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## KILLING VECTORS ON CONTACT RIEMANNIAN MANIFOLDS AND FIBERINGS RELATED TO THE HOPF FIBRATIONS

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**1. Introduction.** Let  $(M, g)$  be a Riemannian manifold. Then  $K$ -contact Riemannian structures and Sasakian structures (=normal contact Riemannian structures) on  $M$  are defined by Killing vectors  $\xi$  of unit length satisfying some conditions (cf. §2). Hence we denote by  $(M, \xi, g)$  a  $K$ -contact Riemannian manifold or a Sasakian manifold.

Every  $(M, \xi, g)$  is odd dimensional.

In this paper, after preliminaries in §2 and §3, we first try to give conditions for Killing vectors to be infinitesimal automorphisms of  $(M, \xi, g)$  in terms of curvature of  $(M, \xi, g)$  in §4~§8.

**THEOREM A.** *Let  $(M, \xi, g)$  be a 3-dimensional  $K$ -contact Riemannian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism of  $(M, \xi, g)$ .*

By  $\phi = -\nabla\xi$ , we have a  $(1,1)$ -tensor field on  $M$ .  $\phi$  satisfies  $\phi\phi X = -X + g(\xi, X)\xi$  for each vector field  $X$  on  $M$ .

**THEOREM B.** *Let  $(M, \xi, g)$  be a 7-dimensional compact Sasakian manifold which is not of constant curvature. Assume that  $\phi$ -holomorphic sectional curvature  $H(X) < 3$ . Then every Killing vector is an infinitesimal automorphism of  $(M, \xi, g)$ .*

For general  $(4r+3)$ -dimensional cases, we need stronger conditions on curvature than those in Theorem B,  $r$  being an integer  $\geq 1$ .

**THEOREM C.** *Let  $(M, \xi, g)$  be a  $(4r+3)$ -dimensional compact Sasakian manifold which is not of constant curvature. Assume that curvature is positive (more generally,  $\phi$ -holomorphic special bisectional curvature is positive). Then*

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every Killing vector is an infinitesimal automorphism of  $(M, \xi, g)$ .

The remaining cases are  $(4r+1)$ -dimensional,  $r$  being an integer  $\geq 1$ .

**THEOREM D.** *Let  $(M, \xi, g)$  be a  $(4r+1)$ -dimensional complete Sasakian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism of  $(M, \xi, g)$ .*

As we have seen in [22], discussions on these problems concern Sasakian 3-structures on  $(M, g)$ .

In §9, we give slightly general statements of the above theorems.

Analogously to the Hopf fibrations of spheres and the Boothby-Wang's fiberings of regular contact manifolds, we consider fibrations of  $(M, g)$  admitting a  $K$ -contact 3-structure in §11 and §12.

**THEOREM E.** *Let  $(M, g)$  be a complete Riemannian manifold admitting a Sasakian 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . If one of the Sasakian structures, for example  $\xi_{(1)}$ , is regular, then  $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$  is a  $S^3[1]$ - or  $RP^3[1]$ -principal bundle over an Einstein manifold  $(B, h)$ .*

In §13 we show that in many cases results on  $K$ -contact 3-structures are generalized to results on 3- $K$ -contact structures.

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**2. Preliminaries.** Let  $(M, g)$  be a Riemannian manifold. By  $\nabla$  and  $R$  we denote the Riemannian connection and the Riemannian curvature tensor  $(R(X, Y) = \nabla_X Y - [\nabla_X, \nabla_Y])$ , respectively. Let  $\xi$  be a unit Killing vector on  $(M, g)$ , which satisfies

$$(2.1) \quad R(X, \xi)\xi = g(X, \xi)\xi - X$$

for any vector field  $X$  on  $M$ . Define a  $(1, 1)$ -tensor field  $\phi$  by  $\phi = -\nabla\xi$  and a 1-form (= contact form)  $\eta$  by  $\eta = g(\xi, \cdot)$ . Then  $(\phi, \xi, \eta, g)$  is a  $K$ -contact Riemannian structure (cf. [5], etc.). We denote this  $K$ -contact Riemannian manifold by  $(M, \xi, g)$ . On  $(M, \xi, g)$  we have

$$(2.2) \quad \phi\xi = -\nabla\xi = 0,$$

$$(2.3) \quad \phi\phi X = -X + g(\xi, X)\xi,$$

$$(2.4) \quad g(\phi X, \phi Y) = g(X, Y) - g(\xi, X)g(\xi, Y).$$

If a unit Killing vector  $\xi$  satisfies

$$(2.5) \quad R(X, \xi)Y = g(X, Y)\xi - g(\xi, Y)X, \text{ or}$$

$$(2.5)' \quad -\nabla_X(\nabla\xi)Y = g(X, Y)\xi - g(\xi, Y)X$$

for any vector fields  $X$  and  $Y$  on  $M$ , then  $(M, \xi, g)$  is called a Sasakian manifold (=normal contact Riemannian manifold) (cf. [12], [13], etc.). A Sasakian manifold is a  $K$ -contact Riemannian manifold.

On a Sasakian manifold  $(M, \xi, g)$ , by the Ricci identity, we have the following relation (cf. for example, Lemma 3.2 in [21]):

$$(2.6) \quad \begin{aligned} \phi R(X, Y)(\phi Z) &= -R(X, Y)Z - g(Y, Z)X + g(X, Z)Y \\ &\quad + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y. \end{aligned}$$

We define the distribution  $D$  by  $D_p = \{X_p ; g(\xi, X_p) = 0, X_p \in M_p\}$ , where  $M_p$  denotes the tangent space to  $M$  at  $p$ . By  $X \in D$  we understand that  $X$  is a vector field on  $M$  such that  $X_p \in D_p$  for every  $p$  of  $M$ . By  $X \in D_p$ , we understand that  $X$  is a tangent vector belonging to  $D_p$ . By  $K(X, Y)$  we denote the sectional curvature for a 2-plane determined by  $X$  and  $Y$ . By  $H(X)$ ,  $X \in D_p$  (or  $X \in D$ ) we denote the sectional curvature  $K(X, \phi X)$ , called  $\phi$ -holomorphic sectional curvature.

Let  $X$  and  $Y$  be an orthonormal pair in  $D_p$  and put  $g(X, \phi Y) = \cos \alpha$ . Then by a direct calculation we have (cf. E. M. Moskal [8])

$$(2.7) \quad \begin{aligned} K(X, Y) &= (1/8)[3(1 + \cos \alpha)^2 H(X + \phi Y) + 3(1 - \cos \alpha)^2 H(X - \phi Y) \\ &\quad - H(X + Y) - H(X - Y) - H(X) - H(Y) + 6 \sin^2 \alpha]. \end{aligned}$$

Furthermore we have (for (2.7) and (2.8), see also [18])

$$(2.8) \quad \begin{aligned} K(X, Y) + \sin^2 \alpha K(X, \phi Y) &= (1/4)[(1 + \cos \alpha)^2 H(X + \phi Y) \\ &\quad + (1 - \cos \alpha)^2 H(X - \phi Y) + H(X + Y) + H(X - Y) - H(X) - H(Y) + 6 \sin^2 \alpha]. \end{aligned}$$

**3.  $K$ -contact 3-structures and Sasakian 3-structures.** Let  $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$  be three  $K$ -contact structures on  $(M, g)$ . Define  $\phi_{(i)}$  ( $i = 1, 2, 3$ ) by  $\phi_{(i)} = -\nabla \xi_{(i)}$ . Assume that

$$(3.1) \quad g(\xi_{(i)}, \xi_{(j)}) = \delta_{ij}, \quad i, j = 1, 2, 3,$$

$$(3.2) \quad \xi_{(k)} = \phi_{(i)} \xi_{(j)} = -\phi_{(j)} \xi_{(i)},$$

$$(3.3) \quad \phi_{(k)} X = \phi_{(i)} \phi_{(j)} X - g(\xi_{(j)}, X) \xi_{(i)} = -\phi_{(j)} \phi_{(i)} X + g(\xi_{(i)}, X) \xi_{(j)},$$

where  $(i, j, k)$  is an even permutation of  $(1, 2, 3)$ . Then we say that  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is a  $K$ -contact 3-structure on  $(M, g)$ . Similarly, if  $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$  are Sasakian structures and satisfy  $(3.1) \sim (3.3)$ , then  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is called a Sasakian 3-structure on  $(M, g)$ .

