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ABSTRACT

The kinematic constraints on helicity amplitudes are derived directly from basic analyticity properties, without the use of crossing or partial-wave decomposition. The constraints are manifest in a representation of helicity amplitudes used earlier to study their kinematic branch points. That work is completed by extracting from that representation the powers of the kinematic poles.

1. INTRODUCTION

The relationships among helicity amplitudes imposed at thresholds and pseudothresholds by kinematic requirements are important in Regge analysis. They have been studied by extensively,[1] and recently have been derived by a general procedure based on crossing properties,[2] and also by an alternative method based on a partial-wave decomposition.[3] However, the use of crossing or partial-wave decompositions to derive these constraints is roundabout: One should be able to derive them directly from basic analyticity properties. This is indeed the case, for they are manifest in a representation of the helicity amplitudes used earlier[4] to derive their analyticity properties. That representation is discussed here in more detail, and the powers of the kinematic poles at thresholds and pseudothresholds are derived from it. This completes the earlier work, which dealt only with branch point singularities. Only the general unequal-mass case is considered here.

2. A REPRESENTATION OF HELICITY AMPLITUDES

This work is a continuation of Ref. 4. Equation numbers with asterisks refer to that work.

Let $\underset{\sim}{h}$ be a unit 3-vector. According to (2.5*) the spin-$\frac{1}{2}$ boost in the direction $\underset{\sim}{h}$ can be written

$$B(v^0,\epsilon) \equiv \frac{1}{\sqrt{2}} \left[(v^0 + 1)^{\frac{1}{2}} + 2\epsilon\, \underset{\sim}{\sigma} \cdot \underset{\sim}{h}\, (v^0 - 1)^{\frac{1}{2}}\right], \qquad (2.1)$$

where the sign of $\epsilon = \pm\frac{1}{2}$ determines the sign (sense) of the boost, and v^0 is the time component of the covariant velocity of a boosted particle that was originally at rest:

$$v^0 = \frac{p^0}{m} = \gamma . \qquad (2.2)$$

Here γ is the Lorentz contraction factor $(1 - \beta^2)^{-\frac{1}{2}}$ associated with the boost.

According to (2.15*) the helicity amplitudes for the scattering of a spin-$\frac{1}{2}$ particle on a spin-zero particle are matrix elements of the operator

$$H = \sum_{\bar{\epsilon},\epsilon = \pm\frac{1}{2}} B(\bar{v}^0, \bar{\epsilon})\, A(\bar{\epsilon}, \epsilon)\, B(v^0, \epsilon). \qquad (2.3)$$

Here v^0 and $\bar{v}^0$ are the time components of the covariant velocities of the initial and final fermion, respectively, and

$$A(\bar{\epsilon}, \epsilon) = a(\epsilon, \bar{\epsilon}; s, t)\, W^{|\bar{\epsilon}+\epsilon|}\, R(\theta) \quad . \tag{2.4}$$

Here the $a(\epsilon, \bar{\epsilon}; s, t)$ are a set of four invariant amplitudes (parity conservation is not assumed) that are free of kinematic singularities, except possibly on the surface $\phi = 0$, which is where at most two of the four energy-momentum vectors p_α are linearly independent; W is the center-of-mass energy $s^{\frac{1}{2}}$; and

$$R(\theta) = \exp\left[i\,\frac{\theta}{2}\,\underset{\sim}{\sigma} \cdot \underset{\sim}{n} \right] \tag{2.5}$$

is the rotation by θ about the axis $\underset{\sim}{n}$, which is the unit normal to the c.m. plane of scattering.

The helicity amplitude is the matrix element of H in the frame where $\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{h}} \equiv \sigma_h = \sigma_3$ and $\underset{\sim}{\sigma} \cdot \hat{\underset{\sim}{n}} \equiv \sigma_n = \sigma_2$. In this frame the boost factors $B(v^0, \epsilon)$ and $B(\bar{v}^0, \bar{\epsilon})$ become simply numerical functions of the helicities λ and $\bar{\lambda}$ of initial and final particles, respectively, and the helicity amplitudes are

$$H_{\bar{\lambda}\,\lambda} = \sum_{\bar{\epsilon}, \epsilon = \pm \frac{1}{2}} B(\bar{v}^0, \bar{\epsilon}, \bar{\lambda})\, A_{\bar{\lambda}\,\lambda}(\bar{\epsilon}, \epsilon, \theta; s, t)\, B(v^0, \epsilon, \lambda) \ , \tag{2.6}$$

where

$$B(v^0, \epsilon, \lambda) = \frac{1}{\sqrt{2}} \left[(v^0 + 1)^{\frac{1}{2}} + 4\epsilon\,\lambda\, (v^0 - 1)^{\frac{1}{2}} \right] . \tag{2.7}$$

Isolating $R(\Theta)$, one obtains

$$H_{\bar{\lambda}\lambda} = F_{\bar{\lambda}\lambda} R_{\bar{\lambda}\lambda}(\Theta) \quad , \tag{2.8a}$$

where

$$F_{\bar{\lambda}\lambda} \equiv F\left[(\bar{v}^0 + 1)^{\frac{1}{2}}, (v^0 + 1)^{\frac{1}{2}}, \bar{\lambda}(\bar{v}^0 - 1)^{\frac{1}{2}}, \lambda(v^0 - 1)^{\frac{1}{2}}; s, t, ; W\right]$$

$$\equiv \sum_{\bar{\epsilon},\epsilon = \pm\frac{1}{2}} B(\bar{v}^0, \bar{\epsilon}, \bar{\lambda})\, a(\bar{\epsilon}, \epsilon; s, t)\, W^{|\bar{\epsilon}+\epsilon|}\, B(v^0, \epsilon, \lambda) \ . \tag{2.8b}$$

This form of the helicity amplitude for a single spin-$\frac{1}{2}$ particle was the basis of the analysis of Ref. 4. The key point is that the dependence of $F_{\bar{\lambda}\lambda}$ on λ and $\bar{\lambda}$ occurs only through the factors $\lambda(v^0 - 1)^{\frac{1}{2}}$ and $\bar{\lambda}(\bar{v}^0 - 1)^{\frac{1}{2}}$, respectively, whereas $R_{\bar{\lambda}\lambda}(\Theta)$ is known.

If one chooses the frame where $\sigma_n = \sigma_3$ and $\sigma_h = -\sigma_2$, then the matrix elements of H are the "transversity amplitudes" of Kotanski.[5] Then the rotation matrix $R(\Theta)$ is diagonal, instead of $B(v^0, \epsilon)$ and $B(\bar{v}^0, \bar{\epsilon})$. This representation of H is denoted by $H_{\bar{\tau}\tau}$.

Note that if $(v^0 - 1)$ $\left[\text{or } (\bar{v}^0 - 1)\right]$ is zero then the boost factor $B(v_0, t)$ $\left[\text{or } B(\bar{v}_0, \bar{\epsilon})\right]$ becomes unity. Then the dependence of H on λ or τ $\left[\text{or on } \bar{\lambda} \text{ or } \bar{\tau}\right]$ is determined by the matrix elements of the known rotation operator $R(\Theta)$. This immediately gives the kinematic constraints, as we shall see in the next section.

