Kinetic Theory of Coupled Oscillators

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We present an approach for the description of fluctuations that are due to finite system size induced correlations in the Kuramoto model of coupled oscillators. We construct a hierarchy for the moments of the density of oscillators that is analogous to the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy in the kinetic theory of plasmas and gases. To calculate the lowest order system size effect, we truncate this hierarchy at second order and solve the resulting closed equations for the two-oscillator correlation function around the incoherent state. We use this correlation function to compute the fluctuations of the order parameter, including the effect of transients, and compare this computation with numerical simulations.

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Systems of coupled oscillators appear as models for the dynamics of a wide range of phenomena [1–8]. The Kuramoto model is a simple and oft-studied description of coupled oscillators which, in the limit of an infinite number of oscillators, exhibits a phase transition from an incoherent state to phase locked dynamics [9–12]. However, numerical simulations show the appearance of fluctuations that are due to finite system size effects even in the absence of any external noise. Because the system is deterministic, these fluctuations are a manifestation of multioscillator correlations and are expected to vanish in the infinite oscillator limit, with potentially divergent behavior near the transition [13]. While there has been some effort towards an analytic treatment of the fluctuations in the Kuramoto model [14,15], there is at present no systematic approach. Here, we present a statistical formalism which draws upon the kinetic theory of plasmas [16,17]. Our methods are generalizable to any oscillator model.

The Kuramoto model describes the phase evolution of N oscillators and is given by

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} f(\theta_j - \theta_i), \qquad i = 1, \ldots, N, \tag{1}$$

where K is the coupling strength; the ω_i are drawn from a distribution $g(\omega)$, assumed to be symmetric and of zero mean. The coupling function $f(\theta)$ can be any function. In the original Kuramoto model $f(\theta) = \sin\theta$, which we use for our simulations.

In the $N \to \infty$ limit, Kuramoto showed [9] that as the coupling K is increased from 0, this model exhibits a phase transition described by the order parameter $Z = \frac{1}{N} \times \sum_{j=1}^{N} e^{i\theta_j} \equiv re^{i\psi}$, which is a measure of the level of synchrony in the population. Kuramoto found a continuous transition from a phase of complete incoherence ($r = 0$) in the population to a relative degree of coherence ($r > 0$) for K greater than $K_c = 2/\pi g(0)$. However, for a finite number of oscillators, r will fluctuate. One of our goals is to calculate $\langle r^2 \rangle$, where $\langle \cdot \rangle$ represents an ensemble average over initial angles and frequencies. At low K, $\langle r^2 \rangle \approx 1/N$, consistent with the finite size effects for the free ($K = 0$) model. As we will show, typical of phase transitions, the correlations become enhanced near the onset of the transition (critical point).

Strogatz and Mirollo [18] analyzed the stability of the incoherent state using a Fokker-Planck formalism. In the absence of external additive noise, their Fokker-Planck equation has the form of a continuity equation. They found that the incoherent state has a continuum of marginally stable modes, which are made stable by additive noise. In the ensuing, we will generate a series of equations analogous to the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy (BBGKY) for which the Strogatz-Mirollo continuity equation is the truncation at first order. Our strategy is to consider an expansion using $1/N$ as a small parameter.

The complete oscillator probability density

$$n(\theta, \omega, t) = \frac{1}{N}\sum_{i=1}^{N} \delta(\theta - \theta_i(t))\delta(\omega - \omega_i) \tag{2}$$

satisfies the continuity equation

$$\frac{\partial n}{\partial t} + \omega\frac{\partial n}{\partial \theta} = -K\frac{\partial}{\partial \theta}\int_{-\infty}^{\infty}\int_{0}^{2\pi} f(\theta' - \theta) n(\theta', \omega', t) \times n(\theta, \omega, t) d\theta' d\omega'. \tag{3}$$

Equation (3) is analogous to the Klimontovich equation in the plasma context and is still an exact description of the microscopic dynamics. Solving the Klimontovich equation for the complete distribution is equivalent to solving the original system and is equally difficult. The strategy of kinetic theory is to consider the smoothed probability density functions of the oscillators by taking ensemble averages.

