CARNEGIE-MELLON UNIVERSITY

Robotics Institute
Vision Laboratory

Kinematics of DDArm II

Pradeep K. Khosla and Takeo Kanade

6 January 1986

Robotics Institute Document RVL017

Location of source file: /usrr0/pkk/project/reports/kin.mss

1. Introduction

The kinematics of a robot are completely described by the Forward solution. the Reverse solution, the Jacobian and the inverse Jacobian. This paper describes the derivation of the kinematics of the CMU-DDArm II robot.

The Forward solution is a 4 by 4 matrix that specifies the position and orientation of the end effector with respect to the base frame. This solution is denoted by the T_6 matrix and is a function of the six joint variables only [3]. We normally know where we want to move the manipulator in terms of the T_6 matrix and it is desired to obtain the joint coordinates in order to make the move. The transformation relating the T_6 matrix to the values of the joint coordinates is called the Reverse solution. The Forward and the Reverse solutions are derived in Sections 3 and 4 respectively.

Differential relationships are important to a manipulator in many ways. The transformation relating the differential changes in the joint coordinates to the differential changes in the world coordinates is called the Jacobian and is specified by a 6 by 6 matrix. The Jacobian of the DDArm II is derived in Section 5.

2. Assignment of Coordinate Frames

The coordinate frames for each link on DDArm II have been assigned according to the Denavit and Hartenberg convention [1] and are depicted in Figure 1. The link parameters of the arm are shown in Table 1. These parameters are used as an input to the ARM program [2] and the A matrices and the Forward solution generated

3. The Forward solution

The relationship between successive frames $n-1$ and n (assigned according to the Denavit and Hartenberg convention) can be established by the following relationship:

- rotate about z_{n-1}, an angle, θ;
- translate along z_{n-1}, a distance, d_n;
- translate along rotated x_{n-1}, a length a_n;
- rotate about x_n, the twist angle α_n;

The product of the above four homogeneous transformations relates the coordinate frame of link n to the coordinate frame of link $n-1$ and is called the A matrix. This is represented as,

$$A_n = \mathrm{Rot}(z,\theta)\ \mathrm{Trans}(0,0,d)\ \mathrm{Trans}(a,0,0)\ \mathrm{Rot}(x,\alpha)$$

The matrices A_1 through A_6 are computed using the link parameters listed in Table 1. These are:

$$A_1 = \begin{bmatrix} C_1 & -S_1 & 0 & a_1C_1 \\ S_1 & C_1 & 0 & a_1S_1 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$A_2 = \begin{bmatrix} C_2 & 0 & S_2 & a_2C_2 \\ S_2 & 0 & -C_2 & a_2S_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$A_3 = \begin{bmatrix} C_3 & 0 & -S_3 & 0 \\ S_3 & 0 & C_3 & 0 \\ 0 & -1 & 0 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

$$A_4 = \begin{bmatrix} C_4 & 0 & S_4 & 0 \\ S_4 & 0 & -C_4 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

$$A_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

$$A_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Multiplying (1) through (6), we obtain the description of the end effector of the manipulator with respect to the base frame.

$$T_6 = A_1A_2A_3A_4A_5A_6 \tag{7}$$

The column vectors of the T_6 matrix are given as:

$$T_6[*][1] = \begin{bmatrix} C_4S_{12}S_6 + C_{12}C_3S_4S_6 - C_{12}C_6S_3S_5 - C_5C_6S_{12}S_4 + C_{12}C_3C_4C_5C_6 \\ -C_{12}C_4S_6 + C_3S_{12}S_4S_6 - C_6S_{12}S_3S_5 + C_{12}C_5C_6S_4 + C_3C_4C_5C_6S_{12} \\ S_3S_4S_6 + C_3C_6S_5 + C_4C_5C_6S_3 \\ 0 \end{bmatrix} \tag{8}$$

$$T_6[*][2] = \begin{bmatrix} C_4S_{12}C_6 + C_{12}C_3S_4C_6 + C_{12}S_6S_3S_5 + C_5S_6S_{12}S_4 - C_{12}C_3C_4C_5S_6 \\ -C_{12}C_4C_6 + S_3S_{12}S_5S_6 + C_6S_{12}C_3S_4 - C_{12}C_5S_6S_4 - C_3C_4C_5S_6S_{12} \\ S_3S_4C_6 - C_3S_6S_5 - C_4C_5S_6S_3 \\ 0 \end{bmatrix} \tag{9}$$