Processes with higher spins are dealt with by constructing their amplitudes from tensor products of spin-$\frac{1}{2}$ amplitudes. For the purpose of this (purely mathematical) construction one can consider a particle of spin J and velocity v to be a composite system (in a purely mathematical sense) of $\mathcal{M} \geqslant 2J$ spin-$\frac{1}{2}$ particles of velocity v. Let the labels on the particles of $a + b \rightarrow c + d$ be chosen so that $J_a = J_c (\mathrm{mod}\ 1)$ and $J_b = J_d (\mathrm{mod}\ 1)$. Let $N_{ac} = \max(2J_a, 2J_c)$ and $N_{bd} = \max(2J_b, 2J_d)$. Then imagine a process with $N_{ac} + N_{bd} \equiv N$ spin-$\frac{1}{2}$ particles, such that first N_{ac} particles come in with velocity v_a and leave with velocity v_c, and last N_{bd} particles come in with velocity v_b and leave with velocity v_d. Let $\mathcal{C}_a$ represent the set of Clebsch-Gordan operators that combine the last $2J_a$ of the N_{ac} particles constituting particle a into a particle with spin J_a. And let $\mathcal{C}_b$, $\mathcal{C}_c$, and $\mathcal{C}_d$ be similarly defined. Let $\mathcal{S}_a$ be the operator that projects each of the first $(\frac{1}{2} N_{ac} - J_a)$ pairs of particles from the set of N_{ac} particles constituting particle a onto a spin-zero system. That is, $\mathcal{S}_a$ is a tensor product of $\frac{1}{2} N_{ac} - J_a$ singlet projection operators, acting on these $\frac{1}{2} N_{ac} - J_a$ pairs of particles. And let $\mathcal{S}_b$, $\mathcal{S}_c$, and $\mathcal{S}_d$ be similarly defined. Then H for the composite system is written as

$$H = \left[(\mathcal{S}_c \otimes \mathcal{C}_c) \otimes (\mathcal{S}_d \otimes \mathcal{C}_d)\right] \prod_{i=1}^{N} \otimes H_i$$

$$\times \left[(\mathcal{S}_a \otimes \mathcal{C}_a) \otimes (\mathcal{S}_b \times \mathcal{C}_b)\right] . \tag{2.9a}$$

This equation is schematic, for it does not make explicit the particular way that the N variables for the operator in the center are separated into the four spaces of the outer operators. But this separation has already been explained. [See also the Appendix.] Also, (2.9a) does not convey the information that in forming the tensor product $[\Pi \otimes H_i]$, each of the four terms corresponding to the four different possible values of $(\bar{\epsilon}_i, \epsilon_i)$ in H_i is to be combined independently with each of the four terms of each of the other H_i, to give altogether 4^N terms, which have independent coefficients $a(\bar{\epsilon}_1, \cdots \bar{\epsilon}_N; \epsilon_1, \cdots \epsilon_N, s, t)$. This fact is exhibited in the explicit definition

$$\left[\prod_{i=1}^{N} \otimes H_i\right] = \sum_{\epsilon_{\alpha i} = \pm \frac{1}{2}} a(\epsilon_{c1} \cdots \epsilon_{d1} \cdots ; \epsilon_{a1} \cdots \epsilon_{b1} \cdots ; s, t)$$

$$\prod_{i=1}^{N_{ac}} \otimes B_i(v_c^{\,0}, \epsilon_{ci})\, W^{|\epsilon_{ci} + \epsilon_{ai}|}\, R_i(\theta)\, B_i(v_a^{\,0}, \epsilon_{ai})$$

$$\times \prod_{j=1}^{N_{bd}} \otimes B_j(v_d^{\,0}, \epsilon_{dj})\, W^{|\epsilon_{dj} + \epsilon_{bj}|}\, R_j(\theta)\, B_j(v_b^{\,0}, \epsilon_{bj}) . \tag{2.9b}$$

The summation on the right is over the 4^N combinations of signs of the various $\epsilon_{\alpha i}$, where $\alpha = a, b, c,$ or d. It was shown in Ref. 4 that the coefficients $a(\epsilon_{\alpha i}; s, t)$ can be made functions of the invariants s and t that are free of kinematic singularities at $\phi \neq 0$.

One can write the equation analagous to (2.9a) for either the M function or the S matrix by simply replacing the H and H_i either by M and M_i or by S and S_i, respectively. The conversions between the three forms go through because both the boosts and the rotations are converted in passage through the $\mathcal{S}_\alpha \otimes \mathcal{C}_\alpha$ to the form appropriate to the space on the other side.[6]

Going to the helicity representation and regrouping factors, one obtains from (2.9) the analogue of (2.8):

$$H_{\lambda_d\lambda_c\lambda_b\lambda_a} = \sum_\gamma F^\gamma_{\lambda_d\lambda_c\lambda_b\lambda_a} R^\gamma_{\lambda_d\lambda_c\lambda_b\lambda_a}(\theta) \quad , \tag{2.10a}$$

where the sum over γ is a sum arising from the linear combinations implied by the factors $\mathcal{S}_\alpha \otimes \mathcal{C}_\alpha$. The factor $R^\gamma_{\lambda_d\lambda_c\lambda_b\lambda_a}(\theta)$ is a linear combination of products of N elementary rotation operator matrix elements $R_{\lambda_{dj}\,\lambda_{bj}}(\theta)$ and $R_{\lambda_{ci}\,\lambda_{ai}}(\theta)$, which must satisfy

$$\sum \lambda_{\alpha i} = \lambda_\alpha \, . \tag{2.10b}$$

The function $F^{\gamma}_{\lambda_d\lambda_c\lambda_b\lambda_a}$ is a linear combination of products of N factors like $F_{\lambda_{dj}\lambda_{bj}}$ and $F_{\lambda_{ci}\lambda_{ai}}$ of (2.8b) which also must satisfy (2.10b). In particular, we have

$$F^{\gamma}_{\lambda_d\lambda_c\lambda_b\lambda_a} = F^{\gamma}\left[(v_\alpha{}^0 + 1)^{\frac{1}{2}}, \lambda_{\alpha i}(v_\alpha{}^0 - 1)^{\frac{1}{2}}; s, t; W\right]\Bigg| , \qquad (2.10c)$$

where the $\lambda_{\alpha i}$ on the right satisfy (2.10b).

The representation (2.10) is discussed in detail in the Appendix. The main pertinent features are that the dependence of $F^{\gamma}_{\lambda_d\lambda_c\lambda_b\lambda_a}$ upon λ_α enters only through the factors $\lambda_{\alpha i}(v_\alpha{}^0 - 1)^{\frac{1}{2}}$, and that $R^{\gamma}_{\lambda_d\lambda_c\lambda_b\lambda_a}(\theta)$ is a linear combination of products of matrix elements of N elementary rotations (2.5) having the correct total helicities λ_α, as specified by (2.10b).

The arguments that follow hold for each term of the sum in (2.10a). Thus the index γ will be omitted.

3. KINEMATIC CONSTRAINTS

Equations (2.10b) and (2.10c) show that F is independent of λ_α at $(v_\alpha^0 - 1) = 0$. Thus the dependence on λ_α is given completely by the rotation operator $R(\theta)$. This gives kinematic constraints.

