Kindergarten Quantum Mechanics

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(or Google ‘‘Bob Coecke’’)

THE CHALLENGE

Why did discovering quantum teleportation take 60 year?

Claim: **bad formalism** since 'too low level' cf.

$$\frac{\text{“}\textbf{GOOD QM}\text{”}}{\texttt{von Neumann QM}} \simeq \frac{\textbf{HIGH-LEVEL language}}{\texttt{low-level language}}$$

Wouldn't it be nice to have a such a 'good' formalism, in which discovering teleportation would be trivial?

Claim: **it exists**! And I'll present it to you.

Isn't it absurdly abstract coming from you guys?

Claim: It could be taught in **kindergarten**!

THE APPROACH

1. **Analyse** quantum compoundness.

⇒ A notion of **quantum information-flow** emerges.

- Physical Traces. Abramsky & Coecke (2003) CTCS'02; `cs/0207057`
- The Logic of Entanglement. Coecke (2003) PRG-RR; `quant-ph/0402014`
- Quantum Information-flow, Concretely, and Axiomatically. `quant-ph/0506132`

2. **Axiomatize** quantum compoundness.

⇒ ... full **quantum mechanics** emerges!

- A Categorical Semantics of Quantum Protocols. Abramsky & Coecke (2004) IEEE-LiCS'04; `quant-ph/0402130`
- Abstract Physical Traces. Abramsky & Coecke (2005) TAC'05.

⇒ ... & **quantum logic** ... & **open systems/CPM's**!

- De-linearizing Linearity I: Projective Quantum Axiomatics from SCC. Coecke (2005) QPL'05; `quant-ph/0506134`.
- †-CCC's and Completely Positive Maps. Selinger (2005) QPL'05.

EXPLICIT OPERATIONALISM

Primitive data are **processes/operations** $f, g, h, \ldots$ which are **typed** as $A \to B, B \to C, A \to A, \ldots$ where $A, B, C, \ldots$ are **kinds/names** of **systems**.

Sequential composition is a primitive connective on processes/operations cf.

$$f \circ g : A \to C \quad \text{for} \quad f : A \to \underline{B} \ \& \ g : \underline{B} \to C$$

Parallel composition is a primitive connective both on systems and processes/operations cf.

$$f \otimes g : A \otimes C \to B \otimes D \quad \text{for} \quad f : A \to B \ \& \ g : C \to D$$

NO DOGMAS nor TABOOS!

Do you want ...

- states to be ontological or empirical?
- vectorial, projective, POVM-/CPM-/open system-style?
- hidden variables, quantum potential, contextuality, (non-)locality, Bayesianism, ... ?

The bulk of the developments ignores these choices, but, they can be implemented formally since we both have

- **great axiomatic freedom**
- **great expressiveness**

CATEGORY THEORY!

Audience: "Seriously, you don't expect us to learn that?"

Bob: "**No! Of course not!**"

"We are gonna go far back in time, ...
to the time you were all still at kindergarten, ..."

"**We're gonna draw pictures!**"

The sheer magic of the kind of category theory we need here is that it formally justifies its own formal absence.

A NEW FORMALISM

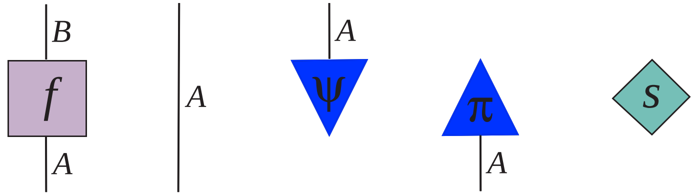
Language and calculus: purely graphical

Behind the scene: categorical algebra

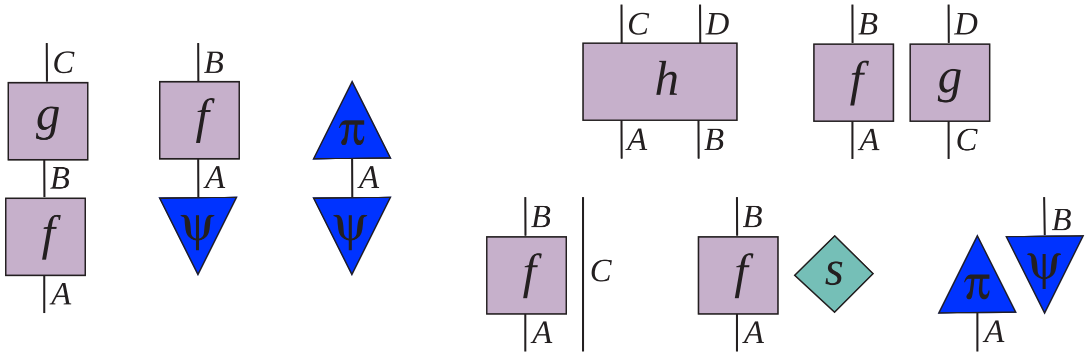
Concrete model: Hilbert space QM, ...
and also many others, ...

Not assumed: some number field, any kind of matrix calculus, vectors and sums thereof, elements of objects/types (cf. state space) and corresponding mappings, ...

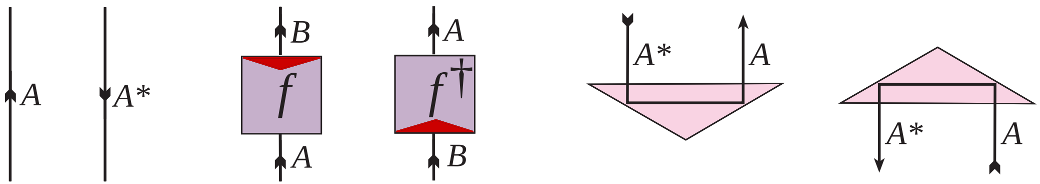
Primitive data:



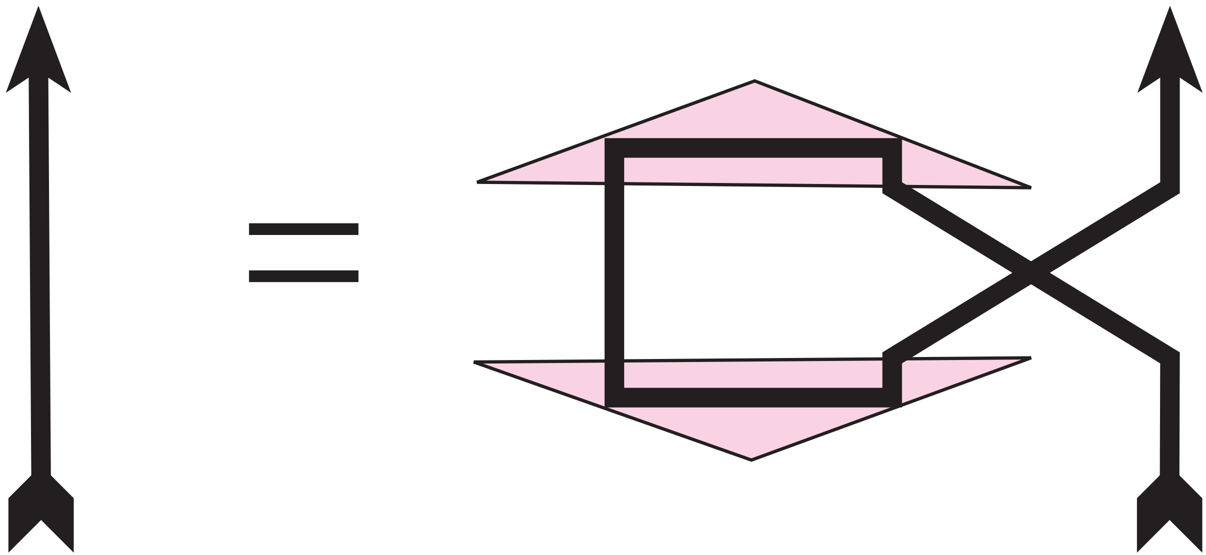
Sequential and parallel composition:



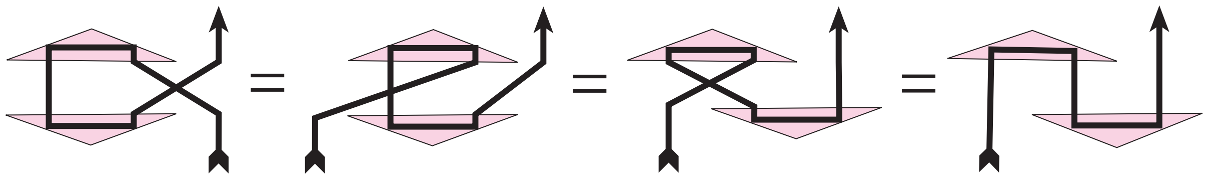
Duals, adjoints and EPR-states:



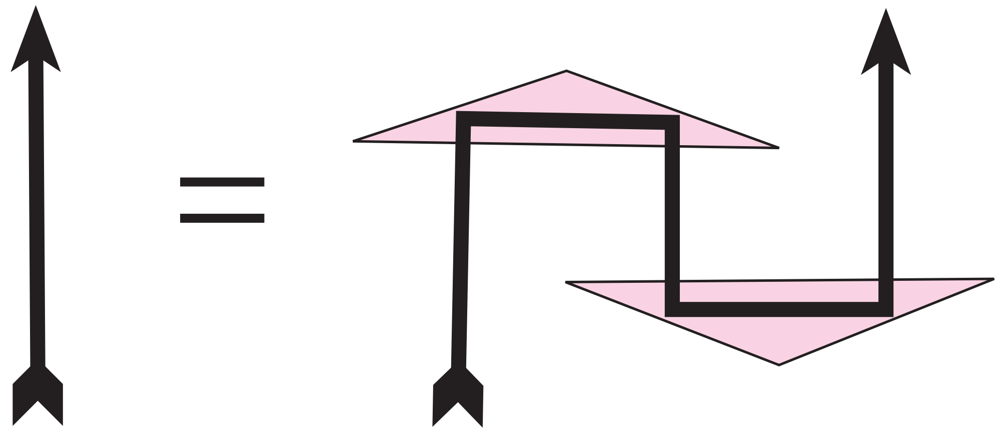
THE SOLE AXIOM



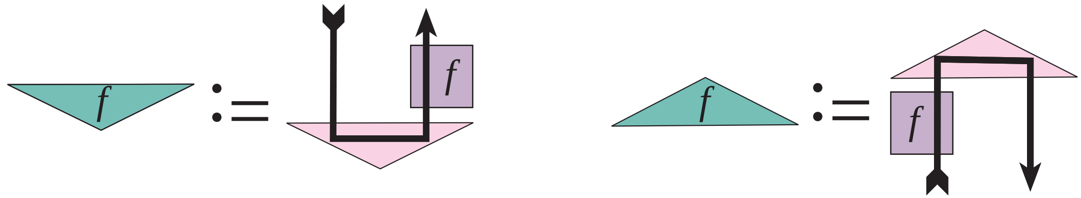
Since



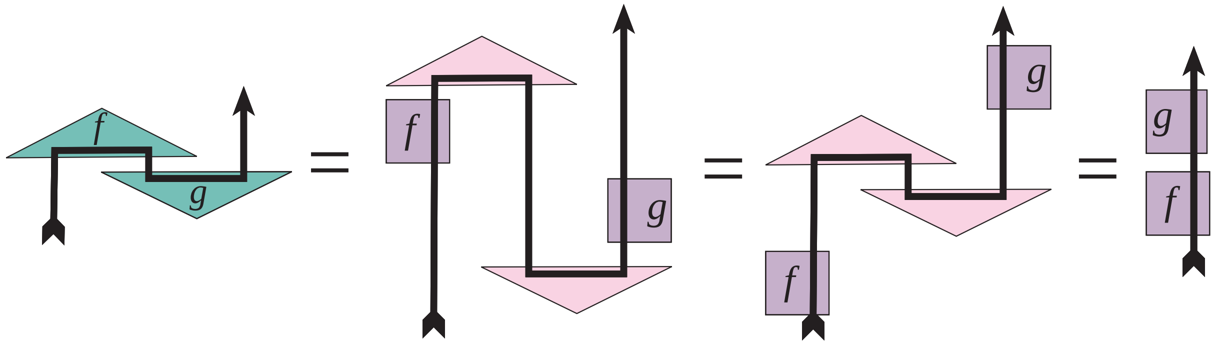
the axiom is equivalent to



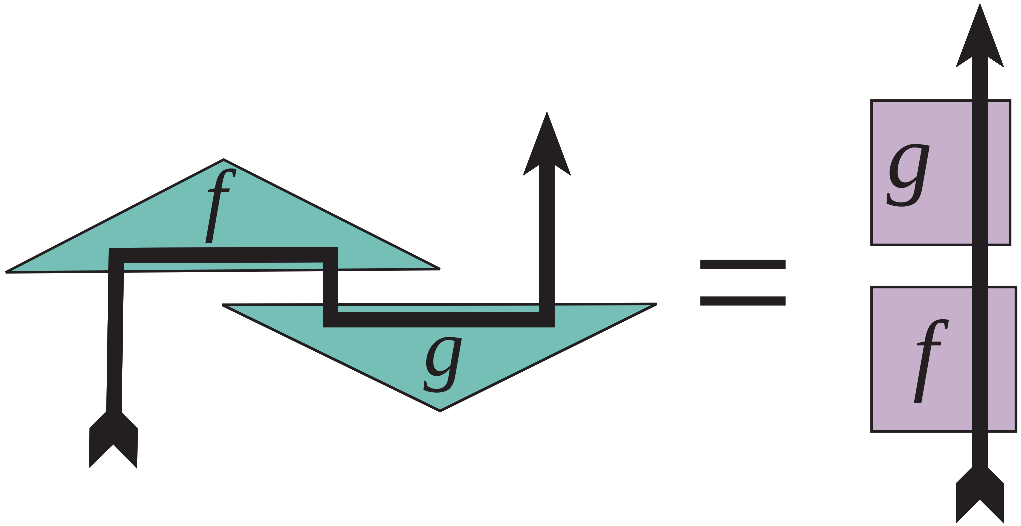
When setting



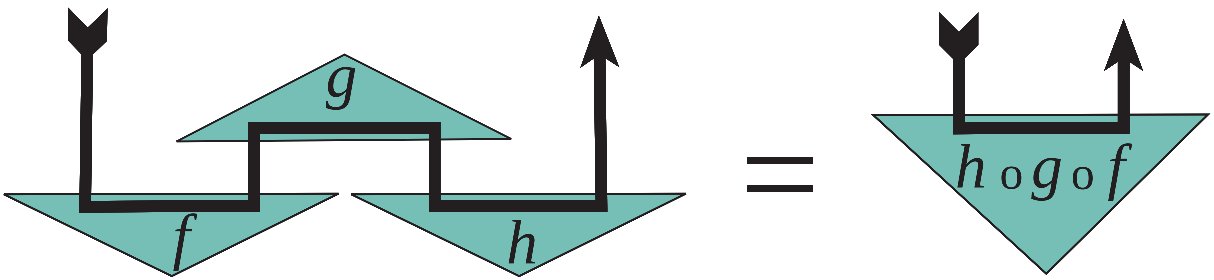
we obtain



COMPOSITIONALITY



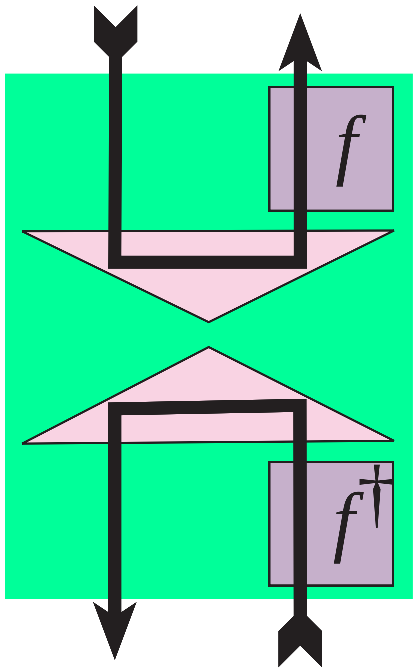
COMPOSITIONALITY bis



We define bipartite projectors as

$$\mathrm{P}_f : A^* \otimes B \longrightarrow A^* \otimes B$$

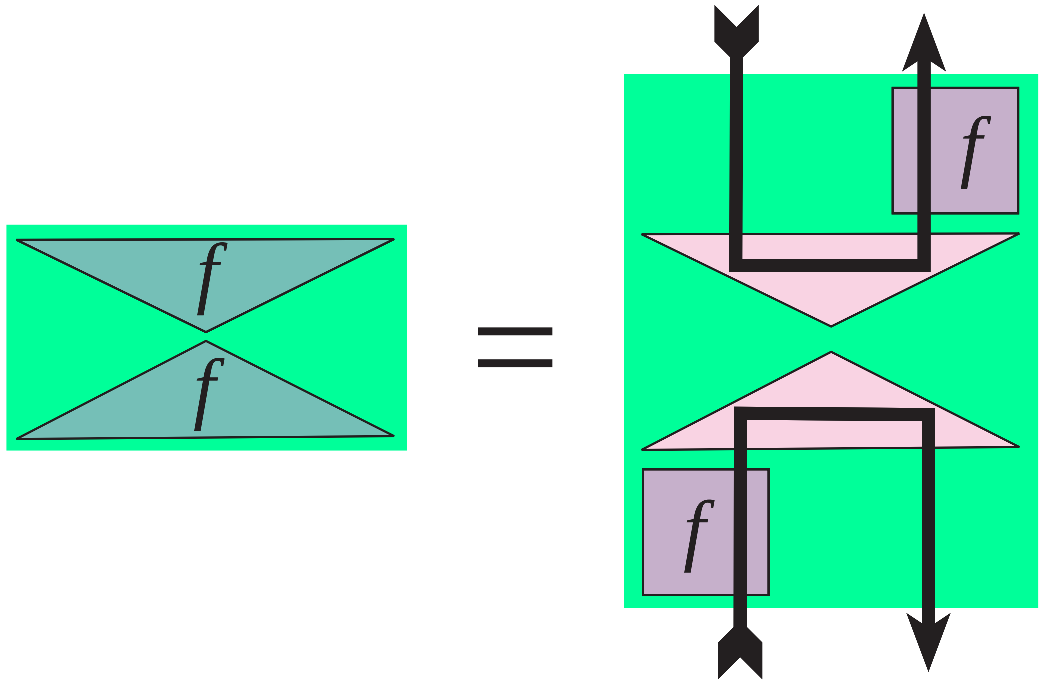
as



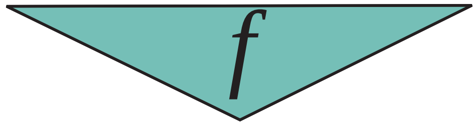
that is, approximately, as

$$\mathrm{P}_f : A \otimes B^* \longrightarrow A^* \otimes B$$

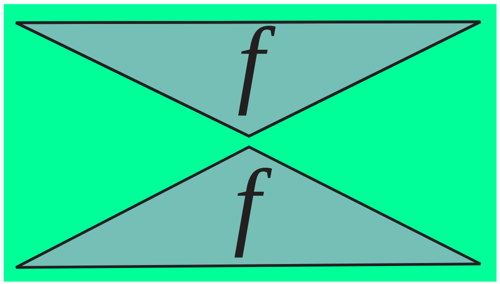
as



The concepts of bipartite state

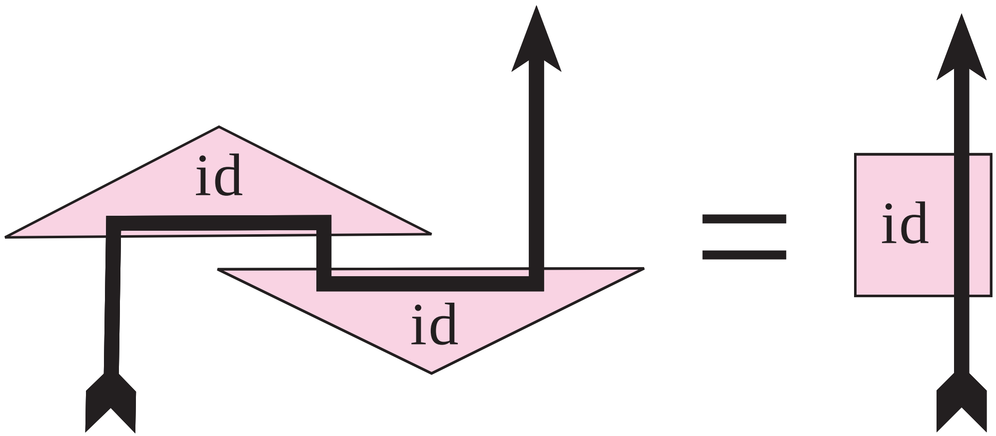


and of bipartite projector



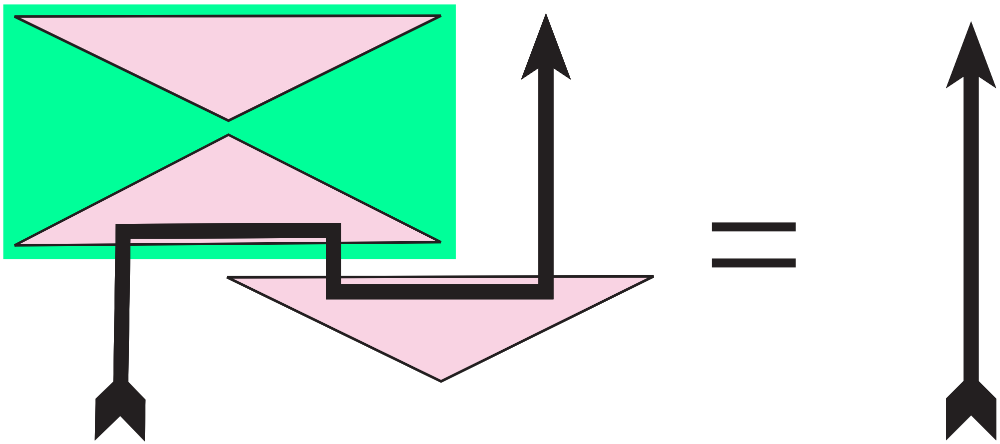
yield the following corrolaries ...

