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# Kinematic Dynamo Problem

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*Summary*

The properties of the dynamo equation are discussed to show that the dynamo equation is self-adjoint for a curl-free velocity under a suitable restriction. The impossibility of a steady dynamo by a motion having only a radial velocity component which vanishes at the boundary surface is proved, and an attempt has been made to verify the possibility of an oscillating toroidal field dynamo. Also it is shown that any poloidal magnetic field cannot be amplified by a uniform stellar contraction, while quasi-steady toroidal fields can be maintained by a homologous contraction.

## 1. Introduction

Since the first suggestion made by Larmor (1919), the dynamo theory for the maintenance of a magnetic field by a regular fluid motion has been investigated by many researchers (Elsasser 1946a, b, 1947; Alfvén 1950, 1961; Bullard & Gellman 1954; Runcorn 1954; Herzenberg 1958; Backus 1958; Davis 1958; Namikawa 1961; Taylor 1963; Braginskiy 1964a, b, 1965a, b, 1967; Gibson & Roberts 1965; Stevenson 1965; Stevenson & Wolfson 1966; Childress 1967; Tough 1967; Lortz 1968b; Jayanthan 1968; Cowling 1968; Tough & Roberts 1968; Kato & Nakagawa 1969). Turbulent dynamo theory has also been studied by a number of investigators (Batchelor 1950; Biermann & Schlüter 1950, 1951; Zel'Dovich 1957; Moffatt 1961; Saffman 1963; Malkus 1963, 1968; Kraichnan 1967; Thomas 1968; Tsytovich 1969; Vainshtein 1969; Steenbeck & Krause 1969). In spite of these efforts, no definite conclusion on the dynamo maintenance of the terrestrial and the stellar magnetic field has been obtained. The impossibility theory of a steady dynamo with axial symmetry has first been presented by Cowling (1934), and generalized by Backus & Chandrasekhar (1956), Cowling (1957) and Lortz (1968a).

In the present paper, the kinematic dynamo problem is considered; our concern is only the generation of the magnetic field for a specified regular motion of the fluid. The general properties of the dynamo equation are discussed in Section 2 and the impossibility of a steady dynamo with a radial velocity field which vanishes on the boundary surfaces is treated in Section 3. The possibility of the oscillating toroidal field dynamo is discussed in Section 4. Impossibility of the amplification of a poloidal magnetic field in a uniformly contracting star and the possible maintenance of quasi-steady toroidal fields by a homologous contraction in a star are discussed in Section 5. The adjoint equation of the dynamo theory is derived in Section 6.

## 2. The dynamo equation

The basic equations of the dynamo problems are given by Maxwell's equations, Ohm's law and the equations for the fluid motion and for the energy. Because of the mathematical difficulties in solving these equations simultaneously (Stevenson & Wolfson 1966), we consider here the dynamo problem only in the kinematic formulation, namely, we are interested only in the generation of the magnetic field for a specified regular motion of fluid (Elsasser 1946a, b, 1947; Bullard & Gellman 1954; Backus 1958; Herzenberg 1958; Braginskiy 1964b, 1965a, b, 1967; Childress 1967; Jayanthan 1968; Gibson & Roberts 1965; Lortz 1968a, b; Tough 1967).

Then, the hydromagnetic dynamo can be described by the induction equation:

$$\nabla \times \mathbf{B} = 4\pi\mu\sigma(\mathbf{E}+\mathbf{v} \times \mathbf{B}), \tag{1}$$

or equivalently,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{1}{4\pi\mu\sigma}\nabla \times (\nabla \times \mathbf{B}), \tag{2}$$

where $\mathbf{B}$, $\mu$, $\sigma$, $\mathbf{E}$ and $\mathbf{v}$ are the magnetic induction, magnetic permeability, electrical conductivity, electric field and velocity, respectively. The electromagnetic units are used through the calculations. The non-dimensional forms of equations (1) and (2) are

$$\eta\nabla \times \mathbf{B} = \mathbf{E}+\mathbf{v} \times \mathbf{B}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B}.\nabla)\,\mathbf{v}-(\mathbf{v}.\nabla)\,\mathbf{B}-\mathbf{B}\nabla.\mathbf{v}-\eta\nabla \times (\nabla \times \mathbf{B}), \tag{4}$$

where $\mathbf{v}$, $\mathbf{E}$ and time $t$ are measured in units of $V$, $VB$ and $L/V$, respectively. $V$, $B$, and $L$ are the characteristic velocity, magnetic field, and the length associated with the system.

$$\eta = \frac{1}{4\pi\mu\sigma VL} \tag{5}$$

is the reciprocal of magnetic Reynolds number $R_m$.

At the boundary surface $S$, we assume that

$$v_n = 0,\ \mathbf{B} = 0 \text{ and } \mathbf{E} = (\eta\nabla \times \mathbf{B}) \text{ is continuous,} \tag{6}$$

or

$$v_n = 0,\ \mathbf{B} \text{ and } \mathbf{E} \text{ are continuous,} \tag{6a}$$

or

$$v = 0,\ \mathbf{B} \text{ and } \mathbf{E} = (\eta\nabla \times \mathbf{B}) \text{ are continuous,} \tag{7}$$

where $n$ denotes outward 'normal' at the boundary surface. The boundary condition (6) may be applied to the surface of a planetary inviscid liquid core and to a stellar surface having only a toroidal field; equation (6a) may be applied to a surface of a planetary inviscid core and to a stellar surface having both a toroidal and a poloidal field, while the condition (7) may be applied to a planetary viscous liquid core, having both a toroidal and a poloidal field. We denote the dynamo-acting region by $\tau$, the whole space by $\varepsilon$, and the outer space of $\tau$ by $\varepsilon-\tau$.

Let $\mathbf{B}'^*$ be the complex conjugate of a second field $\mathbf{B}'$, corresponding to an arbitrary current system $\mathbf{j}'$ within the volume $\tau$ of the conducting fluid. Then, taking the dot product of equation (1) by $\nabla \times \mathbf{B}'^*$ and integrating it through the volume $\tau$ gives

$$\eta\int_\tau \nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^*\,d\tau = \int_\tau \mathbf{E}.\nabla \times \mathbf{B}'^*\,d\tau+\int_\tau \mathbf{v}.\{\mathbf{B} \times (\nabla \times \mathbf{B}'^*)\}\,d\tau. \tag{8}$$

Making use of the identity of

$$\nabla.(\mathbf{E}\times\mathbf{B}'^*) = \mathbf{B}'^*.\nabla\times\mathbf{E}-\mathbf{E}.\nabla\times\mathbf{B}'^*, \tag{9}$$

together with Gauss's theorem and Maxwell's equations, the first of the right-hand side of (8) becomes

$$\int_\tau \mathbf{E}.\nabla\times\mathbf{B}'^*\,d\tau = -\int_\tau \mathbf{B}'^*.\frac{\partial\mathbf{B}}{\partial t}\,d\tau-\int_S(\mathbf{E}\times\mathbf{B}'^*).d\mathbf{S}. \tag{10}$$

Since $\nabla\times\mathbf{B}'^* = 0$ in the outer space, (9) gives

$$-\int_S(\mathbf{E}\times\mathbf{B}'^*).d\mathbf{S} = -\int_{\varepsilon-\tau}\mathbf{B}'^*.\frac{\partial\mathbf{B}}{\partial t}\,d\tau. \tag{11}$$

Combining (10) and (11) presents

$$\int_\tau \mathbf{E}.\nabla\times\mathbf{B}'^*\,d\tau = -\int_\varepsilon\frac{\partial\mathbf{B}}{\partial t}.\mathbf{B}'^*\,d\tau. \tag{12}$$

The second term of the right-hand side of (8) is expressed as a sum of a symmetric part and an anti-symmetric part with respect to $\mathbf{B}$ and $\mathbf{B}'^*$:

$$\tfrac{1}{2}\int_\tau \mathbf{v}.\{\mathbf{B}\times(\nabla\times\mathbf{B}'^*)+\mathbf{B}'^*\times(\nabla\times\mathbf{B})\}\,d\tau$$

$$+\tfrac{1}{2}\int_\tau \mathbf{v}.\{\mathbf{B}\times(\nabla\times\mathbf{B}'^*)-\mathbf{B}'^*\times(\nabla\times\mathbf{B})\}\,d\tau. \tag{13}$$

Making use of the identity of

$$\nabla(\mathbf{B}.\mathbf{B}'^*) = \mathbf{B}\times(\nabla\times\mathbf{B}'^*)+\mathbf{B}'^*\times(\nabla\times\mathbf{B})+(\mathbf{B}.\nabla)\mathbf{B}'^*+(\mathbf{B}'^*.\nabla)\mathbf{B}, \tag{14}$$

the symmetric part in (13) is

$$\tfrac{1}{2}\int_\tau[\mathbf{v}.\nabla(\mathbf{B}.\mathbf{B}'^*)-\mathbf{v}.\{(\mathbf{B}.\nabla)\mathbf{B}'^*+(\mathbf{B}'^*.\nabla)\mathbf{B}\}]\,d\tau. \tag{15}$$

Applying Gauss's theorem to (15) with the boundary conditions (6) or (7) gives

$$\tfrac{1}{2}\int_\tau a_{ij}(B_{ij}B_j'^*+B_i'B_j)\,d\tau, \tag{16}$$

where

$$a_{ij} = \begin{cases} -\nabla.\mathbf{v}+2\dfrac{\partial v_i}{\partial x_i} \quad (i=j), & (17)\\[2ex] \left(\dfrac{\partial v_j}{\partial x_i}+\dfrac{\partial v_i}{\partial x_j}\right) \quad (i\neq j). & (18)\end{cases}$$

Equation (16) indicates that for the dynamo maintenance of a magnetic field, either the normal strain rate or shear strain rate or both should not vanish.