$$T_6[*][3] = \begin{bmatrix} C_{12}C_5S_3 - S_{12}S_4S_5 + C_{12}C_3C_4S_5 \\ S_{12}C_5S_3 + C_{12}S_4S_5 + S_{12}C_3C_4S_5 \\ -C_3C_5 + C_4S_3S_5 \\ 0 \end{bmatrix} \tag{10}$$

$$T_6[*][3] = \begin{bmatrix} a_2C_{12} + a_1C_1 - d_4C_{12}S_3 + d_3S_{12} \\ a_2S_{12} + a_1S_1 - d_4S_{12}S_3 - d_3C_{12} \\ d_1 + d_4C_3 \\ 1 \end{bmatrix}$$

Equations (8)-(11) specify the four column vectors of the T_6 matrix and hence the Forward solution of the manipulator is completely determined. The Reverse solution of the manipulator is derived in the next section.

4. The Reverse solution

We usually know the moves of the end-effector in terms of the T_6 matrices and it is required to obtain the values of the joint variables corresponding to a given T_6 matrix. The closed-form analytical expressions for the joint variables, in terms of the elements of the T_6 matrix, is obtained by isolating each variable by pre-multiplication by a number of the transforms in 7.

Let the given T_6 matrix be specified as:

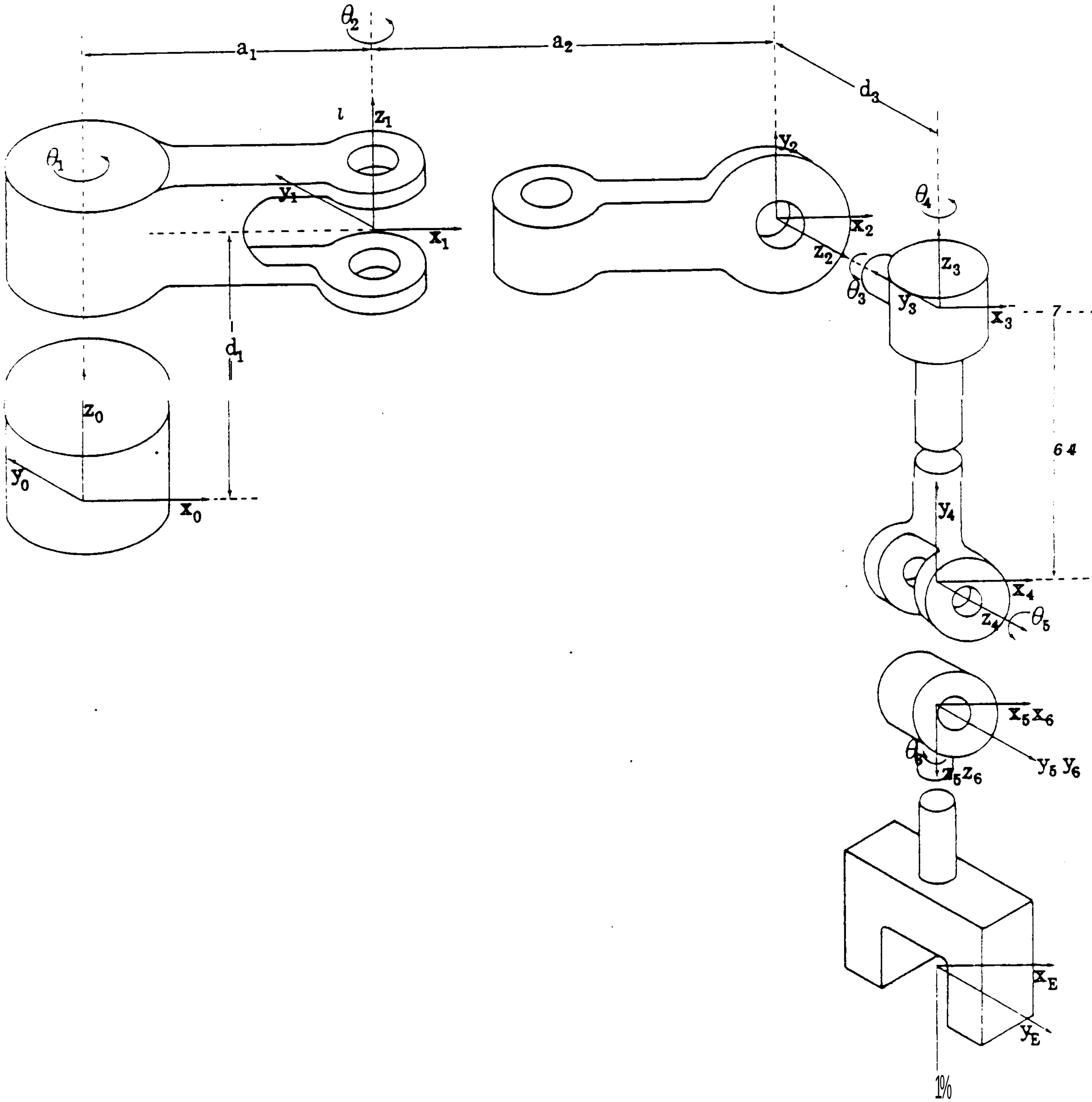

Figure 3-1: Link Coordinates of DDArm II (at the home position).

$$T_6 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (12)$$

The left-hand-side of (12) is completely specified by (8)-(11).

(a) solution for θ_3:

Comparing the (3.4) elements on both sides of (12) we obtain *the* following equation:

$$p_z = d_1 + d_4 C_3.$$

Solving for C_3 and S_3 the following expressions are obtained:

$$C_3 = \frac{p_z - d_1}{d_4}$$

and

$$S_3 = \pm(1 - C_3^2)^{1/2}.$$

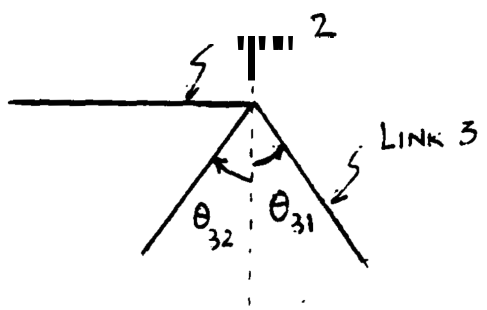

Figure 4-1: Physical Interpretation of Multiple Solutions for θ_3

Therefore the two values of θ_3 are:

$$\theta_{31} = \text{atan2}(S_3, C_3)$$

or

$$\theta_{32} = \text{atan2}(-S_3, C_3) = -\theta_{31}$$

The two values of θ_3 correspond to the *elbow out* and *elbow in* positions of the manipulator are arc depicted in Figure 2. The correct value of θ_3 is selected from the above two values based on some criteria. In the present program the correct value is selected by the user by specifying *elbow out* or *elbow in*.

(b) solution for θ_2:

Comparing the (1,4) elements of (12) we get

$$-d_4C_{12}S_3 + a_2C_{12} + a_1C_1 + d_3S_{12} = p_x \quad (13)$$

and comparing (2,4) elements of (12) we get

$$-d_4S_{12}S_3 + a_2S_{12} + a_1S_1 - d_3C_{12} = p_y \quad (14)$$

Now let $-d_4S_3 = d'_4$ and $d'_4 + a_2 = d$. Therefore, (13) and (14) reduce to

$$d'C_{12} + a_1C_1 + d_3S_{12} = p_x \quad (15)$$

$$d'S_{12} + a_1S_1 - d_3C_{12} = p_y \quad (16)$$

Squaring (15) and (16) and adding, we obtain

$$d'C_2 + d_3S_2 = \frac{p_x^2 + p_y^2 - d'^2 - a_1^2 - d_3^2}{2a_1} = A_0$$

Upon substituting $d' = rS_\varphi$ and $d_3 = rC_\varphi$, in the above equation, we obtain the expression for θ_2:

$$\theta_2 = \mathrm{atan2}\left[\frac{A_0}{\pm(r^2 - A_0^2)^{0.5}}\right] - \mathrm{atan2}\left[\frac{d'}{d_3}\right]$$

The two values of θ_2 correspond to the *right* and *left* shoulder configurations of the manipulator and are depicted in Figure 3. The correct value of θ_2 must be selected based on some criteria. In the present program the user selects this value by specifying the right-shoulder or *left-shoulder* configuration.

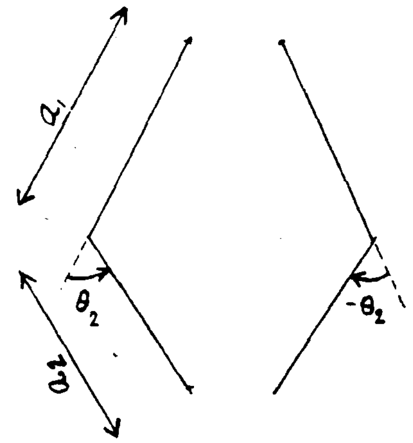

Figure 4-2: Physical Intcrprctation of Multiple Solutions for 8,.

