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KINK-ANTIKINK INTERACTION FOR SEMILINEAR WAVE EQUATIONS WITH A SMALL PARAMETER

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ABSTRACT. We consider a class of semi-linear wave equations with a small parameter and nonlinearities which provide the equations having exact kink-type solutions. As a main result we to obtain sufficient conditions for the nonlinearities under which the kink-antikink collision occurs without changing the waves shape and with only some shifts of the solitary wave trajectories.

1. INTRODUCTION

We consider a class of semi-linear wave equations with a parameter ε

$$\varepsilon^2(u_{tt} - u_{xx}) + F'(u) = 0. \tag{1.1}$$

The nonlinearities $F(u)$ are assumed to be such that (1.1) have self-similar exact solutions of the so-called "kink" ("fluxon") type,

$$\begin{gathered} u(x,t,\varepsilon) = \omega\Big(S\beta\frac{x - Vt}{\varepsilon}\Big), \quad S = \pm 1, \quad \beta = (1 - V^2)^{-1/2}, \quad |V| < 1, \\ \omega(\eta) \in C^\infty(\mathbb{R}), \quad \omega(\eta) \to 0 \quad \text{as } \eta \to -\infty, \quad \omega(\eta) \to 1 \quad \text{as } \eta \to +\infty. \end{gathered} \tag{1.2}$$

More in detail, the function (1.2) is called "kink" if $S = 1$ and "antikink" if $S = -1$. It is easy to establish that under the following conditions such solutions exist and tend to the limit values sufficiently above.

- (A) $F(z) \in C^\infty(\mathbb{R})$, $F(z) > 0$ for $z \in (0, 1)$,
- (B) $F^{(i)}(z_0) = 0$, $i = 0, 1, \dots, k$, $F^{(k+1)}(z_0) > 0$, where $z_0 = 0$ and $z_0 = 1$, and $k = 1$ or $k = 3$.

Under the additional assumption

- (C) $F(1/2 + z) = F(1/2 - z)$,

the function $\omega(\eta) - 1/2$ will be odd and $\omega(\eta) + \omega(-\eta) = 1$.

To consider the superposition of the waves $\omega(\pm\beta_i(x - V_i t - x_i^0)/\varepsilon)$ with large distance between their fronts $x = V_i t + x_i^0$ as two non-interacting kinks or antikinks (for $t \ll 1$) we set the condition of periodicity

- (D) $F(z + 1) = F(z)$.

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As examples of such nonlinearities we consider the following functions:

$$F(z) = \big(1 - \cos(2\pi z)\big)/4\pi^2, \quad \omega = \frac{2}{\pi} \arctan e^{\eta}, \tag{1.3}$$

$$F(z) = \sin^4(\pi z), \quad \omega = \frac{1}{\pi} \operatorname{arccot}(-\sqrt{2}\pi\eta). \tag{1.4}$$

The first example corresponds to the sine-Gordon equation. It is well known that the kinks of the sine-Gordon equation collide without changing their form and the unique result of the interaction is a phase shift appearance (see [17, 18]) or any book about the Inverse Scattering Transform Method).

A natural question appears here: is the sine-Gordon equation the unique one from the class (1.1) for which such scenarios hold? If no, how to find conditions on F under which the kinks and anikinks of (1.1) will collide following the sine-Gordon scenario?

Obviously, traditional functional methods cannot describe the wave collision scenario. We cannot either use exact solutions of Cauchy problems, since the sine-Gordon equation is a unique representative from class (1.1) which can be integrated exactly [19]. Therefore, we will construct an asymptotic solution treating ε as a small parameter. Nevertheless, traditional asymptotic approaches cannot answer the question considered here, since there are problems nowadays unsolved (about the existence of special solutions to nonlinear PDEs and the solvability of the corresponding linearized non homogeneous equations [13, 15, 16].

To avoid these difficulties, we use the weak asymptotic method which is in progress now [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 14]. The main advantage of this approach is the possibility of reducing the problem of describing nonlinear waves interaction to a qualitative analysis of some ordinary differential equations (instead of partial differential equations). This method takes into account the fact that kinks (as well as solitons [5, 6]) which are smooth for $\varepsilon > 0$ become non-smooth in the limit as $\varepsilon \to 0$. So we will treat solutions of (1.1) as a mapping $C^\infty\left(0, T; C^\infty\left(\mathbb{R}^1_x\right)\right)$ for $\varepsilon > 0$ and only as $C\left(0, T; \mathcal{D}'\left(\mathbb{R}^1_x\right)\right)$ uniformly in $\varepsilon \geq 0$.

Next, to construct a definition of an asymptotic solution that admits the passage to the limit as $\varepsilon \to 0$, it is natural to use the standard $\mathcal{D}'$- construction. Recall that $u \in \mathcal{D}'\left((0, T) \times \mathbb{R}^1_x\right)$ is a solution of (1.1) in the weak sense if $F'(u) \in \mathcal{D}'\left((0, T) \times \mathbb{R}^1_x\right)$ and the relation

$$\int_0^T \int_{-\infty}^{\infty} \left\{\varepsilon^2 u(\psi_{tt} - \psi_{xx}) + \psi F'(u)\right\} dx\, dt = 0 \tag{1.5}$$

holds for any test function $\psi(x, t) \in \mathcal{D}\left((0, T) \times \mathbb{R}^1_x\right)$. However, we cannot change the zero in the right-hand side of (1.5) to $O(\varepsilon^2)$ since we will loose all the information about the motion of kink fronts. Indeed, with this accuracy the sine-Gordon type equation (1.5) becomes algebraic. Moreover, Assumptions (C) and (D) imply that for any smooth function $\phi = \phi(t)$ the following equalities hold:

$$\begin{aligned}
\int_{-\infty}^{\infty} \psi(x) F'\Big(\omega\big(\beta\frac{x-\phi}{\varepsilon}\big)\Big) dx &= \int_{-\infty}^{\infty} F'\Big(\omega\big(\beta\frac{x-\phi}{\varepsilon}\big)\Big) \frac{d}{dx} \int_{-\infty}^{x} \psi(x') dx' dx \\
&= -\int_{-\infty}^{\infty} \psi_1\Big(\phi + \eta\frac{\varepsilon}{\beta}\Big) \frac{d}{d\eta} F'(\omega(\eta)) d\eta \\
&= -\psi(\phi)\frac{\varepsilon}{\beta} \int_{-\infty}^{\infty} \eta \frac{d}{d\eta} F'(\omega(\eta)) d\eta + O(\varepsilon^2) = O(\varepsilon^2),
\end{aligned}$$

where $\psi_1(x) = \int_{-\infty}^{x} \psi(x')dx'$, $\psi(x) \in \mathcal{D}(\mathbb{R}^1_x)$.

Moreover, it is impossible to improve this situation by increasing the accuracy to $O(\varepsilon^N)$, $N = \text{const} > 2$. This becomes clear just in the simplest single wave situation when we consider a perturbed sine-Gordon equation (with $a(x)\sin u$ instead of $F'(u) = \sin u$). Indeed, let us try to construct the one-phase asymptotic solution using the weak definition in (1.5). A simple algebraic transformation shows that in this way we obtain an infinitely long chain of connected relations (for parameters of the solution). Moreover, these relations involve smaller-in-ε terms of the asymptotics whereas motion of the actual front is described by an ordinary differential equation (the so-called Hugoniot type condition) which does not depend on smaller corrections [15, 16].

Apparently, this effect was first pointed out in [9], where the passage to the limit (from the phase field system to the Stefan-Gibbs-Thompson problem) was studied. A way to improve the situation has been found as well in [9]. The main idea is to use special test functions, namely, the ones that are rapidly varying there where the solution varies rapidly. For (1.1) this means the choice of the test functions $u'_x\psi(x)$, $\psi(x) \in \mathcal{D}(\mathbb{R}^1)$. Standard transformations of the weak definition for such test functions lead to the following result (see also [5]):

Definition 1.1. A sequence $u(t, x, \varepsilon)$, belonging to $C^\infty\left(0, T; C^\infty\left(\mathbb{R}^1_x\right)\right)$ for $\varepsilon > 0$ and belonging to $C\left(0, T; \mathcal{D}'\left(\mathbb{R}^1_x\right)\right)$ uniformly in ε, is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1.1) if the relation

$$2\frac{d}{dt}\int_{-\infty}^{\infty} \varepsilon^2 u_t u_x \psi dx + \int_{-\infty}^{\infty} \left\{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\right\}\psi_x dx = O(\varepsilon^2) \qquad (1.6)$$

holds for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

Here the right-hand side is a C^∞-function for $\varepsilon > 0$ and a piecewise continuous function uniformly in $\varepsilon \geq 0$. The estimate is understood in the $C(0, T)$ sense:

$$g(t, \varepsilon) = O(\varepsilon^k) \leftrightarrow \max_{t\in[0,T]} |g(t, \varepsilon)| \leq C\varepsilon^k.$$

Note that, in contrast to the equality (1.5), Definition 1.1 involves in the leading term the derivatives of u with arguments x/ε and t/ε. Moreover, it requires the relation $\beta^2 = 1/(1 - V^2)$ for the solitary wave solution (1.2). The left-hand side of (1.6) is the result of multiplication of (1.1) by $\psi(x)u_x$ and integration by parts in case of smooth u. Therefore, it is zero for any exact solution. On the other hand, the relation (1.6) is just the orthogonality condition which appears for single-phase asymptotic [15, 16]. This condition both guarantees the first correction existence and allows to find an equation for the distorted kink's front motion.

In what follows we will use the notation $O_{\mathcal{D}'}(\varepsilon^k)$ for the smallness in the weak sense:

Definition 1.2. We denote by $v(t, x, \varepsilon) = O_{\mathcal{D}'}(\varepsilon^k)$ all sequences $v(t, x, \varepsilon)$ which belong to $C^\infty\left(0, T; C^\infty(\mathbb{R}^1_x)\right)$ for $\varepsilon > 0$ and belong to $C\left(0, T; \mathcal{D}'\left(\mathbb{R}^1_x\right)\right)$ uniformly in ε such that the relation

$$\int_{-\infty}^{\infty} v(t, x, \varepsilon)\psi(x)dx = O(\varepsilon^k)$$

holds for any test function $\psi \in \mathcal{D}\left(\mathbb{R}^1_x\right)$. Here the estimate for the right-hand side is treated in the same way as in Definition 1.1.

The case of two kinks interaction has been studied in [14]. More in detail, (1.1) has been complemented by the initial conditions

$$u|_{t=0} = \sum_{i=1}^{2} \omega\Big(\beta_i \frac{x-x_i^0}{\varepsilon}\Big), \quad \varepsilon \frac{\partial u}{\partial t}\Big|_{t=0} = -\sum_{i=1}^{2} \beta_i V_i \omega'\Big(\beta_i \frac{x-x_i^0}{\varepsilon}\Big), \tag{1.7}$$

were $\beta_i = 1/\sqrt{1-V_i^2}$, $|V_i| \in (0,1)$, and the initial front positions x_i^0 are such that $x_1^0 - x_2^0 > 1$. Obviously it is assumed that the trajectories $x = V_i t + x_i^0$ have a joint point $x = x^*$ at a time instant $t = t^*$.

In [14] has been proved the following statement.

Theorem 1.3. *Let assumptions* (A)-(D) *hold. Moreover, let the function F and the numbers β_i, $i = 1, 2$, be such that*

$$\begin{aligned} &\int_{-\infty}^{\infty} F\left(\omega(\eta) + \omega(\theta\eta)\right) d\eta \\ &\leq \int_{-\infty}^{\infty} \big\{F(\omega(\eta)) + F(\omega(\theta\eta)) + 2\beta_2^2 \sqrt{F(\omega(\eta))F(\omega(\theta\eta))}\big\} d\eta, \quad \theta = \beta_1/\beta_2. \end{aligned} \tag{1.8}$$

Then the interaction of kinks in (1.1), (1.7) *preserves the sine-Gordon scenario with accuracy* $\mod O_{\mathcal{D}'}(\varepsilon^2)$ *in the sense of Definition 1.1.*

In fact, the asymptotic solution construction implies the appearance of a more delicate than (1.8) condition. However, this condition is very complicated therefore it has been changed to the form (1.8). This rough version of the main assumption can be treated as admissible since (1.8) is fulfilled in the case of the sine-Gordon equation for any value of the parameter θ.

