1 \equiv m_2\ (\mathrm{mod}\ 2),\ m_1 \geqq m_2 \geqq 0\} \quad \text{if} \quad n \geqq 4,$$

(6.3) $$\hat{D} = \{m_1\lambda_1 + m_2\lambda_2\,;\ m_1,\ m_2 \in \boldsymbol{Z},\ m_1 \equiv m_2\ (\mathrm{mod}\ 2),\ m_1 \geqq |m_2|\} \quad \text{if} \quad n = 3.$$

In case $n=2$, we take

$$\hat{\mathfrak{a}} = \mathfrak{t} = \left\{ \left[\begin{array}{c|cc} 0 & \multicolumn{2}{c}{0} \\ \hline 0 & 0 & -\lambda_1 \\ & \lambda_1 & 0 \end{array} \right] ;\ \lambda_1 \in \boldsymbol{R} \right\}.$$

Define $\lambda_1 > 0$. Then we have

(6.4) $$\hat{D} = \{m_1\lambda_1\,;\ m_1 \in \boldsymbol{Z},\ m_1 \geqq 0\}.$$

LEMMA 6.1 (Tsukamoto). *The following sum is equal to*

$$\frac{1}{n}\binom{n+p}{p+1}\binom{n+p-1}{p} \quad (= \dim K^p(S^n)).$$

$$\sum_{0 \leq k \leq l \leq [p/2]} \deg \rho_{(p-2k)\lambda_1 + (p-2l)\lambda_2} \qquad \textit{if} \quad n \geqq 4,$$

$$\sum_{0 \leq k \leq l \leq [p/2]} (\deg \rho_{(p-2k)\lambda_1 + (p-2l)\lambda_2} + \deg \rho_{(p-2k)\lambda_1 - (p-2l)\lambda_2}) \qquad \textit{if} \quad n = 3,$$

$$\sum_{0 \leq k \leq [p/2]} \deg \rho_{(p-2k)\lambda_1} \qquad \textit{if} \quad n = 2.$$

This can be proved by Weyl's degree formula and an elementary calculation.

LEMMA 6.2. *The representation ρ^p of G on $P^p(M)^C$ is decomposed as follows.*

$$\rho^p = \sum_{0 \leq k \leq [p/2]} \rho_{p\lambda_1 + (p-2k)\lambda_2} \qquad \textit{if} \quad n \geqq 4,$$

$$\rho^p = \sum_{0 \leq k \leq [p/2]} (\rho_{p\lambda_1 + (p-2k)\lambda_2} + \rho_{p\lambda_1 - (p-2k)\lambda_2}) \qquad \textit{if} \quad n = 3,$$

$$\rho^p = \rho_{p\lambda_1} \qquad \textit{if} \quad n = 2.$$

PROOF. Assume first $n \geqq 4$. Then, by Theorem 5.7, (6.1) and (6.2) we have

(6.5) $$\sum_{p \geq 0} \rho^p = \sum_{\substack{m \geq 0, \\ 0 \leq k \leq [m/2]}} \rho_{m\lambda_1 + (m-2k)\lambda_2}.$$

Now we prove the assertion by the induction on p. Since $P^0(M)^C = \boldsymbol{C}$ and $\rho^0 = \rho_0$, the required decomposition holds for $p=0$. Let $p \geqq 1$. Let ρ_λ be an irreducible component of ρ^p. Then, by (6.5) λ is of the form $\lambda = m\lambda_1 + (m-2k)\lambda_2$, $m \geqq 0$, $0 \leqq k \leqq [m/2]$. Moreover, the induction hypothesis and (6.5) imply that