The anti-symmetric part of (13) can be written

$$\tfrac{1}{2}\int_\tau \mathbf{v}.(\mathbf{L}+\mathbf{M})\,d\tau, \tag{19}$$

where

$$\mathbf{L} = \nabla \times (\mathbf{B} \times \mathbf{B}'^*), \tag{20}$$

and

$$\mathbf{M} = B_j \frac{\partial B_j'^*}{\partial x_i} - B_j'^* \frac{\partial B_j}{\partial x_i}. \tag{21}$$

From (20) and (21), we find

$$\nabla . \mathbf{L} = 0, \tag{22}$$

$$\nabla \times \mathbf{M} = 2\nabla B_j \times \nabla B_j'^*, \tag{23}$$

and

$$\nabla . \mathbf{M} = B_j \nabla^2 B_j'^* - B_j'^* \nabla^2 B_j. \tag{24}$$

**M** can be expressed by Helmholtz's theorem

$$\mathbf{M} = \nabla \times \mathbf{N} + \nabla\Psi, \tag{25}$$

where

$$\mathbf{N} = 2 \int_\tau \frac{\nabla B_j \times \nabla B_j'^*}{4\pi R} d\tau, \tag{26}$$

and

$$\Psi = \int_\tau \frac{B_j'^* \nabla^2 B_j - B_j \nabla^2 B_j'^*}{4\pi R} d\tau. \tag{27}$$

Here $B_j$ and $B_j'^*$ are functions of $(r', \theta', \phi')$, $d\tau = r'^2 \sin\theta' \, d\theta' \, d\phi' \, dr'$, and

$$R = [r^2 + r'^2 - 2rr'\{\cos\theta \cos\theta' + \sin\theta \sin\theta' \cos(\phi - \phi')\}]^{\frac{1}{2}},$$

in a spherical co-ordinate system $(r, \theta, \phi)$. But **M** cannot be expressed in a simple form such as **L** in (20). From (8), (12), (16) and (19), we get

$$\int_\tau \frac{\partial \mathbf{B}}{\partial t} . \mathbf{B}'^* d\tau = \tfrac{1}{2} \int_\tau a_{ij}(B_i B_j'^* + B_i'^* B_j) d\tau - \eta \int_\tau \nabla \times \mathbf{B} . \nabla \times \mathbf{B}'^* d\tau$$
$$+ \tfrac{1}{2} \int_\tau \nabla \times \mathbf{v} . (\mathbf{B} \times \mathbf{B}'^*) d\tau - \tfrac{1}{2} \int_\tau \{\mathbf{B}'^* . (\mathbf{v} . \nabla) \mathbf{B} - \mathbf{B} . (\mathbf{v} . \nabla) \mathbf{B}'^*\} d\tau. \tag{8a}$$

The same equation can be obtained from (4) (see Appendix A).

Making use of Gauss's theorem and the boundary conditions (6) or (7), the anti-symmetric part (19) can be written

$$\tfrac{1}{2} \int_\tau [\nabla \times \mathbf{v} . \{(\mathbf{B} \times \mathbf{B}'^*) + \mathbf{N}\} + \Psi \nabla . \mathbf{v}] d\tau. \tag{28}$$

If $\nabla . \mathbf{v} = 0$ and $\nabla \times \mathbf{v}$ is perpendicular to $\mathbf{B} \times \mathbf{B}'^* + \mathbf{N}$, or if $\nabla \times \mathbf{v} = 0$ and $\mathbf{v} . \nabla\Psi = 0$, the anti-symmetric part of (8) vanishes when $\partial \mathbf{B}/\partial t = 0$ and (8) becomes a Hermitian form. When the symmetrical part of (8) does not vanish, we can solve this eigenvalue problem.

For a general velocity field,

$$\mathbf{v} = \nabla U + \nabla \times \mathbf{W},$$

the anti-symmetric part can be written

$$\tfrac{1}{2} \int_\tau \{\nabla \times \mathbf{v} . (\mathbf{B} \times \mathbf{B}'^*) + U(\mathbf{B}'^* . \nabla^2 \mathbf{B} - \mathbf{B} \nabla^2 \mathbf{B}'^*) + \mathbf{W} . (\nabla B_j \times \nabla B_j'^*)\} d\tau. \tag{29}$$

The first term of (29) comes from the term $(\mathbf{B} . \nabla) \mathbf{v}$, and the second and third terms come from the advection term $(\mathbf{v} . \nabla) \mathbf{B}$ of (4). The first term vanishes for arbitrary **B** and $\mathbf{B}'^*$ when the velocity is curl-free, but it is difficult to find the condition of vanishing of other terms for arbitrary $\mathbf{B}$ and $\mathbf{B}'^*$. Therefore, the advection term $(\mathbf{v}.\nabla)\mathbf{B}$ causes a difficulty in solving the dynamo problem. For the boundary condition (6a), the derivations of (16) from (15) and of (28) and (29) from (19) are impossible. In this case, however, the same discussions mentioned above may hold, because $\nabla \times \mathbf{v} = 0$ must be true for the vanishing of the integral

$$\int_\tau \mathbf{v}.\nabla \times \{(\mathbf{B} \times \mathbf{B}'^*)+\mathbf{N}\}\,d\tau.$$

The equation (8a) can be written, for the boundary condition (6a), in the following form:

$$\begin{aligned}\int_\tau \frac{\partial \mathbf{B}}{\partial t}.\mathbf{B}'^*\,d\tau = &-\tfrac{1}{2}\int_\tau [(\mathbf{B}.\mathbf{B}'^*)\nabla.\mathbf{v}+\mathbf{v}.\{(\mathbf{B}.\nabla)\mathbf{B}'^*+(\mathbf{B}'^*.\nabla)\mathbf{B}\}]\,d\tau \\ &-\eta\int_\tau \nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^*\,d\tau+\tfrac{1}{2}\int_\tau \mathbf{v}.\nabla \times (\mathbf{B} \times \mathbf{B}'^*)\,d\tau \\ &-\tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}-\mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*\}\,d\tau. \end{aligned} \tag{8b}$$

(*a*) *Irrotational fluid dynamo*

In a steady state, the equation of continuity is

$$\rho\nabla.\mathbf{v}+\mathbf{v}.\nabla\rho = 0, \tag{30}$$

where $\rho$ is the density of fluid. The estimation of the order of magnitude of the first and second terms of equation (30) gives $\rho v/L$ and $(v/L)\,\delta\rho$ respectively, where $L$ is the radius of the Earth's core or a star and $\delta\rho$ is the difference in density between the centre and the surface of the Earth's core or a star. In the Earth's core $\rho$ and $\delta\rho$ are approximately 10 and 2, while $\delta\rho \sim \rho$ in a star. Since the ratio of the second term to the first is about 1/5 in the Earth's core and is nearly equal to unity in a star, the velocity field which actually exists inside the Earth and in stars must be at least partly curl free. Furthermore, in the Earth's core, the forces associated with convection are much less than the static force. Non-homogeneity in the Earth's core far exceeds the deviations associated with convection (Braginskiy 1964b). Therefore, a curl-free velocity field may be important for the dynamo maintenance of a magnetic field in the Earth's core and a magnetic star.