The aim of the present paper is the consideration of the kink - antikink collision,

$$u|_{t=0} = \sum_{i=1}^{2} \omega\Big(S_i\beta_i \frac{x-x_i^0}{\varepsilon}\Big), \quad \varepsilon \frac{\partial u}{\partial t}\Big|_{t=0} = -\sum_{i=1}^{2} S_i\beta_i V_i \omega'\Big(S_i\beta_i \frac{x-x_i^0}{\varepsilon}\Big), \tag{1.9}$$

where $S_1 = 1$, $S_2 = -1$, $x_1^0 - x_2^0 > 1$, and we use the same notation as in (1.7).

The main result consists of obtaining sufficient conditions for the nonlinearities under which the kink–antikink interaction occurs without changing the waves shape. It is very important that the main condition, which appears instead of (1.8), is verified numerically for any specific nonlinearity and for any initial velocities V_i of the noninteracting waves. However, it has an extremely complicated form and we can not simplify it in any reasonable way. For example, one of the possible simplifications is of the form

$$\int_{-\infty}^{\infty} \big\{\Big(\eta F'(\omega(\eta))\Big)^2 - \frac{3}{4} F(\omega(\eta))\big\} d\eta > 0. \tag{1.10}$$

This inequality guaranties the preservation of the sine-Gordon scenario, although only for the case $|\theta - 1| \ll 1$. For this reason we will present the main theorem and the main assumption at the end of the paper.

Let us note also that the conditions ((1.8), (1.10) or others) which appear in the asymptotics constructions are necessary ones only. Therefore, their breakdown does not lead, generally speaking, to the breakdown of the sine-Gordon interaction scenario. Actually, we indicate in the present paper how to improve the situation and how to obtain some less burdensome conditions. Namely, in such situation it should be fruitful to add some special corrections to the asymptotics leading term and check again the safety of the sine-Gordon scenario. At the same time it is clear that these additional terms make our calculations much more complicated. So in what follows we restrict ourselves to the simplest version of the asymptotic anzatz.

Let us describe the content of the present paper. In Section 2 we present the leading term of the weak asymptotic solution for (1.1), (1.9). After some technical calculations we obtain a system of ordinary differential equations for the parameters of the asymptotics. Next, we pass to a 2×2 dynamical system for auxiliary functions W and σ. It turned out that the system has a singularity at the line $\sigma = 0$. Therefore, the existence of the required asymptotic solution is equivalent to the existence of a trajectory γ_s such that: 1. it passes from one half-plane to another one, and 2. its W-coordinate tends to zero as $\sigma \to \pm\infty$. The investigation of the dynamical system (Section 3) shows that the first property can be realized only under some specific assumption. In fact, this is the required condition additional to (A)-(D). At the same time the second property fails. On the other hand, the leading term of the asymptotic solution does not have any additional free parameter to change γ_s. For this reason, in Section 4 we supplement the leading term of the asymptotic solution by some small corrections which are localized near the origin $W = 0, \sigma = 0$. The aim of this transformation is to rotate the trajectories preserving the main properties of the phase portrait and obtaining the trajectory γ_s with the required properties. The proof of the existence of such corrections completes the construction of the asymptotic solution. The main statement (Theorem 4.2) is formulated in the end of Section 4. There we present also Theorem 4.3 which shows that the required conditions for the nonlinearity can be obtained directly from two balance laws of (1.1). The realizability of the assumptions of Theorem 4.2 is considered in Conclusion. Appendix contains some technical proofs.

2. Construction of the asymptotic solution

Let us write the leading term of the weak asymptotic $\mod O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1.1), (1.9) as the sum of distorted solitary waves:

$$u = \sum_{i=1}^{2} \omega\Big(S_i\beta_i \frac{x - \Phi_i(t,\tau,\varepsilon)}{\varepsilon}\Big). \tag{2.1}$$

Here $\Phi_i = \phi_{i0}(t) + \varepsilon\phi_{i1}(\tau)$, $\phi_{i0} = V_i t + x_i^0$, $\tau = \frac{\psi_0(t)}{\varepsilon}$, $\psi_0(t) = \phi_{20}(t) - \phi_{10}(t)$, $\phi_{i1} = \phi_{i1}(\tau)$ are smooth functions such that

$$\phi_{i1} \to 0 \quad \text{as } \tau \to -\infty, \quad \phi_{i1} \to \phi_{i1}^{\infty} = \text{const}_i \quad \text{as } \tau \to +\infty \tag{2.2}$$

with a rate not less than $1/|\tau|$, and where the constants β_i, V_i, and S_i are the same as in (1.9).

Let us consider briefly this anzatz. Since $x_1^0 > x_2^0$, the difference $\phi_{20}(t) - \phi_{10}(t)$ remains negative during some $t < t^*$. Thus, for this time interval, the "fast time" $\tau = (\phi_{20}(t) - \phi_{10}(t))/\varepsilon \to -\infty$ as $\varepsilon \to 0$. Therefore, the first assumption in (2.2) implies that the function (2.1) describes the motion of two noninteracting solitary waves. After the interaction $\phi_{20}(t) - \phi_{10}(t) > 0$ and we obtain the situation with $\tau \to +\infty$ as $\varepsilon \to 0$. In the case when the second assumption (2.2) is fulfilled, the kink–antikink collision occurs without changing the shape of the waves. By contrast, if the functions ϕ_{i1} are unbounded as $\tau \to +\infty$, the phase shift will change in time. Finally, non-existence of ϕ_{i1} for some $\tau \geq 0$ means that the travelling waves loose kink form during interaction. For simplicity, we assume that the properties (2.2) hold and will justify this assumptions in the proof.

Let us pass to the construction of the weak asymptotic solution. Obviously, to this aim we should calculate weak asymptotics with discrepancy $O_{\mathcal{D}'}(\varepsilon^2)$ for the expressions $(\varepsilon u_x)^2$, $(\varepsilon u_t)^2$ and others, which appear in the left-hand side of the relation (1.6). We clarify the techniques using as an example the expression $(\varepsilon u_x)^2$. Differentiating the anzatz we obtain:

$$(\varepsilon u_x)^2 = \sum_{i=1}^{2} \beta_i^2 \omega_0^2\Big(S_i\beta_i \frac{x-\Phi_i}{\varepsilon}\Big) - 2\beta_1\beta_2\omega_0\Big(S_1\beta_1\frac{x-\Phi_1}{\varepsilon}\Big)\omega_0\Big(S_2\beta_2\frac{x-\Phi_2}{\varepsilon}\Big).$$

Here and in what follows

$$\omega_0(\eta) := \frac{d\omega(\eta)}{d\eta}$$

is an even function in view of the assumption (C).

Obvious changes of variables in integrals imply the equality

$$\begin{aligned}\int_{-\infty}^{\infty} (\varepsilon u_x)^2\psi(x)dx &= \varepsilon\sum_{i=1}^{2}\beta_i^2\int_{-\infty}^{\infty}\omega_0^2(\beta_i\xi)\psi(\Phi_i+\varepsilon\xi)d\xi\\ &\quad - 2\varepsilon\beta_1\beta_2\int_{-\infty}^{\infty}\omega_0(\beta_2\xi)\omega_0(\beta_1\xi+\beta_1\sigma)\psi(\Phi_2+\varepsilon\xi)d\xi,\end{aligned}$$

where $\psi(x)\in\mathcal{D}(\mathbb{R}^1)$ is an arbitrary function and

$$\sigma := \frac{\Phi_2-\Phi_1}{\varepsilon} = \tau + \phi_{21}(\tau) - \phi_{11}(\tau). \tag{2.3}$$

Next, note that Assumption (B) implies the following estimate, uniform in $\eta\in\mathbb{R}^1$,

$$\eta^2\omega_0(\eta) \le \text{const.} \tag{2.4}$$

Therefore, applying the Taylor formula

$$\psi(\Phi_i+\varepsilon\xi) = \psi(\Phi_i) + \varepsilon\xi\psi'(\xi^*),$$

where ξ^* is an intermediate point, we obtain the relation

$$\begin{aligned}\int_{-\infty}^{\infty} (\varepsilon u_x)^2\psi(x)dx &= \varepsilon\sum_{i=1}^{2}\beta_i^2\psi(\Phi_i)\int_{-\infty}^{\infty}\omega_0^2(\beta_i\xi)d\xi\\ &\quad - 2\varepsilon\beta_1\beta_2\psi(\Phi_2)\int_{-\infty}^{\infty}\omega_0(\beta_2\xi)\omega_0(\beta_1\xi+\beta_1\sigma)d\xi \;+\; O(\varepsilon^2),\end{aligned}$$

which is correct for any test function. Let us introduce the notation

$$\lambda_1(\sigma) = \frac{1}{a_2}\int_{-\infty}^{\infty}\omega_0(\eta)\omega_0(\theta\eta+\beta_1\sigma)d\eta, \quad a_2 = \int_{-\infty}^{\infty}\omega_0^2(\eta)d\eta, \tag{2.5}$$

where $\theta = \beta_1/\beta_2$. Then we can rewrite the last formula in the following form

$$\int_{-\infty}^{\infty} (\varepsilon u_x)^2\psi(x)dx = \varepsilon a_2\big\{\sum_{i=1}^{2}\beta_i\psi(\Phi_i) - 2\beta_1\psi(\Phi_2)\lambda_1(\sigma)\big\} + O(\varepsilon^2). \tag{2.6}$$

It remains to note that estimate (2.4) implies the same rate of the convolution λ_1 vanishing. Therefore, we can use the Taylor formula again. Definitions of Φ_i and σ result in the formulas

$$\Phi_i = x^* + V_i(t-t^*) + \varepsilon\phi_{i1}, \quad t - t^* = \frac{\varepsilon}{\nu}\left(\sigma + \phi_{11} - \phi_{21}\right), \quad \nu := V_2 - V_1.$$

Thus, $\Phi_i = x^* + \varepsilon\chi_i$, $i = 1, 2$, where

$$\chi_i = b_i\sigma + b_2\phi_{11} - b_1\phi_{21}, \quad b_i = V_i/\nu. \tag{2.7}$$

Since $|\sigma\lambda_1(\sigma)| \le \text{const}$, we can transform the relation (2.6) into the final form

$$\int_{-\infty}^{\infty} (\varepsilon u_x)^2\psi(x)dx = \varepsilon a_2\big\{\sum_{i=1}^{2}\beta_i\psi(\Phi_i) - 2\beta_1\lambda_1\psi(x^*)\big\} + O(\varepsilon^2).$$

Obviously, this means

$$(\varepsilon u_x)^2 = \varepsilon a_2\big\{\sum_{i=1}^{2}\beta_i\,\delta(x-\Phi_i) - 2\beta_1\lambda_1\,\delta(x-x^*)\big\} + O_{\mathcal{D}'}(\varepsilon^2), \tag{2.8}$$

where $\delta(x)$ is the Dirac δ-function.