The one-particle probability density function (PDF) is given by $\rho_1(\theta, \omega, t) \equiv \langle n(\theta, \omega, t)\rangle$, where brackets denote the ensemble average over initial conditions and frequencies. The density $\rho_1 d\theta d\omega$ represents the mean fraction of oscillators within frequency range $(\omega, \omega + d\omega)$ and angle range $(\theta, \theta + d\theta)$. We note that $\int_0^{2\pi} \rho_1(\theta, \omega, t)d\theta = g(\omega)$. Henceforth, we will use the compact notation $x = (\theta, \omega)$. Taking the expectation value of Eq. (3) gives

$$\frac{\partial \rho_1}{\partial t} + \omega \frac{\partial \rho_1}{\partial \theta} + K\frac{\partial}{\partial \theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta' - \theta)\rho_1(x, t)\rho_1(x', t)d\theta' d\omega' = -K\frac{\partial}{\partial \theta}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta' - \theta)C(x; x', t)d\theta' d\omega', \quad (4)$$

where

$$C(x; x', t) = \langle n(x, t)n(x', t)\rangle - \rho_1(x, t)\rho_1(x', t) - \frac{1}{N}\delta(x - x')\rho_1(x, t) \quad (5)$$

is the "connected" two-oscillator correlation or moment function. The self-fluctuation term drops out in Eq. (4) because we consider $f(0) = 0$.

The right-hand side of (4) describes two-oscillator interactions and is comparable to the collision integral from the kinetic theory of gases and plasmas. Neglecting the collision integral leads to the Vlasov equation, which amounts to a mean field approximation. The Vlasov equation and corresponding Fokker-Planck equation, which includes a diffusive term when external noise is included, has been studied for coupled oscillators previously in many contexts [3,18–24]. Although the Vlasov equation has the same form as Eq. (3), the two should not be confused. $\rho_1(x, t)$ is a smooth function representing the expectation value of the number density over initial conditions and frequencies, whereas $n(x, t)$ is an operator-valued distribution and contains *all* statistical information about the system.

We obtain an equation for $C(x; x', t)$ by multiplying Eq. (3) by $n(x', t)$ and taking the expectation value. This will result in an equation that depends on the three oscillator moment function. Continuing this process for higher moments results in the BBGKY hierarchy [16,17]. We truncate the hierarchy at second order, expecting the correlation $C(x; x', t)$ to be $O(1/N)$ and a general connected n-point function to be $O(1/N^{n-1})$ as is consistent with previous simulations [13,15].

Using Eq. (4) and removing terms expected to be $O(1/N^2)$ yields

$$\left\{\frac{\partial}{\partial t} + \omega_1\frac{\partial}{\partial \theta_1} + \omega_2\frac{\partial}{\partial \theta_2} + K\int_{-\infty}^{\infty}\int_0^{2\pi}\left[\frac{\partial}{\partial \theta_1}f(\theta_3 - \theta_1) + \frac{\partial}{\partial \theta_2}f(\theta_3 - \theta_2)\right]\rho_1(x_3, t)d\theta_3 d\omega_3\right\}C(x_1, x_2, t)$$
$$+ K\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{\partial}{\partial \theta_1}f(\theta_3 - \theta_1)\rho_1(x_1, t)C(x_2, x_3, t)d\theta_3 d\omega_3$$
$$+ K\int_{-\infty}^{\infty}\int_0^{2\pi}\frac{\partial}{\partial \theta_2}f(\theta_3 - \theta_2)\rho_1(x_2, t)C(x_3, x_1, t)d\theta_3 d\omega_3 = -\frac{K}{N}\left[\frac{\partial}{\partial \theta_1}f(\theta_2 - \theta_1) + \frac{\partial}{\partial \theta_2}f(\theta_1 - \theta_2)\right]\rho_1(x_1, t)\rho_1(x_2, t). \quad (6)$$