The term $(\mathbf{v}.\nabla)\mathbf{B}$ in equation (4) represents the advection of magnetic induction due to mass motion. The term $(\mathbf{B}.\nabla)\mathbf{v}$ shows that a fluid flow stretches the magnetic lines of force and thus increases the magnetic energy of the system. The term $-\mathbf{B}\nabla.\mathbf{v}$ expresses the increase of magnetic energy by compression. In a system moving with fluid, the variation of magnetic field is given by the term

$$D\mathbf{B}/Dt \equiv \partial\mathbf{B}/\partial t+(\mathbf{v}.\nabla)\mathbf{B},$$

and the terms $(\mathbf{B}.\nabla)\mathbf{v}$ and $-\mathbf{B}\nabla.\mathbf{v}$ compensate the Ohmic dissipation term

$$\eta\nabla \times (\nabla \times \mathbf{B}).$$

Therefore, the essential terms for dynamo maintenance of a magnetic field are the terms $(\mathbf{B}.\nabla)\mathbf{v}$ and $-\mathbf{B}\nabla.\mathbf{v}$ in (4). The importance of the terms $(\mathbf{B}.\nabla)\mathbf{v}$ and $-\mathbf{B}\nabla.\mathbf{v}$ can be seen from the similarity between the heat conduction equation of ideal gas

$$\frac{\partial T}{\partial t}+v_j\frac{\partial T}{\partial x_j} = \frac{Q}{\rho C_v}-\frac{R}{C_v}T\frac{\partial v_k}{\partial x_k}+\frac{\kappa}{\rho C_v}\frac{\partial^2 T}{\partial x_j{}^2} \tag{31}$$

and equation (4) in the tensor form

$$\frac{\partial B_i}{\partial t} + v_j \frac{\partial B_i}{\partial x_j} = B_j \frac{\partial v_i}{\partial x_j} - B_i \frac{\partial v_k}{\partial x_k} + \eta \frac{\partial^2 B_i}{\partial x_j^2}, \tag{32}$$

where $T$, $Q$, $C_v$, $\kappa$ and $R$ are temperature, quantity of heat generated by sources in unit volume of the fluid per unit time, specific heat at constant volume, thermal conductivity, and universal gas constant, respectively. $B_i$ and $T$ have similar behaviour, and the term $(\mathbf{B}.\nabla)\mathbf{v}$ in (32) corresponds to the heat source term in (31). There is also a similarity between the equation of continuity and (4) as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho v_j) = 0,$$

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j}(B_i v_j) = B_j \frac{\partial v_i}{\partial x_j} + \eta \frac{\partial^2 B_i}{\partial x_j^2}.$$

$B_i$ and $\rho$ have similar behaviour, and the terms $(\mathbf{B}.\nabla)\mathbf{v}$ and $\eta\nabla^2\mathbf{B}$ correspond to a mass source and a mass sink.

When $\nabla \times \mathbf{v} = 0$, $\partial\mathbf{B}/\partial t = 0$, and

$$(\mathbf{v}.\nabla)\mathbf{B} = 0, \tag{33}$$

or

$$\nabla \times \{(\mathbf{v}.\nabla)\mathbf{B}\} = 0, \tag{34}$$

the anti-symmetrical part of (8) vanishes. Thus, when the symmetrical part of (8) does not vanish, (8) becomes a Hermitian form. Eigenfunctions and eigenvalues can be obtained (see Appendix B) by the calculation of the minimum value of

$$\eta = \frac{\int_\tau \mathbf{B}'^*.\{(\mathbf{B}.\nabla)\mathbf{v} - \mathbf{B}\nabla.\mathbf{v}\}\,d\tau}{\int_\tau \nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^*\,d\tau}. \tag{35}$$

In this case, we must solve simultaneously the equation of motion, the equation of continuity, and the induction equation under a restraint (33) or (34), as a complete dynamo problem. The solution is more restrictive than usual.

(b) *Incompressible fluid dynamo*

Integrating the vector identity

$$\nabla.(\mathbf{B}.\mathbf{B}'^*\mathbf{v}) = (\mathbf{B}.\mathbf{B}'^*)\nabla.\mathbf{v} + \mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B} + \mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*,$$

through a volume $\tau$ and making use of Gauss's theorem and the boundary condition of the vanishing of the normal velocity, we obtain

$$\int_\tau \nabla.(\mathbf{B}.\mathbf{B}'^*\mathbf{v})\,d\tau = \int_S (\mathbf{B}.\mathbf{B}'^*)\mathbf{v}\,d\mathbf{S}$$

$$= 0 - \int_\tau (\mathbf{B}.\mathbf{B}'^*)\nabla.\mathbf{v}\,d\tau + \int_\tau \{\mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B} + \mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*\}\,d\tau. \tag{36}$$

The first term of the right-hand side vanishes in this incompressible case. Thus, $\int_\tau \mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}\,d\tau$ is anti-symmetric. Therefore, we see from (8a) the advection term indicates oscillations of magnetic field. Only the $(\mathbf{B}.\nabla)\mathbf{v}$ term can compensate the Joule loss term. Therefore it is necessary to equate to zero the sum of the anti-symmetric part of $\int_\tau \mathbf{B}'^*.(\mathbf{B}.\nabla)\mathbf{v}\,d\tau$ and $\int_\tau \mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}\,d\tau$.

As a very special case, we consider a toroidal field dynamo in an incompressible fluid. Sufficient conditions for a zero value of the antisymmetrical part (19) or (28) for arbitrary $\mathbf{B}$ and $\mathbf{B}'^*$ when $\nabla.\mathbf{v} = 0$, are that $\mathbf{B}$ and $\mathbf{B}'^*$ are toroidal fields, $\mathbf{v}$ is poloidal, and (33) or (34) is satisfied. Moreover, a condition

$$(\mathbf{B}.\nabla)\, v_r = 0, \tag{37}$$

or equivalently

$$\frac{\partial(T, v_r)}{\partial(\theta, \phi)} = 0, \tag{38}$$

is necessary to produce only toroidal fields from the interaction of a poloidal velocity and the toroidal magnetic field $\mathbf{B} = \nabla T \times \hat{r}$, where $v_r$ is the radial component of the velocity and $(\theta, \phi)$ are spherical angular co-ordinates. It can be seen that this dynamo is very restrictive.

## 3. Impossibility of the maintenance of a steady magnetic field by a radial velocity vanishing on boundary surfaces

The Earth's liquid core is probably a mixture of iron and silicon; silicon floats up to the mantle, while iron precipitates to the inner core (Verhoogen 1960; Braginskiy 1964b). Accordingly, in the liquid core the density distribution changes. Due to the rotation of the Earth, however, other velocity components may also be generated. For simplicity, we consider here a dynamo with only a radial velocity which vanishes at the boundary surfaces. In this case, the dynamo equation (4) can be represented by two very simple equations which govern a poloidal field $\mathbf{B}_P$ and a toroidal field $\mathbf{B}_T$, respectively, as follows:

$$v\frac{dP_n}{dr} - \eta\left\{\frac{d^2 P_n}{dr^2} - \frac{n(n+1)}{r^2} P_n\right\} = 0, \tag{39}$$

and

$$\frac{d}{dr}(vT_n) - \eta\left\{\frac{d^2 T_n}{dr^2} - \frac{n(n+1)}{r^2} T_n\right\} = 0, \tag{40}$$

where

$$\mathbf{B}_P = \nabla \times \left[\nabla\left\{P_n(r)\left(\sum_{m=0}^{n} A_n^m Y_n^m\right)\right\} \times \hat{r}\right], \tag{41}$$

$$\mathbf{B}_T = \nabla\left\{T_n(r) \sum_{m=0}^{n} \left(A_n^m Y_n^m\right)\right\} \times \hat{r}, \tag{42}$$

and

$$\mathbf{v} = v(r)\,\hat{r}.$$

Here $Y_n^m(\theta, \phi)$ and $\hat{r}$ are the spherical surface harmonics and the unit vector of radial direction, respectively. The velocity obeys the continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0. \tag{43}$$

(a) *Poloidal magnetic field*

From (39), we obtain

$$v = \frac{\eta\left\{\dfrac{d^2 P_n}{dr^2} - \dfrac{n(n+1)}{r^2} P_n\right\}}{\dfrac{dP_n}{dr}}. \tag{44}$$

When an electrically-conducting fluid exists in a sphere of unit radius, and the exterior of the sphere is a non-conducting medium or a vacuum, the boundary condition of the continuity of magnetic field is given by

$$\left.\begin{aligned} &P_n = 0 \text{ at } r = 0, \\ \text{and} \quad &\frac{dP_n}{dr} + nP_n = 0 \text{ at } r = 1. \end{aligned}\right\} \tag{45}$$

Because $P_n(0) = 0$ and $dPn(1)/dr \gtrless 0$ according to whether $P_n(1) \lessgtr 0$, $P_n(r)$ has, at least, either a minimum or a maximum in the interval $0 < r < 1$. Therefore, $dP_n(r)/dr = 0$ at a point in $0 < r < 1$, and $d^2 P_n(r)/dr^2 \gtrless 0$ for $P_n(r) \lessgtr 0$ at the point of $d\, P_n(r)/dr = 0$. Thus, $d^2 P_n(r)/dr^2 - [n(n+1)/r^2]\, P_n(r) \gtrless 0$ according to whether $P_n(r) \lessgtr 0$. Hence $v$ tends to infinity at the point of $dP_n(r)/dr = 0$, and no steady poloidal magnetic field can be maintained by a radial velocity which is finite in the interval $0 \leqslant r \leqslant 1$ and vanishes on the boundary surfaces.