(c) solution for θ_1:

Multiplying (4) by S_{12} and (5) by $-C_{12}$, we obtain:

$$p_x S_{12} - P_y C_{12} = a_1 S_2 + d_3$$

Upon substituting $p_x = lC_\varphi$ and $P_y = lS_\varphi$, in the above cquation. the expression for θ_2 is obtained as:

$$\theta_1 = \text{atan2}[\frac{a_1 S_2 + d_3}{(l^2 - (a_1 S_2 + d_3)^2)^{0.5}}] - \theta_2 + \text{atan2}[\frac{p_y}{p_x}]$$

The two values for 8, correspond to the lcft and right shoulder configurations. Having chosen the correct value of θ_2, the value of θ_1 is unique.

(d) solution for 8,: Having obtaincd thc values of θ_1, θ_2 and θ_3 it now rcmains to find thc values of θ_4, θ_5 and 8,. Premultiplying both sidcs of (1) by $({}^0T_3)^{-1}$ wc get

$$({}^0T_3)^{-1}T_6 = {}^4T_6.$$

In cxpandcd form the above cquation is written as:

$$\begin{bmatrix} C_{12}C_3 & C_3S_{12} & S_3 & - \\ -S_{12} & C_{12} & 0 & - \\ -C_{12}S_3 & -S_{12}S_3 & C_3 & - \\ (& 0 & 0 & 1 \end{bmatrix} T_6 = \begin{bmatrix} S_4S_6 + C_4C_5C_6 & C_6S_4 - C_4C_5S_6 & C_4S_5 & 0 \\ -C_4S_6 + C_5C_6S_5 & -C_4C_6 - C_5S_4S_6 & S_4S_5 & 0 \\ S_5C_6 & -S_5S_6 & d_4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Comparing the (3,3) elements on both sides of (17) we obtain: (17)

$$-C_{12}S_3a_x - S_{12}S_3a_y + C_3a_z = -C_5$$

or

$$C, = C_{12}S_3a_x + S_{12}S_3a_y - C_3a_z.$$

Comparing the (1.3) and (2,3) elements on both sides of (17)

$$C_4S_5 = C_{12}C_3a_x + C_3S_{12}a_y + S_3a_z = A$$

$$S_4S_5 = - S_{12}a_x + C_{12}a_y = B \qquad (18)$$

Squaring and adding (18) and (19) we obtain the expression for S, as (19)

$$S_5 \pm (A^2 + B^2)^{0.5}$$

Evaluating 8, using the double argument atan2 function the following two expressions are obtained:

$$\theta_{51} = \mathrm{atan2}(S_5,C_5)$$

or

$$\theta_{52} = \mathrm{atan2}(-S_5,C_5).$$

As in the case of 8, and θ_3, the correct value of 8, is chosen based on some criterion. In the present case this is selected by the user.

(e) solution for 8,:

Upon comparing the (1.3) and (2.3) elements on both sides of (17) we get:

$$C_4S_5 = C_{12}C_3a_x + C_3S_{12}a_y + S_3a_z$$

$$S_4S_5 = -S_{12}a_x + C_{12}a_y.$$

Therefore,

$$\theta_4 = \mathrm{atan2}(S_4S_5, C_4S_5) \qquad \text{if} \quad \theta_5 > 0$$

or

$$e, = \theta_4 + \pi \qquad \text{if} \quad \theta_5 < 0$$

The manipulator becomes degenerate when 8, = 0.

(f) solution for θ_6:

Upon comparing the (3.1) and (3.2) elements on both sides of (17), the following equations are obtained

$$C_6S_5 = -C_{12}S_3n_x - S_{12}S_3n_y + C_3n_z$$

$$-S_6S_5 = -C_{12}S_3o_x - S_{12}S_3o_y + C_3o_z.$$

$$\theta_6 = \text{atan2}(-S_5S_6, S_5C_6) \qquad \text{if} \qquad \theta_5 > 0$$

$$\theta_6 = \theta_6 + \pi \qquad \text{if} \qquad \theta_5 < 0$$

When 8, = 0 the manipulator is degenerate and only the sum of $(\theta_4 + \theta_6)$ is important. At this point one of the angles is given an arbitrary value (usually the present value) and the other computed accordingly.