$\frac{1}{4}$th-TELEPORTATION



since $\mathrm{id} \circ \mathrm{id} = \mathrm{id}$

$\frac{1}{4}$th-TELEPORTATION



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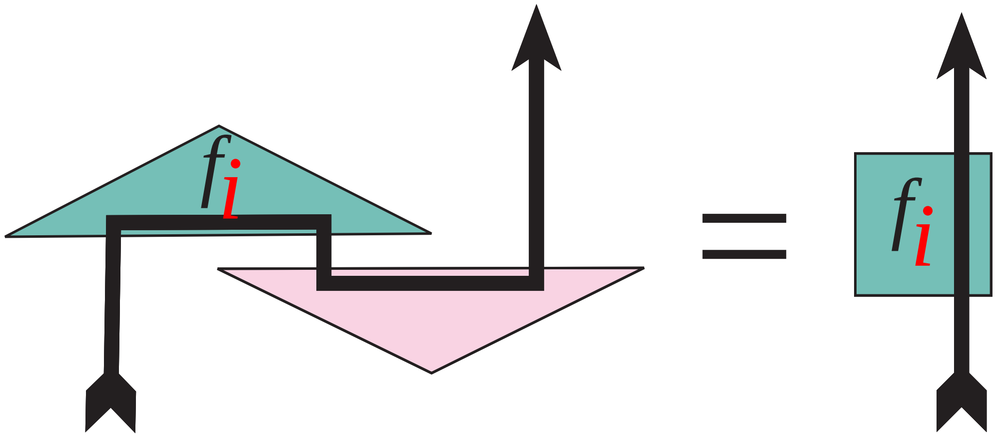
FULL TELEPORTATION



for $1 \leq i \leq 4$

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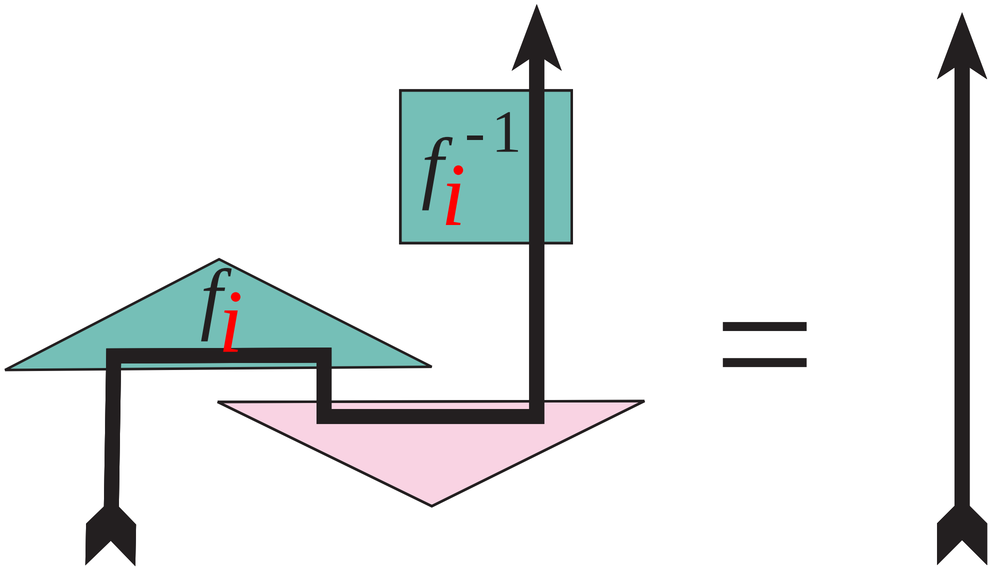
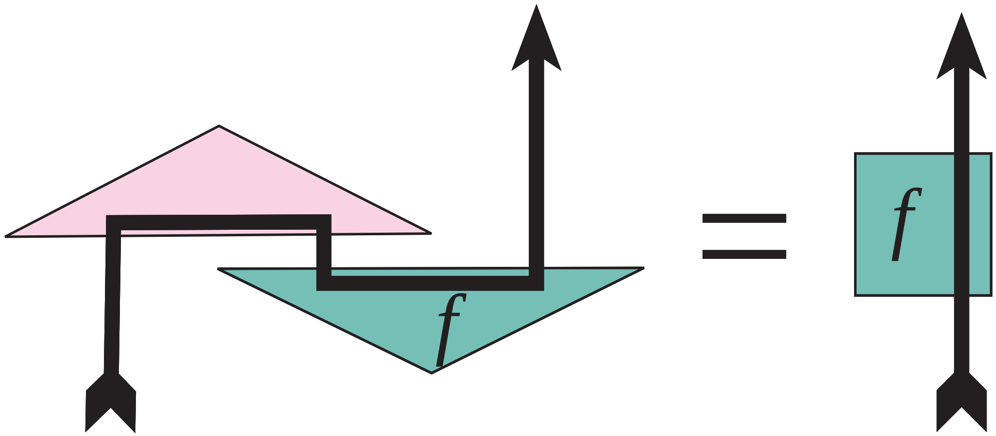


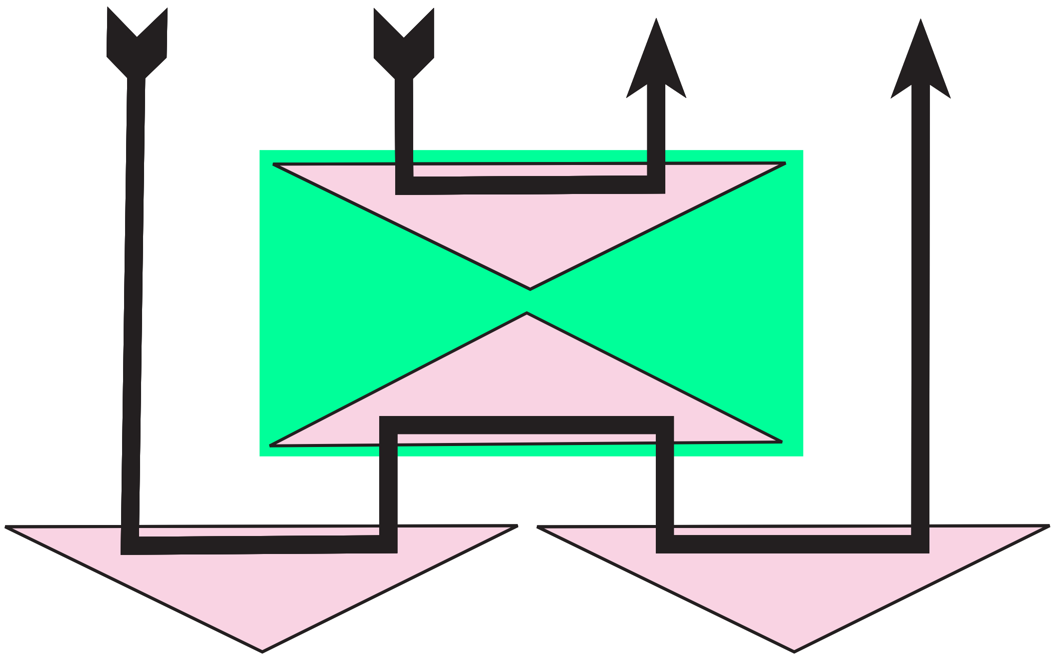
for $1 \leq i \leq 4$

LOGIC GATE TELEPORTATION

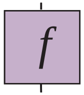


since $f \circ \mathrm{id} = f$

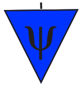
ENTANGLEMENT SWAPPING



HILBERT SPACE QM



$: \mathcal{H}_1 \to \mathcal{H}_2$ is a linear map

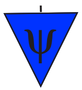


$: \mathbb{C} \to \mathcal{H}$ cf. $\psi(1) \in \mathcal{H}$

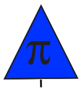
$: \mathbb{C} \to \mathbb{C}$ cf. $s(1) \in \mathbb{C}$