$m\geqq p$. On the other hand, since $\varepsilon K^p(M)^C=\iota_0^* S^p(\mathfrak{g}^C)^*$ by Lemma 4.2 and Theorem 4.6, $K^p(M)^C$ is G-isomorphic to a G-submodule of $S^p(A_{n+1}(\boldsymbol{C})^*)$. But $A_{n+1}(\boldsymbol{C})$ is an $SU(n+1)$-module and there exist a Cartan subalgebra $\mathfrak{s}$ of $\mathfrak{su}(n+1)$ with $\mathfrak{t}\subset\mathfrak{s}$ and linear forms $x_1, \cdots, x_{n+1}$ on $\mathfrak{s}$ such that the set of weights of $SU(n+1)$-module $A_{n+1}(\boldsymbol{C})$ is $\{x_i+x_j\,;\ i<j\}$ and that $x_i|\mathfrak{t}=\lambda_i$, $x_{l+i}|\mathfrak{t}=-\lambda_i$ $(1\leqq i\leqq l)$ $(x_{n+1}|\mathfrak{t}=0$ if n is even). Therefore we must have $m\leqq p$. Thus we have proved that λ is of the form $\lambda=p\lambda_1+(p-2k)\lambda_2$, $0\leqq k\leqq[p/2]$.

On the other hand, by Lemma 6.1 we have

$$\deg \rho^p=\sum_{0\leq k\leq[p/2]}\deg \rho_{p\lambda_1+(p-2k)\lambda_2}\,.$$

Since the multiplicity of ρ_λ in ρ^p is 1, we get the required decomposition for p.

The assertion for $n=3$, 2 is proved in the same way, making use of (6.3), (6.4). q. e. d.

Now, for the determination of Spec $\varDelta$ on $K^p(S^n)$, it is sufficient to determine Spec $\varDelta$ on each $P^p(S^n)$, since (5.2) holds. The latter spectrum is given by the following theorem.

THEOREM 6.3 (Sumitomo-Tandai [11]). *We define*

$$\mu_{p,k}=(n+p-1)p+(n+p-2k-3)(p-2k) \qquad \textit{if} \quad n\geqq 3,$$

$$\mu_p=p(p+1) \qquad \textit{if} \quad n=2.$$

Then Spec $\varDelta$ *on* $P^p(S^n)$ *is given by*

$$\{\mu_{p,k}\,;\ 0\leqq k\leqq[p/2]\} \qquad \textit{if} \quad n\geqq 3,$$

$$\{\mu_p\} \qquad \textit{if} \quad n=2.$$

PROOF. By Theorem 5.7 the problem reduces to the determination of the eigenvalue c_λ of the operator C acting on the representation space of each irreducible component ρ_λ of ρ^p. The eigenvalue c_λ is given by Freudenthal's formula:

$$c_\lambda=(\lambda+2\delta,\ \lambda)\,,$$

where 2δ denotes the sum of positive roots of $\mathfrak{g}$. In our case, 2δ is given by

$$2\delta=\begin{cases}\sum_{i=1}^{l}(n+1-2i)\lambda_i & \text{if} \quad n\geqq 3,\\ \lambda_1 & \text{if} \quad n=2.\end{cases}$$

If $n\geqq 4$, for $\lambda=p\lambda_1+(p-2k)\lambda_2$ with $0\leqq k\leqq[p/2]$ we have

$$\begin{aligned} c_\lambda &= (p\lambda_1+(p-2k)\lambda_2+(n-1)\lambda_1+(n-3)\lambda_2,\ p\lambda_1+(p-2k)\lambda_2) \\ &= (n+p-1)p+(n+p-2k-3)(p-2k) \\ &= \mu_{p,k}\,. \end{aligned}$$

If $n=3$, for $\lambda=p\lambda_1\pm(p-2k)\lambda_2$ with $0\leqq k\leqq[p/2]$ we have

$$\begin{aligned} c_\lambda &= (p\lambda_1\pm(p-2k)\lambda_2+2\lambda_1,\ p\lambda_1\pm(p-2k)\lambda_2) \\ &= (p+2)p+(p-2k)^2 \\ &= \mu_{p,k}\,. \end{aligned}$$

If $n=2$, for $\lambda=p\lambda_1$ we have

$$c_\lambda=(p\lambda_1+\lambda_1,\ p\lambda_1)=p(p+1)=\mu_p\,.$$

Thus, together with Lemma 6.2 we obtain the theorem. q. e. d.

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