Calculating weak asymptotic for other terms in the left-hand side of (1.6), we obtain some more convolutions. Let us introduce the notation

$$\lambda_2(\sigma) = \frac{1}{a_2}\int_{-\infty}^{\infty}\eta\omega_0(\eta)\omega_0(\theta\eta+\beta_1\sigma)d\eta, \quad \bar\lambda_2(\sigma) = \frac{1}{a_2}\int_{-\infty}^{\infty}\eta\omega_0(\eta)\omega_0\Big(\frac{\eta-\beta_1\sigma}{\theta}\Big)d\eta, \tag{2.9}$$

$$B_\Delta(\sigma) = \frac{2}{a_2}\int_{-\infty}^{\infty}\Big\{F\big(\omega(\eta)-\omega(\theta\eta+\beta_1\sigma)\big) - F\big(\omega(\eta)\big) - F\big(\omega(\theta\eta+\beta_1\sigma)\big)\Big\}d\eta. \tag{2.10}$$

Lemma 2.1 (Properties of convolutions)**.** *Under Assumptions* (A)-(D) *the convolutions* (2.5)*,* (2.9)*, and* (2.10) *exist and have the following properties as $\sigma \to \pm\infty$:*

$$|\sigma^2\lambda_1(\sigma)| \le \text{const}, \quad |\sigma\lambda_2(\sigma)| \le \text{const}, \quad |\sigma^2 B_\Delta(\sigma)| \le \text{const}. \tag{2.11}$$

Moreover, $B_\Delta < 0$ for sufficiently large $|\sigma|$ and

$$\bar\lambda_2(\sigma) = \theta\big(\beta_1\sigma\lambda_1(\sigma) + \theta\lambda_2(\sigma)\big).$$

A sketch of the proof of the above lemma can be found in the Appendix. Now we can complete the calculations of the weak asymptotic.

Lemma 2.2. *Let Assumptions* (A)-(D) *and* (2.2) *be satisfied. Then the following relations hold:*

$$(\varepsilon u_t)^2 = \varepsilon a_2\nu^2\Big\{\sum_{i=1}^{2}\beta_i b_i^2\delta(x-\phi_i) + K_1\delta(x-x^*)\Big\} + O_{\mathcal{D}'}(\varepsilon^2), \tag{2.12}$$

$$F(u) = \varepsilon\frac{a_2}{2}\Big\{\sum_{i=1}^{2}\frac{1}{\beta_i}\delta(x-\Phi_i) + \frac{B_\Delta}{\beta_2}\delta(x-x^*)\Big\} + O_{\mathcal{D}'}(\varepsilon^2), \tag{2.13}$$

$$\frac{\partial}{\partial t}\big(\varepsilon^2 u_t u_x\big) = -a_2\nu^2 \frac{dK_2}{d\tau}\delta(x-x^*) + \varepsilon a_2\nu^2 \frac{d}{d\tau}\Big(\sum_{i=1}^{2}\beta_i b_i \phi_{i1} + W\Big)\delta'(x-x^*) + \varepsilon a_2\nu^2 \sum_{i=1}^{2}\beta_i b_i^2 \delta'(x-\Phi_i) + O_{\mathcal{D}'}(\varepsilon^2). \tag{2.14}$$

Here and in what follows

$$K_1 = \sum_{i=1}^{2}\beta_i \phi'_{i1}(2b_i + \phi'_{i1}) - 2\beta_1\lambda_1 \Phi_{1_t}\Phi_{2_t},$$

$$K_2 = \sum_{i=1}^{2}\{\beta_i\phi'_{i1} - \beta_1\lambda_1\Phi_{i_t}\}, \quad W = K_2\chi_2 + \beta_1\phi'_{11}(\chi_1-\chi_2) - \theta\lambda_2\sum_{i=1}^{2}\Phi_{i_t}, \tag{2.15}$$

where

$$\Phi_{i_t} := b_i + \phi'_{i1}, \quad \phi'_{i1} := \frac{d\phi_{i1}}{d\tau}. \tag{2.16}$$

Relations (2.12) - (2.14) are obtained in the same way as (2.8) but after more complicated calculations. Note only that assumptions (B) and (2.2) are the weakest under which we can guarantee both the convolutions existence with the properties (2.11) and the boundedness of the remainders.

Furthermore, Definition 1.1 implies the necessity of the relation

$$2\frac{\partial}{\partial t}\varepsilon^2 u_t u_x - \frac{\partial}{\partial x}\big\{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\big\} = O_{\mathcal{D}'}(\varepsilon^2).$$

Using formulas (2.8) and (2.12)-(2.16) we obtain that the left-hand side of the last relation is a linear combination of the functions $\delta(x-x^*)$, $\delta'(x-x^*)$, and $\delta'(x-\Phi_i)$. Since these functions are linearly independent, the coefficients of δ and δ'-functions have to be zero. Thus, we obtain the following system of equations:

$$\beta_i\Big(b_i^2 - \frac{1}{\nu^2}\Big) + \frac{1}{\beta_i\nu^2} = 0, \quad i=1,2, \tag{2.17}$$

$$\frac{dK_2}{d\tau} = 0, \tag{2.18}$$

$$2\frac{dW}{d\tau} + 2\sum_{i=1}^{2}\beta_i b_i\phi'_{i1} = K_1 - \frac{1}{\nu^2}\Big(2\beta_1\lambda_1 + \frac{B_\Delta}{\beta_2}\Big). \tag{2.19}$$

Equality (2.17) imply the same relations $\beta_i^2 = 1/(1-V_i^2)$ as for the solitary waves. Next, taking into account the condition $\phi'_{i1} \to 0$ as $\tau \to \pm\infty$ and integrating (2.18), we obtain

$$\sum_{i=1}^{2}\big\{\beta_i\phi'_{i1} - \beta_1\lambda_1\Phi_{i_t}\big\} = 0. \tag{2.20}$$

This and (2.3), for the function $\sigma = \sigma(\tau)$, allow us to rewrite ϕ'_{i1} in terms of $\sigma'_\tau := d\sigma/d\tau$. Indeed, from (2.3) we have

$$\frac{d\sigma}{d\tau} = 1 + \phi'_{21} - \phi'_{11}. \tag{2.21}$$

Solving the system (2.20), (2.21) with respect to ϕ'_{11}, ϕ'_{21}, we obtain the following equations

$$\frac{d\phi_{11}}{d\tau} = -\mathcal{F}G_1, \quad \frac{d\phi_{21}}{d\tau} = \theta\mathcal{F}G_2, \tag{2.22}$$

where

$$\begin{gathered} \mathcal{F} = (1+\theta-2\theta\lambda_1)^{-1}, \quad G_1 = (1-\theta\lambda_1)\sigma'_\tau - 1 - 2\theta b_1\lambda_1, \\ G_2 = (1-\lambda_1)\sigma'_\tau - 1 + 2b_2\lambda_1. \end{gathered} \tag{2.23}$$

Note that the right-hand sides of the equations (2.22) depend on σ which is treated now as an unknown function. To complete the system we have to use the last equality in the system (2.17) - (2.19). Let us simplify it. Firstly we note that the equalities (2.20) and (2.22) allow to rewrite the expression for W defined in (2.15) in the following form:

$$W = (\beta_1\sigma + \theta\lambda_2)\mathcal{F}G_1 - \theta^2\lambda_2\mathcal{F}G_2 - \theta(b_1+b_2)\lambda_2.$$

Simple algebraic manipulations lead to

$$\frac{W}{\mathcal{F}} = L\frac{d\sigma}{d\tau} - \beta_1\sigma - 2\left(b_1\bar{\lambda}_2 + \theta b_2\lambda_2\right), \tag{2.24}$$

where

$$L = \beta_1\sigma + \theta\lambda_2 - \bar{\lambda}_2. \tag{2.25}$$

Similar transformations of the right-hand side of (2.19) and (2.24) allow us to rewrite the second order equation (2.19) as the desired system of the first order autonomous equations

$$\frac{d\sigma}{d\tau} = Q, \quad \frac{dW}{d\tau} = P, \tag{2.26}$$

where

$$Q = \frac{1}{L}\Big\{\frac{W}{\mathcal{F}} + \beta_1\sigma + 2(b_1\bar{\lambda}_2 + \theta b_2\lambda_2)\Big\}, \tag{2.27}$$

$$\begin{aligned} P = \frac{\mathcal{F}\beta_1}{2}&\Big\{(1-\theta\lambda_1^2)Q^2 - 2[1-\lambda_1(b_2-\theta b_1)]Q + 1 - 2\lambda_1(\theta b_1^2 + b_2^2)\Big\} \\ &- \frac{\beta_1}{\nu^2}\Big(\lambda_1 + \frac{B_\Delta}{2\beta_1\beta_2}\Big). \end{aligned} \tag{2.28}$$

Let us define additional conditions for the system (2.26). The first assumption in (2.2) and (2.3) imply that $\sigma \to \tau$ as $\tau \to -\infty$. Next, the equality (2.24) implies that $W \to 0$ for such τ. Therefore we obtain the "initial" condition

$$\frac{\sigma}{\tau} \to 1, \quad W \to 0 \quad \text{as } \tau \to -\infty. \tag{2.29}$$

The second assumption in (2.2) and (2.3), (2.24) imply that σ and W need to have the same limiting values as $\tau \to \infty$. Thus, the fulfillment of our assumptions is equivalent to the existence of a trajectory $\gamma_s = \big(\sigma = \sigma_s(\tau), W = W_s(\tau)\big)$ which satisfies the condition (2.29) and

$$\frac{\sigma_s}{\tau} \to 1, \quad W_s \to 0 \quad \text{as } \tau \to \infty.$$

3. Analysis of the basic dynamical system

Passing to the variables $\tau' = \beta_1\tau$, $\sigma' = \beta_1\sigma$ and omitting primes, we rewrite the system (2.26) in the same form but with the simpler right-hand sides:

$$Q = \frac{1}{L}\Big\{\frac{W}{\mathcal{F}} + \sigma + 2(b_1\bar\lambda_2 + \theta b_2\lambda_2)\Big\}, \tag{3.1}$$

$$\begin{aligned} P = \frac{\mathcal{F}}{2}\Big\{(1-\theta\lambda_1^2)Q^2 - 2[1-\lambda_1(b_2-\theta b_1)]Q + 1 - 2\lambda_1(\theta b_1^2 + b_2^2)\Big\} \\ - \frac{1}{\nu^2}\Big(\lambda_1 + \frac{B_\Delta}{2\beta_1\beta_2}\Big). \end{aligned} \tag{3.2}$$

Here L, $\mathcal{F}$, and the convolutions have the form (2.23), (2.25), (2.5), (2.9), (2.10) but with arguments as if $\beta_1 = 1$. For example,

$$L = \sigma - \bar\lambda_2 + \theta\lambda_2, \quad \lambda_1 = \frac{1}{a_2}\int_{-\infty}^{\infty}\omega_0(\eta)\omega_0(\theta\eta+\sigma)d\eta.$$

Let us perform a more detailed analysis with description of the convolutions' properties (for a sketch of the proof see Appendix).

Lemma 3.1. *Under the assumption of Lemma 2.1 the following relations hold:*

$$\begin{gathered} \lambda_1(-\sigma) = \lambda_1(\sigma) > 0, \quad \lambda_1(\sigma) = \lambda_1^0 + O(\sigma^2) \quad \text{as } \sigma\to 0, \\ \operatorname{sign}\sigma\lambda_2(\sigma) = -1, \quad \operatorname{sign}\sigma\bar\lambda_2(\sigma) = 1, \\ \lambda_2(-\sigma) = -\lambda_2(\sigma), \quad \bar\lambda_2(-\sigma) = -\bar\lambda_2(\sigma), \\ \lambda_2(\sigma) = \sigma\lambda_2^1 + O(\sigma^3), \quad \bar\lambda_2(\sigma) = \sigma\bar\lambda_2^1 + O(\sigma^3) \quad \text{as } \sigma\to 0, \\ B_\Delta(-\sigma) = B_\Delta(\sigma), \quad B_\Delta(\sigma) = B_\Delta^0 + O(\sigma^2) \quad \text{as } \sigma\to 0, \end{gathered}$$

where $\lambda_1^0 := \lambda_1(0)$, $B_\Delta^0 := B_\Delta(0)$,

$$\begin{gathered} \lambda_2^1 := \frac{1}{a_2}\int_{-\infty}^{\infty}\eta\omega_0(\eta)\frac{d\omega_0(z)}{dz}\Big|_{z=\theta\eta}d\eta < 0, \\ \bar\lambda_2^1 := -\frac{1}{a_2\theta}\int_{-\infty}^{\infty}\eta\omega_0(\eta)\frac{d\omega_0(z)}{dz}\Big|_{z=\eta/\theta}d\eta > 0. \end{gathered}$$

These properties and the formulas (2.23), (2.25), (3.1), (3.2) imply the following statements:

Corollary 3.2. *System* (2.26) *is invariant with respect to change of variables* $\tau\to -\tau$, $\sigma\to-\sigma$, $W\to -W$.