(i) If  $(M, g)$  admits a  $K$ -contact 3-structure, then  $\dim M = 4r+3$  for some integer  $r \geq 0$  (Y. Y. Kuo [7]).

(ii)  $(M, g)$  admitting a Sasakian 3-structure is an Einstein manifold (T. Kashiwada [6]).

(iii) Let  $\xi_{(1)}$  and  $\xi_{(2)}$  be two Sasakian structures on  $(M, g)$  such that  $g(\xi_{(1)}, \xi_{(2)}) = 0$ . Then  $\xi_{(3)} = (1/2)[\xi_{(1)}, \xi_{(2)}]$  is also a Sasakian structure and orthogonal to  $\xi_{(1)}$  and  $\xi_{(2)}$ . Hence  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is a Sasakian 3-structure (Y. Y. Kuo [7]).

If the inner product  $g(\xi, \xi')$  of two Sasakian structures  $\xi$  and  $\xi'$  on  $(M, g)$  is constant ( $\neq 1, \neq -1$ ), we can find Sasakian structure  $\xi_{(3)}$  so that  $\xi_{(1)} = \xi$  and  $\xi_{(2)}$  are orthogonal. Hence  $(M, g)$  admits a Sasakian 3-structure.

In the case where  $g(\xi, \xi')$  is not constant, we have

**LEMMA 3.1.** (S. Tachibana and W. N. Yu [15]) *Let  $(M, g)$  be a complete Riemannian manifold of  $m$ -dimension. If  $(M, g)$  admits two Sasakian structures  $\xi$  and  $\xi'$  with  $g(\xi, \xi') = \text{non-constant}$ , then  $(M, g)$  is of constant curvature 1.*

Originally, Lemma 3.1 was proved for complete and simply connected  $(M, g)$  with conclusion that  $(M, g)$  is isometric to a unit sphere  $S^m$ .

Let  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  be a  $K$ -contact 3-structure on  $(M, g)$ . By  $E$  we denote the distribution defined by (putting  $\xi_{(1)} = \xi$ )

$$(3.4) \quad E_p = \{X_p \in D_p; g(X_p, \xi_{(2)}) = g(X_p, \xi_{(3)}) = 0\}.$$

Since  $\dim M = 4r+3$ , we have  $\dim E_p = 4r$ . If  $X \in E_p$ , we have

$$(3.5) \quad \phi_{(k)} X = \phi_{(i)} \phi_{(j)} X = -\phi_{(j)} \phi_{(i)} X,$$

where  $(k, i, j)$  is an even permutation of  $(1, 2, 3)$ .

We define  $\phi_{(i)}$ -holomorphic sectional curvature for  $X \in E_p$  by

$$H(X) = H_{(1)}(X) = K(X, \phi_{(1)} X),$$

$$H_{(2)}(X) = K(X, \phi_{(2)} X), \quad H_{(3)}(X) = K(X, \phi_{(3)} X).$$

In the remainder of this section we assume that  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is a Sasakian 3-structure.

**PROPOSITION 3.2.** *For  $X \in E_p$ , we have*

$$(3.6) \quad H_{(1)}(X) + H_{(2)}(X) + H_{(3)}(X) = 3.$$

**PROOF.** In (2.6) we put  $\phi=\phi_{(i)}$  and take  $X, Y, Z$  (of unit length)  $\in E_p$  and consider the inner product with  $W \in E_p$ . Then we get

$$\begin{aligned} (3.7) \quad g(R(X, Y)\phi_{(i)}Z, \phi_{(i)}W) &= g(R(X, Y)Z, W) + g(Y, Z)g(X, W) \\ &\quad - g(X, Z)g(Y, W) - g(\phi_{(i)}Y, Z)g(\phi_{(i)}X, W) \\ &\quad + g(\phi_{(i)}X, Z)g(\phi_{(i)}Y, W), \end{aligned}$$

where we have used (2.3) and (2.4), and  $i = 1, 2, 3$ . If we put  $i = 1$ ,  $Z = X$ , and  $Y = W = \phi_{(3)}X$  in (3.7), we get

$$(3.8) \quad g(R(X, \phi_{(3)}X)\phi_{(1)}X, \phi_{(1)}\phi_{(3)}X) = g(R(X, \phi_{(3)}X)X, \phi_{(3)}X) - 1,$$

that is,

$$(3.9) \quad -g(R(X, \phi_{(3)}X)\phi_{(1)}X, \phi_{(2)}X) = H_{(3)}(X) - 1.$$

Then we have two relations by even permutations of  $(1, 2, 3)$  from (3.9). Hence, (3.6) follows from the Bianchi identity.

**PROPOSITION 3.3.** *For  $X \in E_p$  and for real numbers  $a, b$  ( $a^2 + b^2 = 1$ ), we have*

$$(3.10) \quad H_{(1)}(X) = H_{(1)}(\phi_{(2)}X) = H_{(1)}(a\phi_{(2)}X + b\phi_{(3)}X).$$

**PROOF.** By a permutation  $(1 \rightarrow 2 \rightarrow 3 \rightarrow 1)$  in (3.9), we have

$$\begin{aligned} (3.11) \quad H_{(1)}(X) - 1 &= -g(R(X, \phi_{(1)}X)\phi_{(2)}X, \phi_{(3)}X) \\ &= -g(R(\phi_{(2)}X, \phi_{(3)}X)X, \phi_{(1)}X) \\ &= g(R(\phi_{(2)}X, \phi_{(3)}X)\phi_{(3)}\phi_{(2)}X, \phi_{(3)}\phi_{(3)}X) \quad \text{by (3.5).} \end{aligned}$$

On the other hand, in (3.7) we put  $i = 3$  and replace  $X, Y, Z, W$  by  $\phi_{(2)}X, \phi_{(3)}X, \phi_{(2)}X, \phi_{(3)}X$ . Then we have

$$(3.12) \quad g(R(\phi_{(2)}X, \phi_{(3)}X)\phi_{(3)}\phi_{(2)}X, \phi_{(3)}\phi_{(3)}X) = g(R(\phi_{(2)}X, \phi_{(3)}X)\phi_{(2)}X, \phi_{(3)}X) - 1.$$

By (3.11) and (3.12), we have

$$H_{(1)}(X) = g(R(\phi_{(2)}X, \phi_{(1)}\phi_{(2)}X)\phi_{(2)}X, \phi_{(1)}\phi_{(2)}X) = H_{(1)}(\phi_{(2)}X).$$

Since  $a\phi_{(2)}X + b\phi_{(3)}X = a\phi_{(2)}X + b\phi_{(1)}\phi_{(2)}X$ , we have (3.10).

LEMMA 3.4. Let  $X \in E_p$ . For real numbers  $a, b$  ( $a^2 + b^2 = 1$ ) we have ( $i=2, 3$ )

$$(3.13) \quad H_{(1)}(a\xi_{(i)} + bX) = a^4 + 2a^2b^2 + b^4H_{(1)}(X).$$

PROOF. By a straightforward calculation using (2.5) for  $\xi_{(2)}$  and  $\xi_{(3)} = \phi\xi_{(2)}$ , we have

$$\begin{aligned} & g(R(a\xi_{(2)} + bX, a\phi\xi_{(2)} + b\phi X)(a\xi_{(2)} + bX), a\phi\xi_{(2)} + b\phi X) \\ &= a^4 g(R(\xi_{(2)}, \phi\xi_{(2)})\xi_{(2)}, \phi\xi_{(2)}) + b^4 g(R(X, \phi X)X, \phi X) \\ &+ a^2b^2 g(R(\xi_{(2)}, \phi X)\xi_{(2)}, \phi X) + a^2b^2 g(R(X, \phi\xi_{(2)})X, \xi\phi_{(2)}), \end{aligned}$$

from which we have (3.13) for  $i = 2$ , and the case of  $i = 3$  is similar.