(b) *Toroidal magnetic field*

Multiplying equation (40) by $T_n'$ and integrating with respect to $r$, gives

$$\int_{r_1}^{r_2} T_n' \frac{d}{dr}(vT_n)\, dr - \int_{r_1}^{r_2} \eta \left\{ \frac{d^2 T_n}{dr^2} - \frac{n(n+1)}{r^2} T_n \right\} T_n'\, dr = 0. \tag{46}$$

Integration by parts presents

$$[vT_n T_n']_{r_1}^{r_2} + \eta \left[ T_n \frac{dT_n'}{dr} - T_n' \frac{dT_n}{dr} \right]_{r_1}^{r_2} - \int_{r_1}^{r_2} \left[ \eta \left\{ \frac{d^2 T_n'}{dr^2} - \frac{n(n+1)}{r^2} T_n' \right\} + v \frac{dT_n'}{dr} \right] T_n\, dr = 0. \tag{47}$$

(1) If we consider a dynamo in a fluid sphere of unit radius, we must take $r_1 = 0$ and $r_2 = 1$. When the boundary conditions are given by

$$v(0) = v(1) = T_n(0) = T_n(1) = T_n'(0) = T_n'(1) = 0, \tag{48}$$

the integrated parts of (47) vanish; the adjoint equation of the dynamo equation (40) is

$$\eta \left\{ \frac{d^2 T_n'}{dr^2} - \frac{n(n+1)}{r^2} T_n' \right\} + v \frac{dT_n'}{dr} = 0. \tag{49}$$

This equation is essentially the same as that of poloidal field except with a reversed velocity.

(2) When a spherical conducting fluid shell is surrounded by a conducting solid shell and has an inner solid conducting sphere as shown in Fig. 1, the equation and solutions for regions (a) and (c) are

$$\eta_a \nabla \times \mathbf{B}_a = -\nabla\phi_a, \tag{50}$$

$$\eta_c \nabla \times \mathbf{B}_c = -\nabla\phi_c, \tag{51}$$

$$\mathbf{B}_a = \nabla \left\{ r^{n+1} \sum_{m=0}^{n} (A_n^m Y_n^m) \right\} \times \hat{r}, \tag{52}$$

and

$$\mathbf{B}_c = \nabla \left\{ (r^{n+1} - r^{-n}) \sum_{m=0}^{n} C_n^m Y_n^m \right\} \times \hat{r}, \tag{53}$$

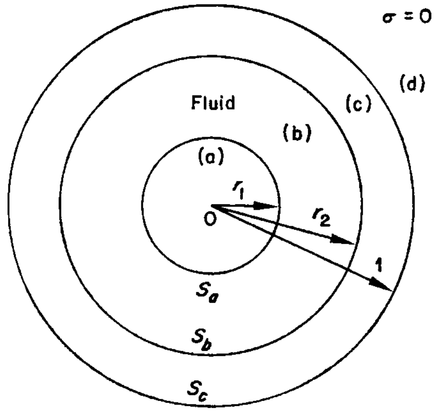


FIG. 1

where $\phi$ is an electric potential. The region (d) is the vacuum or an electrically insulating solid. From the continuity of the magnetic induction and tangential components of the electric field, we get

$$\frac{dT_n(r_1)}{dr} = \frac{\eta_a}{\eta_b}\frac{n+1}{r_1} T_n(r_1), \tag{54}$$

and

$$\frac{dT_n(r_2)}{dr} = \frac{\eta_c}{\eta_b}\frac{(n+1)r_2^{\,n}+nr_2^{\,-n-1}}{r_2^{\,n+1}-r_2^{\,-n}} T_n(r_2). \tag{55}$$

If equation (40) is satisfied and $T_n$ and $T_n'$ satisfy the boundary conditions (54) and (55), the integrated parts of (47) vanish and the adjoint equation (49) can be defined also in this case as in category (1).

Suppose the solution of the dynamo equation (40) satisfying the boundary condition (48) [or (54) and (55)] could be obtained for a finite velocity in the interval $r_1 \leqslant r \leqslant r_2$, the adjoint equation should then have a solution which satisfies the same boundary conditions. In this case, we get from (49)

$$v = -\eta \frac{\dfrac{d^2 T_n'}{dr^2} - \dfrac{n(n+1)}{r^2} T_n'}{\dfrac{dT_n'}{dr}}. \tag{56}$$

The coefficient of $T_n$ in (54) is positive, while that of $T_n$ in (55) is negative. Therefore, from (48) [or (54) and (55)], we can see that $dT_n'/dr = 0$ at a point in $r_1 \leqslant r \leqslant r_2$. At the point $dT_n'(r)/dr = 0$, $d^2 T_n'(r)/dr^2 \gtrless 0$ for $T_n'(r) \gtrless 0$, hence

$$\frac{d^2 T_n'}{dr^2} - \frac{n(n+1)}{r^2} T_n' \gtrless 0$$

according to $T_n' \gtrless 0$. Therefore, $v$ becomes infinite at the point $dT_n'(r)/dr = 0$ and contradicts the assumptions. Thus, we conclude that no steady toroidal field can be maintained. Although we have studied here a non-axially symmetric field in general, equations (39) and (40) have a resemblance to (6) and (8) in Lortz's paper (1968a) of an axially symmetric field, and (39) and (40) may be discussed in the same way as Lortz's theory.

## 4. Oscillating dynamo

The magnetic variable star changed its magnetic polarity in the order of days (Babcock 1958; Ledoux & Renson 1966), the Sun changed its polarity of dipole magnetic field during 1957–1958 (Babcock 1959), and the Earth's dipole magnetic field often changed its polarity during geological time (Hide & Roberts 1961; Cox 1969). Since the time scale of reversals in the Earth's dipole magnetic field is very long, the reversals might have occurred incidentally as a result of changes in the structure of the Earth's interior. If these reversals are caused by a manifestation of non-linear hydromagnetic oscillations in the magnetic variable star and the Earth's core, we should consider a non-steady dynamo. From the study of the oscillations of a pair of coupled disc dynamos each of which is excited by the other, Rikitake (1958) and Allan (1958) showed that both the angular velocities and magnetic fields were reversed in the general large amplitude case. When a magnetic field oscillates, some of the terms in (4) compensate the Joule loss term and the others make the magnetic field oscillation.

### (a) *Oscillating incompressible toroidal field dynamo*

When

$$\frac{D\mathbf{B}}{Dt} \equiv \frac{\partial \mathbf{B}}{\partial t} + (\mathbf{v}.\nabla)\mathbf{B} = 0, \tag{57}$$

and

$$\nabla.\mathbf{v} = 0, \tag{58}$$

the dynamo equation reduces to

$$\eta\nabla \times (\nabla \times \mathbf{B}) = (\mathbf{B}.\nabla)\mathbf{v}. \tag{59}$$

Equations (57) and (59) are equivalent to

$$\int_\tau \frac{\partial \mathbf{B}}{\partial t}.\mathbf{B}'^* d\tau + \int_\tau \mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}\, d\tau = 0, \tag{60}$$

and

$$\tfrac{1}{2}\int_\tau a_{ij}(B_i B_j'^* + B_i'^* B_j)\, d\tau + \tfrac{1}{2}\int_\tau \nabla \times \mathbf{v}.(\mathbf{B} \times \mathbf{B}'^*)\, d\tau - \eta \int_\tau \nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau = 0. \tag{61}$$

From equation (58) and the condition that the normal component of the velocity vanishes, the symmetrical part of the second term of (60) vanishes as shown in (36). Then, the toroidal field oscillates and the total magnetic energy of the volume $\tau$ does not change. Therefore, this dynamo is a steady total energy dynamo,

$$\frac{d}{dt}\int_\tau |\mathbf{B}|^2\, d\tau = 0,$$

contrasting to a steady field dynamo, $\partial\mathbf{B}/\partial t = 0$. When $\mathbf{B}$ and $\mathbf{B}'^*$ are toroidal fields and $\mathbf{v}$ is poloidal, the second term of (61) vanishes, because $\mathbf{B} \times \mathbf{B}'^*$ has only a radial component and $\nabla \times \mathbf{v}$ is toroidal (with only $\theta$ and $\phi$ components which are perpendicular to the radius vector). Equation (61) becomes a Hermitian form and this eigenvalue problem can be solved.