Equations (4) and (6) form a Gaussian closure of a kinetic theory describing the Kuramoto model. We use this to calculate the fluctuations about the incoherent state. We start with the ansatz [16]:

$$C(x_1, x_2, t) = -\frac{K}{N}\int_{-\infty}^{\infty} d\omega_1' d\omega_2' \int_0^{2\pi} d\theta_1' d\theta_2' \int_{t_0}^{t} dt P(x_1, x_1', t - t')P(x_2, x_2', t - t')[\frac{\partial}{\partial \theta_1'}f(\theta_2' - \theta_1') + \frac{\partial}{\partial \theta_2'}f(\theta_1' - \theta_2')]\rho_1(x_1', t')\rho_1(x_2', t') \quad (7)$$

where the initial conditions are imposed at t_0 and $t_0 < t' < t$. Using Eq. (7) in Eq. (6), we obtain the dynamics for the propagator P,

$$\left\{\frac{\partial}{\partial t} + \omega_1\frac{\partial}{\partial \theta_1} + K\frac{\partial}{\partial \theta_1}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_2 - \theta_1)\rho_1(x_2, t)d\theta_2 d\omega_2\right\} \times P(x_1, x_1', t - t') + K\frac{\partial}{\partial \theta_1}\int_{-\infty}^{\infty}\int_0^{2\pi} f(\theta_2 - \theta_1)\rho_1(x_1, t) \times P(x_2, x_1', t - t')d\theta_2 d\omega_2 = 0, \quad (8)$$

where the initial condition is $P(x; x', 0) = \delta(x - x')$.

The fluctuations in the order parameter are given by $\langle r^2\rangle \equiv \langle ZZ^*\rangle = \int d\omega d\omega' d\theta d\theta' \langle n(\omega, \theta, t)n(\omega', \theta', t)\rangle \times e^{i(\theta - \theta')}$. We consider fluctuations in the incoherent state and thus seek solutions to (4) and (6) such that $\rho_1(x, t) = \frac{1}{2\pi}g(\omega)$. From Eqs. (5) and (6), we see that a computation of the fluctuations amounts to a calculation of the connected correlation function, which is phase invariant (because ρ_1 is independent of θ), so that $C(\theta_1, \theta_2, \omega_1, \omega_2, t) = C(\theta_1 - \theta_2, \omega_1, \omega_2, t)$. Hence, the collision integral in Eq. (4) is zero, making $\rho_1(\theta, \omega) = g(\omega)/2\pi$ an exact solution of the equations. Taking the Fourier and Laplace transforms in θ and time of Eq. (7) gives

$$C_n(\omega_1, \omega_2, s) = \frac{nK\mathrm{Im}[f_n]}{2\pi^2 N}\int_{-\infty}^{\infty} d\omega_1' d\omega_2' d\tau \int_{\mathcal{L}_1} ds_1 \int_{\mathcal{L}_2} ds_2 g(\omega_1')g(\omega_2')\hat{P}_{-n}(\omega_1, \omega_1', s_1)\hat{P}_n(\omega_2, \omega_2', s_2)\frac{1}{s}e^{(s_1 + s_2 - s)\tau}, \quad (9)$$

where n is the Fourier mode index, $s_{1,2}$ is a Laplace transform variable and $\tau = t - t'$. Using Eq. (5), in the definition of $\langle r^2 \rangle$ gives

$$\langle r^2(\tau) \rangle = 4\pi^2 \int_{-\infty}^{\infty} d\omega d\omega' C_{-1}(\omega, \omega', \tau) + \frac{1}{N} \tag{10}$$

because $\langle Z \rangle = 0$.

We can obtain a general expression for $\langle r^2 \rangle$ without explicitly solving for the correlation function. From the Laplace transform of Eq. (8), we can derive the relation

$$\int_{-\infty}^{\infty} \hat{P}_n(\omega, \omega', s) d\omega = \frac{1}{(s + in\omega')\Lambda_n(s)}, \tag{11}$$

where

$$\Lambda_n(s) \equiv 1 + inKf_n^* \int_{-\infty}^{\infty} \frac{g(\omega)d\omega}{s + in\omega} \tag{12}$$

is the analog of the dielectric response function. Using Eqs. (9) and (11) in Eq. (10) yields

$$\langle r^2(\tau) \rangle = \frac{2}{iKN\pi} \int_{\mathcal{L}} ds \frac{\Lambda_1(s - s_0) - 1}{\Lambda_1(s - s_0)} \times \text{Res}\left[\frac{-1}{\Lambda_1(s)}\right]_{s=s_0} \frac{1}{s} e^{s\tau} + \frac{1}{N}, \tag{13}$$

where s_0 is the zero of $\Lambda_n(s)$. The strategy of the calculation leading to Eq. (13) is similar to the calculation of the Lenard-Balescu collision integral [16,17].