REMARK. Since  $c\xi_{(2)} + d\xi_{(3)}$  for constant  $c, d$  ( $c^2 + d^2 = 1$ ) is also Sasakian, Lemma 3.4 shows that

$$(3.13)' \quad H_{(1)}(a(c\xi_{(2)} + d\xi_{(3)}) + bX) = a^4 + 2a^2b^2 + b^4H_{(1)}(X).$$

**4. Theorem A.** A 3-dimensional  $K$ -contact Riemannian manifold  $(M, \xi, g)$  is necessarily Sasakian and it is a  $D$ -Einstein manifold, i. e.,

$$(4.1) \quad R_1(X, Y) = ag(X, Y) + bg(\xi, X)g(\xi, Y),$$

where  $a$  and  $b$  are functions on  $M$  and  $R_1$  denotes the Ricci curvature tensor (cf. [16], [17]). Consequently the scalar curvature  $S$  is given by  $S = 3a + b$ .

**THEOREM A.** Let  $(M, \xi, g)$  be a 3-dimensional  $K$ -contact Riemannian manifold which is not of constant curvature. Then every Killing vector is an infinitesimal automorphism.

To prove Theorem A, it suffices to show the following.

**PROPOSITION 4.1.** *Let  $(M, \xi, g)$  and  $(M', \xi', g')$  be two 3-dimensional K-contact Riemannian manifolds. If they admits an isometry  $\varphi (\varphi^* g' = g)$  such that  $\varphi \xi \neq \xi'$  and  $\varphi \xi \neq -\xi'$ , then  $(M, g)$  is of constant curvature.*

**PROOF.** Let  $x$  be an arbitrary point of  $M$  and put  $y=\varphi x$ . Since  $\varphi$  is an isometry, we have  $S_x=S_y'$  and

$$(4.2) \quad R_{1x}(X, Y) = (\varphi^* R_1')_x(X, Y) = R'_{1y}(\varphi X, \varphi Y).$$

By (4.1) we get

$$(4.3) \quad 3a_x + b_x = 3a_y' + b_y',$$

$$(4.4) \quad a_x g_x(X, Y) + b_x g_x(\xi, X) g_x(\xi, Y) = a_y' g_y'(\varphi X, \varphi Y) + b_y' g_y'(\xi', \varphi X) g_y'(\xi', \varphi Y).$$

Since  $\dim M=3$ , we have  $Z \in D_x$  such that  $g_y'(\xi', \varphi Z)=0$ . Putting  $X=Y=Z$  in (4.4), we get  $a_x=a_y'$ . Then (4.3) implies  $b_x=b_y'$ . If we put  $X=Y=\xi$  in (4.4), we have  $b_x=b_y'[g_y'(\xi', \varphi \xi)]^2$ . Hence, if  $b_x \neq 0$ , we have  $[g_y'(\xi', \varphi \xi)]^2=1$ . If  $(M, g)$  is not of constant curvature, we have a non-empty open set  $U$  where  $b$  is non-vanishing. Then we have  $\varphi \xi=\xi'$  on  $U$  or  $\varphi \xi=-\xi'$  on  $U$ . Since  $\varphi \xi, \xi'$  (or  $-\xi'$ ) are Killing vectors on  $(M', g')$ , and since they coincide on  $U$ , they coincide on  $M'$ . This contradicts the assumption of  $\varphi$ , and hence,  $b=0$  on  $M$ . Consequently,  $(M, g)$ ,  $(M', g')$  are of constant curvature 1.

By  $I(M, g)$  and  $A(M, \xi, g)$ , we denote the isometry group and the automorphism group of  $(M, \xi, g)$ , respectively.

**COROLLARY 4.2.** *Let  $(M, \xi, g)$  be a 3-dimensional K-contact Riemannian manifold. Then we have either*

(i)  $(M, g)$  is of constant curvature, or

(ii-1)  $I(M, g) = A(M, \xi, g)$  or

-2)  $I(M, g) = A(M, \xi, g) \cup A'(M, \xi, g)$ ,

where  $A'(M, \xi, g) = \{\varphi f ; f \in A(M, \xi, g), \varphi \in I(M, g) : \varphi \xi = -\xi\}$ .

**5. Einstein-Kählerian manifolds.** Let  $(N, J, G)$  be a  $2n$ -dimensional Kählerian manifold with (almost) complex structure tensor  $J$  and Kählerian metric tensor  $G$ . Holomorphic sectional curvature is defined by  $'H(\sigma) = 'H(u) = 'K(u, Ju)$ , where  $\sigma$  denotes the holomorphic section determined by  $u$ . For two holomorphic sections  $\sigma$  and  $\sigma'$ , holomorphic bisectional curvature  $'H(\sigma, \sigma')$  is defined in [4]. In this paper we consider holomorphic special bisectional curvature  $'H(\sigma, \sigma')$ , where the word "special" means  $\sigma \perp \sigma'$ . In this case

$$'H(\sigma, \sigma') = 'K(u, v) + 'K(u, Jv),$$

where  $u \in \sigma$  and  $v \in \sigma'$ . Generalizing a result of M. Berger [1], S. I. Goldberg and S. Kobayashi [4] proved the followings : On an Einstein-Kählerian manifold  $(N, J, G)$  assume that the maximum value ' $H_1$ ' of holomorphic sectional curvature is attained at  $x$  of  $N$ . Let  $u$  be a unit tangent vector at  $x$  such that ' $H_1 = H(u)$ '.

(i) For an orthonormal basis  $(u_1, \dots, u_n, u_{1*} = Ju_1, \dots, u_{n*} = Ju_n)$  at  $x$  such that

$$(5.1) \quad u_1 = u, \quad \text{and}$$

$$(5.2) \quad 'R_{11*ia} = G('R(u_1, Ju_1)u_i, u_a) = 0$$

for all  $i$  and  $a$  such that [ $a \neq i^*$ ;  $2 \leq i \leq n$ ,  $2 \leq a \leq n$  or  $n+2 \leq a \leq 2n$ ], if ' $R_{11*ii*}$ ' (holomorphic special bisectional curvature) is positive, then  $(N, J, G)$  has constant holomorphic sectional curvature ' $H_1$ '.

Especially,

(ii) If  $(N, J, G)$  is of positive holomorphic bisectional curvature, then it is of constant holomorphic sectional curvature.