Corollary 3.3. *Under the assumptions of Lemma 2.1*

$$L|_{\sigma=0} = 0 \tag{3.3}$$

and the following estimates hold, uniformly in σ,

$$(1+\theta)^{-1} \le \mathcal{F} \le \big(1+\theta-2\theta\lambda_1^0\big)^{-1} := \mathcal{F}_0 \le (1-\sqrt{\theta})^{-2}. \tag{3.4}$$

Equality (3.3) shows that the system (2.26), (3.1), (3.2) has a singularity at the line $(0, W)$ on (σ, W)-plane. We will assume that this singularity is of the type $1/\sigma$ and that there are not other points of singularity. Therefore we set the additional assumption

(E1) Let the function F and the number $\theta = \beta_1/\beta_2$ be such that

$$L(\sigma) > 0 \quad \text{for} \quad \sigma > 0, \quad L_1 := \frac{dL}{d\sigma}\Big|_{\sigma=0} > 0.$$

Let us note that the last condition implies in particular the inequality

$$\theta \neq 1 \tag{3.5}$$

since

$$L_1|_{\theta=1} = \{1 - \bar{\lambda}_2^1 + \theta\lambda_2^1\}|_{\theta=1} = 0.$$

Conversely, under the condition (3.5) the estimates (3.4) guarantee that $\mathcal{F} \leq \text{const}$ uniformly in σ. Thus the singularity of Q is of the type $1/\sigma$. In fact, for $\theta = 1$ the singularities of Q and P are of the type $1/\sigma$ again. However, this case needs to be considered separately and here we assume the fulfilment of the condition (3.5).

The next remark is such that Assumption (E1) is verifiable for any specific nonlinearity F and for any number θ defined in initial data. Moreover, this assumption is realizable. We will present some examples in Conclusion.

Let us investigate when a trajectory would be able to pass from the left half-plane to the right one. Obviously, it can occur only through the point $(0, 0)$.

Let us consider a sufficiently small neighborhood of the origin of coordinates. Lemma 3.1 implies that the dynamical system has the following representation for $|\sigma| \ll 1$:

$$\begin{gathered}\frac{d\sigma}{d\tau} = \frac{1}{L_1}\Big\{\frac{1}{\mathcal{F}_0}\frac{W}{\sigma} + 1 + 2\left(b_1\bar{\lambda}_2^1 + \theta b_2\lambda_2^1\right)\Big\} + O\left(\sigma^2 + |\sigma W|\right), \\ \frac{dW}{d\tau} = \frac{\mathcal{F}_0}{2}\Big\{\big(1 - \theta{\lambda_1^0}^2\big)\big(\frac{d\sigma}{d\tau}\big)^2 - 2\Big(1 - \lambda_1^0(b_2 - \theta b_1)\Big)\frac{d\sigma}{d\tau} + R\Big\} + O(\sigma^2),\end{gathered} \tag{3.6}$$

where

$$R = 1 - 2\lambda_1^0\left(b_2^2 + \theta b_1^2\right) - \frac{2}{\nu^2\mathcal{F}_0}\Big(\lambda_1^0 + \frac{B_\Delta^0}{2\beta_1\beta_2}\Big). \tag{3.7}$$

Next, the system (3.6) can be easily transformed into

$$L_1\frac{d^2}{d\tau^2}(\sigma^2) = \Big(1 - \theta{\lambda_1^0}^2\Big)\big(\frac{d\sigma}{d\tau}\big)^2 + 2N\frac{d\sigma}{d\tau} + R + O\Big(\sigma^2\Big(\big|\frac{d\sigma}{d\tau}\big| + \big|\frac{dW}{d\tau}\big|\Big)\Big), \tag{3.8}$$

$$W = \mathcal{F}_0\,\sigma\big\{L_1\frac{d\sigma}{d\tau} - 1 - 2\left(b_1\bar{\lambda}_2^1 + \theta b_2\lambda_2^1\right)\big\} + O\left(|\sigma^3| + \sigma^2|W|\right), \tag{3.9}$$

where

$$N = (\lambda_1^0 + 2\theta\lambda_2^1)(b_2 + \theta b_1). \tag{3.10}$$

Applying the Cauchy-Kovalevskaya method and taking into account the oddness of $\sigma = \sigma(\tau)$, we write:

$$\sigma = a_1\tau + a_3\tau^3 + \dots \tag{3.11}$$

with arbitrary coefficients a_i. Then (3.8) and simple algebra imply the following equations:

$$Ma_1^2 - 2Na_1 - R = 0, \tag{3.12}$$

$$\{(M + 2nL_1)a_1 - N\}\,a_{2n+1} = f_{2n+1}(a_1, \dots, a_{2n-1}), \quad n \geq 1, \tag{3.13}$$

where

$$M = 2L_1 - (1 - \theta{\lambda_1^0}^2). \tag{3.14}$$

If $M \neq 0$, for a_1 we obtain two roots

$$a_1^{\pm} = \frac{1}{M}\left(N \pm \sqrt{N^2 + MR}\right). \tag{3.15}$$

Consequently, the function $\sigma(\tau)$ has the representation (3.11) only under the conditions

$$N^2 + MR \geq 0 \quad \text{and} \quad (M + 2nL_1^0)a_1^\pm - N \neq 0 \quad \text{for all} \quad n = 1, 2, \dots. \tag{3.16}$$

However, M can be arbitrary (including the case $M = 0$ since $\theta\lambda_1^{0^2} < 1$ for $\theta < 1$ and $\theta\lambda_1^{0^2} > 1$ for $\theta > 1$) and we have to investigate all the possible situations. Let $M > 0$. Then

$$a_1^- < 0, \quad a_1^+ > 0, \quad \text{and } Ma_1^- < N < Ma_1^+.$$

Let $R > 0$. Then both conditions (3.16) are fulfilled since

$$(M + 2nL_1^0)a_1^+ - N > 2nL_1^0 a_1^+ > 0 \quad \text{and} \quad (M + 2nL_1^0)a_1^- - N < 2nL_1^0 a_1^- < 0.$$

Next, if $R < 0$, then we need to assume the fulfillment of the first condition (3.16). Furthermore, the second condition (3.16) can be easily transformed into the following form:

$$N \neq \sqrt{N^2 + MR}\,(1 + q_n) \quad \text{for all } n = 1, 2, \dots, \tag{3.17}$$

where the numbers q_n,

$$q_n := \frac{M}{2nL_1} = \frac{1}{n}\Big(1 - \frac{1 - \theta\lambda_1^{0^2}}{2L_1}\Big),$$

are positive here. Obviously, for $M > 0$ and $R < 0$ both of the roots $a_1^\pm$ are positive. It is clear that if $R = 0$ then there exists $a_1^+ > 0$ if and only if $N > 0$.

Considering in the same way the case $M < 0$ we obtain the condition:

- (E2) Let $M \neq 0$. Moreover, let the function F and the velocities V_i be such that the one of the following assumptions holds:
  - (E2a) $M > 0$ and $R > 0$,
  - (E2b) $M > 0$, $R < 0$ with
  $$N^2 + MR \geq 0. \tag{3.18}$$
  Moreover, let $N > 0$ under the assumption (3.17),
  - (E2c) $M > 0$, $R = 0$ and $N > 0$ under the assumption (3.17),
  - (E2d) $M < 0$, $R < 0$ and the inequality (3.17) is satisfied,
  - (E2e) $M < 0$ and $R > 0$. Moreover, let $N < 0$ and the assumptions (3.17), (3.18) are satisfied,
  - (E2f) $M < 0$, $R = 0$, $N < 0$, and $1 - \theta\lambda_1^{0^2} \neq 2L_1(1 + 2n)$ for $n = 1, 2, \dots$.

Remark. Theoretically, there exists also a specific case when $M = 0$. However, we will not consider it.

Calculating the W-coordinates of the trajectories in the form similar to (3.11), that is

$$W = w_1\tau + w_3\tau^3 + \dots, \tag{3.19}$$

we pass to the following statement.

Lemma 3.4. *Let Assumptions* (A)-(E2) *be satisfied. Then there exists at least one trajectory* $\gamma_s = \{(\sigma = \sigma_s(\tau), W = W_s(\tau))\}$ *of the system* (2.26), (3.1), (3.2) *which goes from the left half-plane* (W, σ) *to the right one when* τ *grows from* $-\tau_0$ *to* τ_0 *for sufficiently small* τ_0.

Consider now thin domains where

$$|W| \sim 1, \quad |\sigma| \ll 1. \tag{3.20}$$

Relation (3.9) shows that in the case (3.20) $\sigma\sigma'_\tau = W/L_1\mathcal{F}_0 + O(\sigma) = O(1)$, whereas $\sigma'_\tau = O(1/\sigma)$. Thus, conserving in (3.8) the leading terms $O(1/\sigma^2)$ only, we obtain the following model equation:

$$\sigma\frac{d^2\sigma}{d\tau^2} + r\big(\frac{d\sigma}{d\tau}\big)^2 = 0, \tag{3.21}$$

where $r = M/2L_1$. The solution of (3.21) and the function $W = L_1\mathcal{F}_0\sigma\sigma'_\tau$ have the form

$$\sigma = \sigma_0\big\{1 + \frac{W_0}{\alpha\sigma_0^2}(\tau - \tau_0)\big\}^{1/\kappa}, \quad W = W_0\big\{1 + \frac{W_0}{\alpha\sigma_0^2}(\tau - \tau_0)\big\}^{2/\kappa - 1}, \tag{3.22}$$

where $\kappa = 1 + r$, $\alpha = \mathcal{F}_0L_1/\kappa$, $W_0 = W|_{\tau=\tau_0} = O(1)$, $|\sigma_0| = \big|\sigma|_{\tau=\tau_0}\big| \ll 1$.

For $M > 0$ we have $\kappa \in (1, 2)$. It is clear that for $W_0 > 0$ functions (3.22) exist for any $\tau \geq \tau_0$ and they grow with τ. Conversely, for $W_0 < 0$ these functions vanish.

For $M < 0$ the behavior of the solution (3.22) is more complicated. If $\kappa \in (1/2, 1)$, both of the coordinates vanish when the arguments in the braces vanish. If $\kappa \in (0, 1/2)$, the σ-coordinate vanishes and the W-coordinate grows when the arguments in the braces vanish and for $\kappa < 0$ both of the coordinates vanish. In the case $r = -1$ the behavior of the trajectories is described by the formulas

$$\sigma = \sigma_0\, e^{\frac{W_0}{\alpha_1\sigma_0^2}(\tau-\tau_0)}, \quad W = W_0\, e^{2\frac{W_0}{\alpha_1\sigma_0^2}(\tau-\tau_0)},$$

where $\alpha_1 = \mathcal{F}_0L_1 > 0$.

It should be noted also that the first assumption (3.20) is very important since the above analysis does not hold in the case $|W| \ll 1$.