(g) Solution for 8, and 8, when 8, = 0:

When 8, = 0, $C_5 = 1$. Comparing the (1.1) of (1.2) elements on both sides of (6) **wc** get

$$\sin(\theta_4 - \theta_6) = C_{12}C_3o_x + C_3S_{12}o_y + S_3o_z = A$$

and

$$\cos(\theta_4 - \theta_6) = C_{12}C_3n_x + C_3S_{12}n_y + S_3n_z = B$$

Thus the value of $(\theta_4 - \theta_6)$ is

$$(\theta_4 - \theta_6) = \text{atan2}(A, B)$$

At this point 8, or 8, is given an arbitrary value (usually the present value) and the other computed accordingly.

The analytical expressions Fa the six joints of the DDArm **II are** outlined in paragraphs (a)-(g). The multiple solutions Fa **joints** 2, 3 and 5 **give rise to** 8 sets of Reverse solutions. These are represented diagrammatically in **Figure 4.**

5. The Jacobian

In a manipulator, differential changes in position and orientation of T, are caused by differential **changes** (dq_i) in the joint coordinates. The **transformation** relating the differential changes in **the** joint coordinates to the differential changes in the T_6 frame is called the Jacobian **(a** 6 x 6 **matrix). Each** column of the Jacobian corresponds to a differential translation and rotation **vector** corresponding to the differential change in each joint coordinate.

The elements of the 6 column vectors of the Jacobian mamx are:

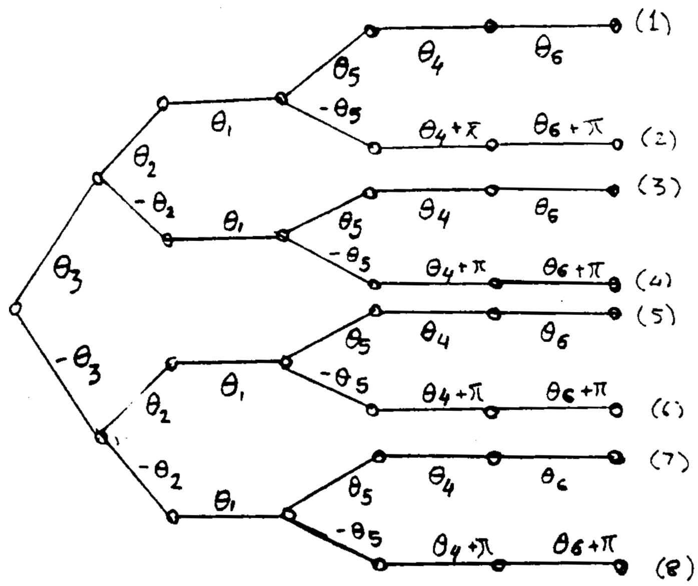

Figure 4-3: Graph Depicting Multiple Sets of Reverse Solutions

$$d_{1x} = -(- C_5C_6S_{12}S_4 + C_4S_{12}S_6 + C_{12}C_3C_4C_5C_6 + C_{12}C_3S_4S_6 - C_{12}C_6S_3S_5)py1 + (C_3C_4C_5C_6S_{12} + C_3S_{12}S_4S_6 - C_6S_{12}S_3S_5 + C_{12}C_5C_6S_4 - C_{12}C_4S_6)px1;$$

$$d_{1y} = -(C_5S_6S_{12}S_4 + C_4S_{12}C_6 - C_{12}C_3C_4C_5S_6 + C_{12}C_3S_4C_6 + C_{12}S_6S_3S_5)py1 + (- C_3C_4C_5S_6S_{12} + C_3S_{12}S_4C_6 + S_6S_{12}S_3S_5 - C_{12}C_5S_6S_4 - C_{12}C_4C_6)px1;$$

$$d_{1z} = -(- S_{12}S_4S_5 + C_{12}C_3C_4S_5 + C_{12}C_5S_3)\,py1 + (C_3C_4S_{12}S_5 + C_5S_{12}S_3 + C_{12}S_4S_5)px1;$$