Only the toroidal field $\nabla T \times \hat{r}$ is self-excited under the condition (37) or equivalently (38). Taking dot product of (57) by $\mathbf{B}^*$ and integrating through the volume $\tau$ give

$$\int_\tau \frac{\partial \mathbf{B}}{\partial t}.\mathbf{B}^* d\tau = -\int_\tau \{(\mathbf{v}.\nabla)\mathbf{B}\}.\mathbf{B}^* d\tau.$$

The right side of this equation is anti-symmetric from (36). When $v$ is steady, we can assume $\mathbf{B}$ is proportional to $e^{\omega t}$, and this equation becomes

$$\omega \int_\tau \mathbf{B}.\mathbf{B}^* d = -\tfrac{1}{2}\left[\int_\tau \{(\mathbf{v}.\nabla)\mathbf{B}\}.\mathbf{B}^* d\tau - \int_\tau \{(\mathbf{v}.\nabla)\mathbf{B}^*\}.\mathbf{B}\, d\tau\right]$$

showing $\omega$ is imaginary. Therefore the magnetic field moves with convection according to (57) along closed stream lines and oscillates even if the velocity is steady. When a conducting layer with finite electrical conductivity, zero velocity and non-vanishing toroidal field surrounds the dynamo region, the magnetic field in this layer obeys the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{62}$$

The toroidal field is continuous at the inner boundary and must vanish at the outer boundary of this layer. When a strong toroidal magnetic field moves by convection at the inner boundary of this layer, its radial gradient of the magnetic field becomes large. If, however, no such layer exists, the magnetic field must vanish along the stream lines part of which lie on the boundary, because the toroidal field, which must vanish on the boundary, moves with the fluid in this case. If this dynamo is stable and a poloidal field is generated as a small perturbation due to a velocity perturbation, the polarity of a dipole magnetic field can easily be reversed.

## 5. Quasi-steady dynamo due to contraction of a star

As a simple example of non-steady dynamo, we consider solutions of the dynamo equations of uniformly contracting stars, which are non-steady cases of equations (39) and (40)

$$\frac{\partial P_n}{\partial t} + v\frac{\partial P_n}{\partial r} - \eta\left\{\frac{\partial^2 P_n}{\partial r^2} - \frac{n(n+1)}{r^2} P_n\right\} = 0, \tag{63}$$

and

$$\frac{\partial T_n}{\partial t} + r\frac{\partial T_n}{\partial r} + \frac{\partial v}{\partial r} T_n - \eta\left\{\frac{\partial^2 T_n}{\partial r^2} - \frac{n(n+1)}{r^2} T_n\right\} = 0. \tag{64}$$

From (63) we obtain

$$v = \frac{\eta\left\{\dfrac{\partial^2 P_n}{\partial r^2} - \dfrac{n(n+1)}{r^2} P_n\right\} - \dfrac{\partial P_n}{\partial t}}{\dfrac{\partial P_n}{\partial r}}. \tag{65}$$

In the same way as discussed in Section 3, we have

$$\frac{\partial^2 P_n(r,t)}{\partial r^2} - \frac{n(n+1)}{r^2} P_n(r,t) \gtrless 0$$

for $P_n(r, t) \lesseqgtr 0$ at the point of $\partial P_n(r, t)/\partial r = 0$. Also $\partial P_n(r, t)/\partial t \gtrless 0$ according to whether $P_n(r, t) \gtrless 0$ for a growing magnetic field. Hence $v$ becomes infinite at the point of $\partial P_n(r, t)/\partial r = 0$. Therefore we may conclude that uniformly contracting stars cannot amplify any poloidal magnetic fields.

If we assume homologous contraction (Rosseland 1949) given by

$$\mathbf{v} = -\frac{dk_l(t)/dt}{k_l(t)} r, \tag{66}$$

the solutions of (64) are given by

$$T_n(r, t) = A_l(t)\, J_{n+\frac{1}{2}}\,(k_l(t)\, r)\, \sqrt{(r)} \tag{67}$$

and

$$k_l(t)\, R(t) = \alpha_l, \tag{68}$$

where $R(t)$ and $\alpha_l$ are the radius of the star and the $l$th root of $J_{n+\frac{1}{2}}(\alpha_l) = 0$. We took $V$ in (5) as the magnitude of the initial velocity on the stellar surface and $L$ in (5) as the initial radius of the star. Initial conditions are $R(0) = 1$ and $k_l(0) = \alpha_l$. $A(t)$ is determined by the equation

$$\frac{dA_l(t)/dt}{A_l(t)} = \frac{3(dk_l(t)/dt)}{2k_l(t)} - \eta k_l^2(t). \tag{69}$$

For $k_l(t)$ making the right side of (69) positive, $|A_l(t)|$ increases; we have growing toroidal magnetic fields. The estimation of the order of magnitude of (69) gives

$$V > \frac{\alpha_l^2}{4\pi\mu\sigma L}, \tag{70}$$

for growing magnetic fields. Thus a toroidal magnetic field given by (67) is amplified when the magnitude of velocity satisfies the condition (70). As a numerical example, let us take the value of $\sigma = 10^{-4}$ e.m.u. (Cowling 1953), and $L = 7 \times 10^{10}$ cm for the Sun. Using the value of $\alpha_1 = 4{\cdot}49$ for $n = 1$, we get $V = 2{\cdot}4 \times 10^{-7}\,\mathrm{cm\,s^{-1}}$ and $L/V = 10^{10}$ yr. The radius of the Sun decreases by only 1/10 during $10^9$ yr. Therefore quasi-toroidal fields may be maintained in a slowly contracting star. As a simple example we get an exponentially increasing solution

$$A_l(t) = A(0) \exp(\lambda t) \quad (\lambda = \tfrac{3}{4}(1+\eta\alpha_l^2)^{-1}), \tag{71}$$

for

$$V = \frac{-2\lambda(1+\eta\alpha_l^2) \exp(-4\lambda t/3)}{3\{(1+\eta\alpha_l^2) \exp(-4\lambda t/3) - \eta\alpha_l\}} r, \tag{72}$$

$$k_l(t) = \alpha_l\{(1+\eta\alpha_l^2) \exp(-4\lambda t/3) - \eta\alpha_l^2\}^{-\frac{1}{2}} \tag{73}$$

and

$$R(t) = \{(1+\eta\alpha_l^2) \exp(-4\lambda t/3) - \eta\alpha_l^2\}^{\frac{1}{2}}. \tag{74}$$

At first sight it seems that there is a solution making the right side of (69) zero. Since $A_l$ is constant in this case, we have from (67)

$$\frac{\partial T_n}{\partial t} = Ar^{\frac{3}{2}} \frac{dk_l(t)}{dt} J'_{n+\frac{1}{2}}(k_l(t)\, r).$$

Making use of the boundary conditions, we get from this equation

$$\frac{d}{dt} \int_0^r \tfrac{1}{2} T_n^2\, dr = -A_l^2 \frac{dk_l(t)/dt}{k_l(t)} \int_0^r r\{J_{n+\frac{1}{2}}(k_l(t)\, r)\}^2\, dr. \tag{75}$$

Because

$$\frac{dk_l(t)/dt}{k_l(t)} = \tfrac{2}{3}\eta k_l^2(t) > 0,$$

this equation shows that the total magnetic energy decreases, and no magnetic fields can be maintained.

We obtain an oscillating solution

$$A_l(t) = A_l(0) \exp(\sin \omega t), \tag{76}$$

for

$$k_l(t) = \alpha_l \exp(2 \sin \omega t/3) \left\{ 1 - \tfrac{4}{3}\eta\alpha_l^2 \int \exp(4 \sin \omega t/3)\, dt \right\}^{-\frac{1}{2}}, \tag{77}$$

$$R(t) = \exp(-2\omega t/3) \left\{ 1 - \tfrac{4}{3}\eta\alpha_l^2 \int \exp(4 \sin \omega t/3)\, dt \right\}^{\frac{1}{2}} \tag{78}$$

and

$$V = \tfrac{2}{3} \left[ \omega \cos \omega t + \eta\alpha_l^2 \exp(4 \sin \omega t/3) \left\{ 1 - \tfrac{4}{3}\eta\alpha_l^2 \int \exp(4 \sin \omega t/3)\, dt \right\} \right]. \tag{79}$$

This solution represents that for pulsating stars contracting very slowly, toroidal magnetic fields also oscillate but are not reversed.

## 6. Adjoint equation

We seek here an equation adjoint to the steady dynamo equation

$$\eta \nabla \times (\nabla \times \mathbf{B}_b) = \nabla \times (\mathbf{v} \times \mathbf{B}_b), \tag{80}$$

in a region of fluid $\tau_b$ surrounded by two conducting regions (a) and (c) as shown in Fig. 1. Suppose the fluid velocity $\mathbf{v}$ vanishes on $S_a$ and $S_b$. In the regions (a) and (c), the fields $\mathbf{B}_a$ and $\mathbf{B}_c$ obey

$$\nabla . \mathbf{B} = 0, \tag{81}$$

and Ohm's law for a stationary conductor:

$$\left.\begin{aligned} \eta \nabla \times \mathbf{B} &= -\nabla \phi, \\ \text{or} \quad \eta \nabla \times (\nabla \times \mathbf{B}) &= 0. \end{aligned}\right\} \tag{82}$$

In the vacuum region (or non-conducting solid region) $\tau_d$, $\mathbf{B}_d$ obeys (81) and

$$\nabla \times \mathbf{B}_d = 0. \tag{83}$$

The regions (a), (b), (c) and (d) in the Earth represent the inner solid core, the liquid core, the lower mantle layer which has a small electrical conductivity, and the mantle, respectively. The region (a) does not exist in a star. The regions (b), (c) and (d) are the stellar body, conducting stellar atmosphere, and the vacuum. Across the boundary surfaces $S_a$, $S_b$ and $S_c$ (see Fig. 1), $\mathbf{B}$ and $\phi$ are continuous.