These constraints take a neat form in the transversity representation. For at $v_\alpha^0 - 1 = 0$ the boost factors associated with particle α all become unity. Thus the transversity index τ_α applies directly to $R(\theta)$. There are several cases, which are discussed separately.

At $W = m_a + m_b$ both $(v_a^0 - 1)$ and $(v_b^0 - 1)$ vanish. [See (3.1*).]. Therefore the boost factors for the particles that constitute particles a and b all become unity, and the transversity indices τ_a and τ_b apply directly to $R(\theta)$. Thus the behavior near $W = m_a + m_b$ is dominated by the factor $e^{i\theta(\tau_a+\tau_b)}$ coming from $R_{\tau_d\tau_c\tau_b\tau_a}(\theta)$:

$$H_{\tau_d\tau_c\tau_b\tau_a} \sim e^{i\theta(\tau_a+\tau_b)} \quad (W \simeq m_a + m_b) \ . \tag{3.1a}$$

Similarly, we obtain

$$H_{\tau_d\tau_c\tau_b\tau_a} \sim e^{i\theta(\tau_c+\tau_d)} \quad (W \simeq m_c + m_d) \ . \tag{3.1b}$$

The factor $e^{i\theta} = \cos\theta + i\sin\theta$ behaves like $\left(W - (m_a + m_b)\right)^{\pm 1/2}$ near $W = m_a + m_b$. [See (2.10*).].

At $W = m_a - m_b > 0$ the factors $(v_a^0 - 1)$ and $(v_b^0 + 1)$ vanish. Thus the boosts for the particles that constitute particle a all become unity, whereas the boosts for the particles that constitute b all become (in the transversity representation) proportional to $i\sigma_2$. The boost factors for particle b will therefore be screw-diagonal. Thus we have

$$H_{\tau_d \tau_c \tau_b \tau_a} \sim e^{i\theta(\tau_a - \tau_b)} \qquad (W \simeq m_a - m_b > 0) , \tag{3.2}$$

and similarly for the other mass cases. The constraints (3.1) and (3.2) were derived in Ref. 2 from crossing properties, and were shown to give all the known kinematic constraints.

4. THRESHOLD AND PSEUDOTHRESHOLD BRANCH POINTS

A continuation of H around a small circle centered at $W = m_c + m_d$ reverses the sign of $(v_c^0 - 1)^{\frac{1}{2}}$ and $(v_d^0 - 1)^{\frac{1}{2}}$ [see (3.1*)] and carries θ to $\theta \pm \pi$. [See (2.10*).]. Reversal of the signs of $(v_c^0 - 1)^{\frac{1}{2}}$ and $(v_d^0 - 1)^{\frac{1}{2}}$ is equivalent to reversal of the signs of the λ_{ci} and λ_{dj} in the boost factors $B(v_c^0, \epsilon_{ci}, \lambda_{ci})$ and $B(v_d^0, \epsilon_{dj}, \lambda_{di})$, and it leads to a reversal of the signs of λ_c and λ_d on $F_{\lambda_d\lambda_c\lambda_b\lambda_a}$ in (2.10a). Moreover,

$$R_{\lambda_d\lambda_c\lambda_b\lambda_a}(\theta \pm \pi) = \Big(R(\pm \pi)\, R(\theta)\Big)_{\lambda_d\lambda_c\lambda_b\lambda_a} = (\pm 1)^{2J_c+2J_d}$$

$$(-1)^{J_c+J_d-\lambda_c+\lambda_d}\; R_{-\lambda_d-\lambda_c\lambda_b\lambda_a}(\theta)\,. \tag{4.1}$$

[The identity $\lambda_d \equiv -\mu_d$ is used here. It is assumed in (4.1) that $R(\theta)$ transforms on the left according to the $J_c \otimes J_d$ representation. This is justified in the Appendix.]

The effect of the continuation on $H \equiv FR(\theta)$ is therefore

$$H_{\lambda_d\lambda_c\lambda_b\lambda_a} \longrightarrow H^{c_f}_{\lambda_d\lambda_c\lambda_b\lambda_a}$$

$$= (\pm 1)^{2J_c+2J_d} (-1)^{J_c+J_d-\lambda_c+\lambda_d} H_{-\lambda_d-\lambda_c\lambda_b\lambda_a} . \tag{4.2}$$

Now the continuation $\theta \to \theta \pm \pi$ effects also the change

$$\sin\frac{\theta}{2}^{|\mu-\lambda|} \cos\frac{\theta}{2}^{|\mu+\lambda|} \to (\pm 1)^{2\lambda} (-1)^{-\lambda-\mu} \sin\frac{\theta}{2}^{|\mu+\lambda|} \cos\frac{\theta}{2}^{|\mu-\lambda|} . \tag{4.3}$$

Thus setting λ equal to $\lambda_a - \lambda_b$, and μ equal to $\lambda_c - \lambda_d$, and defining

$$\bar{H}_{\lambda_d\lambda_c\lambda_b\lambda_a} \equiv \frac{H_{\lambda_d\lambda_c\lambda_b\lambda_a}}{\sin\frac{\theta}{2}^{|\mu-\lambda|} \cos\frac{\theta}{2}^{|\mu+\lambda|}} , \tag{4.4}$$

we find that

$$\bar{H}_{\lambda_d\lambda_c\lambda_b\lambda_a} \longrightarrow \bar{H}^{c_f}_{\lambda_d\lambda_c\lambda_b\lambda_a} = \eta_f \bar{H}_{-\lambda_d-\lambda_c\lambda_b\lambda_a} , \tag{4.5a}$$

where

$$\eta_f = (-1)^{J_c+J_d+\lambda} = \eta_f(\lambda) ; \tag{4.5b}$$

Thus the factor $(\pm 1)^{2J_c+2J_d} = (\pm 1)^{2\lambda} = (\pm 1)^{2\mu}$ drops out when $\bar{H}$, rather than H, is considered.

The same arguments give also

$$\bar{H}_{\lambda_d\lambda_c\lambda_b\lambda_a} \longrightarrow \bar{H}^{c_i}_{\lambda_d\lambda_c\lambda_b\lambda_a} = \eta_i\, \bar{H}_{\lambda_d\lambda_c-\lambda_b-\lambda_a} , \qquad (4.5c)$$

where

$$\eta_i = (-1)^{J_a+J_b-\mu} = \eta_i(\mu). \qquad (4.5d)$$

[The signs of λ and μ in (4.5b) and (4.5d) correspond to the case where the initial particles have lower dotted indices and the final particles have lower undotted indices. According to the conventions of Ref. 6, which are adopted here, rotations act by multiplication from the right or left on lower dotted and undotted indices respectively.]

