QUARTERLY OF APPLIED MATHEMATICS

Vol. XXXII OCTOBER 1974 No. 3

KIRCHHOFF'S PROBLEM FOR NONLINEARLY ELASTIC RODS*

BY

STUART S. ANTMAN

University of Maryland

1. Introduction. In 1859 Kirchhoff [15] extended earlier work of Euler [13] by developing a semilinear theory for the spatial deformation of initially straight elastic rods (cf. Love [16, Chs. 18, 19]). This work was clarified by Clebsch [8]. Kirchhoff's theory rests upon the following constitutive assumptions:

i) The stress couple depends linearly upon the curvature and twist.
ii) The axis of the rod is inextensible.
iii) There can be no shear of the cross-section with respect to the axis.
iv) There can be no deformation within the cross-section.

The material constraints (iii), (iv) are termed *Kirchhoff's hypotheses*. For this theory Kirchhoff showed that an initially straight prismatic rod with a cross-section having equal principal moments of inertia admits helical solutions solely under the action of applied terminal loads ([16, §270]).

In this article we formulate a general theory of nonlinearly elastic rods with sufficient geometric structure to allow not only for flexure and torsion as in the Kirchhoff theory, but also to allow for axial extension and shear of the cross-section with respect to the axis. We thereby remove constraints (ii) and (iii). Moreover, we can replace constraint (iv) by a much larger class of constraints without affecting the mathematical structure of the resulting equations. We merely assume that the cross-sectional deformation depends uniquely upon the state of flexure, torsion, axial extension, and shear of the cross-section with respect to the axis. (Constraint (iv) is a special case of this.) In particular, quantities that measure the thickness of the cross-section may be given as functions of the axial strain in such a way that a lateral contraction results from an axial extension and vice versa. (In Sec. 7, we briefly comment on more sophisticated models in which the deformation within the cross-section is independent of the other kinds of deformation.)

In our theory, the constitutive equations give the stress resultants and couples as arbitrary nonlinear functions of appropriate strain variables. These functions need not be derivable from a strain energy density function, i.e. the material need not be hyperelastic.

* Received March 22, 1973. This research was supported by the National Science Foundation under Grant GP-37770.

For this general theory, we show that an initially straight prismatic rod with a sufficiently symmetric cross-section admits helical solutions solely under the action of applied terminal loads. The additional kinematic structure and nonlinearity of our theory give rise to a far wider variety of solutions than does the Kirchhoff theory. In particular, a number of our solutions manifest interesting nonlinear effects.

The Kirchhoff problem of finding helical solutions is a semi-inverse problem: certain features of the solution are prescribed, while others are left free. It must be shown that the free variables can be chosen so that the governing equations and boundary conditions are satisfied. This sort of problem, so successfully treated by St. Venant [18] in linear elasticity, has also received much attention in nonlinear elasticity [21, §59], although in the latter case much of the analysis is purely formal. Within various theories of the elastica, semi-inverse problems have been treated in [13, 7, 20, 1] (cf. [3, §27].)

For a general class of hyperelastic materials, Ericksen [12] showed that certain invariance requirements characterizing a "uniform state" imply that the solutions must be helical, but, as he observes, "it is impossible to say much about the existence or multiplicity of [such] solutions without introducing some assumptions concerning the form of [the strain energy density function]." In this article we do treat such existence and multiplicity questions by introducing mild assumptions on the constitutive relations. Our theory is not quite comparable to Ericksen's in that ours has less geometric structure but is not restricted to hyperelastic materials.

Our work represents a considerable generalization of that of Whitman and De Silva [22], who treated the problem only for a special set of linear constitutive equations of hyperelastic type. They consequently did not obtain the variety of nonlinear effects that we do. We employ the same scalar variables as they.