**6. Local fiberings.** Let  $p$  be a point of a  $K$ -contact Riemannian manifold  $(M, \xi, g)$ . We have a sufficiently small coordinate neighborhood  $U$  of  $p$ , which is cubical and flat with respect to  $\xi$  (cf. [10]). Then  $U$  is a regular  $K$ -contact Riemannian manifold with the induced structure and we have a fibering

$$(6.1) \quad \pi : U \rightarrow U/\xi = N.$$

Since  $U$  is a  $K$ -contact Riemannian manifold,  $N$  is an almost Kählerian manifold. We denote the almost Kählerian structure tensors by  $J$  and  $G$ . Then we have

$$(6.2) \quad \phi u^* = (Ju)^*,$$

$$(6.3) \quad g = \pi^*G + \eta \otimes \eta,$$

where  $u^*$  on  $U$  is the horizontal lift of a vector field  $u$  on  $N$  with respect to the contact form  $\eta$ . Further

$$(6.4) \quad d\eta(u^*, v^*) = 2g(u^*, \phi v^*) = 2G(u, Jv) \cdot \pi.$$

Denoting by ' $R$ ' the Riemannian curvature tensor on  $N$ , we have

$$(6.5) \quad \begin{aligned} R(u^*, v^*)z^* &= ('R(u, v)z)^* + 2g(u^*, \phi v^*)\phi z^* \\ &\quad + g(u^*, \phi z^*)\phi v^* - g(v^*, \phi z^*)\phi u^* + \langle u, v, z \rangle \xi, \end{aligned}$$

where  $\langle u, v, z \rangle$  denotes some function depending on  $u, v, z$  and  $u, v, z$  are vector fields on  $N$  (cf. [9], [17], [18], etc.). The relation between holomorphic sectional curvature ' $H(u)$ ' on  $N$  and  $\phi$ -holomorphic sectional curvature  $H(u^*)$  on  $U$  is

$$(6.6) \quad H(u^*) = 'H(u) \cdot \pi - 3.$$

The relation between  $\phi$ -holomorphic special bisectional curvature  $H(\rho, \rho') = K(X, Y) + K(X, \phi Y)$  ( $X \in \rho \subset D, Y \in \rho' \subset D$ ) on  $U$  and holomorphic special bisectional curvature ' $H(\pi\rho, \pi\rho')$ ' on  $N$  is

$$(6.7) \quad H(\rho, \rho') = 'H(\pi\rho, \pi\rho') \cdot \pi.$$

$U$  is a  $D$ -Einstein space if and only if  $N$  is an Einstein space ([17]). If  $(M, \xi, g)$  is Sasakian, then  $(N, J, G)$  is Kählerian.

**7. Theorem B.** Now we prove the following Proposition.

**PROPOSITION 7.1.** *Let  $(\xi = \xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  be a Sasakian 3-structure on a compact Riemannian manifold  $(M, g)$  of dimension 7. If*

$$H(X) = H_{(1)}(X) = K(X, \phi X) < 3$$

*for any non-zero vector  $X \in E$ , then  $(M, g)$  is of constant curvature.*

**PROOF.** Let  $x$  be a point of  $M$ . Put

$$H_x^* = \max\{H(X) = H_{(1)}(X), X \in E_x\}.$$

Case I, where  $H_x^* \leq 1$  for any  $x$  of  $M$ . Let  $X \in E_x$  be any unit vector. Take a  $\phi$ -basis  $(\xi = \xi_{(1)}, \xi_{(2)}, \xi_{(3)} = \phi\xi_{(2)}, X, \phi X, Y = \phi_{(2)}X, \phi Y = \phi_{(3)}X)$ . Since  $\cos \alpha = g(X, \phi Y) = 0$ , by (2.8) we have

$$\begin{aligned} 4(K(X, Y) + K(X, \phi Y)) &= H(X + \phi Y) + H(X - \phi Y) \\ &\quad + H(X + Y) + H(X - Y) - H(X) - H(Y) + 6. \end{aligned}$$

Noticing  $K(X, Y) = H_{(2)}(X)$  and  $K(X, \phi Y) = H_{(3)}(X)$ , and applying (3.6) and (3.10), we have

$$6 = H(X + \phi Y) + H(X - \phi Y) + H(X + Y) + H(X - Y) + 2H(X).$$

Since  $H_x^* \leq 1$ , we have  $H(X + \phi Y) = H(X - \phi Y) = H(X + Y) = H(X - Y) = H(X)$

=1. By (3.13)',  $(M, \xi, g)$  has constant  $\phi$ -holomorphic sectional curvature 1. Therefore  $(M, g)$  is of constant curvature 1 (cf. [18]).

Case II, where  $1 < H_p^*$  for some  $p$ . Since  $M$  is compact, we can assume that  $H_p^*$  is the maximum value on  $M$ . Let  $V \in E_p$  such that  $H_p^* = H(V)$ . Let  $U$  be a regular neighborhood of  $p$  and let  $\pi: U \rightarrow U/\xi = N$  be a (local) fibering. Let  $u_1 = \pi_p V$ . Then, by (6.6), we see that  $'H(u_1)_q = H_p^* + 3$  is the maximum on  $N$ , where  $q = \pi p$ . We define a vector  $u_3$  by  $u_3 = \pi_p \xi_{(2)}$ . Then  $Ju_3 = \pi_p \phi \xi_{(2)} = \pi_p \xi_{(3)}$ . In (6.5), if we replace  $u, v, z$  by  $u_1, Ju_1, u_3$ , we have

$$\begin{aligned} R(u_1^*, \phi u_1^*) \xi_{(2)} &= ('R(u_1, Ju_1)u_3)^* + 2g(u_1^*, \phi \phi u_1^*) \phi \xi_{(2)} \\ &\quad + 0 - 0 + \langle u_1, Ju_1, u_3 \rangle \xi_{(1)} \end{aligned}$$

at  $p$ . Projecting this, we have

$$R(u_1, Ju_1)u_3 = 2Ju_3.$$

This shows that  $u_3$  and  $Ju_3$  are characteristic vectors of a symmetric bilinear form  $\alpha_{u_1}$  defined by  $\alpha_{u_1}(y, z) = G('R(u_1, Ju_1)y, Jz)$ . Hence, a  $J$ -basis :

$$u_1, Ju_1, u_2 = \pi_p \phi_{(2)} u_1^*, Ju_2 = \pi_p \phi_{(3)} u_1^*, u_3, Ju_3$$

satisfies the conditions (5.1) and (5.2). We define three holomorphic sections by  $\sigma = (u_1, Ju_1)$ ,  $\sigma' = (u_2, Ju_2)$  and  $\sigma'' = (u_3, Ju_3)$ . Then, by (6.7), we have

$$\begin{aligned} 'H(\sigma, \sigma') \cdot \pi &= H((u_1^*, \phi u_1^*), (\phi_{(2)} u_1^*, \phi_{(3)} u_1^*)) \\ &= K(u_1^*, \phi_{(2)} u_1^*) + K(u_1^*, \phi_{(3)} u_1^*) \\ &= H_{(2)}(u_1^*) + H_{(3)}(u_1^*) \\ &= 3 - H_{(1)}(u_1^*) \quad \text{by (3.6).} \end{aligned}$$

Therefore,  $'H(\sigma, \sigma') > 0$ , which implies  $'R_{11*33*} > 0$  in §5. Next, by (2.1), we have

$$'H(\sigma, \sigma'') \cdot \pi = K(u_1^*, \xi_{(2)}) + K(u_1^*, \xi_{(3)}) = 2,$$

which implies  $'R_{11*33*} = 2 > 0$  in §5. Since  $(U, g)$  admits a Sasakian 3-structure, it is an Einstein manifold and  $(N, J, G)$  is an Einstein-Kählerian manifold. By (i) of §5,  $(N, J, G)$  is of constant holomorphic sectional curvature  $H_p^* + 3$ . Therefore  $(U, \xi, g)$  is of constant  $\phi$ -holomorphic sectional curvature  $H_p^*$ . In particular we have  $H_p^* = K(\xi_{(2)}, \phi \xi_{(2)}) = 1$ , which is a contradiction.

Hence, only case I is possible, and  $(M, g)$  is of constant curvature.

LEMMA 7.2. (Theorem 4.4, [22]) *Let  $(M, \xi, g)$  be a complete Sasakian manifold which is not of constant curvature. Then we have either*

- (i)  $\dim I(M, g) = \dim A(M, \xi, g)$   
 $\iff (M, g) \text{ admitting no Sasakian 3-structure, or}$
- (ii)  $\dim I(M, g) = \dim A(M, \xi, g) + 2$   
 $\iff (M, g) \text{ admitting a Sasakian 3-structure.}$

THEOREM B. *Let  $(M, \xi, g)$  be a 7-dimensional compact Sasakian manifold which is not of constant curvature. Assume that  $\phi$ -holomorphic sectional curvature  $H(X) < 3$ . Then every Killing vector is an infinitesimal automorphism of  $(M, \xi, g)$ , i.e.,*

$$\dim I(M, g) = \dim A(M, \xi, g).$$

PROOF. By Lemma 7.2, if  $\dim I(M, g) \neq \dim A(M, \xi, g)$ , we have a Sasakian 3-structure such that  $\xi_{(1)} = \xi$ . By the assumption  $H(X) < 3$ , Theorem B follows from Proposition 7.1.