$\mathcal{H}^*$:= conjugate Hilbert space of $\mathcal{H}$

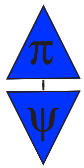
$f^\dagger$:= linear adjoint of f



$= |\psi\rangle$

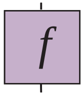


$= \langle\phi|$ for $\pi := \phi^\dagger$

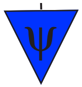


$= \langle\phi \mid \psi\rangle$

HILBERT SPACE QM



$: \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ is a linear map



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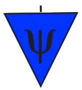

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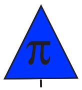

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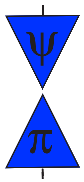
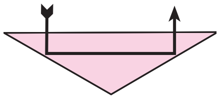
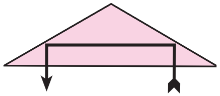


$= |\psi\rangle\langle\phi|$

EPR-states and their adjoints:



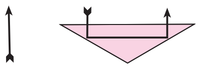
$$: \mathbb{C} \to \mathcal{H}^* \otimes \mathcal{H} :: 1 \mapsto \left| \sum_i e_i \otimes e_i \right\rangle$$



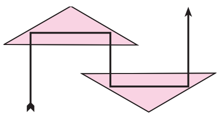
$$: \mathcal{H}^* \otimes \mathcal{H} \to \mathbb{C} :: \Phi \mapsto \left\langle \sum_i e_i \otimes e_i \,\middle|\, \Phi \right\rangle$$

$$:: \phi_1 \otimes \phi_2 \mapsto \langle \phi_1 \mid \phi_2 \rangle$$

We verify the axiom:

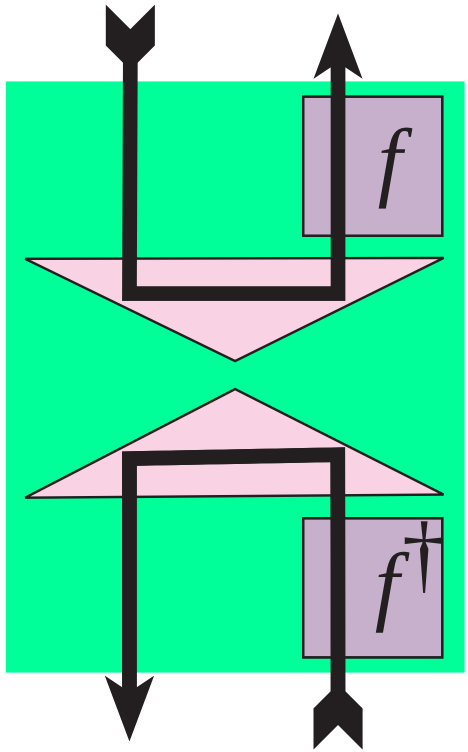


$$= (-) \otimes \left(\sum_i e_i \otimes e_i \right) = \sum_i (- \otimes e_i) \otimes e_i$$



$$= \sum_i \langle - \mid e_i \rangle \cdot e_i \;=\; \text{id}$$

Exercise. Verify that in Hilbert space bipartite projectors on one-dimensional subspaces indeed factor as



A key role is played by

$$\mathcal{H}_1^* \otimes \mathcal{H}_2 \quad \simeq \quad \mathcal{H}_1 \to \mathcal{H}_2$$

i.e. bipartite states $\Psi \in \mathcal{H}_1^* \otimes \mathcal{H}_2$ are representable by linear functions $f : \mathcal{H}_1 \to \mathcal{H}_2$ and vice versa. Indeed

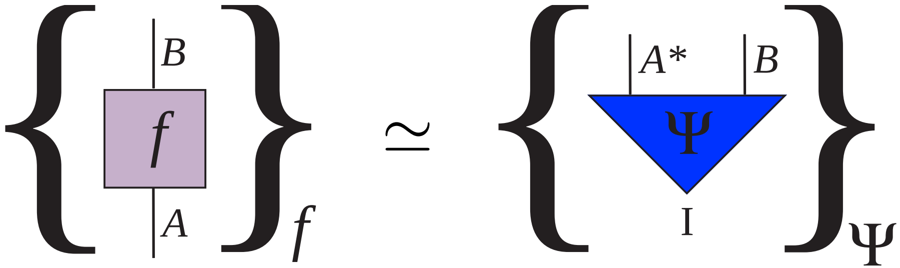
$$\Psi = \sum_{ij} m_{ij} \, |ij\rangle \quad \stackrel{\simeq}{\longleftrightarrow} \quad \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kn} \end{pmatrix}$$

$$\stackrel{\simeq}{\longleftrightarrow} \quad f = \sum_{ij} m_{ij} \, |j\rangle\langle i|$$

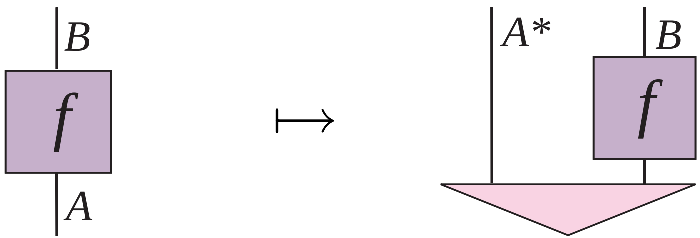
e.g.

$$|00\rangle + |11\rangle \quad \stackrel{\simeq}{\longleftrightarrow} \quad \mathsf{id} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

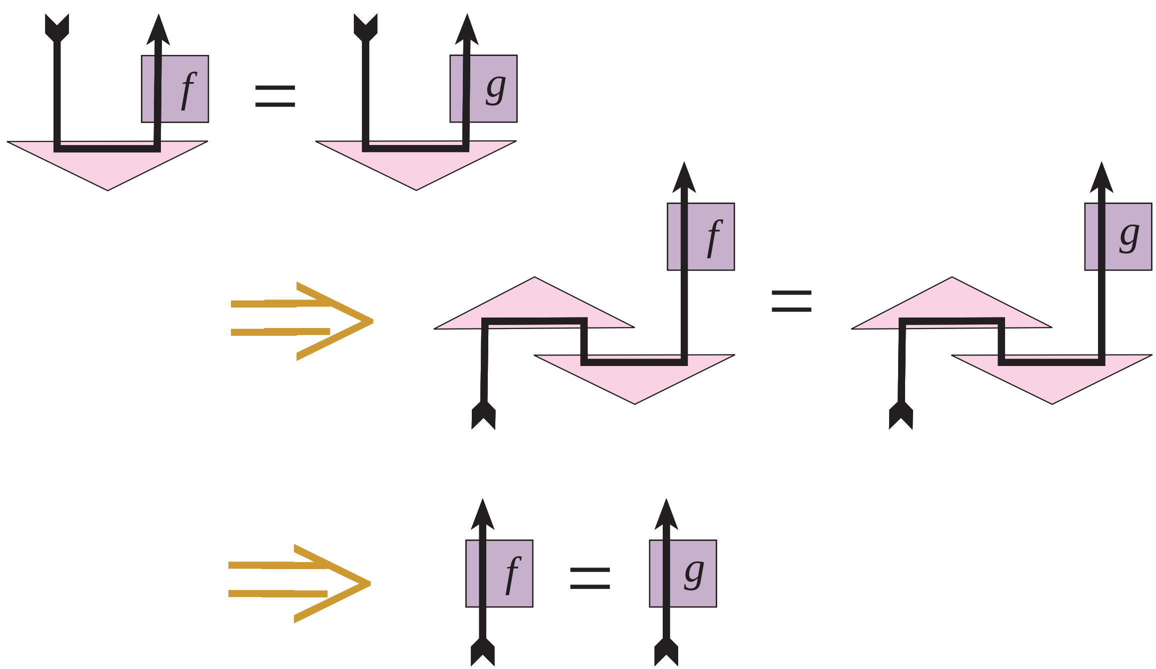
PROCESSES $\simeq$ 2-STATES



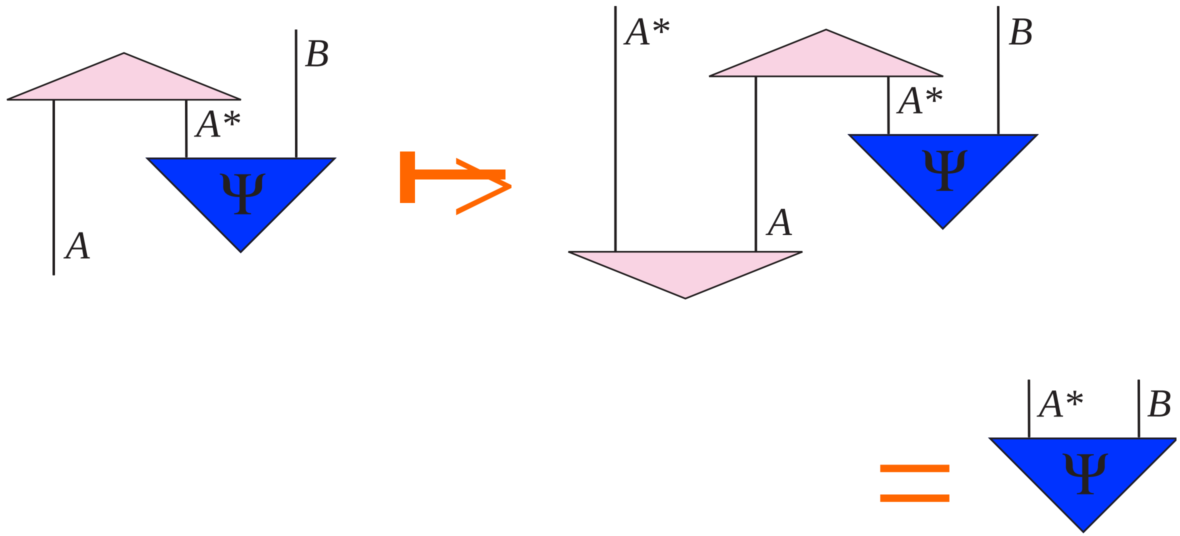
for the bijection $f \mapsto \ulcorner f \urcorner$ i.e.