2. Formulation of the governing equations. *Geometry of deformation.* To construct a rod theory with sufficient geometric structure to allow for flexure, torsion, axial extension, and shear of cross-sections with respect to the axis, we assume that the configuration of a rod is specified by an arbitrary vector function $\mathbf{r}$ and by a pair of orthonormal vector functions $\mathbf{d}_1$, $\mathbf{d}_2$ of the variable S with

$$0 \leq S \leq L. \tag{2.1}$$

We suppose that the rod is prismatic in its reference configuration. Then we interpret S as the arc length parameter of the line of centroids of the cross-sections in this configuration and $\mathbf{r}(S)$ as the position of the particle S in an arbitrary configuration. The curve defined by $\mathbf{r}$ is called the *axis*. We interpret the vectors $\mathbf{d}_1(S)$ and $\mathbf{d}_2(S)$, which define a plane and a line in this plane at $\mathbf{r}(S)$, as representing the deformation of the particles forming the cross-section at S in the reference configuration. The plane passing through $\mathbf{r}(S)$ with normal

$$\mathbf{d}_3(S) \equiv \mathbf{d}_1(S) \times \mathbf{d}_2(S) \tag{2.2}$$

is called the *section* at S. (Theories of this type were introduced by the Cosserats [10] and have recently been studied in [9, 14, 11, 3], etc.)

The orthonormality of $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3$ permits us to represent them relative to a fixed orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by means of the Euler angles ψ, θ, ϕ (as employed by Love [16, §253]):

$$\mathbf{d}_1 = (-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta)\mathbf{e}_1 + (\cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta)\mathbf{e}_2 - \cos\phi\sin\theta\,\mathbf{e}_3\,, \qquad (2.3a)$$

$$\mathbf{d}_2 = (-\sin\psi\cos\phi - \cos\psi\sin\phi\cos\theta)\mathbf{e}_1 + (\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta)\mathbf{e}_2 + \sin\phi\sin\theta\mathbf{e}_3\,, \qquad (2.3b)$$

$$\mathbf{d}_3 = \cos\psi\sin\theta\mathbf{e}_1 + \sin\psi\sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3\,. \qquad (2.3c)$$

We restrict θ to the interval $[0, \pi]$.

We employ the summation convention for Latin indices that range over 1, 2, 3 and for Greek indices that range over 1, 2. We conventionally replace a statement such as "$\{w_k\}$ are the components of a vector $\mathbf{w}$" with "w_k is a vector." The appearance of w_k as an argument of a function means that the function depends on the three variables w_1, w_2, w_3.

We set

$$\mathbf{r}(S) = x_k(S)\mathbf{e}_k\,, \qquad (2.4)$$

$$\mathbf{r}'(S) = x_k'(S)\mathbf{e}_k \equiv y_a(S)\mathbf{d}_a(S) \equiv y_\rho(S)\mathbf{d}_\rho(S) + z(S)\mathbf{d}_3\,, \qquad y_3 \equiv z, \qquad (2.5)$$

where the prime denotes differentiation with respect to S. The substitution of (2.3) into (2.5) yields

$$x_1' = y_1(-\sin\psi\sin\phi + \cos\psi\cos\phi\cos\theta) - y_2(\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta) + z\cos\psi\sin\theta, \qquad (2.6a)$$

$$x_2' = y_1(\cos\psi\sin\phi + \sin\psi\cos\phi\cos\theta) + y_2(\cos\psi\cos\phi - \sin\psi\sin\phi\cos\theta) + z\sin\psi\sin\theta, \qquad (2.6b)$$

$$x_3' = -y_1\cos\phi\sin\theta + y_2\sin\phi\sin\theta + z\cos\theta. \qquad (2.6c)$$

That no S-interval of positive length can be squeezed to arbitrarily small length in a deformed configuration is ensured by the requirement that the magnitude of $\mathbf{r}'$ be positive. This and the further requirement that a section can never be sheared so severely that it contain the tangent to the axis are both guaranteed by the requirement that

$$z = \mathbf{r}'\cdot\mathbf{d}_3 > 0. \qquad (2.7)$$

We set

$$2u_a = e_{abc}\mathbf{d}_b'\cdot\mathbf{d}_c\,, \qquad u_3 \equiv v, \qquad (2.8)$$

where e_{abc} is the alternating tensor. From (2.3) it follows that

$$u_1 = \theta'\sin\phi - \psi'\cos\phi\sin\theta,$$
$$u_2 = \theta'\cos\phi + \psi'\sin\phi\sin\theta, \qquad (2.9)$$
$$v = \phi' + \psi'\cos\theta.$$

The functions y_a, u_a will constitute the strains for our problem. We assume that in the reference configuration the axis is straight, $\mathbf{d}_3$ coincides with $\mathbf{r}'$, and $\mathbf{d}_1$ and $\mathbf{d}_2$ are constant functions of S. Then the reference values of the strains are

$$y_1 = y_2 = 0, \qquad z = 1, \qquad u_a = 0. \qquad (2.10)$$

We do not require that the reference configuration be stress-free.