**8. Theorems C and D.** By a theorem of E. M. Moskal [8] (for proof, also see [23], §7) we see that every compact Einstein-Sasakian manifold with positive curvature (or positive  $\phi$ -holomorphic special bisectional curvature) is of constant curvature 1. Therefore, Lemma 7.2 and the fact that  $(M, g)$  admitting a Sasakian 3-structure is an Einstein manifold imply the following theorem.

THEOREM C. *Let  $(M, \xi, g)$  be a  $(4r+3)$ -dimensional compact Sasakian manifold which is not of constant curvature. Assume that every sectional curvature is positive (more generally, every  $\phi$ -holomorphic special bisectional curvature is positive). Then we have*

$$\dim I(M, g) = \dim A(M, \xi, g).$$

For  $\dim M = 4r+1$  ( $r$ : an integer  $\geq 1$ ), there is no Sasakian 3-structure on  $(M, g)$ . Hence,

THEOREM D. *Let  $(M, \xi, g)$  be a  $(4r+1)$ -dimensional complete Sasakian manifold which is not of constant curvature. Then*

$$\dim I(M, g) = \dim A(M, \xi, g).$$

**9. Infinitesimal translations.** In this section, we give more general statements of Theorems B and C. The Riemannian curvature tensor of  $(M, g)$  of constant curvature  $k$  satisfies

$$(9.1) \quad R(X, Z)Y = k[g(X, Y)Z - g(Z, Y)X].$$

A Killing vector of constant length is called an infinitesimal translation (cf. for example, K. Yano [24]).

**THEOREM 9.1.** *Let  $(M, g)$  be a compact Riemannian manifold. Assume that on  $(M, g)$  there are two (non-proportional) infinitesimal translations  $\xi$  and  $\xi'$ , satisfying*

$$(9.2) \quad R(X, \xi)Y = k[g(X, Y)\xi - g(\xi, Y)X],$$

$$(9.3) \quad R(X, \xi')Y = k[g(X, Y)\xi' - g(\xi', Y)X]$$

for a positive constant  $k$ .

- (i) If  $\dim M=7$  and sectional curvature is smaller than  $3k$ , then  $(M, g)$  is of constant curvature  $k$ .
- (ii) If  $\dim M=4r+3$  and sectional curvature is positive, then  $(M, g)$  is of constant curvature  $k$ .
- (iii) If  $\dim M=3$  or  $\dim M=4r+1$ , then  $(M, g)$  is of constant curvature  $k$ .

**Proof.** By a homothetic deformation, we can assume that  $k=1$ . Since (9.2) and (9.3) are linear homogeneous in  $\xi$  and  $\xi'$ , we can assume that they are of unit length. Then, if  $g(\xi, \xi')$  is constant,  $(M, g)$  admits a Sasakian 3-structure, and (i), (ii), (iii) hold by Theorem 7.1, etc. If  $g(\xi, \xi')$  is not constant,  $(M, g)$  is of constant curvature by Lemma 3.1.

**10. The Hopf-fibrations.** Let  $S^{2n+1}[1]$  be a unit sphere with the natural Sasakian structure of constant ( $\phi$ -holomorphic sectional) curvature 1. Since  $\xi$  on  $S^{2n+1}[1]$  is regular, we have the fibering:

$$(10.1) \quad \pi : S^{2n+1}[1] \longrightarrow S^{2n+1}[1]/\xi = CP^n[4],$$

where  $CP^n[4]$  denotes a complex  $n$ -dimensional projective space with Fubini-Study metric of constant holomorphic sectional curvature 4. The map  $\pi : S^3 \rightarrow S^2 = CP^1$  is the classical Hopf map.

For  $S^{4r+3}[1]$ , we have a Sasakian 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . The 3-dimensional distribution defined by  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is completely integrable. Each maximal integral

submanifold is isomorphic to  $S^3[1]$ . In this case, the Hopf fibration is :

$$(10.2) \quad \pi: S^{4r+3}[1] \longrightarrow S^{4r+3}[1]/(\xi_{(1)}, \xi_{(2)}, \xi_{(3)}) = QP^r,$$

where  $QP^r$  denotes the quaternionic projective space (cf. N. Steenrod [14], p. 105-).

(10.1) and (10.2) are principal bundles with group  $S^1$  and  $S^3$ , respectively. A generalization of (10.1) for regular contact manifolds is the Boothby-wang's fiberings [2].

In the next section, we give a generalization of (10.2).

**11. Fiberings of  $(M, g)$  admitting a  $K$ -contact 3-structure.** Let  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  be a  $K$ -contact 3-structure on  $(M, g)$  (cf. §13). We define the 3-dimensional distribution by  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . Since we have

$$[\xi_{(1)}, \xi_{(2)}] = \nabla_{\xi_{(1)}} \xi_{(2)} - \nabla_{\xi_{(2)}} \xi_{(1)} = 2\phi_{(1)} \xi_{(2)} = 2\xi_{(3)},$$

etc. by (3.2), etc., it is completely integrable. Each maximal integral submanifold (leaf)  $L$  is totally geodesic and of constant curvature 1. By the restriction,  $L$  admits a  $K$ -contact 3-structure (and hence, a Sasakian 3-structure, since  $\dim L=3$ ). Now we assume that  $\xi_{(1)}$  is regular and that  $(M, g)$  is complete. Then we show that all leaves are isomorphic. To begin with,

**LEMMA 11.1.** *In the classification of 3-dimensional space forms  $(M, g)$  admitting a Sasakian 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  (cf. S. Sasaki [11]), only  $S^3[1]$  and  $RP^3[1]$  are regular with respect to  $\xi_{(1)}$ .*

**PROOF.** Each  $(M, g)$  of the classification is of the form  $S^3[1]/\Gamma$ , where  $\Gamma$  is a finite subgroup of the automorphism group of the Sasakian 3-structure. By  $I$  and  $-I$  ( $I^\Delta$  and  $-I^\Delta$ , resp.) we denote the identity and the anti-podal map of  $S^3[1]$  (of  $S^2$ , resp.). Assume that  $(M, g)$  is neither  $S^3[1]$  nor a real projective space  $RP^3[1] = S^3[1]/\{I, -I\}$ . Then,  $\Gamma$  contains  $\varphi$  such that  $\varphi \neq I$  and  $\varphi \neq -I$ . Since  $\varphi$  is an automorphism of  $(S^3[1], \xi, g)$ , it induces an automorphism  $\varphi^\Delta$  of the Kählerian manifold  $S^3[1]/\xi = CP^1[4] = S^2$ , where  $\xi = \xi_{(1)}$ .

(i) If  $\varphi^\Delta = I^\Delta$ , we have  $\varphi = \exp r\xi$  for some  $r$ . Since  $[\xi_{(1)}, \xi_{(2)}] = 2\xi_{(3)}$  and  $[\xi_{(1)}, \xi_{(3)}] = -2\xi_{(2)}$ , we have

$$(\exp r\xi)\xi_{(2)} = (\cos 2r)\xi_{(2)} - (\sin 2r)\xi_{(3)}.$$

$\varphi\xi_{(2)} = \xi_{(2)}$  implies  $r = \pi$  and  $\varphi = \exp \pi\xi = -I$  on  $S^3[1]$ , which is a contradiction to the assumption of  $\varphi$ .