For the specific frequency distribution $g(\omega) = (\gamma/\pi) \times [1/(\omega^2 + \gamma^2)]$ (i.e., a Lorentz distribution), Eq. (13) evaluates to

$$\langle r^2(\tau) \rangle = \frac{1}{N} \frac{K_c}{K_c - K} - \frac{1}{N} \frac{K}{K_c - K} e^{-(K_c - K)\tau}, \tag{14}$$

where $K_c = 2\gamma$ for the Lorentz frequency distribution. $\langle r^2(0) \rangle = 1/N$ because the initial conditions for Eqs. (4) and (6) are such that $\rho_1(x, 0)$ is the equilibrium incoherent state and $C(x_1, x_2, 0) = 0$. For the uncoupled system $K = 0$, so $\langle r^2 \rangle = 1/N$ as expected. We also see that the amplitude of the fluctuations and the transient decay time become singular at the critical point $K = K_c$. At criticality, we obtain the expression $\langle r^2(\tau) \rangle = (1/N)(1 + K_c\tau)$. The closer K is to criticality, the less this calculation should be valid. Near critical behavior requires an analysis of all orders in the $1/N$ expansion. Dynamically, the implication is that as the coupling strength nears criticality, oscillators will interact more strongly and higher order correlations will become more important. The result $\langle r^2(\infty) \rangle = K_c/[N(K_c - K)]$ was first derived by Daido [15] with a completely different approach. Our method facilitates a systematic expansion in $1/N$, in addition to providing an examination of the transient behavior of $\langle r^2(\tau) \rangle$.

We can examine the transient behavior of the correlations by solving Eq. (9) for the Lorentz distribution. We first solve for the propagator in Eq. (8) by taking a Fourier series expansion in θ and Laplace transform in time, to obtain

$$\hat{P}_n(\omega_1, \omega_1', s) = \frac{1}{s + in\omega_1} \frac{\delta(\omega_1 - \omega_1')}{2\pi} - \frac{inKf_n^* g(\omega_1)}{2\pi(s + in\omega_1)(s + in\omega_1')\Lambda_n(s)}, \tag{15}$$

where s is the Laplace transform variable and $\Lambda_n(s) = 1 - (K/2)|n|/(s + |n|\gamma)$. The propagator (15) has poles along the imaginary axis corresponding to the continuous spectrum of marginally stable modes as well as those given by the discrete zeros of $\Lambda_n(s)$, which for $K < K_c = 2\gamma$ are real and negative [18].

An expression of the correlation function in Eq. (9) can be obtained by inserting Eq. (15) and performing the integrals. The only surviving modes are C_1 and $C_{-1} = C_1^*$ since $f(\theta) = \sin\theta$. The correlation function will thus have the form

$$C(\omega, \omega', \theta - \theta', \tau) = C_1 e^{i(\theta - \theta')} + C_{-1} e^{-i(\theta - \theta')}. \tag{16}$$

The correlation function contains modes which consist of all possible pairings of marginal oscillating modes with decaying modes. Although the correlation function has marginal modes that do not decay, $\langle Z(\tau) \rangle$ does not because