At $W = m_c - m_d > 0$ it is $v_c^0 - 1$ and $v_d^0 + 1$ that vanish. The argument is just the same, except that the $N_{cd} = 2J_d (\text{mod } 2)$ boost factors associated with d all get an additional overall sign change. Thus one obtains, instead of (4.5), rather

$$\bar{H}_{\lambda_d\lambda_c\lambda_b\lambda_a} \longrightarrow \eta_f (-1)^{2J_d}\, \bar{H}_{-\lambda_d-\lambda_c\lambda_b\lambda_a} . \qquad (4.6a)$$

For continuation around $W = m_a - m_b > 0$ one gets

$$\bar{H}_{\lambda_d\lambda_c\lambda_b\lambda_a} \longrightarrow \eta_i\, (-1)^{2J_b}\, \bar{H}_{\lambda_d\lambda_c-\lambda_b-\lambda_a} . \qquad (4.6b)$$

Equations (4.5) and (4.6) were derived in Ref. 2 from an analysis of Williams' representation of M functions in terms of invariant functions. They are the basic equations for the analysis of threshold and pseudothreshold kinematic branch points given there. [Certain sign differences are due to the unorthodox definition of helicity states used in Ref. 2.]

5. THRESHOLD AND PSEUDOTHRESHOLD POLES

Define

$$\left\{[S - m_a + m_b)^2]\ [S - (m_a - m_b)^2]\right\}^{\frac{1}{2}} = S_{ab} = S_i \qquad (5.1a)$$

and

$$\left\{[S - (m_c + m_d)^2]\ [S - (m_c - m_d)^2]\right\}^{\frac{1}{2}} = S_{cd} = S_f\ . \qquad (5.1b)$$

And define

$$\bar{J}_\alpha \equiv \max\,\{J_\alpha, J_\gamma\} = \frac{1}{2} N_{\alpha\gamma} \equiv J_{\alpha\gamma}\ , \qquad (5.2)$$

where γ is the mate of α in the pairing (a,c) or (b,d).

The rotations $R(\Theta)$ in $H \equiv FR(\Theta)$ are constructed as linear combinations of terms of the form

$$\left(\sin\frac{\Theta}{2}\right)^\rho \left(\cos\frac{\Theta}{2}\right)^{N-\rho} = \left(\frac{1-\cos\Theta}{2}\right)^{\rho/2} \left(\frac{1+\cos\Theta}{2}\right)^{(N-\rho)/2}\ ,$$

where

$$N = N_{ac} + N_{bd} = 2\bar{J}_a + 2\bar{J}_b = 2\bar{J}_c + 2\bar{J}_d = 2\bar{J}\ . \qquad (5.3)$$

Thus near $S_i\,S_f = 0$ each of these terms has a singularity of the form [see (2.10*)]

$$(S_i\,S_f)^{-N/2} = (S_i\,S_f)^{-\bar{J}}\ . \qquad (5.4)$$

The operator $\mathcal{S}_\alpha$ consists of $\bar{J}_\alpha - J_\alpha$ singlet projection. It is shown in the Appendix that the action of $\mathcal{S}_\alpha$ upon the boost factors in F produces an effective factor

$$(S_{\alpha\beta})^{\bar{J}_\alpha - J_\alpha} = \left(S_{(i,f)}\right)^{\bar{J}_\alpha - J_\alpha} , \qquad (5.5)$$

where β is the mate of α in the pairing (a,b) or (c,d), and (i,f) is i or f according to whether α is initial or final. The four factors (5.5), for α = a, b, c, and d, reduce (5.4) to the form

$$S_i^{-J_i} S_f^{-J_f} \equiv S_{ab}^{-J_a - J_b} S_{cd}^{-J_c - J_d} , \qquad (5.6)$$

where $J_i \equiv J_a + J_b$ and $J_f \equiv J_c + J_d$. This is the worst possible singularity in H; cancellations conceivably could reduce the magnitude of the exponents.

Define

$$\bar{R}_{\lambda_d \lambda_c \lambda_b \lambda_a}(\theta) \equiv \frac{R_{\lambda_d \lambda_c \lambda_b \lambda_a}(\theta)}{\sin \frac{\theta}{2}^{|\mu-\lambda|} \cos \frac{\theta}{2}^{|\mu+\lambda|}} \qquad (5.7)$$

with $\lambda = \lambda_a - \lambda_b$ and $\mu = \lambda_c - \lambda_d$. The matrix elements of $\bar{R}(\theta)$ are polynomials in $\cos\theta$ of order at most $\bar{J} - \bar{M}$, where $\bar{J} \equiv N/2$, as before, and $\bar{M} \equiv \mathrm{Max}\,\{|\lambda|, |\mu|\}$. This follows from the fact that

$R(\theta)$ is a linear combination of factors $(\sin\theta/2)^{\rho}(\cos\theta/2)^{N-\rho}$, where $\rho - |\mu - \lambda| = \text{(even integer)} \geqslant 0$ and $(N - \rho) - |\mu + \lambda| = \text{(even integer)} \geqslant 0$. These conditions on ρ follow easily from an examination of the form of $[\pi \otimes R_i]$.

In terms of $\bar{R}$, $\bar{H}$ becomes

$$\bar{H} = F\,\bar{R}(\theta)\ , \tag{5.8}$$

where the indices are now suppressed. The worst possible singularity of $\bar{R}(\theta)$ at $S_i\,S_f = 0$ is evidently $(S_i\,S_f)^{-(\bar{J}-\bar{M})}$. Thus, in view of (5.5), the worst possible singularity in $\bar{H}$ is

$$S_i^{-(J_i-\bar{M})}\,S_f^{-(J_f-\bar{M})} = S_{ab}^{-(J_a+J_b-\bar{M})}\,S_{cd}^{-(J_c+J_d-\bar{M})}, \tag{5.9}$$

as follows also directly from (5.6).

6. COMBINED RESULTS

The various results given above are the ingredients from which regularized helicity amplitudes are constructed in Ref. 2. One can therefore follow their procedure. Alternatively one can use the procedure of Ref. 4.

Invariance under space reflection was assumed, in Ref. 4, but this is unnecessary. In the general case one uses in place of (3.9*) the definition

$$F^{\pm\,\pm}_{\Lambda_f\Lambda_i} \equiv \frac{1}{2}\left[\left(1 \pm (-1)^{N_f^-}\right)\left(1 \pm (-1)^{N_i^-}\right)\right] F_{\Lambda_f\Lambda_i} \,, \qquad (6.1)$$

where N_i^- and N_f^- are the operators that give the number of factors $(v_\alpha^{\,0} - 1)^{\frac{1}{2}}$ associated with the initial and final particles, respectively. The two ± signs on the right side of (6.1) are identified with the two ± signs on the left, in the same order, so that N_f^- is even or odd according to whether the sign on the left in $F^{\pm\,\pm}$ is plus or minus, and N_i^- is related in the same way to the sign on the right. Then (3.10*) shows that the function

$$F^{\sigma_f\sigma_i}/G_f(\sigma_f)\; G_i(\sigma_i) \quad , \qquad (6.2)$$

is free of kinematic branch points at sums and difference of masses. Here $G_i(\sigma_i)$ is the function on the right of (3.11*) corresponding to the sign σ_i on the left, for the case BB, FF, or FB at hand, and $G_f(\sigma_F)$ is the corresponding function for the final particles.