The next step in the analysis is the consideration of system (2.26), (3.1), (3.2) for $\sigma \to \pm\infty$. Since the convolutions vanish as $\sigma \to \pm\infty$, we have

$$Q = 1 + (1+\theta)\frac{W}{\sigma} + O\Big(\frac{1}{\sigma^2}\big(1 + \frac{W}{\sigma}\big)\Big), \quad P = \frac{1}{2(1+\theta)}(Q-1)^2 + O\big(\frac{1}{\sigma^2}\big). \tag{3.23}$$

Thus, solving the system (2.26), (3.23) in the leading term for large $|\sigma|$, we obtain the solution

$$\sigma = \frac{1+\theta}{2}\frac{W^2}{c - W}, \quad c = W_0\Big(1 + \frac{1+\theta}{2}\frac{W_0}{\sigma_0}\Big), \tag{3.24}$$

where $W_0 = W|_{\sigma=\sigma_0}$. Therefore, $W = c + O(1/|\sigma|)$ as $|\sigma| \gg 1$. Moreover,

$$W = c - \frac{1+\theta}{2\tau} + O\left(1/|\tau|^2\right).$$

The last formulas imply the stabilization of the W-coordinate of any trajectory for $\tau \to \pm\infty$, if $|W_0| = |W(\tau_0)|$ is bounded by a constant and $|\sigma_0| = |\sigma(\tau_0)|$ is sufficiently large.

The last step of the analysis of system (2.26), (3.1), (3.2) is the consideration of the isoclines. The isocline $\gamma_Q = \{(\sigma, W), Q(\sigma, W) = 0\}$ is the curve

$$W_Q = -\mathcal{F}\left(\sigma + 2(b_1\bar{\lambda}_2 + \theta b_2\lambda_2)\right),$$

which turns into the line $W_Q^\infty = -\sigma/(1+\theta)$ for sufficiently large $|\sigma|$. We stress that the curve $W_Q = W_Q(\sigma)$ intersects the trajectory γ_s at the origin since

$$\Big\{\frac{dW_s}{d\sigma} - \frac{dW_Q}{d\sigma}\Big\}\Big|_{\sigma=0} = \mathcal{F}_0 L_1 a_1 > 0, \tag{3.25}$$

where W_s denotes γ_s as the function $W_s(\sigma)$.

To find the isocline $\gamma_P = \{(\sigma, W), P(\sigma, W) = 0\}$ we need to solve firstly the equation

$$\big(1 - \theta\lambda_1^2\big)\, Q^2 - 2D_1 Q + D_2 = 0, \tag{3.26}$$

where

$$D_1 = 1 - \lambda_1(b_2 - \theta b_1), \quad D_2 = 1 - 2\lambda_1(b_2^2 + \theta b_1^2) - \frac{2}{\nu^2 \mathcal{F}}\Big(\lambda_1 + \frac{B_\Delta}{2\beta_1\beta_2}\Big).$$

Denote by $Q_\pm(\sigma)$ the roots of the equation (3.26). Then we obtain the following expressions for the branches $W_P^\pm = W_P^\pm(\sigma)$ of the isocline γ_P

$$W_P^\pm = \mathcal{F}\Big(L\, Q_\pm(\sigma) - \sigma - 2(b_1\bar{\lambda}_2 + \theta b_2\lambda_2)\Big).$$

For sufficiently large $|\sigma|$ the equation (3.26) has the unique root $Q_+ = Q_- = 1$. Therefore γ_P turns into the line $\{(\sigma, 0)\}$ as $\sigma \to \pm\infty$. Furthermore, since $\lambda_1 \to 0$ as $|\sigma| \to \infty$, for sufficiently large $|\sigma|$ we can transform the discriminant D_P of the equation (3.26) as follows

$$D_P = \frac{1+\theta}{\nu^2}\Big\{2\lambda_1(1 + V_1V_2) + \frac{B_\Delta}{\beta_1\beta_2}\Big\} + O(\lambda_1^2).$$

An asymptotic analysis of the convolutions implies the following statement (for a sketch of the proof see Appendix).

Lemma 3.5. *Under Assumptions* (A)-(D)*, $D_P \to +0$ as $|\sigma| \to \infty$.*

Then, we obtain the following result.

Corollary 3.6. *The isocline γ_P consists on two branches $W_P^\pm$ which are defined at least for sufficiently large $|\sigma|$ and stick together as $|\sigma| \to \infty$. If $D_P \geq 0$ for all $\sigma \in \mathbb{R}$, then the branches $W_P^\pm$ pass through the point* $(0,0)$.

Let us assume now that (3.24) has real roots near the origin. Then

$$\Big\{\frac{dW_P^\pm}{d\sigma} - \frac{dW_Q}{d\sigma}\Big\}\Big|_{\sigma=0} = \mathcal{F}_0 L_1 Q_\pm^0. \tag{3.27}$$

Formulas (3.25), (3.26) show that, depending on the initial velocities V_i, there can be realized one of the following possibilities:
(i) if $Q_- < Q_+ < 0$, then the curves $\gamma_P^\pm$ and γ_Q have two intersection points for $\sigma > 0$. Therefore, the phase portrait contains five singular points,
(ii) if $Q_- < 0$, $Q_+ > 0$, then the curves γ_P^- and γ_Q have one intersection point for $\sigma > 0$. Therefore, the phase portrait contains three singular points, and
(iii) if $Q_+ \geq Q_- > 0$, then the curves $\gamma_P^\pm$ and γ_Q have not intersection points for $\sigma > 0$. Thus, the phase portrait contains the unique singular point $(0,0)$. Let us note that the second possibility is realized for $R < 0$, whereas the others are realized for $R > 0$ and $D_1|_{\sigma=0} > 0$ or $D_1|_{\sigma=0} < 0$ respectively.

Furthermore, the equality (3.24) shows that the W-coordinate of trajectories grows with σ with the rate not more than σ, whereas the convolutions vanish with the rate not less then $1/\sigma^2$. Therefore, all trajectories, which started near the origin, stabilize with bounded W-coordinate as $\sigma \to \pm\infty$. However, concerning the desired trajectory γ_s, there is not any reason to assume that $W_s(\tau) \to 0$ as $\tau \to \pm\infty$ As a result of this analysis we obtain the following statement.

Lemma 3.7. *Let Assumptions* (A)-(E2) *hold. Then the dynamical system* (2.26), (3.1), (3.2) *has at least one trajectory* γ_s *which lies in a strip* $\{\sigma \in \mathbb{R}^1, |W| \leq const\}$ *and goes from* $(-W_s^\infty, -\infty)$ *to* $(W_s^\infty, +\infty)$ *through the point* $(0,0)$.

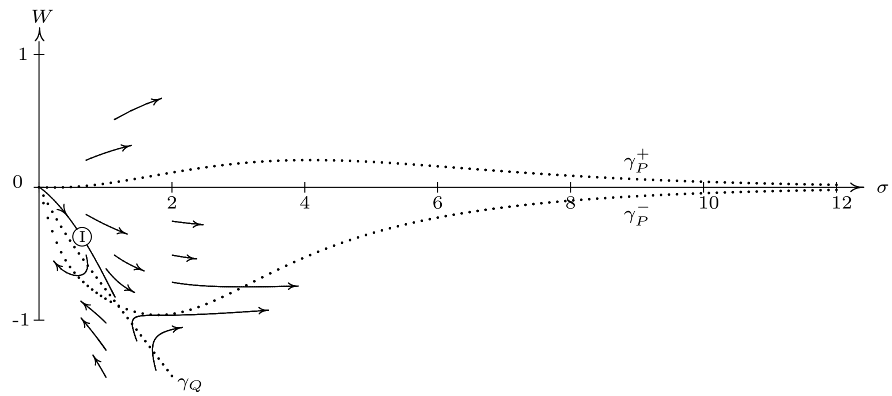

Figure 1. The phase portrait of the basic dynamical system for the case of three singular points. The curve I is the separatrix, which joins the saddle point with the origin. Arrows show the direction of motion along the trajectories.

The phase portrait of the basic dynamical system (2.26), (3.1), (3.2) is shown on Fig. 1. More in detail, Fig. 1 presents results of numerical simulations for the sine-Gordon equation (3) in the case $V_1 = -\sqrt{3}/2$, and $V_2 = \sqrt{15}/4$. Since $R < 0$ for such velocities, there appear two additional saddle points (approximately $(\sigma^*_\pm, W^*_\pm) = (\pm 1.19, \mp 0.84)$). Respectively, there appear separatrixes which join the saddle points with the origin. For the scale which has been used, the trajectories $\gamma_s^\pm = (W_s^\pm, \sigma_s^\pm)$ coincide with the σ-axis. That is why we show the behavior of these curves at the additional picture Fig.2. More in detail, the trajectories γ_s^- and γ_s^+ are situated in the regions bounded by the curves $1, 2$ and $3, 4$ respectively.

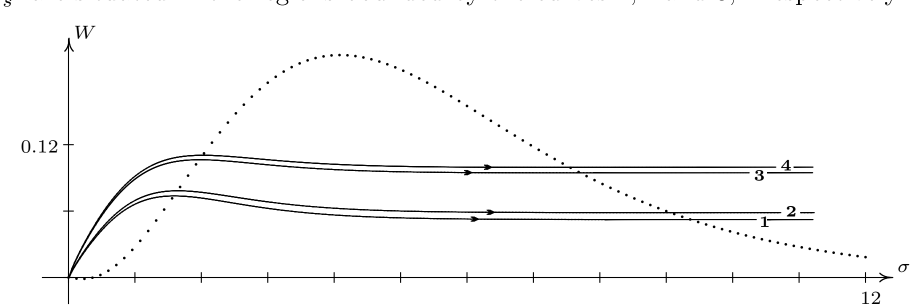

Figure 2. Trajectories $\gamma_s^\pm$

Let us apply now the obtained results for calculations of the phase shifts ϕ_{i1}. Passing to $\pm\infty$ along one of the trajectories $\gamma_s^\pm$ and using the formulas (2.22), (2.23), (2.26), (3.1) we obtain that

$$\frac{d\phi_{11}}{d\tau} = -\frac{1}{\theta}\frac{d\phi_{21}}{d\tau} = -\frac{W_s^\infty}{\sigma_s} \quad \text{as} \quad |\tau| \gg 1.$$

Let $W_s^\infty \neq 0$. Since $\sigma_s(\tau) = \tau + O(1)$ for such τ, this implies a logarithm-type behavior of ϕ_{i1} as $\tau \to \pm\infty$. Obviously, we obtain a contradiction with the second assumption (2.2). On the other hand, the leading term (2.1) of the asymptotic solution does not contain any free parameter to change the value of W_s^∞. Thus, we need to change the anzatz.

4. Influence of arbitrary small perturbations. Completion of the asymptotic solution construction

To simplify formulas, in what follows we will write again τ and σ instead of $\beta_1\tau$ and $\beta_1\sigma$.

As it has been written in Introduction, the main idea of the anzatz correction is to rotate the phase trajectories near the origin with the aim to change the limiting value W_s^∞. To this end we add to the leading term (2.1) some small, localized near the origin, corrections. For definiteness, let the trajectory $\gamma_s = \{W = W_s(\sigma)\}$ defined from the basic dynamical system be such that $dW_s/d\sigma|_{\sigma\to-\infty} > 0$. Then we need to rotate γ_s in the clockwise direction.

The main result of this section is the following lemma.

Lemma 4.1. *Under the assumptions of Lemma 3.7 there exist such corrections of the leading term* (2.1) *that both of the conditions* (2.2) *are fulfilled.*

Proof. We look for the asymptotic solution in the following form:

$$u = \sum_{i=1}^{2}\Big\{\omega\Big(S_i\beta_i\frac{x-\Phi_i(t,\tau,\varepsilon)}{\varepsilon}\Big) + \frac{A_i}{\sqrt{a_2'}}U\Big(S_i\mu\beta_i\frac{x-\Phi_i(t,\tau,\varepsilon)}{\varepsilon}\Big)\Big\}. \tag{4.1}$$

Here the functions ω, Φ_i, and notation τ, σ, S_i are the same as in (2.1), $A_i = A_i(\beta_i\tau)$, where $A_i(z) \in C^\infty$ are exponentially vanishing as $|z| \to \infty$ functions, μ is a sufficiently small parameter, $\varepsilon < \mu < 1$, and

$$U(\eta) = \frac{d^m U_0(\eta)}{d\eta^m}, \quad a_2' = \frac{1}{a_2}\int_{-\infty}^{\infty}\big(U(\eta)\big)^2 d\eta,$$

where $m \geq 1$ is an arbitrary number and $U_0(\eta) \in C^\infty$ is a sufficiently fast vanishing function as $\eta \to \infty$. Moreover, we will assume that $U(\eta)$ is an odd function.