(ii) If  $\varphi^\Delta = -I^\Delta$ , and if  $S^3[1]/\Gamma$  is regular with respect to  $\xi$ , then  $(S^3[1]/\Gamma)/\xi$  is Kählerian and orientable. However, since  $\xi$  is invariant by  $\Gamma$ , we have

$$(S^3[1]/\Gamma)/\xi = (S^3[1]/\xi)/\Gamma^\Delta = (S^3[1]/\xi)/(\varphi^\Delta, **) = RP^2/(**) .$$

Because every complete Riemannian manifold of even dimension with constant curvature ( $> 0$ ) is  $S^m$  or  $RP^m$ ,  $(**)$  = (identity). Since  $RP^2$  is not orientable, this is a contradiction.

(iii) If  $\varphi^\Delta \neq I^\Delta$  and  $\varphi^\Delta \neq -I^\Delta$ , then  $\varphi^\Delta$  has fixed points. Since  $\Gamma$  is a finite group, the set of all such points is composed of finite number of points. Therefore, on  $S^3[1]/\Gamma$ ,  $\xi$  is not regular (cf. also, S. Tanno [20]).

**LEMMA 11.2.** *Assume that a complete Riemannian manifold  $(M, g)$  admits a K-contact 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . If  $\xi_{(1)}$  is regular, then  $\xi_{(2)}$  and  $\xi_{(3)}$  are regular, and all leaves  $L$  are isomorphic to  $S^3[1]$  or  $RP^3[1]$ .*

**PROOF.** This follows from Lemma 11.1.

**REMARK.**  $S^3[1]$  and  $RP^3[1]$  are Lie groups (cf. [14], p. 37, p. 115). In fact, let  $Q$  be the space of quaternions ( $\mathbf{q} = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ ) and let  $S^3 = \{\mathbf{q} \in Q; |\mathbf{q}| = 1\}$ . Then the right translation  $R_q$  and the left translation  $L_q$  by  $\mathbf{q} \in S^3$  are defined by  $R_q\mathbf{q}' = \mathbf{q}' \cdot \mathbf{q}$  and  $L_q\mathbf{q}' = \mathbf{q} \cdot \mathbf{q}'$ , respectively. We define a Sasakian 3-structure  $(\xi_{(1)}^0, \xi_{(2)}^0, \xi_{(3)}^0)$  such that

$$(\exp t\xi_{(1)})\mathbf{q}' = (\cos t)\mathbf{q}' + (\sin t)\mathbf{q}' \cdot \mathbf{i}, \quad \mathbf{q}' \in S^3$$

etc. ( $\xi_{(2)}^0$  for  $\mathbf{j}$ ,  $\xi_{(3)}^0$  for  $\mathbf{k}$ ). Then  $\xi_{(1)}^0, \xi_{(2)}^0, \xi_{(3)}^0$  are left invariant vector fields. We denote by  $\mathfrak{g}$  the Lie algebra of  $S^3[1]$  or  $RP^3[1]$ .

**THEOREM 11.3.** *Let  $(M, g)$  be a complete Riemannian manifold admitting a K-contact 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . Assume that  $\xi_{(1)}$  is regular. Then  $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$  is a  $S^3[1]$ - or  $RP^3[1]$ -principal bundle over a Riemannian manifold  $(B, h)$ .  $h$  and  $g$  are related by*

$$(11.2) \quad g(X, Y) = h(\pi X, \pi Y) \cdot \pi + \sum_{i=1}^3 g(\xi_{(i)}, X)g(\xi_{(i)}, Y).$$

A  $\mathfrak{g}$ -valued 1-form  $w$  defined by

$$(11.3) \quad w(X) = \sum_{i=1}^3 g(\xi_{(i)}, X)\xi_{(i)}^0$$

is an infinitesimal connection form.

**PROOF.** By Lemmas 11.1 and 11.2, we see that  $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$  is a  $S^3[1]$ - or  $RP^3[1]$ -principal bundle over a manifold  $B$ . First we show that  $w$  defined by (11.3) is an infinitesimal connection form. Since  $S^3[1]$  or  $RP^3[1]$  acts to the

right,  $\xi_{(i)}$  are considered as the fundamental vector fields corresponding to  $\xi_{(i)}^0$ , respectively. Clearly,  $w(\xi_{(i)}) = \xi_{(i)}^0$ . To prove  $R_a^*w = ad(a^{-1})w$ , it suffices to show it for  $a = \exp r\xi_{(1)}$ . For this  $a$  we have  $R_a^{-1}\xi_{(1)} = \xi_{(1)}$ , and

$$\begin{aligned} R_a^{-1}\xi_{(2)} &= \lambda\xi_{(2)} + \mu\xi_{(3)}, \quad R_a^{-1}\xi_{(3)} = -\mu\xi_{(2)} + \lambda\xi_{(3)}, \\ R_a\xi_{(2)}^0 &= \lambda\xi_{(2)}^0 - \mu\xi_{(3)}^0, \quad R_a\xi_{(3)}^0 = \mu\xi_{(2)}^0 + \lambda\xi_{(3)}^0, \end{aligned}$$

where  $\lambda$  and  $\mu$  are constants depending on  $a$  ( $\lambda^2 + \mu^2 = 1$ ). Then we have

$$\begin{aligned} (R_a^*w)_p(X) &= w_{pa}(R_aX) = \sum_{i=1}^3 g_{pa}(\xi_{(i)}, R_aX)\xi_{(i)}^0 \\ &= \sum g_p(R_a^{-1}\xi_{(i)}, X)\xi_{(i)}^0 \\ &= g_p(\xi_{(1)}, X)\xi_{(1)}^0 + g_p(\lambda\xi_{(2)} + \mu\xi_{(3)}, X)\xi_{(2)}^0 + g_p(-\mu\xi_{(2)} + \lambda\xi_{(3)}, X)\xi_{(3)}^0 \\ &= \sum g_p(\xi_{(i)}, X)R_a\xi_{(i)}^0 = ad(a^{-1})w_p(X). \end{aligned}$$

Hence,  $w$  is an infinitesimal connection form on the principal bundle. Let  $x$  and  $y$  be vector fields on  $B$  and let  $x^*$  and  $y^*$  be their horizontal lifts with respect to  $w$ . We define a  $(0, 2)$ -tensor  $h$  on  $B$  by  $h(x, y) = g(x^*, y^*)$ . Since  $\xi_{(i)}$  are Killing vectors,  $h$  is well defined and satisfies (11.2).

**REMARK.** The map  $\pi : (M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g) \rightarrow (B, h)$  is harmonic in the sense of Eells-Sampson [3] (cf. Proposition, p. 127). This is the same for the Boothby-Wang's fiberings.

**12. The Riemannian curvature tensors.** We consider the fibering of Theorem 11.3. By  $\nabla$  we denote the Riemannian connection of  $(B, h)$ . Let  $x, y, z$  be vector fields on  $B$ , and let  $x^*, y^*, z^*$  be their horizontal lifts. First we note that

$$(12.1) \quad [\xi_{(i)}, x^*] = L_{\xi_i}x^* = 0,$$

because the horizontal distribution is invariant and  $x^*$  is the horizontal lift of  $x$ . Now we have

$$\begin{aligned} (12.2) \quad 2g(\nabla_x y^*, Z) &= x^* \cdot g(y^*, Z) + y^* \cdot g(x^*, Z) - Z \cdot g(x^*, y^*) \\ &\quad + g([x^*, y^*], Z) + g([Z, x^*], y^*) - g(x^*, [y^*, Z]). \end{aligned}$$

Putting  $Z = z^*$ , projecting this identity on  $B$ , and noticing  $\pi[x^*, y^*] = [x, y]$ , we have

$$\begin{aligned}
(12.3) \quad 2h(\pi(\nabla_x^* y^*), z) &= x \cdot h(y, z) + y \cdot h(x, z) - z \cdot h(x, y) \\
&\quad + h([x, y], z) + h([z, x], y) - h(x, [y, z]) \\
&= 2h(''\nabla_x y, z).
\end{aligned}$$