The same argument works if F is replaced by $\bar{H}$. The operator $N_\alpha = N_\alpha^- + N_\alpha^+$ is the total number of factors $(v_\alpha^0 - 1)^{\frac{1}{2}}$ and $(v_\alpha^0 + 1)^{\frac{1}{2}}$. This number includes a term $2\bar{J}_\alpha$ [see (5.2)] coming from the $2\bar{J}_\alpha$ boost factors associated with particle α. It also includes contributions from the powers of $\cos\Theta$ in $\bar{R}$ of (5.8). Each power of $\cos\Theta$ has one factor of $S_{\alpha\beta}$ in the denominator. Now

$$S_{\alpha\beta} \simeq (v_\alpha^0 - 1)^{\frac{1}{2}} (v_\alpha^0 + 1)^{\frac{1}{2}} \simeq (v_\beta^0 - 1)^{\frac{1}{2}} (v_\beta^0 + 1)^{\frac{1}{2}} , \qquad (6.3)$$

where $\simeq$ means equal up to factors regular at sums and differences of masses. Thus each power of $\cos\Theta$ can be considered to subtract two from either N_α or N_β. Since only the evenness and oddness of N_α and N_β are relevant to the arguments, these contributions from $\bar{R}$ can be ignored.

Thus if one defines, in analogy to (6.1), the function

$$\bar{H}^{\pm\,\pm} \equiv \frac{1}{2}\left[\left(1 \pm (-1)^{N_f^-}\right)\left(1 \pm (-1)^{N_i^-}\right)\right]\bar{H} , \qquad (6.4)$$

then (3.10*) shows that

$$\bar{H}^{\sigma_f\sigma_i}/G_f(\sigma_f)\, G_i(\sigma_i) , \qquad (6.5)$$

is free of kinematic branch points at sums and differences of the (unequal) masses. And the arguments leading to (5.2*) show that

$$\hat{H}^{\sigma_f\sigma_i} \equiv \bar{H}^{\sigma_f\sigma_i} \; W^{|\mu|+|\lambda|}/G_f(\sigma_f)\; G_i(\sigma_i) \tag{6.6}$$

is free of all kinematic branch points. [The variable is W for the FB case].

The operator $(-1)^{N_f^-}$ acting on $\bar{H}$ changes the sign of each factor $(v_c^{\,0} - 1)^{\frac{1}{2}}$ and $(v_d^{\,0} - 1)^{\frac{1}{2}}$. This is equivalent to a continuation of $\bar{H}$ around $W = m_c + m_d$. Thus (4.5) gives

$$(-1)^{N_f^-}\, \bar{H}_{\Lambda_f\Lambda_i} = \bar{H}^{c_f}_{\Lambda_f\Lambda_i} = \eta_f\, \bar{H}_{-\Lambda_f\Lambda_i} \;, \tag{6.7a}$$

and similarly

$$(-1)^{N_i^-}\, \bar{H}_{\Lambda_f\Lambda_i} = \bar{H}^{c_i}_{\Lambda_f\Lambda_i} = \eta_i\, \bar{H}_{\Lambda_f-\Lambda_i} \;. \tag{6.7b}$$

These identities allow $\bar{H}^{\sigma_f\sigma_i}_{\Lambda_f\Lambda_i}$ to be expressed as linear combinations of the $\bar{H}_{\pm\Lambda_f\pm\Lambda_i}$, with the appropriate coefficients $\eta_f(\lambda)$ and $\eta_i(\mu)$:

$$\bar{H}^{\sigma_f\sigma_i} = \frac{1}{2} \sum_{\epsilon_f,\epsilon_i=\pm 1} \Big(\sigma_f\, \eta_f(\epsilon_i\lambda)\Big)^{(1+\epsilon_f)/2} \Big(\sigma_i\, \eta_i(\epsilon_f\,\mu)\Big)^{(1+\epsilon_i)/2} \; H_{\epsilon_f\Lambda_f\epsilon_i\Lambda_i} \;. \tag{6.8}$$

The functions $\hat{H}^{\pm\,\pm}$ defined in (6.6) have no kinematic branch points, but they may have kinematic poles. According to (5.9) the functions $\bar{H}^{\pm\,\pm}\,(S_{ab})^{J_a+J_b-\bar{M}}$ will be bounded at $S_{ab} = 0$. Thus if $J_a + J_b - M$ is even then $\hat{H}^{\pm}\,S_{ab}^{J_a+J_b-M}$ must free of kinematic branch points and poles at $S_{ab} = 0$, since it is free of branch points, and the denominator function $G_i(\sigma_i)$ cannot lead to poles. If $J_a + J_b - M$ is odd, then

$$\hat{H}^{\sigma_f\sigma_i}\quad S_{ab}^{J_a+J_b-M-1}\quad G_i^2$$

must be free of kinematic branch points and poles at $S_{ab} = 0$. For if the factor G_i is regular at any point of $S_{ab} = 0$, then $\bar{H}^{\pm\,\pm}$ can have no branch point at that point, since $\hat{H}^{\pm\,\pm}$ has none, hence the factor $(S_{ab})^{J_a+J_b-M-1}$ is sufficient to ensure boundedness. If the factor G_i is singular at a point of $S_{ab} = 0$, then

$$\hat{H}^{\pm\,\pm}\,(S_{ab})^{J_a+J_b-M-1}$$

would not necessarily be bounded at that point, both because of the singularity of G_i in the denominator, and because of the one missing power of S_{ab}, which is needed to ensure the boundedness of $\bar{H}$. The factor G_i^2 supplies the two needed powers.

The same arguments apply to the final particles. Thus we may conclude that

$$\tilde{H}^{\sigma_f \sigma_i} = \hat{H}^{\sigma_f \sigma_i} \; S_f^{J_f - \bar{M} - e_f} \; S_i^{J_i - \bar{M} - e_i} \; ,$$

$$G_i^{2e_i} \; G_f^{2e_f} \qquad (6.9)$$

is free of kinematic singularities. Here e_i is zero or one depending on whether $J_i - \bar{M} = J_a + J_b - \bar{M}$ is even or odd, and e_f is related to $J_f - \bar{M} = J_c + J_d - \bar{M}$ in the same way. This is the result obtained in Ref. 2.

APPENDIX

DISCUSSION OF THE REPRESENTATION (2.10)

From a set of n spin-$\frac{1}{2}$ states one can form, by Clebsch-Gordan composition, states of various spin $J \leqslant n/2$. Let the original spin-$\frac{1}{2}$ states be numbered in a particular way, and let a sequence of states be formed by first combining the first and second spin-$\frac{1}{2}$ states to get a state of spin zero or one; then combining this with the third spin-$\frac{1}{2}$ state in the various ways consistent with the vector sum rules; and so on. The particular mode of composition is described by a set $\mathcal{J} = (J_1, J_2, \cdots J_{n-1}, J_n)$ of n spins J_i, where J_i is the total spin of the system consisting of the first i particles. (Thus J_1 is $1/2$, and can be suppressed if desired; or $J_0 \equiv 0$ can be included.)