Taking the scalar product of (80) with an arbitrary solenoidal vector $\mathbf{B}_d$ and integrating over $\tau_b$ gives

$$\int \hat{\mathbf{B}}_b . \nabla \times (\eta_b \nabla \times \mathbf{B}_b)\, d\tau_b - \int \nabla \times (\mathbf{v} \times \mathbf{B}) . \hat{\mathbf{B}}_b\, d\tau_b = 0$$

$$= \int \nabla . \{\eta_b (\nabla \times \mathbf{B}_b) \times \hat{\mathbf{B}}_b\}\, d\tau_b + \int \eta_b \nabla \times \mathbf{B}_b . \nabla \times \hat{\mathbf{B}}_b\, d\tau_b$$

$$- \int \nabla . \{(\mathbf{v} \times \mathbf{B}) \times \hat{\mathbf{B}}_b\}\, d\tau_b - \int \mathbf{v} \times \mathbf{B}_b . \nabla \times \hat{\mathbf{B}}_b\, d\tau_b$$

$$= \int_{S_a^+, S_b^-} \{\eta_b(\nabla \times \mathbf{B}_b) \times \hat{\mathbf{B}}_b\}.d\mathbf{S} - \int \nabla.\{\eta_b(\nabla \times \hat{\mathbf{B}}_b) \times \mathbf{B}_b\}\, d\tau_b$$

$$+ \int \nabla \times (\eta_b \nabla \times \hat{\mathbf{B}}_b).\mathbf{B}_b\, d\tau_b - \int_{S_a, S_b} \{(\mathbf{v} \times \mathbf{B}_b) \times \hat{\mathbf{B}}_b\}.d\mathbf{S}$$

$$+ \int (\mathbf{v} \times \nabla \times \hat{\mathbf{B}}_b).\mathbf{B}_b\, d\tau_b$$

$$= \int_{S_a^+, S_b^-} \eta_b\{(\nabla \times \mathbf{B}_b) \times \hat{\mathbf{B}}_b - (\nabla \times \hat{\mathbf{B}}_b) \times \mathbf{B}_b\}.d\mathbf{S}$$

$$+ \int \{\nabla \times (\eta_b \nabla \times \hat{\mathbf{B}}_b) + \mathbf{v} \times \nabla \times \hat{\mathbf{B}}_b\}.\mathbf{B}_b\, d\tau_b, \tag{84}$$

where $S_a^+$ and $S_b^-$ denote the surfaces just outside $S_a$ and inside $S_b$. The surface integral containing $v$ vanishes by the boundary condition of zero velocity on the boundaries. Similarly, assuming that $\mathbf{B}_a = 0(r)$ and $\phi_a = 0(r)$ when $r \to 0$ in $\tau_a$, we obtain from (82)

$$\int \hat{\mathbf{B}}_a.\nabla \times (\eta_a \nabla \times \mathbf{B}_a)\, d\tau_a = 0$$

$$= -\int_{S_a^-} \eta_a\{(\nabla \times \mathbf{B}_a) \times \hat{\mathbf{B}}_a - (\nabla \times \hat{\mathbf{B}}_a) \times \mathbf{B}_a\}.d\mathbf{S} + \int \nabla \times (\eta_a \nabla \times \mathbf{B}_a).\mathbf{B}_a\, d\tau_a, \tag{85}$$

and

$$\int \hat{\mathbf{B}}_c.\nabla \times (\eta_c \nabla \times \mathbf{B}_c)\, d\tau_c = 0$$

$$= -\int_{S_b^\bullet} \eta_c\{(\nabla \times \mathbf{B}_c) \times \hat{\mathbf{B}}_c - (\nabla \times \hat{\mathbf{B}}_c) \times \mathbf{B}_c\}.d\mathbf{S}$$

$$+ \int \nabla \times (\eta_c \nabla \times \hat{\mathbf{B}}_c).\mathbf{B}_c\, d\tau_c$$

$$+ \int_{S_c^-} \eta_c\{(\nabla \times \mathbf{B}_c) \times \hat{\mathbf{B}}_c - (\nabla \times \hat{\mathbf{B}}_c) \times \mathbf{B}_c\}.d\mathbf{S}, \tag{86}$$

because no current flows in the region $\tau_d$, the field $\hat{\mathbf{B}}_d$ must satisfy

$$\nabla \times \hat{\mathbf{B}}_d = 0. \tag{87}$$

In the region $\tau_d$, we have

$$\int \nabla.(\mathbf{E}_d \times \hat{\mathbf{B}}_d)\, d\tau_d = \int_{S_c^+} (\mathbf{E}_d \times \hat{\mathbf{B}}_d).d\mathbf{S}$$

$$= -\int \hat{\mathbf{B}}_d.\frac{\partial \mathbf{B}_d}{\partial t}\, d\tau_d - \int \mathbf{E}_d.\nabla \times \mathbf{B}_d\, d\tau_d = 0, \tag{88}$$

because $\partial \mathbf{B}/\partial t = 0$, $\nabla \times \mathbf{B}_d = 0$, and $\mathbf{E}_d$ and $\hat{\mathbf{B}}_d$ must vanish at infinity. Similarly, we have

$$\int_{S_c^+} (\hat{\mathbf{E}}_d \times \mathbf{B}_d).d\mathbf{S} = 0. \tag{89}$$

The last term of (86) is equal to the difference between (88) and (89), because the tangential components of $\mathbf{E}_c (= \eta_c \nabla \times \mathbf{B}_c)$, $\hat{\mathbf{E}}_c (= \eta \nabla \times \hat{\mathbf{B}}_c)$, $\mathbf{B}_c$ and $\hat{\mathbf{B}}_c$ are continuous across $S_c$ and equal to $\mathbf{E}_d$, $\hat{\mathbf{E}}_d$ and $\hat{\mathbf{B}}_d$. Thus the last term of (86) vanishes.

Adding (84) through (86) gives

$$\int \nabla \times (\eta_a \nabla \times \hat{\mathbf{B}}_a).\mathbf{B}_a \, d\tau_a + \int \{\nabla \times (\eta_b \nabla \times \hat{\mathbf{B}}_b) + \mathbf{v} \times \nabla \times \hat{\mathbf{B}}_b\}.\mathbf{B}_d \, d\tau_b + \int \nabla \times (\eta_c \nabla \times \mathbf{B}_c).\mathbf{B}_c \, d\tau_c = 0, \quad (90)$$

because the tangential components of $\mathbf{E} = (\eta \nabla \times \mathbf{B})$ and $\mathbf{B}$ are continuous across the surfaces $S_a$, $S_b$ and $S_c$. The volume integrals vanish for every eigenvector $\mathbf{B}$. Assuming these form a complete set, we have

$$\nabla \times (\eta_a \nabla \times \hat{\mathbf{B}}_a) = 0 \text{ in } \tau_a, \quad (91)$$

$$\nabla \times (\eta_b \nabla \times \hat{\mathbf{B}}_b) + \mathbf{v} \times \nabla \times \hat{\mathbf{B}}_b = 0 \text{ in } \tau_b, \quad (92)$$

or

$$\left.\begin{aligned} \eta_b \nabla \times \hat{\mathbf{B}}_b &= -\nabla \hat{\phi}_b + \mathbf{F}(\mathbf{v}, \hat{\mathbf{B}}_b), \\ \nabla \times \mathbf{F}(\mathbf{v}, \hat{\mathbf{B}}_b) &= -\mathbf{v} \times \nabla \times \hat{\mathbf{B}}_b, \end{aligned}\right\} \quad (92a)$$

where

and

$$\nabla \times (\eta_c \nabla \times \hat{\mathbf{B}}_c) = 0 \text{ in } \tau_c. \quad (93)$$

These equations determine the adjoint system together with (81), (87) and the condition that $\hat{\mathbf{B}}$ and the tangential components of $\hat{\mathbf{E}}$ are continuous across the boundary surfaces. The adjoint equations in $\tau_a$, $\tau_c$ and $\tau_d$ are the same as those of the given system, but the adjoint equation in $\tau_b$ is quite different from the given equation. Boundary conditions of the adjoint system are the same as those of the given system. Gibson & Roberts (1965) derived the adjoint equation of the dynamo equation, but they had difficulty in the expression of boundary conditions. Their $\mathbf{B}\dagger$ corresponds to our current density $\mathbf{j}$ as follows:

Making use of the relation

$$\nabla \times \hat{\mathbf{B}}_b = 4\pi \hat{\mathbf{j}}_b, \quad (94)$$

we can take

$$\nabla \times \eta_b \hat{\mathbf{j}}_b = -\nabla \phi_b - \mathbf{v} \times \hat{\mathbf{j}}_b, \quad (95)$$

as the adjoint equation instead of (92). Equation (95) is their equation [A6]. $\eta_b$ is obtained from the calculation of the minimum value of

$$\eta_b = \frac{\int \hat{\mathbf{B}}_b.\nabla \times (\mathbf{v} \times \mathbf{B}_b) \, d\tau_b}{\int \hat{\mathbf{B}}_b.\nabla \times (\nabla \times \mathbf{B}_b) \, d\tau_b}, \quad (96)$$

because if $\mathbf{B}_b$ and $\hat{\mathbf{B}}_b$ satisfy the dynamo and the adjoint equations, $\delta\eta_0$ is zero to the first order for all small variations of $\delta\mathbf{B}_b$ and $\delta\hat{\mathbf{B}}_b$, satisfying the boundary conditions and vice versa.