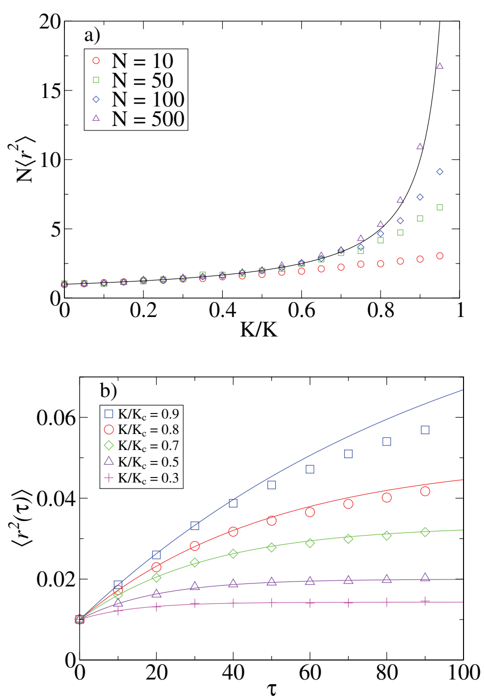

FIG. 1 (color online). (a) Simulated and predicted $N\langle r^2 \rangle$ vs K/K_c evaluated at $\tau = 6/(K_c - K)$ for various values of N and averaged over 1000 different initial conditions and frequencies. The frequency distribution is Lorentz with $\gamma = 0.05$. The simulation was performed with a time step of 0.05. The initial distribution of angles was uniform. (b) Time evolution of $\langle r^2(\tau) \rangle$ vs τ for various values of K and for $N = 100$. At each time point the values are averaged over 10 000 different initial conditions and frequencies. All other parameters as above.

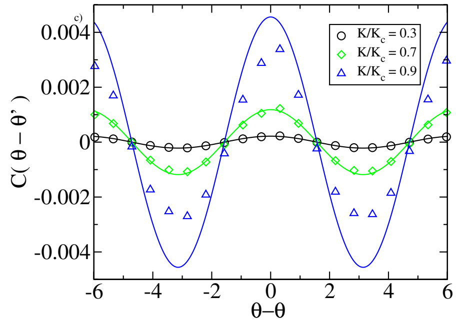

FIG. 2 (color online). $C(\omega, \omega', \theta - \theta')$ integrated over ω and ω' versus $\theta - \theta'$ for $N = 100$ and various values of K. Frequency distribution is Lorentz with $\gamma = 0.05$ and the initial angle distribution is uniform. Results are averaged over 100 000 samples in a 2D histogram with 100×100 bins. The data are then averaged over angle differences and then put into a one-dimensional histogram with bins of width 10. The time step for the evolution is 0.05.

of a Landau dampinglike dephasing effect as described in Ref. [25]. Likewise $\langle r^2(\tau) \rangle$ does not have marginal modes. We should expect a similar result for higher moments. At this order, the marginal modes are not rendered stable by finite size effects as they are with the addition of external additive noise [18]. Should stabilization occur due to the intrinsic fluctuations, it will necessarily be a consequence of higher order effects.

We compare our analytical results to numerical simulations of the Kuramoto system. Figure 1(a) shows the asymptotic value of $\langle r^2 \rangle$ for various values of K and N. The analytical prediction matches extremely well for $N = 500$ and reasonably well for $N = 50$ and $N = 100$. Only at $N = 10$ are there significant deviations from the prediction. Figure 1(b) shows the transient behavior of $\langle r^2 \rangle$. The results match quite well below $K/K_c = 0.8$. Numerical results for the correlation function integrated over ω, ω' are shown in Fig. 2. The simulation agrees well with the prediction Eq. (16) except near the critical point as expected.

Our calculation is the first presentation of a systematic approach to understanding the fluctuations due to finite size effects to an arbitrary order in $1/N$. Although we truncate at lowest order, our approach allows a truncation at any level of the moment hierarchy to produce an expansion in $1/N$. We note that Ref. [26] found that when the oscillators are driven with Gaussian noise, $1/N$ dependence is still seen in the fluctuations of the order parameter.

Some previous work [3,20–22,26,27] for both phase and pulse coupled oscillators also start with a continuity equation similar to Eq. (3) but either go directly to mean field theory, with and without an external noise source to approximate fluctuations, or assume the fluctuations are Gaussian. References [23,24] derive a kinetic theory for a network of integrate-and-fire neurons by constructing a moment hierarchy similar to ours that is closed using the maximum entropy principle. However, this work differs from ours in that the hierarchy is built from a Boltzmann-like equation for a one-particle distribution function with stochastic inputs and hence does not capture the same correlation effects that we find by starting from a continuity equation that contains the full statistics of the system.

We feel it is important to stress that the Klimontovich continuity equation [Eq. (3)] is not an approximation. The approximation appears in the method of finding solutions. The mean field limit is equivalent to setting correlations to zero. Computing the moment hierarchy allows for an expansion which accounts for the effects of correlations. We produce a systematic method for deriving such an expansion and show explicitly in what regime higher order correlations can be ignored.

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