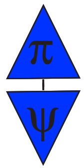
Proof of injectivity.



Proof of injectivity.



The **inner-product** of $\psi, \phi : \mathrm{I} \longrightarrow A$ is

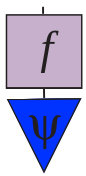
$\langle \phi \mid \psi \rangle :=$ 

$= \phi^{\dagger} \circ \psi : \mathrm{I} \longrightarrow \mathrm{I}$

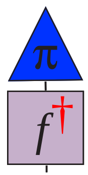
where $\pi := \phi^{\dagger}$ cf.

$\texttt{bra} := \langle \phi \mid \qquad \texttt{ket} := \mid \psi \rangle \qquad \texttt{bra-ket} := \langle \phi \mid \psi \rangle$

e.g. for $f : A \longrightarrow B$ we have

$\mid f \circ \psi \rangle =$ 

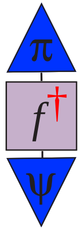
$= f \circ \psi$

$\langle f \circ \phi \mid =$ 

$= \phi^{\dagger} \circ f^{\dagger}$

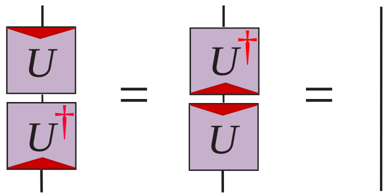
Adjointness implies

$$\langle f \circ \phi \mid \psi \rangle =$$



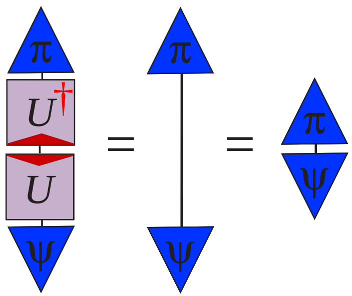
$$= \langle \phi \mid f^{\dagger} \circ \psi \rangle$$

Unitarity means $U^{-1} = U^{\dagger}$ i.e.



hence

$$\langle U \circ \phi \mid U \circ \psi \rangle =$$



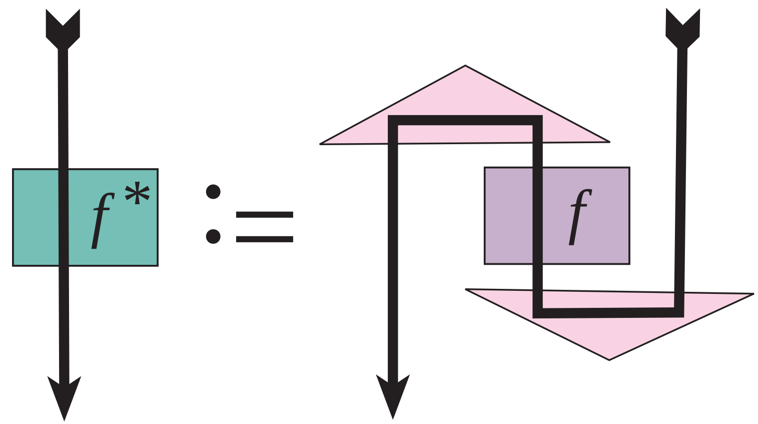
$$= \langle \phi \mid \psi \rangle$$

UPPER STAR STRUCTURE

A "contravariant" Barr-Kelly-Laplaza involution

$$f : A \to B \quad \mapsto \quad f^* : B^* \to A^*$$

called **upper star** arises as

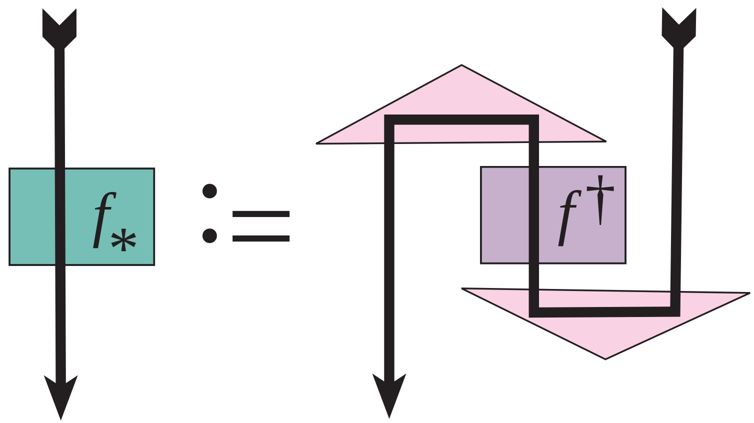


LOWER STAR STRUCTURE

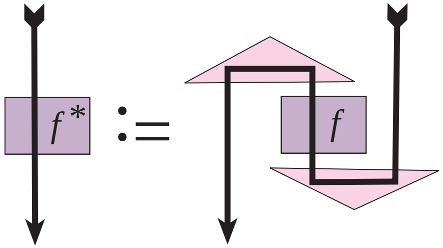
A "covariant" involution

$$f : A \to B \quad \mapsto \quad f_* : A^* \to B^*$$

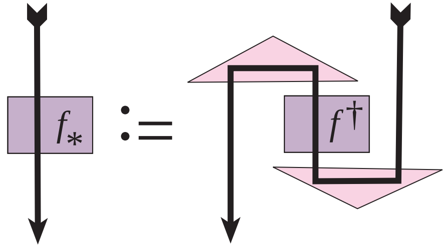
called **lower star** arises as



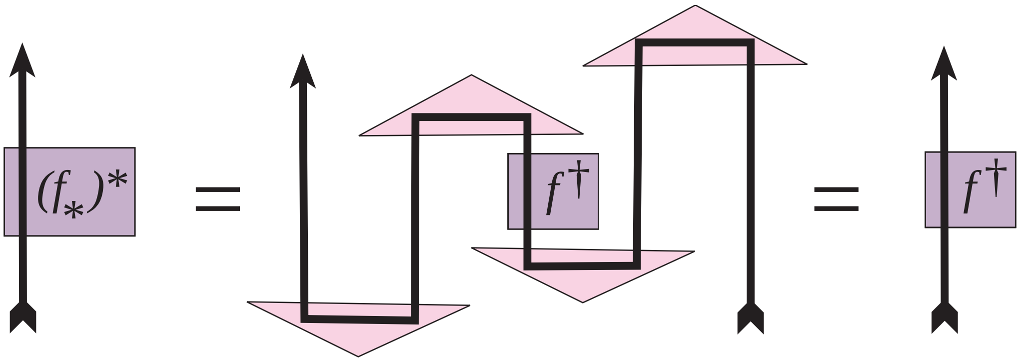
From



and

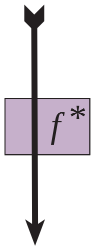


follows



and analogous we can prove that $(f^*)_* = f^\dagger$

Hence the star operations



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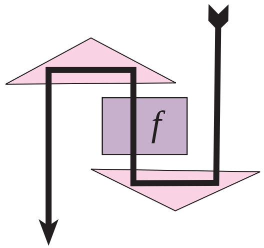
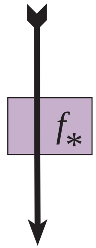
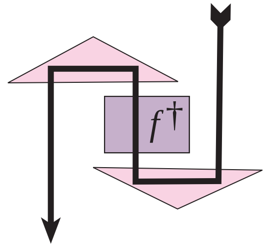


and



:=



provide a decomposition of the adjoint:

$$f^\dagger = (f^*)_* = (f_*)^*$$

In particular, for the Hilbert space model we have

$(-)^* :=$ **transposition**

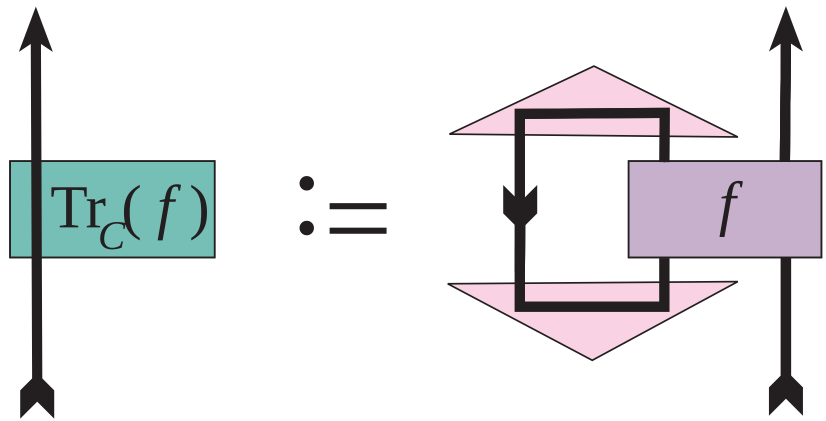
$(-)_* :=$ **complex conjugation**

TRACE STRUCTURE

A Joyal-Street-Verity **partial trace**

$$f : C \otimes A \to C \otimes B \quad \mapsto \quad \mathrm{Tr}_C(f) : A \to B$$