In the C-sense the function U is of the value $O(1)$. However, it is arbitrary small in $\mathcal{D}'$ sense:

$$\int_{-\infty}^{\infty} U\big(\mu\beta\frac{x-\Phi}{\varepsilon}\big)\psi(x)dx = \big(-\frac{\varepsilon}{\mu\beta}\big)^m \int_{-\infty}^{\infty} U_0\big(\mu\beta\frac{x-\Phi}{\varepsilon}\big)\frac{d^m\psi(x)}{dx^m}dx = O\big(\big(\frac{\varepsilon}{\mu}\big)^m\big).$$

To prove the lemma we need to construct the asymptotic solution again. However we run here into technical obstacles calculating the terms of relation (1.6) for anzatz (4.1), since the leading term of asymptotic expansions becomes extremely huge. In particular, the simplest expression $(\varepsilon u_x)^2$ contains now 10 terms. For this reason we will present all the terms with the precision $O_{\mathcal{D}'}(\varepsilon^2 + \varepsilon\mu)$ only.

To calculate the terms of the equation (1.6) we take into account the relations: $\omega_{0i}U_i \sim \omega_{0i}U_i' = O_{\mathcal{D}'}(\varepsilon^2)$ whereas $\omega_{0i}U_j \sim \omega_{0i}U_j' = O_{\mathcal{D}'}(\varepsilon)$ and $U_iU_j \sim U_iU_j' = O_{\mathcal{D}'}(\varepsilon\mu^{-1})$ for $i \neq j$.

It is easy to verify that the expression $(\varepsilon u_x)^2$ has the same $\mod O_{\mathcal{D}'}(\varepsilon\mu)$ form as in (2.8) whereas

$$2F(u) = \varepsilon a_2\Big\{\sum_{i=1}^{2}\frac{1}{\beta_i}\delta(x-\Phi_i) + \frac{1}{\beta_2}B_{\Delta\,A}\,\delta(x-x^*)\Big\}, \quad B_{\Delta\,A} = \frac{1}{\mu}(B+\mu B_\Delta),$$

where B_Δ has been defined in (2.10) and

$$\begin{aligned} B = \frac{2}{a_2}\int_{-\infty}^{\infty}\Big\{&F\Big(\omega(\frac{\theta}{\mu}\eta+\sigma) - \omega(\frac{\eta}{\mu}) + \frac{A_1}{\sqrt{a_2'}}U(\theta\eta+\mu\sigma) - \frac{A_2}{\sqrt{a_2'}}U(\eta)\Big) \\ &- F\Big(\omega(\frac{\theta}{\mu}\eta+\sigma) - \omega(\frac{\eta}{\mu})\Big)\Big\}d\eta. \end{aligned}$$

Furthermore,

$$(\varepsilon u_t)^2 = (\varepsilon u_t)^2|_{A_i=0} + a_2\nu^2\frac{\varepsilon}{\mu}\Big\{\sum_{i=1}^{2}\beta_i {A_i'}^2 - 2\beta_1 A_1'A_2'\Lambda_1\Big\}(1+O(\mu))\delta(x-x^*),$$

$$\begin{aligned} \frac{\partial}{\partial t}\big(\varepsilon^2 u_t u_x\big) &= -a_2\nu^2\frac{dK_{2A}}{d\tau}\delta(x-x^*) + \varepsilon a_2\nu^2\frac{d}{d\tau}\Big(\sum_{i=1}^{2}\beta_i b_i\phi_{i1} + W_A\Big)\delta'(x-x^*) \\ &\quad + \varepsilon a_2\nu^2\sum_{i=1}^{2}\beta_i b_i^2\delta'(x-\Phi_i) + O_{\mathcal{D}'}(\varepsilon^2), \end{aligned}$$

where $A_i' = dA_i(z)/dz|_{z=\beta_i\tau/\beta_1}$, $K_{2A} = K_2 + r_0$,

$$W_A = \chi_2 K_{2A} + W - \chi_2 K_2 + \frac{r_1}{\mu} + r_{11} - r_2\sum_{i=1}^{2}\Phi_{it},$$

the functions K_2 and W are defined in (2.15),

$$r_0 = A_2'(\Lambda_{01} + A_1\Lambda_3) + A_1'(\Lambda_{02} + A_2\Lambda_4), \quad r_{11} = \theta(A_2'\bar\Lambda_{01} + A_1'\bar\Lambda_{02}) - \alpha\sum_{i=1}^{2}A_i',$$

$$r_1 = \theta(A_1A_2'\bar\Lambda_3 + A_2A_1'\bar\Lambda_4) + \frac{1}{2}\sum_{i=1}^{2}A_iA_i', \quad r_2 = a_3\theta A_1A_2\bar\Lambda_2.$$

Here and in what follows we use the notation

$$\Lambda_1 = \frac{1}{a_2a_2'}\int_{-\infty}^{\infty}U(\eta)U(\theta\eta+\mu\sigma)d\eta, \quad \Lambda_3 = \frac{1}{a_2a_2'}\int_{-\infty}^{\infty}U(\eta)U'(\theta\eta+\mu\sigma)d\eta,$$

$$\Lambda_4 = \frac{1}{a_2a_2'}\int_{-\infty}^{\infty}U'(\eta)U(\theta\eta+\mu\sigma)d\eta, \quad \bar\Lambda_2 = \frac{1}{a_2a_2'a_3}\int_{-\infty}^{\infty}\eta U'(\eta)U'\big(\theta\eta+\mu\sigma\big)d\eta,$$

$$\Lambda_{01} = \frac{1}{a_2\sqrt{a_2'}a_3}\int_{-\infty}^{\infty}\omega_0(\theta\eta+\sigma)U(\mu\eta)d\eta,$$

$$\Lambda_{02} = \frac{1}{a_2\sqrt{a_2'}}\int_{-\infty}^{\infty}\omega_0(\eta)U\big(\mu(\theta\eta+\sigma)\big)d\eta,$$

$$a_3 = \frac{1}{a_2a_2'}\int_{-\infty}^{\infty}\big(U'(\eta)\big)^2d\eta, \quad \alpha = \frac{1}{a_2\sqrt{a_2'}}\int_{-\infty}^{\infty}\eta\omega_0(\eta)U(\mu\eta)d\eta,$$

and $\bar\Lambda_3$, $\bar\Lambda_4$, $\bar\Lambda_{0i}$ denotes the convolution similar to Λ_3, Λ_4, Λ_{0i} respectively but with the additional factor η in the integrand (like in λ_j, $\bar\lambda_j$).

Substituting these asymptotic into (1.6) we obtain the relations (2.17) and the following analog of the equations (2.22), (2.26):

$$\frac{d\phi_{11}}{d\tau} = -\mathcal{F}G_{1A}, \quad \frac{d\phi_{21}}{d\tau} = \theta\mathcal{F}G_{2A}, \tag{4.2}$$

$$\frac{d\sigma}{d\tau} = Q_A, \quad \frac{dW}{d\tau} = P_A, \tag{4.3}$$

where $G_{1A} = G_1 + \theta r_0$, $G_{2A} = G_2 - r_0$, $L_A = L + (1-\theta)r_2$,

$$\begin{aligned} Q_A &= \frac{1}{L_A}\Big\{\frac{W + r_1}{\mathcal{F}} + \sigma + 2(b_1\bar\lambda_2 + \theta b_2\lambda_2) - \theta r_0(\sigma + 2\theta\lambda_2) - 2r_2(b2 + \theta b_1 - \theta r_0)\Big\},\\ P_A &= \frac{\mathcal{F}}{2}\Big\{(1-\theta\lambda_1^2)Q_A^2 - 2l_1Q_A + l_0\Big\} + \frac{1}{2\mu}\Big\{{A_1'}^2 + \frac{1}{\theta}{A_2'}^2 - 2A_1'A_2'\Lambda_1\Big\}\\ &\quad - \frac{1}{\nu^2}\Big(\lambda_1 + \frac{B_{\Delta A}}{2\beta_1\beta_2}\Big), \end{aligned}$$

and

$$\begin{gathered} l_1 = 1 - \lambda_1(b_2 - \theta b_1) + (1-\theta\lambda_1)A_2'\Lambda_{01} - \theta(1-\theta\lambda_1)A_1'\Lambda_{02},\\ l_0 = 1 - 2\lambda_1(\theta b_1^2 + b_2^2) + \theta r_0^2 + (b_2 + \theta b_1 - \theta r_0)(A_2'\Lambda_{01} + A_1'\Lambda_{02}). \end{gathered}$$

Let us choose the sign of the coefficients A_i. Denote by z_0 the first positive root of the equation $\Lambda_2(z)|_{\mu=1} = 0$ and let τ_0 be such that $\sigma(\tau_0) = z_0/\mu$. Suppose

$$(1-\theta)A_1(\tau)A_2(\tau)\bar\Lambda_2(\sigma(\tau)) > 0 \quad \text{for } 0 < \tau < \tau_0. \tag{4.4}$$

Since $\tau = O(\sigma)$ for $|\sigma| \gg 1$ and A_i vanish with an exponential rate, $r_2 = O(\mu^\infty)$ for $\tau \geq \tau_0$. Thus, under the assumption (E1), $L_A > 0$ for $\sigma > 0$ and $L_A' := dL_A/d\sigma|_{\tau=\sigma=0} > 0$.

Under the additional assumption

$$A_i'(0) = 0, \quad i = 1, 2, \tag{4.5}$$

there exist isoclines γ_Q, γ_P which pass through the point $(0,0)$.

The last step is the construction of the trajectory γ_s. For small $|\sigma|$ we obtain again the equation similar to (3.12), that is

$$M_A a_1^2 - N_A a_1 - R_A = 0, \tag{4.6}$$

where M_A and N_A coincide $\mod O(\mu)$ with M and N (3.14), (3.10), and with the following R_A:

$$R_A = R - \frac{2\theta}{\mu\mathcal{F}_0}(\zeta_A + \zeta_B).$$

Here

$$\zeta_A = \big\{A_1A_2''\bar\Lambda_3 + A_1''A_2\bar\Lambda_4 + \frac{1}{2}\sum_{i=1}^{2}A_iA_i''\big\}\big|_{\tau=\sigma=0}, \quad \zeta_B = \frac{B}{\nu^2\beta_2}\Big|_{\sigma=0}.$$

The choice of the amplitudes $A_i(\tau)$ depends on the sign of the coefficient M in (3.12). Let $M > 0$. Obviously, the assumption $\zeta_A + \zeta_B < 0$ implies the existence of the real solution

$$a_1 = \frac{1}{\sqrt{\mu}}\Big(\sqrt{\frac{2\theta}{M\mathcal{F}_0}|\zeta_A + \zeta_B|} + O(\sqrt{\mu})\Big).$$

of (4.6). Moreover, condition (3.16) is satisfied now automatically for any n.