## 7. Concluding remarks

We have examined the general properties of the kinematic dynamo equation (see Section 2), and found that the advection term $(\mathbf{v}.\nabla)\mathbf{B}$ in (4) prevents the equation from becoming self-adjoint. If the third and fourth terms of the right-hand side of (8a) cancel each other, however, the equation becomes self-adjoint for $\partial\mathbf{B}/\partial t = 0$. Unfortunately, the fourth term which comes from the advection term cannot be expressed as simply as the third term which comes from the $(\mathbf{B}.\nabla)\mathbf{v}$ term. Thus, we cannot find any condition on the velocity in which these terms may cancel each other. When the advection term is zero or curl-free, the fourth term of (8a) vanishes. Then the equation becomes self-adjoint for a curl-free velocity. But in this case, as a complete dynamo problem, we must solve the equation of motion and the induction equation under a restriction of (33) or (34): then the solution may be restrictive.

In Section 3, the impossibility of a dynamo with zero radial velocity on the boundary surface has been proved. The possibility of an oscillating toroidal magnetic field dynamo is discussed in Section 4. This is a dynamo which has a constant total magnetic energy of the volume $d/dt \int |\mathbf{B}|^2 \, d\tau = 0$ in contrast to a steady field dynamo $\partial \mathbf{B}/\partial t = 0$.

Amplification and maintenance of magnetic fields in a uniformly contracting star is discussed in Section 5. It is proved that no poloidal magnetic field can be amplified in a uniformly contracting star, while quasi-steady toroidal magnetic fields can be maintained by a homologous contraction.

The adjoint equation of the dynamo equation is derived in Section 6. This is quite different from the original equation, although both equations have the same boundary conditions. We have suggested that a non-homogeneous dynamo with a curl-free velocity under a restriction, a homogeneous and oscillating toroidal field dynamo, and a quasi-steady toroidal field dynamo in a uniformly contracting star are possible to exist. However, the complete hydromagnetic problem, which requires the solution of the equation of motion, Maxwell's equation, and the energy equation, is almost impossible to solve analytically. It will require an electronic computer with much larger memory and much higher speeds than the present ones to solve this problem numerically (Stevenson & Wolfson 1966).

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## APPENDIX A

Multiplying equation (4) scalarly with $\mathbf{B}'^*$ and integrating through volume $\tau$, we get

$$\int_\tau \frac{\partial \mathbf{B}}{\partial t}.\mathbf{B}'^*\,d\tau = \tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{B}.\nabla)\mathbf{v}+\mathbf{B}.(\mathbf{B}'^*.\nabla)\mathbf{v}\}\,d\tau$$

$$-\tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}+\mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*\}\,d\tau - \int_\tau (\nabla.\mathbf{v})(\mathbf{B}.\mathbf{B}'^*)\,d\tau$$

$$-\eta\int_\tau \nabla\times(\nabla\times\mathbf{B}).\mathbf{B}'^*\,d\tau+\tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{B}.\nabla)\mathbf{v}-\mathbf{B}.(\mathbf{B}'^*.\nabla)\mathbf{v}\}\,d\tau$$

$$-\tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B}-\mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*\}\,d\tau. \tag{A.1}$$

Making use of (6) [or (7)], Gauss's theorem and the identities

$$\nabla.\{(\mathbf{v}.\mathbf{B})\mathbf{B}'^*\} = \mathbf{v}.(\mathbf{B}'^*.\nabla)\mathbf{B}+\mathbf{B}.(\mathbf{B}'^*.\nabla)\mathbf{v}, \tag{A.2}$$

and

$$\nabla.\{(\mathbf{v}.\mathbf{B}'^*)\mathbf{B}\} = \mathbf{v}.(\mathbf{B}.\nabla)\mathbf{B}^*+\mathbf{B}'^*.(\mathbf{B}.\nabla)\mathbf{v}, \tag{A.3}$$

the first and fifth terms of the right-hand side in (A.1) reduce to

$$-\tfrac{1}{2}\int_\tau \mathbf{v}.\{(\mathbf{B}.\nabla)\mathbf{B}'^*+(\mathbf{B}'^*.\nabla)\mathbf{B}\}\,d\tau, \tag{A.4}$$

and

$$\tfrac{1}{2}\int_\tau \mathbf{v}.\{(\mathbf{B}'^*.\nabla)\mathbf{B}-(\mathbf{B}.\nabla)\mathbf{B}'^*\}\,d\tau. \tag{A.5}$$

With the boundary conditions, Gauss's theorem and the identities

$$v_i B_j \frac{\partial B_i'^*}{\partial x_j} = \frac{\partial}{\partial x_j}(v_i B_j B_i'^*) - \frac{\partial v_i}{\partial x_j} B_j B_i'^*, \tag{A.6}$$

$$\nabla\times(\mathbf{B}\times\mathbf{B}'^*) = (\mathbf{B}'^*.\nabla)\mathbf{B}-(\mathbf{B}.\nabla)\mathbf{B}'^* \tag{A.7}$$

and

$$\nabla.\{\mathbf{v}\times(\mathbf{B}\times\mathbf{B}'^*)\} = (\mathbf{B}\times\mathbf{B}'^*).\nabla\times\mathbf{v}-\mathbf{v}.\nabla\times(\mathbf{B}\times\mathbf{B}'^*), \tag{A.8}$$

(A.4) and (A.5) are reduced to

$$\tfrac{1}{2}\int \frac{\partial v_i}{\partial x_j}(B_i B_j'^*+B_i'^* B_j)\,d\tau, \tag{A.9}$$

and

$$\tfrac{1}{2}\int_\tau \nabla\times\mathbf{v}.(\mathbf{B}\times\mathbf{B}'^*)\,d\tau. \tag{A.10}$$

Making use of Gauss's theorem, (6) [or (7)] and (36), the second term on the right-hand side of (A.1) becomes

$$\tfrac{1}{2}\int_\tau (\mathbf{B}.\mathbf{B}'^*)\,\nabla.\mathbf{v}\,d\tau. \tag{A.11}$$

The fourth term on the right-hand side of (A.1) can be written as

$$-\int_\tau \eta\nabla \times (\nabla \times \mathbf{B}).\mathbf{B}'^* d\tau$$

$$= -\int_\tau \nabla.(\eta\nabla \times \mathbf{B} \times \mathbf{B}'^*) d\tau - \int_\tau \eta\nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau$$

$$= -\int_S \{\eta\nabla \times (\mathbf{B} \times \mathbf{B}'^*)\}.dS - \int_\tau \eta\nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau. \qquad \text{(A.12)}$$

In the exterior of the dynamo-acting region, we have

$$\int_{\varepsilon-\tau} \nabla.(\mathbf{E} \times \mathbf{B}'^*) d\tau$$

$$= -\int_S (\mathbf{E} \times \mathbf{B}'^*).dS = -\int_{\varepsilon-\tau} \mathbf{B}'^*.\frac{\partial \mathbf{B}}{\partial t} d\tau - \int_{\varepsilon-\tau} \mathbf{E}.\nabla \times \mathbf{B}'^* d\tau$$

$$= -\int_{\varepsilon-\tau} \mathbf{B}'^*.\frac{\partial \mathbf{B}}{\partial t} d\tau, \qquad \text{(A.13)}$$

because $\nabla \times \mathbf{B}'^* = 0$ in the region. Since $\mathbf{E}$ ($= \eta\nabla \times \mathbf{B}$) and $\mathbf{B}'^*$ are continuous across the boundary surface, adding (A.12) and (A.13) gives

$$-\int_{\varepsilon-\tau} \mathbf{B}'^*.\frac{\partial \mathbf{B}}{\partial t} d\tau - \int_\tau \eta\nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau = -\int_\tau \eta\nabla \times (\nabla \times \mathbf{B}).\mathbf{B}'^* d\tau. \qquad \text{(A.14)}$$

From (A.1), (A.9), (A.10), (A.11) and (A.14), we have

$$\int_\varepsilon \frac{\partial \mathbf{B}}{\partial t}.\mathbf{B}'^* d\tau = \tfrac{1}{2}\int_\tau a_{ij}(B_i B_j'^* + B_i'^* B_j) d\tau - \int_\tau \eta\nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau$$

$$+ \tfrac{1}{2}\int_\tau \nabla \times \mathbf{v}.(\mathbf{B} \times \mathbf{B}'^*) d\tau - \tfrac{1}{2}\int_\tau \{\mathbf{B}'^*.(\mathbf{v}.\nabla)\mathbf{B} - \mathbf{B}.(\mathbf{v}.\nabla)\mathbf{B}'^*\} d\tau.$$

This agrees with (8a). Here $a_{ij}$'s are given by (17) and (18).