arises as

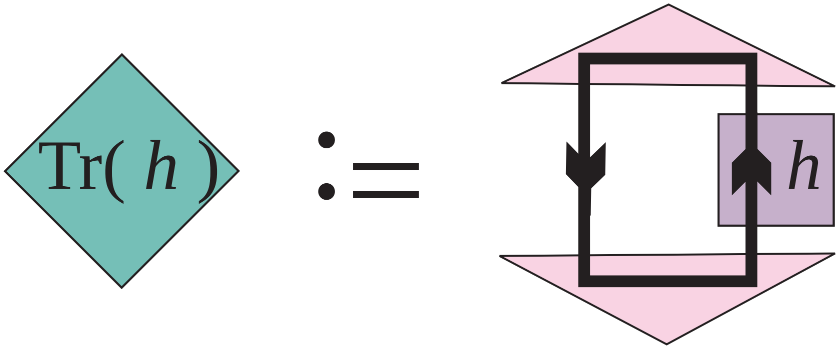


TRACE STRUCTURE bis

A corresponding **full trace**

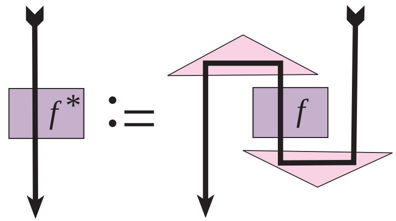
$$h : A \to A \quad \mapsto \quad \mathrm{Tr}(h) : \mathrm{I} \to \mathrm{I}$$

arises as

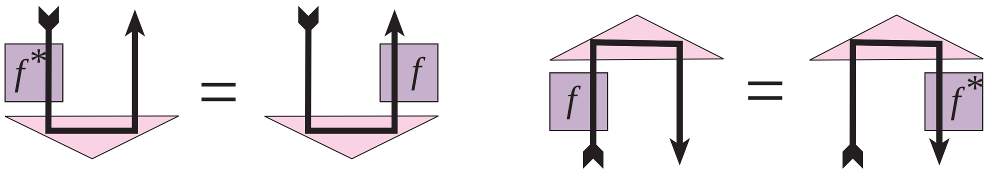


$\Rightarrow h$ "carries a diamond" cf. **probabilistic weight**

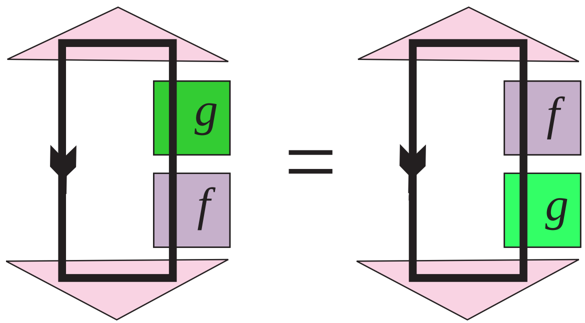
From



follows

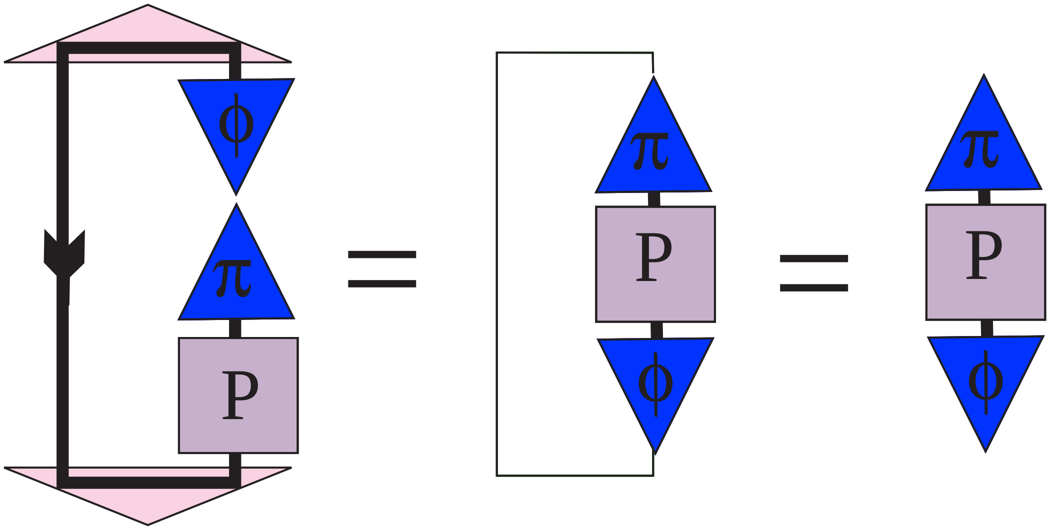


and hence



EQUIVALENT BORN RULES

$$\mathsf{Tr}(\rho_\phi \circ \mathrm{P}) \overset{???}{=} \langle \phi \mid \mathrm{P} \circ \phi \rangle \quad \text{for} \quad \rho_\phi := |\phi\rangle\langle\phi|$$



$$\mathbb{C}^* \otimes \mathbb{C} \simeq \mathbb{C}$$

ALGEBRA BEHIND THE SCENE

Symmetric monoidal bifunctor $-\otimes- : \mathbf{C}\times\mathbf{C} \longrightarrow \mathbf{C}$ and

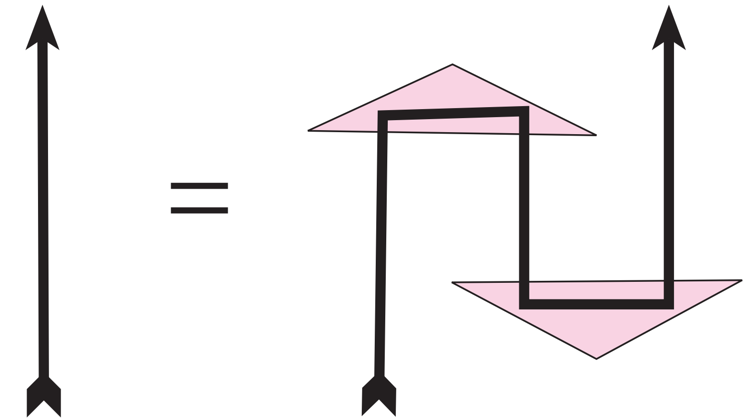
- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A\to B} \mapsto f^\dagger_{B\to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$;

$$\begin{array}{ccccc}
A & \xleftarrow{\;\simeq\;} & \mathrm{I}\otimes A & \xleftarrow{\;\eta^\dagger_{A^*}\otimes 1_A\;} & (A\otimes A^*)\otimes A \\
{\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\
A & \xrightarrow[\;\simeq\;]{} & A\otimes \mathrm{I} & \xrightarrow[\;1_A\otimes\eta_A\;]{} & A\otimes(A^*\otimes A)
\end{array}$$

ALGEBRA BEHIND THE SCENE

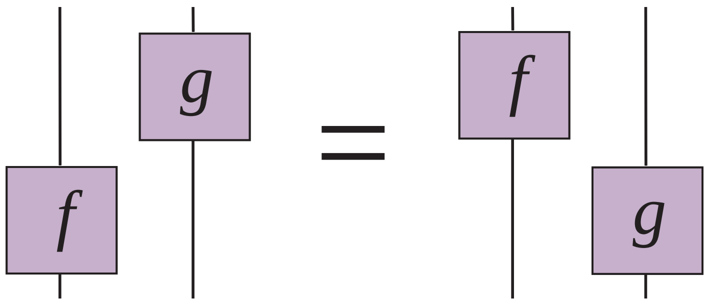
Symmetric monoidal bifunctor $-\otimes- : \mathbf{C}\times\mathbf{C} \longrightarrow \mathbf{C}$ and

- $\otimes$-involution **dual** $A \mapsto A^*$;
- contravariant $\otimes$-involution **adjoint** $f_{A\to B} \mapsto f^{\dagger}_{B\to A}$;
- **Units** $\eta_A : \mathrm{I} \longrightarrow A^* \otimes A$ with $\eta_{A^*} = \sigma_{A,A^*} \circ \eta_A$;



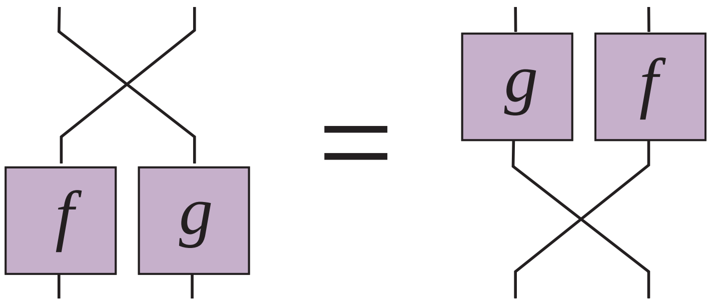
BIFUNCTORIALITY OF $\otimes$

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f_1 \otimes \mathsf{id}} & B_1 \otimes A_2 \\ \Big\downarrow {\scriptstyle \mathsf{id} \otimes f_2} & & \Big\downarrow {\scriptstyle \mathsf{id} \otimes f_2} \\ A_1 \otimes B_2 & \xrightarrow[f_1 \otimes \mathsf{id}]{} & B_1 \otimes B_2 \end{array}$$



NATURAL SYMMETRY

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f_1 \otimes f_2} & B_1 \otimes B_2 \\ {\scriptstyle \sigma_{A_1,A_2}} \downarrow & & \downarrow {\scriptstyle \sigma_{B_1,B_2}} \\ A_2 \otimes A_1 & \xrightarrow[f_2 \otimes f_1]{} & B_2 \otimes B_1 \end{array}$$

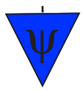


STATES AND NUMBERS

We use the unit I for $-\otimes-$ i.e.