Finally, the second equation in (4.3) allows to define the W-coordinate of the trajectory γ_s. Simple algebra implies the following formula for the leading term of the expansion (3.19):

$$w_1 = -\frac{\theta}{\mu M}\big\{\big(M + (1-\theta{\lambda_1^0}^2)\big)\zeta_B + (1-\theta{\lambda_1^0}^2)\zeta_A + O(\sqrt{\mu})\big\}.$$

Obviously, w_1 will be negative under some special choice of ζ_A, ζ_B only. Since there appears a second assumption for the second order derivatives of A_i, we need to summarize both of them:

$$\zeta_B < |\zeta_A| < q\,\zeta_B, \quad \text{where } q = \frac{M + 1 - \theta{\lambda_1^0}^2}{1-\theta{\lambda_1^0}^2}. \tag{4.7}$$

Note that the inequality $M > 0$ implies that $q > 1$. Thus, the fulfilment of the conditions (4.4), (4.5), and (4.7) is obvious. Therefore, the derivative

$$\frac{dW_s}{d\sigma}\Big|_{\sigma=0} = \frac{w_1}{a_1} := -\frac{1}{\sqrt{\mu}}\, f\left(A_1(0), A_2(0)\right), \quad f > 0,$$

may be of arbitrary value. The variation of the amplitudes $A_i(0)$ and of the parameter μ implies the variation of the limiting value $W_s^\infty := \lim_{\sigma\to\pm\infty} W_s(\sigma)$ from the positive number $W_s^\infty|_{A_i(0)=0}$ to any negative number. This and the structure of the phase portrait imply the existence of such values of $A_i(0)$ and μ that $W_s^\infty = 0$.

Let $M < 0$. Then the assumption $\zeta_A + \zeta_B > 0$ implies the existence of the real solution

$$a_1 = \frac{1}{\sqrt{\mu}}\Big(\sqrt{\frac{2\theta}{|M|\mathcal{F}_0}(\zeta_A+\zeta_B)} + O(\sqrt{\mu})\Big).$$

of (4.6). Therefore,

$$w_1 = -\frac{\theta}{\mu q_1}\big\{\big((q_1-1)\zeta_B - \zeta_A\big) + O(\sqrt{\mu})\big\},$$

where

$$q_1 = \frac{|M|}{1-\theta{\lambda_1^0}^2} > 0.$$

Thus, instead of (4.7) we obtain the following assumption

$$-\zeta_B < \zeta_A < (q_1-1)\zeta_B. \tag{4.8}$$

Since $q_1 > 0$, the fulfilment of the conditions (4.4), (4.5), and (4.8) is obvious. Consequently, there exist such values of $A_i(0)$ and μ that $W_s^\infty = 0$.

It remains to consider the limiting values of ϕ_{i1}. Let us choose $\sigma = \sigma_s(\tau)$ and $W = W_s(\tau)$. Then $|W/\sigma| = o(1/|\tau|)$ as $\tau \to \pm\infty$. Next, let $M < 0$. Then, integrating the equations of the form (4.2), we obtain that ϕ_{i1}^∞ are bounded by a constant for any choice of the parameter $\mu \in [0,1)$. Let $M > 0$. Then the right-hand sides of the equalities (4.2) contain the terms of the type $A_1(\tau)A_2'(\tau/\theta)\Lambda_3(\mu\sigma)$. However, A_i vanish as $\tau \to \pm\infty$ with an exponential rate. Hence the integral of this term is bounded by a constant uniformly in $\mu \geq 0$. This implies the fulfilment of Assumption (2.2) and completes the proof of Lemma 4.1. $\square$

Consequently we obtain our main result.

Theorem 4.2. *Let the assumptions* (A)-(E2) *hold. Then the kink and antikink preserve* $\bmod O_{\mathcal{D}'}(\varepsilon^2)$ *their form after the interaction. The weak asymptotic solution of* (1.1), (1.9) *has the form* (4.1) *with the special choice of the amplitudes* A_i *and of the parameter* μ.

Finally we would like to present a result which shows that our definition of weak solutions for (1.1), (1.9) is equivalent to the fulfilment of the conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_t u_x dx = 0 \tag{4.9}$$

and of the energy relation

$$2\frac{d}{dt}\int_{-\infty}^{\infty} x\varepsilon^2 u_t u_x dx + \int_{-\infty}^{\infty}\big\{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\big\}dx = 0. \tag{4.10}$$

Theorem 4.3. *Let the assumptions of Theorem 4.2 hold. Then the function* (4.1) *is the weak asymptotic* $\bmod O_{\mathcal{D}'}(\varepsilon^2)$ *solution of* (1.1), (1.9) *if and only if relations* (4.9), (4.10) *are fulfilled.*

To prove this statement it is sufficient to substitute the anzatz (4.1) into the balance laws (4.9) and (4.10). For more detail see Appendix.

5. Conclusion

The shape of the kink-type solution (1.2) of the semilinear wave (1.1) is similar to a smoothed shock wave. However, these waves have distinct properties. In particular, we can choose an arbitrary anzatz constructing asymptotic [15, 16] or weak asymptotic [3, 5, 10, 11, 12] for quasilinear parabolic equations with a small viscosity, that is, the shock wave regularization. Indeed, the resulting wave front motion does not depend on the type of regularization. Conversely, kink-type asymptotic solutions require special (in the leading term) anzatz related with the nonlinearity. It becomes clear if we consider the simplest problem of a weak asymptotic for solitary kink motion. Indeed, writing

$$u = \omega\Big(\beta\frac{x-Vt}{\varepsilon}\Big), \quad \omega(\eta)\to 0 \quad \text{as} \quad \eta\to-\infty, \quad \omega(\eta)\to 1 \quad \text{as} \quad \eta\to+\infty$$

and using Definition 1.1 we obtain the relation

$$\beta^2(1-V^2) = 2\int_{-\infty}^{\infty} F(\omega(\eta))d\eta\Big/\int_{-\infty}^{\infty}\big(\omega_0(\eta)\big)^2 d\eta.$$

Obviously, we obtain a coincidence with the exact solitary wave solution only for special anzatz $\omega(\eta)$ which satisfies the Newton equation

$$\frac{d\omega}{d\eta} = \sqrt{2F(\omega(\eta))}. \tag{5.1}$$

The same situation holds for the problem of soliton interaction for the KdV-type equation (with the nonlinearity u^n, $n \geq 2$) [5, 6]. At the same time, to describe the soliton collision for the KdV-type equation it is enough to present the asymptotic solution as the sum of two distorted solitons [5, 6]. The same is true for quasilinear parabolic equations with a small viscosity, that is, to describe the shock wave interaction we simply write two distorted shock-wave regularizations [3, 5, 10, 11, 12]. On the contrary, for the sine-Gordon-type (1.1) we need to add some special corrections to the sum of the distorted kinks/antikinks. We guess that the sense of this difference is the following. It is clear that the shape of the sum of the distorted kinks/antikinks is very far from the shape of the real two-kinks solution during the time of interaction. The sine-Gordon equation is more sensitive as others and it requires to approximate the real solution better as simply the sum of kinks. However, this is nothing more than a speculative hypothesis.

Let us come back to Assumptions (E1), (E2). They are very complicated and we need to verify their realizability at least by examples.

Example 5.1. Let $\theta = 1+\kappa$, $0<|\kappa|\ll 1$. Then for sufficiently small $0<\sigma\ll 1$

$$L = \frac{\sigma}{2}(3c_0-1)\kappa^2(1+O(\kappa)), \quad c_0 = \frac{1}{a_2}\int_{-\infty}^{\infty}\big(\eta\omega_0'(\eta)\big)^2 d\eta.$$

Thus, to satisfy the second assumption (E1) we need to assume

$$3c_0 > 1. \tag{5.2}$$

Furthermore, for bounded σ positive,

$$\begin{aligned} L &= \sigma\big(1-\theta\lambda_1(\sigma,\theta)\big) + \theta(1-\theta)\lambda_2(\sigma,\theta) \\ &= \sigma\big(1-\lambda_1(\sigma,1)\big) - \kappa\Big\{\sigma\int_{-\infty}^{\infty}\omega_0(\eta)\omega_0'(\eta+\sigma)d\eta \\ &\quad + \frac{\sigma}{2}\int_{-\infty}^{\infty}\omega_0(\eta+\frac{\sigma}{2})\omega_0(\eta-\frac{\sigma}{2})d\eta\Big\}(1+O(|\kappa|)) \\ &\geq \sigma\big(1-\lambda_1(\sigma,1)\big) - \text{const}|\kappa|\sigma(1+O(|\kappa|)) > 0 \end{aligned} \tag{5.3}$$

for sufficiently small $|\kappa|$.

To check the fulfilment of the condition E2 we need to take into account the direction of the wave movement. Let the waves move with almost the same velocities in opposite direction, that is $V_1 = -v-\alpha$, $V_2 = v$, where $v\in(0,1)$ is a constant and $|\alpha|\ll 1$. Then with the accuracy $O(|\alpha|^3)$

$$R = -\Big(\frac{\alpha}{2v(1-v^2)}\Big)^2, \quad M = \frac{v^2(8c_0-3)}{4(1-v^2)^2}\,\alpha^2, \quad N = \frac{4c_0-1}{4(1-v^2)^2}\,\alpha^2. \tag{5.4}$$

Obviously, one of the assumptions (E2b), (E2d) is fulfilled for any $c_0\neq 3/8$ and the roots of the equation (3.12) are: $a_1^+ = 1/v^2$, $a_1^- = 1/\big(v^2(8c_0-3)\big)$. Let the waves move with almost the same velocities in the same direction, that is $V_1 = v+\alpha$, $V_2 = v$, where $v\in(0,1)$ is a constant and $|\alpha|\ll 1$. Clearly, α needs to be negative. Then

$$R = \frac{2}{|\alpha|}\frac{2-v^2}{1-v^2}, \tag{5.5}$$

whereas M is of the form (5.4). Therefore, the assumption (E2) is fulfilled if $c_0 > 3/8$ or $c_0 < 1/4$ and it is failed for $c_0\in[1/4,3/8]$. Combining this with (5.2) we obtain the following restrictions for the nonlinearity:

$$c_0 > \frac{1}{3}, \quad c_0\neq\frac{3}{8}, \quad \text{if} \quad V_1V_2<0, \quad \text{and} \quad c_0>\frac{3}{8}, \quad \text{if} \quad V_1V_2>0.$$

Note that $c_0\approx 0.607 > 3/8$, $c_0\approx 0.5 > 3/8$ for examples (1.3), and (1.4) respectively. In fact, the condition (1.10) is the inequality $c_0 > 3/8$ rewritten in terms of $F(\omega(\eta))$.

Example 5.2 (Counterexample)**.** Here and in what follows we assume that the number k in condition (B) is equal to 1 which implies an exponential rate of ω_0 vanishing.

Let $\theta \ll 1$. Since $\beta_i > 1$, this requires $1 - V_2^2 \ll 1$, whereas β_1 is a constant. Let, for definiteness, $V_2 > 0$. To check the assumption (E1) it is sufficient to use the obvious estimates $|\lambda_i| \le 1/\sqrt{\theta}$, $i = 1, 2$, and choose θ to be small enough.

After some simple calculations we obtain the relations:

$$\lambda_1^0 = \zeta + O(\theta), \quad \lambda_2^1 = c_1\theta\frac{\omega_0''(0)}{a_2} + O(\theta^3), \quad \zeta := \frac{\omega_0(0)}{a_2}, \quad c_1 = \frac{1}{a_2}\int_{-\infty}^{\infty}\eta^2\omega_0(\eta)d\eta. \tag{5.6}$$

Therefore, $L_1 = 1 + O(\theta) = M$, $N = b_2\zeta + O(\theta)$, $R = 1 - \zeta\frac{4}{\nu^2} + O(\theta)$, which implies

$$N^2 + MR = 1 - \zeta\frac{3}{\nu^2} + O(\theta).$$

Respectively, to satisfy the condition (E2) we need to assume:

$$\sqrt{F\Big(\frac{1}{2}\Big)} < (1 - V_1)^2\frac{\sqrt{2}}{3}\int_{-\infty}^{\infty}F\big(\omega(\eta)\big)d\eta. \tag{5.7}$$

Since

$$\int_{-\infty}^{\infty}F\big(\omega(\eta)\big)d\eta \le \frac{1}{\sqrt{2}}\sqrt{F\Big(\frac{1}{2}\Big)},$$

there appears the restriction $-V_1 > \sqrt{3} - 1$ for any nonlinearity.