Let the generalized Clebsch-Gordan coefficient that connects the set of n spin-$\frac{1}{2}$ indices to the single spin-J_n index α via the sequence $\mathcal{J}$ be denoted by $\mathcal{C}(\mathcal{J}, \alpha; \alpha_1 \cdots \alpha_n) = \langle \mathcal{J} \alpha | \alpha_1 \cdots \alpha_n \rangle$. They are defined inductively in terms of Clebsch-Gordan coefficients by the formula

$$\langle J_1 \cdots J_n, \alpha | \alpha_1 \cdots \alpha_n \rangle = \sum_\beta \langle J_1 \cdots J_{n-1}, \beta | \alpha_1 \cdots \alpha_{n-1} \rangle \, C_{J_{n-1} \frac{1}{2}}(J_n, \alpha; \beta \alpha_n). \tag{A.1}$$

They are real and obey the orthogonality relations

$$\sum_{\alpha_i} \langle \mathcal{G}\alpha|\alpha_1 \cdots \alpha_n\rangle \langle \alpha_1 \cdots \alpha_n|\mathcal{G}'\alpha'\rangle = \delta_{\mathcal{G}\mathcal{G}'}\,\delta_{\alpha\alpha'} \tag{A.2a}$$

and

$$\sum_{\mathcal{G}\alpha} \langle \alpha_1 \cdots \alpha_n|\mathcal{G}\alpha\rangle \langle \mathcal{G}\alpha|\alpha'_1 \cdots \alpha'_n\rangle = \prod_i \delta_{\alpha_i\alpha'_i} \ . \tag{A.2b}$$

← By means of these coefficients quantities in any space of n variables $\alpha_i = \pm\frac{1}{2}$ can be transformed to the $\mathcal{G}\alpha$ representation.

To compactify the formulas we define, for $\epsilon = \pm\frac{1}{2}$,

$$B(v^0, \epsilon) \equiv \frac{1}{\sqrt{2}}\left[(v^0+1)^{\frac{1}{2}} + 2\epsilon\,\sigma_h(v^0-1)^{\frac{1}{2}}\right]$$

$$\equiv B(v^0)|\epsilon\rangle \equiv \langle\epsilon|B(v^0) \ , \tag{A.3}$$

and similarly

$$B(v^0)|\epsilon_1, \cdots \epsilon_n\rangle = \prod_{i=1}^{n} \otimes\, B_i(v^0, \epsilon_i). \tag{A.4}$$

The coefficients $a(\epsilon_{\alpha i}; s, t)$ of (2.9b) are also represented in this bracket form:

$$a(\epsilon_{c1}\cdots\epsilon_{d1},\cdots;\ \epsilon_{a1}\cdots\epsilon_{b1},\cdots;\ s,t)$$

$$\equiv \langle\epsilon_{c1}\cdots\epsilon_{d1}\cdots|a(s,t)|\epsilon_{a1}\cdots\epsilon_{b1}\cdots\rangle\ . \qquad (A.5)$$

Then (2.9b) becomes

$$\left[\prod_{i=1}^{N}\otimes H_i\right] = B_c(v_c^{\,0})\,B_d(v_d^{\,0})\,\mathcal{A}(s,t;W)\,R(\theta)\,B_a(v_a^{\,0})\,B_b(v_b^{\,0}), \qquad (A.6)$$

where the bracket identity $|i\rangle\langle i| = 1$ is used to recover (2.9b). The variable W in $\mathcal{A}(s,t;\,W)$ indicates that the explicit powers of W appearing in (2.9b) are incorporated into it.

The Clebsch-Gordan decomposition is not specifically associated with spins; it is basically a decomposition according to symmetries. In any case, the formal identities (A.1) and (A.2) can be applied just as well to the ϵ variables as to spin variables. Thus we may write

$$\left[\prod_{i=1}^{N}\otimes H_i\right] =$$

$$B_c(v_c^{\,0})\,|\mathcal{J}_c^{\,\epsilon},\ \epsilon_c\rangle \otimes B_d(v_d^{\,0})\,|\mathcal{J}_d^{\,\epsilon},\ \epsilon_d\rangle$$

$$\langle\mathcal{J}_c^{\,\epsilon},\ \epsilon_c\,;\ J_d^{\,\epsilon},\ \epsilon_d|\mathcal{A}(s,\ t;\ W)\ R(\theta)\,|\mathcal{J}_a^{\,\epsilon},\ \epsilon_a;\mathcal{J}_b^{\,\epsilon},\ \epsilon_b\rangle$$

$$\langle\mathcal{J}_a^{\,\epsilon},\ \epsilon_a|B_a(v_a^{\,0}) \otimes \langle\mathcal{J}_b^{\,\epsilon},\ \epsilon_b|B_b(v_b^{\,0}), \qquad (A.7)$$

where the superscripts on the $\mathcal{J}_\alpha^\epsilon$ signify indices of the Clebsch-Gordan decomposition of ϵ space, as opposed to spin space.

Applying the Clebsch-Gordan decomposition also to the spin variables, one obtains

$$\langle \mathcal{J}_c, \lambda_c; \mathcal{J}_d, \lambda_d | \left[\prod_{i=1}^{N} \otimes H_i \right] | \mathcal{J}_a, \lambda_a; \mathcal{J}_b, \lambda_b \rangle$$

$$= \langle \mathcal{J}_c, \lambda_c | B_c(v_c^0) | \mathcal{J}_c', \lambda_c'; \mathcal{J}_c^\epsilon, \epsilon_c \rangle$$

$$\langle \mathcal{J}_d, \lambda_d | B_d(v_d^0) | \mathcal{J}_d', \lambda_d'; \mathcal{J}_d^\epsilon, \epsilon_d \rangle$$

$$\langle \mathcal{J}_c', \lambda_c'; \mathcal{J}_d', \lambda_d' | R(\Theta) | \mathcal{J}_a', \lambda_a'; \mathcal{J}_b', \lambda_b' \rangle$$

$$\langle \mathcal{J}_c^\epsilon, \epsilon_c, \mathcal{J}_d^\epsilon, \epsilon_d | \mathcal{A}(s, t; W) | \mathcal{J}_a^\epsilon, \epsilon_a; \mathcal{J}_b^\epsilon, \epsilon_b \rangle$$

$$\langle \mathcal{J}_a', \lambda_a'; \mathcal{J}_a^\epsilon, \epsilon_a | B_a(v_a^0) | \mathcal{J}_a, \lambda_a \rangle$$

$$\langle \mathcal{J}_b', \lambda_b'; \mathcal{J}_b^\epsilon, \epsilon_b | B_b(v_b^0) | \mathcal{J}_b, \lambda_b \rangle . \qquad \text{(A.8)}$$

The Clebsch-Gordan coefficient $\mathcal{C}(\mathcal{J}, \alpha; \alpha_1 \cdots \alpha_n)$ has the following symmetry property: It is antisymmetric under interchange of α_i and α_{i+1} if, and only if, $J_{i+1} = J_{i-1}$; and it is symmetric under interchange of α_i and α_{i+1} if, and only if, $|J_{i+1} - J_{i-1}| = 1$. This property is used continually in what follows.