Therefore, we have

$$(12.4) \quad \nabla_x^* y^* = (''\nabla_x y)^* + \sum a_i \xi_{(i)},$$

where  $a_i = g(\xi_{(i)}, \nabla_x^* y^*)$ . Putting  $Z = \xi_{(i)}$  in (12.2), we have

$$\begin{aligned}
(12.5) \quad 2a_i &= -\xi_{(i)} \cdot g(x^*, y^*) + g([x^*, y^*], \xi_{(i)}) \\
&= g([x^*, y^*], \xi_{(i)}) \\
&= \eta_{(i)}([x^*, y^*]) = -d\eta_{(i)}(x^*, y^*) \\
(12.6) \quad &= -2g(x^*, \phi_{(i)} y^*).
\end{aligned}$$

By (12.4), (12.5) and (12.6), we have

$$(12.7) \quad [x^*, y^*] = [x, y]^* - 2 \sum_{i=1}^3 g(x^*, \phi_{(i)} y^*) \xi_{(i)}.$$

By ' $R$ ' we denote the Riemannian curvature tensor of  $(B, h)$ .

$$(''\nabla_x ''\nabla_y z)^* = \nabla_x (''\nabla_y z)^* + \sum g(x^*, \phi_{(i)} (''\nabla_y z)^*) \xi_{(i)}.$$

By (12.4), etc., we get

$$\begin{aligned}
(''\nabla_x ''\nabla_y z)^* &= \nabla_x^* \nabla_y^* z^* + \sum g(y^*, \phi_{(i)} z^*) \nabla_x^* \xi_{(i)} \\
&\quad + \sum [g(\nabla_x^* y^*, \phi_{(i)} z^*) + g(y^*, \nabla_x^* \phi_{(i)} \cdot z^*) \\
&\quad + g(y^*, \phi_{(i)} \nabla_x^* z^*) + g(x^*, \phi_{(i)} \nabla_y^* z^*)] \xi_{(i)}.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
(''\nabla_{[x, y]} z)^* &= \nabla_{[x, y]^*} z^* + \sum g([x, y]^*, \phi_{(i)} z^*) \xi_{(i)} \\
&= \nabla_{[x^*, y^*]} z^* + 2 \sum g(x^*, \phi_{(i)} y^*) \nabla_{\xi_i} z^* + \sum g([x^*, y^*], \phi_{(i)} z^*) \xi_{(i)}.
\end{aligned}$$

Therefore, using  $\nabla_{\xi_{(i)}} z^* = \nabla_z \xi_{(i)} = -\phi_{(i)} z^*$ , we have

$$(12.8) \quad (''R(x, y) z)^* = R(x^*, y^*) z^* + \sum [g(y^*, \phi_{(i)} z^*) \phi_{(i)} x^* - g(x^*, \phi_{(i)} z^*) \phi_{(i)} y^*]$$

$$\begin{aligned} & -2g(x^*, \phi_{(i)}y^*)\phi_{(i)}z^*] + \sum [g(x^*, \nabla_y \phi_{(i)} \cdot z^*) \\ & - g(y^*, \nabla_x \phi_{(i)} \cdot z^*)]\xi_{(i)}. \end{aligned}$$

**PROPOSITION 12.1.** *In the fibering of Theorem 11.3, let  $x, y$  be an orthonormal (local) vector fields on  $B$  (or tangent vectors at a point of  $B$ ). Then we have*

$$(12.9) \quad 'K(x, y) \cdot \pi = K(x^*, y^*) + 3 \sum_{i=1}^3 [g(y^*, \phi_{(i)}x^*)]^2.$$

**PROOF.** Putting  $z=x$  in (12.8) and taking the inner products of  $y^*$  and the both sides of (12.8), we get

$$h('R(x, y)x, y) \cdot \pi = g(R(x^*, y^*)x^*, y^*) + 3 \sum [g(y^*, \phi_{(i)}x^*)]^2,$$

from which we have (12.9).

**THEOREM 12.2.** *In the fibering of Theorem 11.3, assume that  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is a Sasakian 3-structure and  $\dim M=7$ . Then  $(M, g)$  is of constant curvature 1 if and only if  $(B, h)$  is of constant curvature 4.*

**PROOF.** Let  $x, y$  be any orthonormal pair in  $B_q$ ,  $q \in B$ . Then  $x^*, y^*$  are orthonormal and  $y^*$  is expressed by

$$y^* = \sum_{i=1}^3 b_i \phi_{(i)} x^*, \quad b_i = g(y^*, \phi_{(i)}x^*).$$

Since  $\sum b_i^2 = 1$ , (12.9) implies  $'K(x, y) \cdot \pi = K(x^*, y^*) + 3$ . Hence, if  $(M, g)$  is of constant curvature 1,  $(B, h)$  is of constant curvature 4. Conversely, if  $(B, h)$  is of constant curvature 4, we have  $H_{(1)}(X) = 1$  for any non-zero  $X \in E_p$ . This implies that  $(M, g)$  has constant  $\phi_{(1)}$ -holomorphic sectional curvature 1 by (3.13)'. Thus,  $(M, g)$  is of constant curvature 1.

**EXAMPLE.** The Hopf fibration of  $S^7$  is;  $\pi: S^7 \rightarrow QP^1 = S^4$ .

**THEOREM 12.3.** *In the fibering of Theorem 11.3,  $(M, g)$  is an Einstein manifold if and only if  $(B, h)$  is an Einstein manifold such that*

$$'R_1(x, y) = (4r + 8)h(x, y), \quad 4r = \dim B.$$

**PROOF.** Let  $p$  be an arbitrary point of  $M$  and put  $q = \pi p$ . Let  $(\xi_{(i)}, X_u, \phi_{(i)}X_u)$ ;  $i = 1, 2, 3, u = 1, \dots, r$  be an orthonormal basis at  $p$ . If we denote  $\pi_p X_u$  by  $\pi X_u$ ,

etc.,  $(\pi X_u, \pi \phi_{(i)} X_u)$  is an orthonormal basis at  $q$ . By (12.8), we have

$$(12.10) \quad h_q(R(x, \pi X_u)y, \pi X_u) = g_p(R(x^*, X_u)y^*, X_u) \\ + 3 \sum_i g_p(\phi_{(i)}x^*, X_u)g_p(\phi_{(i)}y^*, X_u),$$

$$(12.11) \quad h_q(R(x, \pi \phi_{(j)} X_u)y, \pi \phi_{(j)} X_u) = g_p(R(x^*, \phi_{(j)} X_u)y^*, \phi_{(j)} X_u) \\ + 3 \sum_i g_p(\phi_{(i)}x^*, \phi_{(j)} X_u)g_p(\phi_{(i)}y^*, \phi_{(j)} X_u)$$

for  $j = 1, 2, 3$ . On the other hand, by (2.1). we have

$$(12.12) \quad 0 = \sum g_p(R(x^*, \xi_{(i)})y^*, \xi_{(i)}) - 3g_p(x^*, y^*).$$

First we notice that

$$\sum_u g(x^*, X_u)g(y^*, X_u) + \sum_{j,u} g(x^*, \phi_{(j)} X_u)g(y^*, \phi_{(j)} X_u) = g(x^*, y^*).$$

Then by (12.10)~(12.12), we have

$$(12.13) \quad R_{1q}(x, y) = R_{1p}(x^*, y^*) + 6g_p(x^*, y^*).$$

If  $(M, g)$  is an Einstein manifold, we have  $R_1 = (m-1)g = (4r+2)g$  (cf. (2.1)). Therefore, we have  $R_1(x, y) = (4r+8)h(x, y)$ . Conversely, if  $(B, h)$  is an Einstein manifold such that  $R_1 = (4r+8)h$ , then  $R_1(x^*, y^*) = (m-1)g(x^*, y^*)$  holds. Since  $R_1(X, \xi_{(i)}) = (m-1)\eta_{(i)}(X)$  (cf. (1.6) of [21]),  $(M, g)$  is an Einstein manifold.