We see again that the restriction for the nonlinearity depends not only on the value of θ, but also on the sign of the velocity V_1. Moreover, for the examples (1.3) and (1.4) $F(1/2) = 1/2\pi^2$, $a_2 = 2/\pi^2$ and $F(1/2) = 1$, $a_2 = 2\sqrt{2}$ respectively. Thus, the inequality (5.7) is violated always for these examples.

Example 5.3 (Counterexample)**.** *Let $\theta \gg 1$ which requires $V_1 = -1 + \alpha$, $0 < \alpha \ll 1$, whereas β_2 is a constant. To verify the condition (E1) we use the equalities:*

$$\theta\lambda_1(\sigma, \theta) = \lambda_1\Big(-\frac{\sigma}{\theta}, \frac{1}{\theta}\Big), \quad \theta^2\lambda_2(\sigma, \theta) = \frac{1}{a_2}\int_{-\infty}^{\infty}\eta\,\omega_0\Big(\frac{\eta}{\theta}\Big)\omega_0(\eta + \sigma)d\eta := \widetilde{\lambda}_2(\sigma, \theta). \tag{5.8}$$

It is easy to check that $-\sigma\theta\lambda_1(\sigma, \theta) - \widetilde{\lambda}_2(\sigma, \theta) = O(1/\theta^2)$ for bounded σ. Thus $L = \sigma + O(1/\sigma)$ for bounded σ, which implies the fulfilment of the assumption (E1) for sufficiently large θ. Next we use the asymptotic expansion

$$\lambda_1^0 = \zeta + O\Big(\frac{1}{\theta}\Big), \quad \lambda_2^1 = O\Big(\frac{1}{\theta^5}\Big),$$

where ζ is the same as in (5.6). *Then*

$$L_1 = 1 - \zeta + O\Big(\frac{1}{\theta}\Big), \quad N = \zeta\frac{b_2}{\theta} + O(1), \quad M = 1 - 2\zeta + O\Big(\frac{1}{\theta}\Big), \quad R = 1 - \zeta\frac{4}{\nu^2} + O\Big(\frac{1}{\theta}\Big).$$

Consequently,

$$N^2 + MR = \frac{1}{\nu^2}\big(9\zeta^2 - 2(2 + \nu^2)\zeta + \nu^2\big).$$

On the other hand, $\nu = 1 + V_2 + O(1/\theta) \in (1, 2)$. Therefore $N^2 + MR < 0$ and Assumption (E2) is violated always for large θ.

Finally we recall that the assumptions of Theorem 4.2 are sufficient ones only. When the conditions (E1), (E2) are violated we should have to look for the asymptotic solution in the form (4.1) again. However, in this case we should have to analyze the corresponding dynamical system in detail over the complete (σ, W) plane. Obviously, this investigation is much more complicated.

6. Appendix

Proof of Lemma 2.1. Existence of the convolutions is obvious, since ω_0 vanishes with the rate not less as $1/\eta^2$. Next, when $k = 1$ in Assumption (B), the function ω_0 has an exponential rate of vanishing. Obviously, the same holds for the convolutions. Consider the case when $k = 3$ and, consequently, ω_0 vanishes as $|\eta|^{-2}$. For our aim it is enough to estimate the convolution

$$\lambda_{(n)} = \int_{-\infty}^{\infty} \eta^n \omega_0(\eta)\omega_0(\eta+\sigma)d\eta \quad \text{for} \quad \sigma \gg 1,$$

where n is equal to 1 or 2. Let us rewrite $\lambda_{(n)}$ as the sum of integrals

$$\lambda_{(n)} = \Big\{ \int_{-\infty}^{-\sigma/2} + \int_{-\sigma/2}^{\infty} \Big\} \eta^n \omega_0(\eta)\omega_0(\eta+\sigma)d\eta$$

To estimate the last integral we use the following obvious inequality:

$$|\eta|^n \omega_0(\eta+\sigma) \le C\sigma^{-2+n} \frac{|z|^n}{(1+z)^2}\Big|_{z=\eta/\sigma} \le C\sigma^{-2+n} \quad \text{for} \quad z \ge -\frac{1}{2}, \tag{6.1}$$

where C is a constant. For the first integral we change the variable $\eta' = \eta + \sigma$ and estimate it as follows:

$$\begin{aligned} &\frac{1}{n}\Big| \int_{-\infty}^{-\sigma/2} \eta^n \omega_0(\eta)\omega_0(\eta+\sigma)d\eta \Big| \\ &\le \int_{-\infty}^{\sigma/2} |\eta|^n \omega_0(\eta)\omega_0(\eta-\sigma)d\eta + \sigma^n \int_{-\infty}^{\sigma/2} \omega_0(\eta)\omega_0(\eta-\sigma)d\eta. \end{aligned}$$

Now it is clear that it is enough to use an inequality similar to (6.1) for the first integral and the inequality $\omega_0(\sigma-\eta) \le C\sigma^{-2}$, $\eta \le \sigma/2$, for the second one. To prove the last statement of the lemma it is enough to use the periodicity of F and the equalities

$$\omega(\eta) = 1 + v(\eta) \quad \text{as } \eta \to +\infty, \quad \omega(\eta) = v(\eta) \quad \text{as } \eta \to -\infty,$$

where $v(\eta)$ is an exponentially vanishing function in the case $k = 1$ and $|v(\eta)| \le \text{const}/|\eta|$ when $k = 3$. □

Proof of Lemma 3.1. Let us indicate only the proof of the relation $\lambda_2(\sigma) = \sigma\lambda_2^1 + O(\sigma^3)$. Using the Taylor formula and taking into account the evenness of ω_0, we obtain

$$\lambda_2(\sigma) = \sigma\lambda_2^1 + \sigma^3 \frac{\beta_1^3}{6} \int_{-\infty}^{\infty} \eta\omega_0(\eta)\omega_0'''(\theta\eta + \xi\beta_1\sigma)d\eta, \tag{6.2}$$

where $\xi \in [0, 1]$. Let us rewrite the last integral as the following sum

$$\Big\{ \int_{-\infty}^{-1} + \int_{-1}^{1} + \int_{1}^{\infty} \Big\} \eta\omega_0(\eta)\omega_0'''(\theta\eta + \xi\beta_1\sigma)d\eta, \tag{6.3}$$

The second integral is bounded by a constant. For the last integral we take into account that

$$\theta\eta + \xi\beta_1\sigma \geq \text{const}\cdot\eta \quad \text{for } |\sigma| \ll 1$$

uniformly in $\xi \in [0,1]$. Thus, $\eta|\omega_0'''(\theta\eta + \xi\beta_1\sigma)| \leq$ const. This implies the convergence of this integral. Similar consideration of the first integral in (6.3) implies that the remainder in (6.2) is of the value $O(\sigma^3)$. This completes the proof. □

Proof of Lemma 3.5. Clearly, it is enough to consider the case when $\sigma \gg 1$. Let us rewrite the convolution B_Δ defined in (2.10) as the sum of two integrals, one from $-\infty$ to $-\sigma/(1+\theta)$ and another one from $-\sigma/(1+\theta)$ to ∞. For the second integral we note that $\theta\eta+\sigma \geq \sigma/(1+\theta) \to \infty$ as $\sigma \to \infty$. Thus $1-\omega(\theta\eta+\sigma) \ll 1$ and we can apply the Taylor expansion. Transforming in the same manner the first integral and using the equality (5.1), after some manipulations we pass to the following asymptotic expansion

$$B_\Delta = \frac{2}{a_2}\Big\{-\frac{1+\theta}{2\theta}\Big(2\omega_0(\xi)\omega(-\xi) + \int_\xi^\infty \omega_0^2(\eta)d\eta\Big) + \frac{1}{\theta}\lambda_{1,1} + \theta\lambda_{1,2}\Big\} + O\left(\omega_0^3(\xi)\right), \tag{6.4}$$

where

$$\xi = \frac{\sigma}{1+\theta}, \quad \lambda_{1,1} = \frac{1}{\theta}\int_{-\infty}^{\xi} \omega_0(\eta)\omega_0\Big(\frac{\eta-\sigma}{\theta}\Big)d\eta, \quad \lambda_{1,2} = \frac{1}{\theta}\int_{-\xi}^{\infty}\omega_0(\eta)\omega_0(\theta\eta+\sigma)d\eta.$$

Let Assumption (A) be realized for $k=1$. Then $\omega(-\xi) \sim \omega_0(\xi)$ vanish with an exponential rate as $\xi \to \infty$. Therefore, the first two terms in (6.4) are of the value $O(\omega_0^2(\xi))$. To estimate the other terms we assume firstly that $\theta > 1$. Then for any $q > 0$,

$$\int_{-q\sigma}^{\xi} \omega_0(\eta)\omega_0\Big(\frac{\eta-\sigma}{\theta}\Big)d\eta \geq \omega_0\Big(\frac{1+q}{\theta}\,\sigma\Big)\int_{-q\sigma}^{\xi}\omega_0(\eta)d\eta = \text{const}\,\omega_0\Big(\frac{1+q}{\theta}\,\sigma\Big). \tag{6.5}$$

Choosing $q < (\theta-1)/(\theta+1)$ we obtain the estimate

$$\omega_0^2(\xi) \ll \lambda_{1,1} \quad \text{as } \sigma \to \infty.$$

It is obvious that the same is true for $\lambda_{1,2}$. Consequently $B_\Delta > 0$.

If the number k is equal to 3, then $\omega_0(\eta)$ vanishes with the rate $|\eta|^{-2}$ as $\eta \to \infty$. Thus the first two terms in (6.4) are of the value $O(\sigma^{-3})$. On the other hand, in the right-hand side of the inequality (6.5) we have now the estimate $\omega_0(\sigma(1+q)/\theta) \sim \sigma^{-2}$. Thus we conclude again that $\lambda_{1,i}$ are the leading terms of the asymptotic expansion (6.4) and $B_\Delta > 0$.

Finally, for $\theta > 1$ we use the equality $B_\Delta(\sigma,\theta) = \theta^{-1}B_\Delta(-\sigma\theta^{-1},\theta^{-1})$ and the symmetry (5.8). Since $1+V_1V_2 > 0$ for any $|V_i| \in (0,1)$, all the terms of D_P mod $O(\lambda_1^2)$ are positive and we compleat the proof. □

Proof of Theorem 4.3. For simplicity let us calculate the product u_tu_x for the function of the form (2.1). Integrating this expression over x and changing variables $\beta_1(x-\Phi_1) = \varepsilon\eta$ or $\beta_2(x-\Phi_2) = \varepsilon\eta$ we obtain the formula

$$\int_{-\infty}^{\infty} u_tu_x dx = -a_2\frac{\nu}{\varepsilon}\sum_{i=1}^{2}\{\beta_i - \beta_1\lambda_1\}(b_i + \varphi_{i1}') = -a_2\frac{\nu}{\varepsilon}\Big(K_2 + \sum_{i=1}^{2}\beta_ib_i\Big), \tag{6.6}$$

where the notation (2.5), (2.15), and (2.16) has been used. Since the right-hand side in (6.6) is a function of argument τ only, we calculate again the time derivative using the formula $d/dt = \varepsilon^{-1}\nu d/d\tau$.

Now it is obvious that the conservation law (4.9) is precisely the equation (2.18). Similar but more complicated calculations show that the energy relation (4.10) implies the equalities (2.17) and (2.19) for the anzatz (2.1). Conversely, rewriting equations (2.17)-(2.19) in terms of u_t, u_x, $F(u)$ for u of the form (2.1), we arrive to the energy relations (4.9) and (4.10). Calculating the left-hand sides of (4.9), (4.10) for the anzatz (4.1), we obtain again the equalities similar to (2.17) - (2.19). □

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