## APPENDIX B

As seen in Section 2a, the anti-symmetric part of $\int_\tau \{\mathbf{B} \times (\nabla \times \mathbf{B}'^*)\}.\mathbf{v}\, d\tau$ vanishes when $\nabla \times \mathbf{v} = 0$, $\partial \mathbf{B}/\partial t = 0$ and the condition (33) is satisfied. Under these conditions, we calculate the minimum value of

$$\eta = \frac{J_1(\mathbf{B}, \mathbf{B})}{J_2(\mathbf{B}, \mathbf{B})}, \qquad \text{(B.1)}$$

where

$$J_1(\mathbf{B}, \mathbf{B}) = \int_\tau \{(\mathbf{B}.\nabla)\mathbf{v} - \mathbf{B}\nabla.\mathbf{v}\}.\mathbf{B}'^* d\tau, \qquad \text{(B.2)}$$

and

$$J_2(\mathbf{B}, \mathbf{B}) = \int_\tau \nabla \times \mathbf{B}.\nabla \times \mathbf{B}'^* d\tau. \qquad \text{(B.3)}$$

Then, putting $\delta\eta = 0$, we get

$$\delta J_1 - \eta\delta J_2 = 0. \qquad \text{(B.4)}$$

Denoting the minimum value of $\eta$ as $\eta_1$, the corresponding $\mathbf{B}$ as $\mathbf{B}_1$, and

$$\mathbf{B} = \mathbf{B}_1 + \varepsilon_1 \mathbf{b}, \tag{B.5}$$

we get, to first order of $\varepsilon_1$

$$J_1(\mathbf{B}, \mathbf{B}) = J_1(\mathbf{B}_1, \mathbf{B}_1) + \varepsilon_1{}^* J_1(\mathbf{B}_1, \mathbf{b}) + \varepsilon_1 J_1(\mathbf{b}, \mathbf{B}_1). \tag{B.6}$$

Since

$$\{(\mathbf{B}_1{}^*.\nabla)\mathbf{v}\}.\mathbf{b} = \{(\mathbf{b}.\nabla)\mathbf{v}\}.\mathbf{B}_1{}^* \text{ for } \nabla \times \mathbf{v} = 0,$$

we see

$$J_1(\mathbf{b}, \mathbf{B}_1) = \int_\tau \{(\mathbf{b}.\nabla)\mathbf{v} - \mathbf{b}\nabla.\mathbf{v}\}.\mathbf{B}_1{}^* d\tau$$

$$= \int_\tau \{(\mathbf{B}_1{}^*.\nabla)\mathbf{v} - \mathbf{B}_1{}^* \nabla.\mathbf{v}\}.\mathbf{b}\, d\tau$$

$$= J_1{}^*(\mathbf{B}_1, \mathbf{b}). \tag{B.7}$$

Thus, we have

$$J_1(\mathbf{B}, \mathbf{B}) = J_1(\mathbf{B}_1, \mathbf{B}_1) + 2R\varepsilon_1{}^* J_1(\mathbf{B}_1, \mathbf{b}), \tag{B.8}$$

where $R$ denotes the real part. Similarly,

$$J_2(\mathbf{B}, \mathbf{B}) = J_2(\mathbf{B}_1, \mathbf{B}_1) + 2\varepsilon_1{}^* J_2(\mathbf{B}_1, \mathbf{b}), \tag{B.9}$$

as

$$J_2(\mathbf{b}, \mathbf{B}_1) = J_2{}^*(\mathbf{B}_1, \mathbf{b}). \tag{B.10}$$

Inserting (B.8) and (B.9) in (B.4), we have

$$2R\varepsilon_1{}^*\{J_1(\mathbf{B}_1, \mathbf{b}) - \eta_1 J_2(\mathbf{B}_1, \mathbf{b})\} = 0. \tag{B.11}$$

Since $\varepsilon_1$ is arbitrary, we get

$$J_1(\mathbf{B}_1, \mathbf{b}) - \eta_1 J_2(\mathbf{B}_1, \mathbf{b}) = 0. \tag{B.12}$$

We see

$$\eta_1 J_2(\mathbf{B}_1, \mathbf{b}) = \int_\tau \eta_1 \nabla \times \mathbf{B}_1 . \nabla \times \mathbf{b}^* d\tau$$

$$= \int_\tau \nabla.(\mathbf{b}^* \times \eta_1 \nabla \times \mathbf{B}_1) d\tau + \int_\tau \eta_1 \nabla \times (\nabla \times \mathbf{B}_1).\mathbf{b}^* d\tau$$

$$= \int_S (\mathbf{b}^* \times \eta_1 \nabla \times \mathbf{B}_1).d\mathbf{S} + \int_\tau \eta_1 \nabla \times (\nabla \times \mathbf{B}_1).\mathbf{b}^* d\tau. \tag{B.13}$$

In the exterior of $\tau$, we have

$$\int_{s-\tau} \nabla.(\mathbf{b}^* \times \mathbf{E}_1) d\tau = \int_{s-\tau} \mathbf{E}_1.\nabla \times \mathbf{b}^* d\tau - \int_{s-\tau} \mathbf{b}^*.\nabla \times \mathbf{E}_1 d\tau$$

$$= -\int_S (\mathbf{b}^* \times \mathbf{E}_1).d\mathbf{S} = 0, \tag{B.14}$$

because $\nabla \times \mathbf{b}^* = \nabla \times \mathbf{E}_1 = 0$. Therefore, the first term of the last expression in (B.13) vanishes from the continuity of $\mathbf{E}_1 = \eta_1 \nabla \times \mathbf{B}_1$ and $\mathbf{b}^*$. Furthermore, from (B.12) we have

$$\int_\tau \{(\mathbf{B}_1.\nabla)\mathbf{v} - \mathbf{B}_1 \nabla.\mathbf{v} - \eta_1 \nabla \times (\nabla \times \mathbf{B}_1)\}.\mathbf{b}^* d\tau = 0. \tag{B.15}$$

As **b** is arbitrary, we get

$$\eta_1 \nabla \times (\nabla \times \mathbf{B}_1) = (\mathbf{B}_1 . \nabla)\mathbf{v} - \mathbf{B}_1 \nabla . \mathbf{v}. \tag{B.16}$$

Thus, we can see that as a solution of minimum value to the problem of (B.1), $\eta_1$ and $\mathbf{B}_1$ are the eigenvalue and the corresponding eigenfunction of (B.16), respectively. Next, under the condition of

$$\int \mathbf{B}.\mathbf{B}_1 \, d\tau = 0, \tag{B.17}$$

calculating the minimum value of (B.1) gives

$$\eta_2 \nabla \times (\nabla \times \mathbf{B}_2) = (\mathbf{B}_2 . \nabla)\mathbf{v} - \mathbf{B}_2 \nabla . \mathbf{v}, \tag{B.18}$$

where $\eta_2$ is the second eigenvalue and $\mathbf{B}_2$ is the second eigenfunction. Similarly, we can obtain the eigenvalues $\eta_1 \leqslant \eta_2 \leqslant \eta_3 \ldots$ and eigenfunctions $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, ....

When the condition (34) is satisfied instead of (33), we can write

$$\int_\tau \{(\mathbf{B}_1 . \nabla)\mathbf{v} - \mathbf{B}_1 \nabla . \mathbf{v} - (\mathbf{v} . \nabla)\mathbf{B}_1 - \eta_1 \nabla \times (\nabla \times \mathbf{B}_1)\} \, d\tau = 0, \tag{B.15a}$$

and

$$\eta_1 \nabla \times (\nabla \times \mathbf{B}_1) = (\mathbf{B}_1 . \nabla)\mathbf{v} - \mathbf{B}_1 \nabla . \mathbf{v} - (\mathbf{v} . \nabla)\mathbf{B}_1, \tag{B.16a}$$

instead of (B.15) and (B.16), because $\int (\mathbf{v}.\nabla)\mathbf{B}_1 . \mathbf{b}^* \, d\tau = 0$. We can obtain all eigenvalues and eigenfunctions of (B.16a) in the same manner.