By definition

$$\langle \mathcal{J}\lambda | B(v^0) | \mathcal{J}'\lambda' ; \mathcal{J}^\epsilon \epsilon\rangle$$

$$= \sum_{\lambda_i \lambda_i' \epsilon_i} \mathcal{C}(\mathcal{J}, \lambda; \lambda_1 \cdots \lambda_n) \prod_i \otimes B_i(v^0, \epsilon_i; \lambda_i, \lambda_i')$$

$$\times \mathcal{C}(\mathcal{J}', \lambda'; \lambda_1' \cdots \lambda_n') \mathcal{C}(\mathcal{J}^\epsilon, \epsilon; \epsilon_1 \cdots \epsilon_n). \qquad (A.9)$$

The factor $\prod_i \otimes B_i$ is completely symmetric under interchanges of labels i. Thus the product of the three factors $\mathcal{C}$ in (A.9) must also be symmetric under the simultaneous interchange of labels $i \longleftrightarrow i + 1$, if the sum on the right is to be nonzero. Thus if any two of the three sets $\mathcal{J}, \mathcal{J}', \mathcal{J}^\epsilon$ are given, the other is uniquely determined. The relationship between the three J's is visualized by drawing plots of J_i, J_i' and J_i^ϵ versus i, with straight line segments in the intervals between integral i. The plot J_i is said to have a "break" at position i if, and only if, $J_{i-1} = J_{i+1}$, and similarly for J_i' and J_i^ϵ. If one of the three curves has a

$$d^J_{-\lambda-\mu}(\theta) = d^J_{\lambda\mu}(-\theta) = (-1)^{\mu-\lambda}\, d^J_{\lambda\mu}(\theta) = d^J_{\mu\lambda}(\theta) , \tag{A.13}$$

and the Jacobi polynomials have the symmetry[7]

$$P^{(|\lambda-\mu|,\ |\lambda+\mu|)}_{J-M} (\cos\theta) = (-1)^{J-M}\, P^{(|\lambda+\mu|,\ |\lambda-\mu|)}_{J-M} (-\cos\theta) . \tag{A.14}$$

The explicit formula for the boost factor in (A.8) is

$$\langle \mathcal{J}\, \lambda | B(v^0) | \mathcal{J}'\, \lambda' ; \mathcal{J}^{\epsilon}\, \epsilon \rangle = \langle \mathcal{J}\, \lambda ; \mathcal{J}^{\epsilon}\, \epsilon | B(v^0) | \mathcal{J}'\, \lambda' \rangle =$$

$$= \sum_{\lambda_i \lambda_i'} \mathcal{C}(\mathcal{J}, \lambda;\ \lambda_1 \cdots \lambda_n) \prod_{i=1}^{n} \frac{1}{\sqrt{2}} \left[(v^0 + 1)^{\frac{1}{2}} + 2\epsilon_i\ \sigma_{h\lambda_i\lambda_i'} (v^0 - 1)^{\frac{1}{2}} \right]$$

$$\times\ \mathcal{C}(\mathcal{J}', \lambda';\ \lambda_1' \cdots \lambda_n')\, \mathcal{C}(\mathcal{J}^{\epsilon}, \epsilon;\ \epsilon_1 \cdots \epsilon_n) . \tag{A.15}$$

In the helicity frame, where $\sigma_h = \sigma_3$, this reduces to

$$B_{\pm}(v^0, \lambda, \epsilon; \mathcal{J}, \mathcal{J}', \mathcal{J}^{\epsilon}) =$$

$$\sum_{\lambda_i} \mathcal{C}(\mathcal{J}, \lambda;\ \lambda_1 \cdots \lambda_n) \prod_{i=1}^{n} \frac{1}{\sqrt{2}} \left[(v^0 + 1)^{\frac{1}{2}} \pm 4\epsilon_i \lambda_i (v^0 - 1)^{\frac{1}{2}} \right]$$

$$\times\ \mathcal{C}(\mathcal{J}', \lambda;\ \lambda_1 \cdots \lambda_n)\, \mathcal{C}(\mathcal{J}^{\epsilon}, \epsilon;\ \epsilon_1, \cdots \epsilon_n) . \tag{A.16}$$

The $\pm$ sign is plus for particles a and c, whose λ's are plus the z-component of spin, and minus for particles b and d, whose λ's are minus the z component.

Consider the case $n = 2$. Then $\mathcal{J}^{\epsilon} = 1$ or 0. [The fixed $J_1 = \frac{1}{2}$ is suppressed in $\mathcal{J} = (J_1, J_2)$.] For $\mathcal{J}^{\epsilon} = 1$ the Clebsch-Gordan coefficient $\mathcal{C}(\mathcal{J}^{\epsilon}, \epsilon; \epsilon_1, \epsilon_2)$ is the symmetrizer:

$$\mathcal{C}(1, \epsilon; \epsilon_1, \epsilon_2) = \mathcal{C}(1, \epsilon; \epsilon_2, \epsilon_1) = \begin{cases} 1 & \text{for } \epsilon = \epsilon_1 + \epsilon_2 = 1 \\ \frac{1}{\sqrt{2}} & \text{for } \epsilon = \epsilon_1 + \epsilon_2 = 0 \\ 1 & \text{for } \epsilon = \epsilon_1 + \epsilon_2 = -1 . \end{cases} \tag{A.17}$$

For $\mathcal{J}^{\epsilon} = 0$, it is the antisymmetrizer:

$$\mathcal{C}(0, \epsilon; \epsilon_1, \epsilon_2) = -\mathcal{C}(1, \epsilon; \epsilon_2, \epsilon_1) = \begin{cases} \frac{1}{\sqrt{2}} \operatorname{sign} \epsilon_1 & \text{for } \epsilon = \epsilon_1 + \epsilon_2 = 0 \\ 0 & \text{for } \epsilon = \epsilon_1 + \epsilon_2 = 0 . \end{cases} \tag{A.18}$$

These same formulas hold with ϵ replaced by λ. Direct computation then gives

$$B_{\pm}(v^0, \lambda, \epsilon; J, J', 1)_{|\lambda| \leqslant J}$$

$$= \delta_{JJ'} \begin{cases} \frac{1}{2}\left[(v^0 + 1)^{\frac{1}{2}} \pm \epsilon\lambda(v^0 - 1)^{\frac{1}{2}}\right]^2 & \text{for } |\lambda| + |\epsilon| = 2 \\ \sqrt{1 + |\lambda|} & \text{for } |\lambda| + |\epsilon| = 1 \\ \sqrt{2}\, v^0 & \text{for } |\lambda| + |\epsilon| = 0 \end{cases} \tag{A.19}$$

and

$$B_{\pm}(v^0, \lambda, 0; J, J', 0)$$

$$= \delta_{J+J',1} \begin{cases} \pm\sqrt{2}\,(v^0 + 1)^{\frac{1}{2}}\,(v^0 - 1)^{\frac{1}{2}} & \text{for } \lambda = 0 \\ 0 & \text{for } \lambda = 0\,. \end{cases} \tag{A.20}$$

The factors $\delta_{JJ'}$ and $\delta_{J+J',1}$ are in accord with the general connection between $\mathcal{J}, \mathcal{J}'$ and $\mathcal{J}^{\epsilon}$ discussed above.