$$\delta_{1x} = C_3C_6S_5 + S_3S_4S_6 + C_4C_5C_6S_3;$$

$$\delta_{1y} = - C_3S_5S_6 + C_6S_3S_4 - C_4C_5S_3S_6;$$

$$\delta_{1z} = - C_3C_5 + C_4S_3S_5;$$

$$d_{2x} = -(C_4S_2S_6 - C_5C_6S_2S_4 - C_2C_6S_3S_5 + C_2C_3S_4S_6 + C_2C_3C_4C_5C_6)py2 + (-C_2C_4S_6 + C_2C_5C_6S_4 - C_6S_2S_3S_5 + C_3S_2S_4S_6 + C_3C_4C_5C_6S_2)px2;$$

$$d_{2y} = -(C_4S_2C_6 + C_5S_6S_2S_4 + C_2C_6S_3S_5 + C_2C_3S_4C_6$$
$$- C_2C_3C_4C_5S_6)py2 + (-C_2C_4C_6 - C_2C_5S_6S_4 + S_6S_2S_3S_5$$
$$+ C_3S_2S_4C_6 - C_3C_4C_5S_6S_2)px2;$$

$$d_{2z} = -(-S_2S_4S_5 + C_2C_5S_3 + C_2C_3C_4S_5)py2 + (C_2S_4S_5 + C_5S_2S_3 +$$
$$C_3C_4S_2S_5)px2;$$

$$6, = C_3C_6S_5 + S_3S_4S_6 + C_4C_5C_6S_3;$$

$$\delta_{2y} = -C_3S_5S_6 + C_6S_3S_4 - C_4C_5S_3S_6;$$

$$\delta_{2z} = -C_3C_5 + C_4S_3S_5;$$

$$d_{3x} = -(C_6S_3S_5 + C_3S_4S_6 + C_3C_4C_5C_6)py3 + (C_3C_6S_5 + S_3S_4S_6 +$$
$$C_4C_5C_6S_3)px3;$$

$$d_{3y} = -(S_6S_3S_5 + C_3S_4C_6 - C_3C_4C_5S_6)py3 + (-C_3S_6S_5 + S_3S_4C_6 -$$
$$C_4C_5S_6S_3)px3;$$

$$d_{3z} = -(C_5S_3 + C_3C_4S_5)py3 + (-C_3C_5 + C_4S_3S_5)px3;$$

$$\delta_{3x} = C_4S_6 - C_5C_6S_4;$$

$$\delta_{3y} = C_4C_6 + C_5S_4S_6;$$

$$\delta_{3z} = -S_4S_5;$$

$$d_{4x} = 0;$$

$$d_{4y} = 0;$$

$$d_{4z} = 0;$$

$$\delta_{4x} = C_6S_5;$$

$$\delta_{4y} = -S_5S_6;$$

$$\delta_{4z} = -C_5;$$

$d_{5x} = 0;$

$d_{5y} = 0;$

$d_{5z} = 0;$

$\delta_{5x} = S_6;$

$\delta_{5y} = C_6;$

$\delta_{5z} = 0;$

$d_{6x} = 0;$

$d_{6y} = 0;$

$d_{6z} = 0;$

$\delta_{6x} = 0;$

$\delta_{6y} = 0;$

$\delta_{6z} = 1;$

where,

$px_1 = -d_4C_{12}S_3 + a_2C_{12} + a_1C_1 + d_3S_{12}$

$py_1 = -d_4S_{12}S_3 + a_2S_{12} + a_1S_1 - d_3C_{12}$

$px_2 = a_2C_2 - d_4C_2S_3 + d_3S_2$

$py_2 = a_2S_2 - d_4S_2S_3 - d_3C_2$

$px_3 = -d_4S_3$

$py_3 = d_4C_3.$

A MATRIX PARAMETER

link	variable	θ	*a*	a	d
1	θ_1	θ_1	0°		d_1
2	θ_2	θ_2	90°	a_2	0
3	θ_3	θ_3	-90°	0	d_3
4	θ_4	θ_4	90°	0	d_4
5	θ_5	θ_5	90°	0	0
6	θ_6	θ_6	0°	0	0

Table 1

References

[1] Denavit, J. and Hartenberg, R. S.
A Kinematic Notation for Lower Pair Mechanisms Based on Matrices.
Jour-rid of *Applied Mechanics* 77(2):215-221, June, 1955.

[2] J. J. Murray and C. P. Neuman.
ARM: An Algebraic Robot Dynamic Modeling Program.
1983.
Unpublished.

[3] Paul, R.P.
Robot Manipulators :Mathematics, Programming and Control.
MIT Press, Cambridge, MA, 1981.