In the fibering of Theorem 11.3, if  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$  is a Sasakian 3-structure, then  $(B, h)$  is an Einstein manifold. Hence, we have

**THEOREM E.** *Let  $(M, g)$  be a complete Riemannian manifold admitting a Sasakian 3-structure  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)})$ . If one of the Sasakian structures is regular, then  $(M, \xi_{(1)}, \xi_{(2)}, \xi_{(3)}, g)$  is a  $S^3[1]$ - or  $RP^3[1]$ -principal bundle over an Einstein manifold  $(B, h)$  such that  $R_1 = (4r+8)h$ ,  $4r = \dim B$ .*

**13. 3-K-contact structures.** We define a 3-K-contact structure on  $(M, g)$  by three K-contact structures  $\xi_{(1)}, \xi_{(2)}, \xi_{(3)}$  satisfying (3.1) and (3.2). Some results on K-contact 3-structures are generalized to results on 3-K-contact structures.

**LEMMA 13.1.** *Let  $\xi_{(1)}$  and  $\xi_{(2)}$  be two K-contact structures on  $(M, g)$  such that  $g(\xi_{(1)}, \xi_{(2)}) = 0$ . Then  $(\xi_{(1)}, \xi_{(2)}, \xi_{(3)} = (1/2)[\xi_{(1)}, \xi_{(2)}])$  is a 3-K-contact structure.*

**PROOF.** Since

$$\begin{aligned} [\xi_{(1)}, \xi_{(2)}] &= \nabla_{\xi_{(1)}} \xi_{(2)} - \nabla_{\xi_{(2)}} \xi_{(1)} = 2\nabla_{\xi_{(1)}} \xi_{(2)} = -2\phi_{(2)} \xi_{(1)} \\ &= -2\nabla_{\xi_{(2)}} \xi_{(1)} = 2\phi_{(1)} \xi_{(2)}, \end{aligned}$$

$\xi_{(3)} = \phi_{(1)} \xi_{(2)}$  is also a unit Killing vector. Then we have

$$\begin{aligned} (13.1) \quad [\xi_{(1)}, \xi_{(3)}] &= L_{\xi_{(1)}} \xi_{(3)} = L_{\xi_{(1)}} (\phi_{(1)} \xi_{(2)}) = \phi_{(1)} [\xi_{(1)}, \xi_{(2)}] \\ &= 2\phi_{(1)} \xi_{(3)} = 2\phi_{(1)} \phi_{(1)} \xi_{(2)} = -2\xi_{(2)}, \end{aligned}$$

$$(13.2) \quad [\xi_{(2)}, \xi_{(3)}] = L_{\xi_{(2)}} (-\phi_{(2)} \xi_{(1)}) = 2\xi_{(1)}.$$

Hence,  $\xi_{(i)}$ ,  $i=1, 2, 3$ , satisfy (3.1), (3.2) where  $\phi_{(3)} = -\nabla \xi_{(3)}$ . We show that  $\xi_{(3)}$  is a  $K$ -contact structure. Since  $\xi_{(1)}$  satisfies

$$(13.3) \quad R(X, \xi_{(1)}) \xi_{(1)} = g(X, \xi_{(1)}) \xi_{(1)} - X,$$

operating the Lie derivation  $L_{\xi_{(3)}}$  to (13.3), we have

$$(13.4) \quad R(X, \xi_{(3)}) \xi_{(1)} + R(X, \xi_{(1)}) \xi_{(3)} = g(X, \xi_{(3)}) \xi_{(1)} + g(X, \xi_{(1)}) \xi_{(3)}.$$

Operating  $L_{\xi_{(3)}}$  again to (13.4), and using (13.3), we have

$$(13.5) \quad R(X, \xi_{(3)}) \xi_{(3)} = g(X, \xi_{(3)}) \xi_{(3)} - X$$

Therefore,  $\xi_{(3)}$  is a  $K$ -contact structure.

**PROPOSITION 13.2.** *A 3-K-contact structure on  $(M, g)$  is a K-contact 3-structure if and only if*

$$(13.6) \quad R(X, \xi_{(1)}) \xi_{(2)} = g(X, \xi_{(2)}) \xi_{(1)}.$$

**PROOF.** Operating  $\nabla_X$  to  $\phi_{(1)} \xi_{(2)} = \xi_{(3)}$ , we have

$$\nabla_X \phi_{(1)} \cdot \xi_{(2)} - \phi_{(1)} \phi_{(2)} X = -\phi_{(3)} X.$$

Since  $\nabla_X \phi_{(1)} = -\nabla_X (\nabla \xi_{(1)})$  and  $\nabla_X (\nabla \xi_{(1)}) + R(X, \xi_{(1)}) = 0$ , we have

$$(13.7) \quad R(X, \xi_{(1)}) \xi_{(2)} - \phi_{(1)} \phi_{(2)} X = -\phi_{(3)} X.$$

Hence, if (13.6) holds, we have (3.3)<sub>k=3</sub>. If we operate  $L_{\xi_{(1)}}$  to (13.6), we have  $R(X, \xi_{(1)}) \xi_{(3)} = g(X, \xi_{(3)}) \xi_{(1)}$ , and then we get (3.3)<sub>k=2</sub>. Similarly, we get (3.3)<sub>k=1</sub>.

**REMARK.** In the above discussion, if  $\xi_{(1)}$  and  $\xi_{(2)}$  are Sasakian, then replacing (13.3) by (2.5) for  $\xi_{(1)}$  we see that  $\xi_{(3)}$  is Sasakian. Since we have (13.6) for Sasakian  $\xi_{(1)}$ , we have (iii) in §3.

**PROPOSITION 13.3.** *Theorem 11.3, Proposition 12.1 and Theorem 12.3 are true for a 3-K-contact structure.*

In fact, in proofs of Propositions listed above, (3.3) are not used. Only two points we must notice here are :

(i) we have a basis of the form  $(\xi_{(i)}, X_j, \phi_{(i)}X_j)$  at each point. If  $\dim M=3$ , this is clear. If  $\dim M>3$ , we have a unit  $X_1 \in M_p$ , which is orthogonal to  $\xi_{(i)}, i=1, 2, 3$ . If we put  $X=X_1$  in (13.4), we get  $R(X_1, \xi_{(3)})\xi_{(1)}+R(X_1, \xi_{(1)})\xi_{(3)}=0$ . Similarly, we have

$$(13.8) \quad R(X_1, \xi_{(1)})\xi_{(2)}+R(X_1, \xi_{(2)})\xi_{(1)}=0$$

By (13.7) and (13.7)' ( $\leftarrow \phi_{(2)}\xi_{(1)}=-\xi_{(3)}$ ):

$$(13.7)' \quad R(X, \xi_{(2)})\xi_{(1)}-\phi_{(2)}\phi_{(1)}X=\phi_{(3)}X,$$

(13.8) is written as

$$(13.9) \quad \phi_{(1)}\phi_{(2)}X_1+\phi_{(2)}\phi_{(1)}X_1=0.$$

By (13.9), (13.9)', (13.9)'', we see that  $(\xi_{(1)}, X_1, \phi_{(i)}X_1)$  is orthonormal. These steps complete a basis stated above.

(ii) With respect to (12.11)  $\rightarrow$  (12.13), it is required that  $(\xi_{(i)}, X_j, \phi_{(1)}\phi_{(2)}X_j, \phi_{(1)}\phi_{(3)}X_j, \phi_{(2)}\phi_{(3)}X_j)$  is also an orthonormal basis. This is also assured by (13.9).

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