The maximum possible value of J'_{α} is $\bar{J}_{\alpha}$ [see (5.2)]. If J'_{α} is $\bar{J}_{\alpha}$, then the plot $J'_{\alpha i}$ has no breaks, and $J_{\alpha i} = J^{\epsilon}_{\alpha i}$. In this case the product of the three factors $\mathcal{C}$ in (A.16) will be antisymmetric under interchange of any two variables λ_i with $1 \leqslant i \leqslant 2(\bar{J}_{\alpha} - J_{\alpha}) + 1$, and symmetric under interchange of any two variables λ_i with $2(\bar{J}_\alpha - J_\alpha) + 1 \leqslant i \leqslant 2\bar{J}_\alpha$. This same symmetry holds also for the ϵ_i. In particular, the product of the $\mathcal{C}$'s will be antisymmetric in both λ- and ϵ- space under interchange of the two variables of each of the first $\bar{J}_\alpha - J_\alpha$ pairs of variables. The factors $\mathcal{C}$ will therefore convert the two boost factors of each of these $\bar{J}_\alpha - J_\alpha$ pairs into factors (A.20). This gives (5.5).

If J'_α is less than $\bar{J}_\alpha$, then there can be fewer of the boost factors (A.20). On the other hand, if $\bar{J}'_\alpha$ is less than $\bar{J}_\alpha$, then the $\bar{J}_\alpha$ in (5.4) can be reduced to J'_α, as is seen from (A.11). [$R^J(\theta)$ can be constructed as a linear combination of $2J$ elementary rotation operators.] These changes compensate each other, yielding (5.6) and (5.9) in all cases. In particular, it is shown below that the number of boost factors (A.20) associated with particle α is

$$n_\alpha^{\ \epsilon} = J'_\alpha - J_\alpha + 2I, \qquad \text{(A.21)}$$

where I is a positive integer, or zero. This equation ensures that the boost factors (A.20) combine with the singularities of $R(\theta)$, to give (5.6) and (5.9).

By combining $R(\theta)$ with the $n_\alpha^{\ \epsilon}$ boost factors (A.20) we have effectively replaced the variable index J'_α by the fixed index J_α, in the exponent of the singular factor. Similarly, the exponent

J_α in (4.1), which in general should be J'_α, will be changed to J_α if the n_α^ϵ boost factors are combined with $R(\theta)$. It is shown below that if the n_α^ϵ boost factors are removed from F, then the effect of reversing the signs of all $(v_\alpha^0 - 1)^{\frac{1}{2}}$ is equivalent to reversing the signs of all the indices λ_α. Thus (4.1) leads to (4.2).

The general relationship between $\mathcal{J}_\alpha$, $\mathcal{J}'_\alpha$, and $\mathcal{J}^\epsilon_\alpha$ entails that if there is a break in $J^\epsilon_{\alpha i}$ at i, there must be a break at i in either $J_{\alpha i}$ or $J'_{\alpha i}$, but not both. For such a point i the symmetries under $i \leftrightarrow i + 1$ in the indices of the three $\mathcal{C}$'s in (A.10) are just those that lead to (A.20). That is, these symmetries are such that the two elementary boost factors B_i and B_{i+1} of (A.16) combine into a factor (A.20), which conserves $\lambda_{\alpha j}$ and $\epsilon_{\alpha j}$, in the jump from $j = i - 1$ to $j = i + 1$; conserves also $J_{\alpha j}$ or $J'_{\alpha j}$; and gives a one unit change to $J_{\alpha j} + J'_{\alpha j}$.

The symmetry property of $\mathcal{C}(\mathcal{J}, \alpha; \alpha_i)$ is such that it is symmetric or antisymmetric under the interchange $\alpha_j \leftrightarrow \alpha_k$ $(k > j)$ according to whether there is an even or odd number of breaks in J_i in the range $j \leqslant i < k$. This means that if there is a break in J_j at $j = i$ then the symmetries in the indices α_j with $j \neq (i$ or $i + 1)$ can be obtained by simply eliminating from the curve J_j the two segments incident on the break at $j = i$, and considering the plot of the reduced curve J_j^r, which has these segments taken out. That is, the symmetries in the indices $j \neq (i$ or $i + 1)$ are given as well by J_j^r as by J_j. This permits one to proceed stepwise, first eliminating, in favor of factors (A.20), the pair of boosts incident on any break in J_i^ϵ, then the pair incident on any break in the reduced $J_i^{\epsilon r}$, and so on. Finally one arrives at a curve $J_i^{\epsilon f}$ that has no breaks. The number of boost factors (A.20) introduced during the reduction of $J_{\alpha i}^\epsilon$ to $J_{\alpha i}^{\epsilon f}$ is n_α^ϵ.

To derive (A.21), note that n_α^ϵ is just the number of downward slanted segments of the curve $J_{\alpha i}^\epsilon$. Thus

$$\bar{J}_\alpha = J_\alpha^\epsilon + n_\alpha^\epsilon = J_\alpha' + n_\alpha' = J_\alpha + n_\alpha , \tag{A.22}$$

where $\bar{J}_\alpha$ is defined in (5.2), and n_α' and n_α are the number of downward slanted segments of $J_{\alpha i}'$ and $J_{\alpha i}$, respectively. The relationship between the curves $J_{\alpha i}$, $J_{\alpha i}'$, and $J_{\alpha i}^\epsilon$ demands that the total number of downward slanted segments in each interval i to i + 1 be even. Thus one obtains, by summation,

$$n_\alpha^\epsilon = n_\alpha - n_\alpha' + 2I , \tag{A.23}$$

which in view of (A.22) is equivalent to (A.21).

Once the n_α^ϵ boost factors (A.20) have been factored out of F, the remaining function F' is such that a change of sign of all factors $(v_\alpha^0 - 1)^{\frac{1}{2}}$ is equivalent to a change of the signs of all indices λ_α on F'. An equivalent statement is

$$(-1)^{N_i^- + N_f^-} F_{\lambda_d \lambda_c \lambda_b \lambda_a} = (-1)^{\Sigma n_\alpha^\epsilon} F_{-\lambda_d -\lambda_c -\lambda_b -\lambda_a} \quad . \tag{A.24}$$

What must be shown to prove (A.24) is that the product of the three $\mathcal{C}$'s in (A.16) goes into itself times $(-1)^{n^\epsilon}$ if the signs of all the λ and λ_i are reversed. The reversal of the signs of M, m and m' takes $C_{j\frac{1}{2}}(J, M; m, m')$ into itself times the factor of $2(J - j) = \pm 1$. Thus there will be a net factor of -1 whenever $J_{\alpha i} - J'_{\alpha i}$ changes by a unit. Thus (A.24) follows from (A.21).

The distinction between J'_α and J_α is not indicated in (A.4*), and the J_α's in the expression for the sign in (3.8*) should be primed. Consequently (3.6*) should read

$$F_{\Lambda_f \Lambda_i} = \eta (-1)^{N_i^- + N_f^-} F_{\Lambda_f \Lambda_i} \quad , \tag{A.25}$$

if we take $\eta = \eta_a \eta_b / \eta_c \eta_d$. Then $F^{\sigma_f \sigma_i}$ vanishes unless $\sigma_f = \eta\, \sigma_i$, if reflection invariance is maintained. The equation $p_z^2 = m_j^2$ near the end of Ref. 4 should read $p_z^2 = -m_j^2$.

FOOTNOTES AND REFERENCES

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