$$A \simeq \mathrm{I} \otimes A \simeq A \otimes \mathrm{I}$$

to define states and numbers respectively as



$$: \mathrm{I} \longrightarrow A$$

and

$$: \mathrm{I} \longrightarrow \mathrm{I}$$

NATURAL SCALAR MULTIPLES

Scalars satisfy

$$s \circ t = \mathrm{I} \xrightarrow{\simeq} \mathrm{I} \otimes \mathrm{I} \xrightarrow{s \otimes t} \mathrm{I} \otimes \mathrm{I} \xrightarrow{\simeq} \mathrm{I}$$

and we define **scalar multiplication** as

$$s \bullet f := A \xrightarrow{\simeq} A \otimes \mathrm{I} \xrightarrow{f \otimes s} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

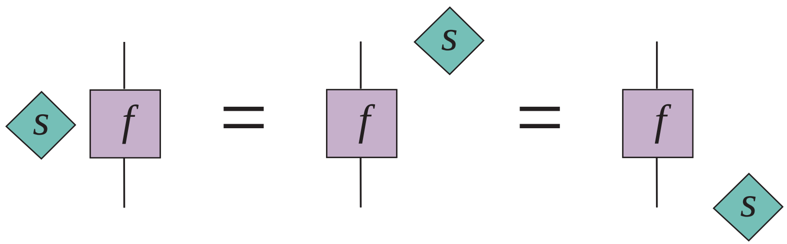
for which we can then prove

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g)$$

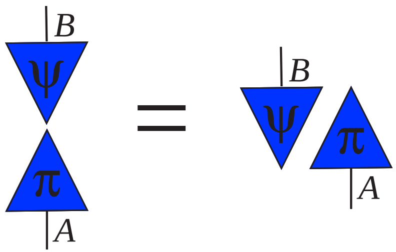
$$(s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

i.e. diamonds can move around freely in 'time' and 'space'

NATURAL SCALAR MULTIPLES



and similarly



i.e.

$$\psi \circ \pi = A \xrightarrow{\simeq} \mathrm{I} \otimes A \xrightarrow{\psi \otimes \pi} B \otimes \mathrm{I} \xrightarrow{\simeq} B$$

NO-CLONING NO-DELETING

Cf. Dieks-Wooters-Zurek **1982** & Pati-Braunstein **2000**

Obviously we do <u>not</u> want to be $- \otimes -$ a **categorical (co-)product** since that would imply existence of

$$A \xrightarrow{\Delta} A \otimes A \qquad\qquad A \otimes B \xrightarrow{p} A$$

i.e. there are no logical rules

$$A \vdash A \wedge A \qquad\qquad A \wedge B \vdash A$$

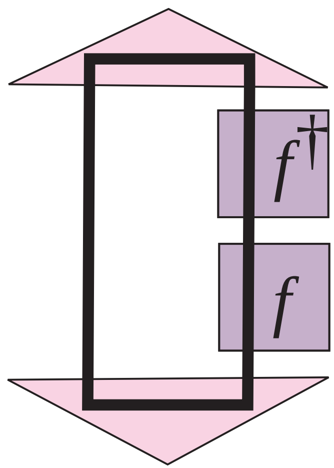
The **squared Hilbert-Schmidt norm**

$$||f|| = \sum_i \langle f(e_i) \mid f(e_i)\rangle$$

exists in the picture formalism as

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

i.e.



The **squared Hilbert-Schmidt norm**

$$||f|| = \sum_i \langle f(e_i) \mid f(e_i) \rangle$$

exists in the picture formalism as

$$||f|| := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$$

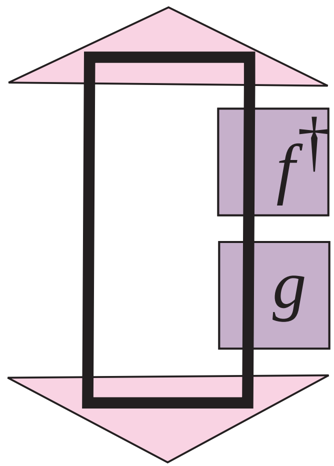
Proof.

$$\begin{aligned}
||f||(1) &= \left(\eta^\dagger \circ (1 \otimes f)^\dagger \circ (1 \otimes f) \circ \eta\right)(1) \\
&= \left(\eta^\dagger \circ (1 \otimes (f^\dagger \circ f))\right)\left(\sum e_i \otimes e_i\right) \\
&= \eta^\dagger \left(\sum e_i \otimes f^\dagger(f(e_i))\right) \\
&= \sum \langle e_i \mid f^\dagger(f(e_i))\rangle \\
&= \sum \langle f(e_i) \mid f(e_i)\rangle \,.
\end{aligned}$$

The corresponding **Hilbert-Schmidt inner-product** also exists in the picture formalism as

$$\langle f \mid g\rangle := (\ulcorner f \urcorner)^\dagger \circ \ulcorner g \urcorner$$

i.e.



and generalizes 'the one on states' since

$$(\ulcorner \psi \urcorner)^\dagger \circ \ulcorner \phi \urcorner = \psi^\dagger \circ \phi$$

ALL IS QUANTITATIVE!

The **squared Hilbert-Schmidt norm** yields:

a canonical norm on processes

The **Hilbert-Schmidt inner-product** yields:

an inner-product on processes

ABSTRACT GLOBAL PHASES

$$f \otimes f^\dagger = e^{i\theta} \cdot g \otimes (e^{i\theta} \cdot g)^\dagger = e^{i\theta} \cdot g \otimes e^{-i\theta} \cdot g^\dagger = g \otimes g^\dagger$$

Proposition 1.

$$s \bullet f = t \bullet g \, , \, s \circ s^\dagger = t \circ t^\dagger = 1_{\mathrm{I}} \quad \Longrightarrow \quad f \otimes f^\dagger = g \otimes g^\dagger$$

Proposition 2.

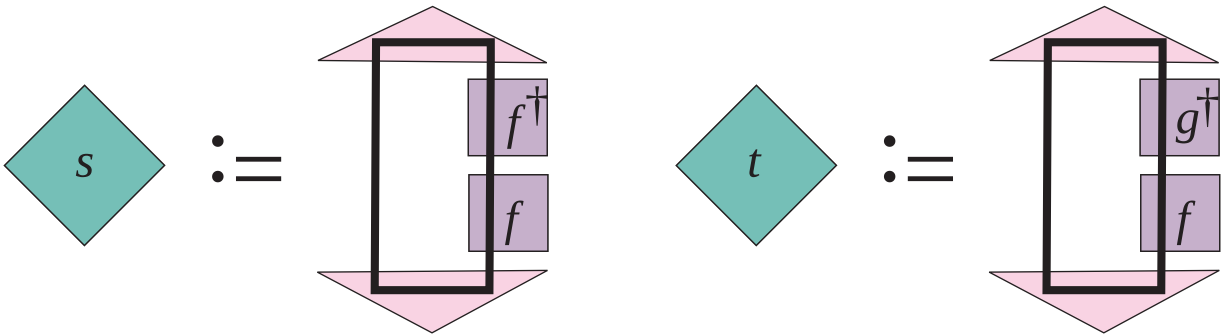
$$f \otimes f^\dagger = g \otimes g^\dagger \quad \Longrightarrow \quad \exists s, t : \; s \bullet f = t \bullet g \, , \, s \circ s^\dagger = t \circ t^\dagger$$

e.g.

$$s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner \qquad \text{and} \qquad t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$$

Proof.

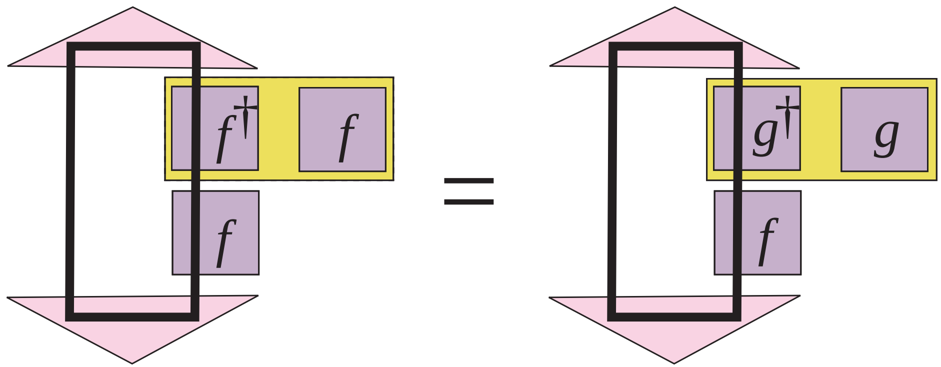
♯1 $s := (\ulcorner f \urcorner)^\dagger \circ \ulcorner f \urcorner$ and $t := (\ulcorner g \urcorner)^\dagger \circ \ulcorner f \urcorner$



♯2 $f \otimes f^\dagger = g \otimes g^\dagger$

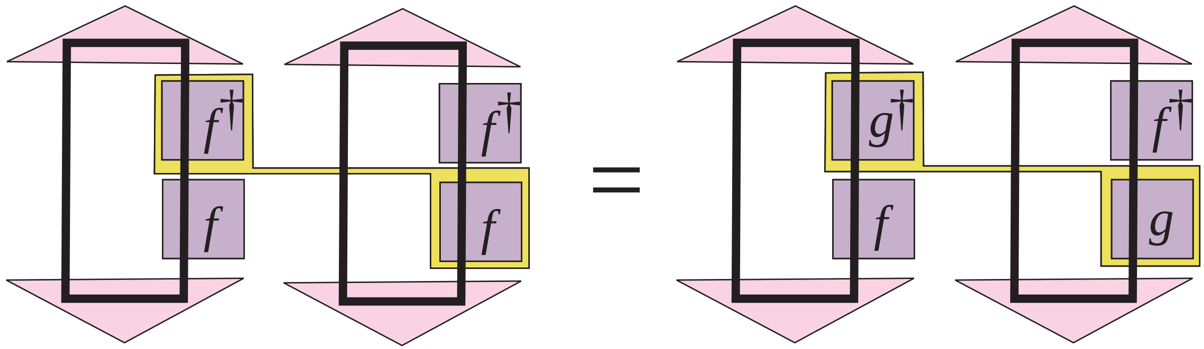
Proof.

$$\sharp 3 \quad s \bullet f = t \bullet g \quad \textsf{with} \quad s/t := (\ulcorner f/g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$$

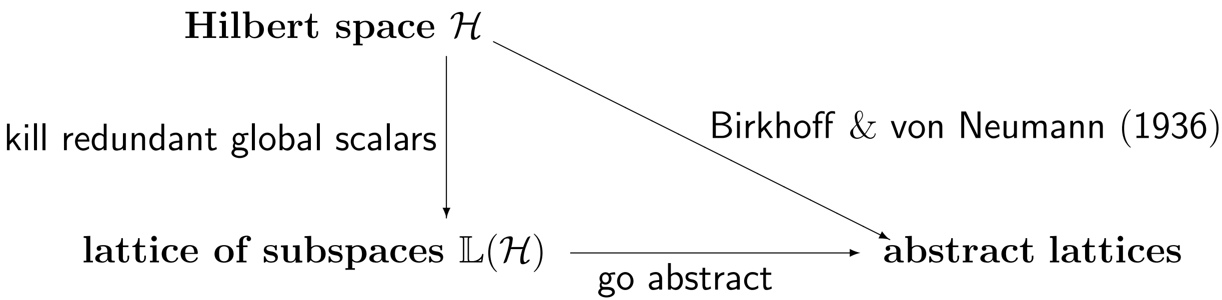


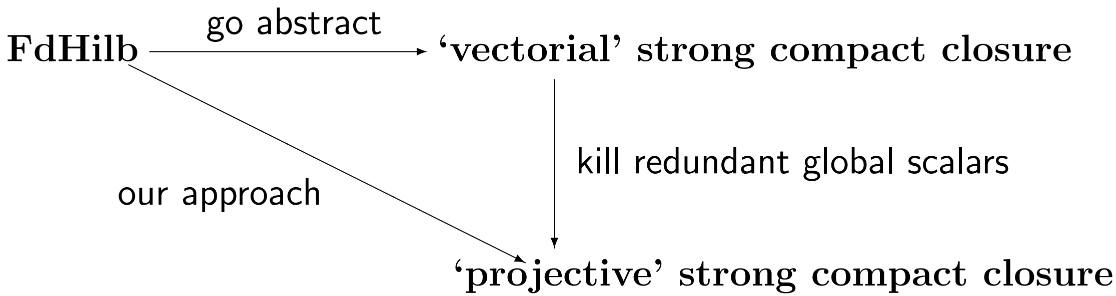
Proof.

$\sharp 4 \quad s \circ s^{\dagger} = t \circ t^{\dagger}$ with $s/t := (\ulcorner f/g \urcorner)^{\dagger} \circ \ulcorner f \urcorner$



PROJECTIVE vs VECTORIAL





ABSENCE OF GLOBAL PHASES

Proposition. $WProj(\mathbf{C}) \simeq \mathbf{C}$ (canonically) iff

$$f \otimes f^\dagger = g \otimes g^\dagger \implies f = g$$

iff

$$\mathrm{P}_f = \mathrm{P}_g \implies \ulcorner f \urcorner = \ulcorner g \urcorner$$

iff

$$\psi \circ \psi^\dagger = \phi \circ \phi^\dagger \implies \psi = \phi$$

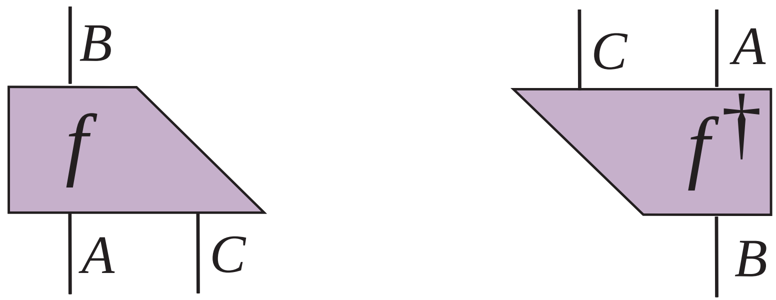
iff

Equal Preparations Produce Equal States

OPEN SYSTEMS AND CPMs

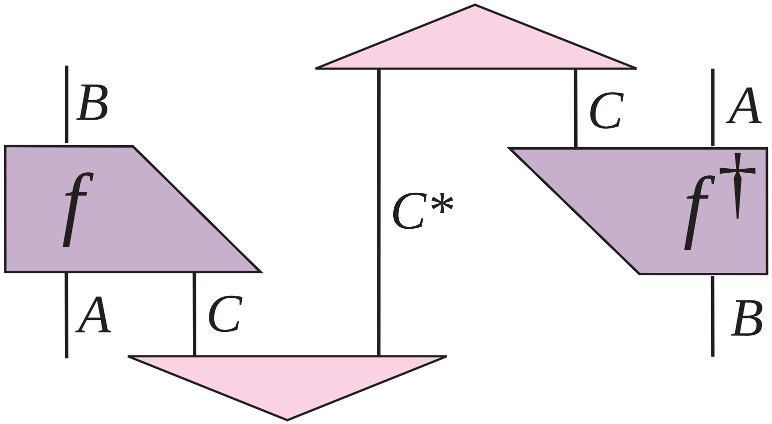
⇒ projective process

OPEN SYSTEMS AND CPMs



⇒ projective process with ancila

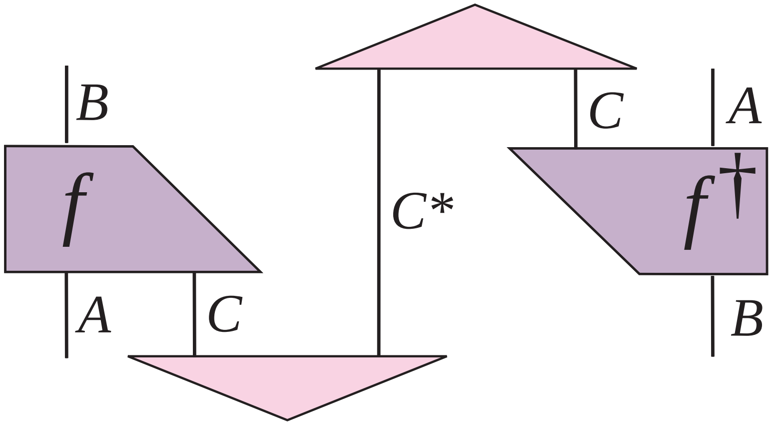
OPEN SYSTEMS AND CPMs



$\Rightarrow$ projective process with hidden ancila

= open process on open system

OPEN SYSTEMS AND CPMs



In the case of Hilbert spaces and linear maps we exactly obtain **completely positive maps** (Selinger 2005)!

ABSTRACT QM

System of type A := Object A

Composite of A and B := Tensor $A \otimes B$

Process of type $A \to B$:= Morphism $f : A \to B$

State of A := Element $\psi : \mathrm{I} \to A$

Evolution of A := Unitary $U : A \to A$

Measurement on A := "Projectors" $\{\mathrm{P}_i : A \to A\}_i$

- Data := $\nu \in \{i\}_i$
- Dynamics := $\psi \mapsto \mathrm{P}_\nu \circ \psi$
- Probability := $\psi^\dagger \circ \mathrm{P}_\nu \circ \psi = \mathrm{Tr}(\mathrm{P}_\nu \circ \rho_\psi) : \mathrm{I} \to \mathrm{I}$

Some extra structure is required both for

- Specification of the families $\{\mathrm{P}_i : A \longrightarrow A\}_i$
- Combining $\{\mathrm{P}_i\}_i$ into a single $M : A \longrightarrow \ldots$

But, **you can pick your favorite**!

For each unitary morphism $U : A \longrightarrow \bigoplus_i A_i$ we have

$$\{\mathrm{P}_j := \pi_j^\dagger \circ \pi_j\}_j \qquad M := \left(\bigoplus_i \pi_i^\dagger\right) \circ U : A \longrightarrow \bigoplus_i A$$

where $\pi_j := p_j \circ U$. Alternatively, $\{f_i\}_i$ has to satisfy $\sum_i f_i = 1_A$ and the corresponding measurement is

$$M := \langle f_i \rangle_i : A \longrightarrow \bigoplus_i A\,.$$

DIGEST

... first full formal descrition of protocols

... types reflect kinds

... classical data-flow is included

... quantum info-flow is explicit

... kindergarten description/correctness proofs

... space for formal/conceptual choices

... the thing people call QM-relationalism?

APPLICATIONS

— "why computer scientists care about this stuff" —

Quantum programing language design

Quantum program logics for verification

Quantum protocol specification

Quantum protocol verfication

Appropriate semantics for new quantum computational paradigms e.g. one-way (Briegel), teleportation based (Gottesman-Chuang), measurement based in general, topological quantum computing (Kitaev et al.) etc.

RELATED WORK

Penrose. *Applications of negative dimensional tensors* (1971) ⇒ Diagramatic reasoning in physics (GR)

Kauffman. *Teleportation topology*. `quant-ph/0407224` ⇒ Independent logic of entanglement observation as Coecke PRG-R-03-12 & `quant-ph/0402014`

Baez. *Quantum quandaries: a category-theoretic perspective*. `quant-ph/0404040` ⇒ Independent Rel-QM connection observation as Abramsky-Coecke `quant-ph/0402130`; also, GR-QM structural connection

Deligne. *Catégories tannakiennes*. In *The Grothendieck Festschrift* (1990). ⇒ Representation Theorem !!!