Equilibrium equations. Let

$$\mathbf{n}(S) \equiv n_a(S)\mathbf{d}_a(S) \equiv n_\rho(S)\mathbf{d}_\rho(S) + p(S)\mathbf{d}_3(S) \tag{2.11}$$

be the resultant force and

$$\mathbf{m}(S) \equiv m_a(S)\mathbf{d}_a(S) \equiv m_\rho(S)\mathbf{d}_\rho(S) + q(S)\mathbf{d}_3(S) \tag{2.12}$$

be the resultant couple acting across the section at S. If the applied loads consist only of forces and couples at the ends $S = 0, L$, then the equilibrium equations are

$$\mathbf{n}' = \mathbf{0}, \tag{2.13}$$

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = \mathbf{0}. \tag{2.14}$$

(Note that the prime denotes differentiation with respect to the arc-length parameter of the *reference* configuration.) By dotting these equations with $\mathbf{d}_a$, we obtain componential forms. In particular, (2.13) yields

$$m_a' + e_{abc}(u_b m_c + y_b n_c) = 0. \tag{2.15}$$

We remark that components with respect to $\mathbf{d}_a$ are most natural both physically and mathematically; components with respect to $\mathbf{r}'/|\mathbf{r}'|$ and two vectors normal to it are far less satisfactory. In particular, it is the components of $\mathbf{n}$ in the deformed section, namely n_1 and n_2, that are responsible for the shearing, whereas components in the plane normal to $\mathbf{r}'$ have no definite connection with the shearing (despite the standard terminology that calls these latter components the "shear resultants"). We illustrate this in the two-dimensional situation of Fig. 1, in which the section has suffered a large shear. We take $n_2 = 0$ and $\mathbf{r}'$, $\mathbf{d}_1$, $\mathbf{d}_3$ coplanar. It is reasonable to expect that an increase in n_1 results in a decrease in the angle between $\mathbf{d}_1$ and $\mathbf{r}'$, whereas an increase in $\mathbf{n}\cdot(\mathbf{d}_2 \times \mathbf{r}')/|\mathbf{r}'|$ need have no such consequence. (We do not show all the components of $\mathbf{n}$ in this figure.) In our discussion of constitutive restrictions below, we shall encounter the mathematical advantages of the basis $\mathbf{d}_a$.

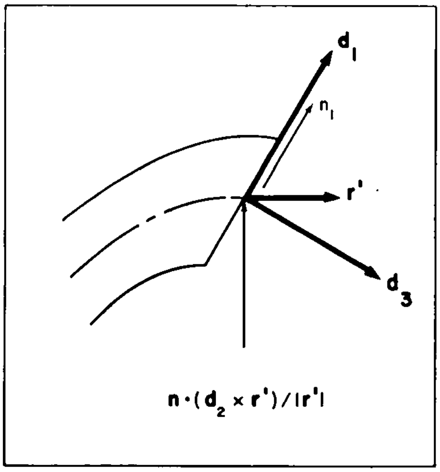

FIG. 1.

Constitutive relations. We assume that $\mathbf{n}$ and $\mathbf{m}$ depend upon $\mathbf{r}'$, $\mathbf{d}_\rho$, $\mathbf{d}_\rho'$. If the resulting constitutive relations are to be unaffected by rigid motions, it can be shown [14] that they must have component forms

$$n_a(S) = \mathrm{n}_a(y_b(S), u_b(S), S), \qquad \mathrm{n}_3 \equiv \mathrm{p}, \tag{2.16}$$

$$m_a(S) = \mathrm{m}_a(y_b(S), u_b(S), S), \qquad \mathrm{m}_3 \equiv \mathrm{q}. \tag{2.17}$$

We take n_a, m_a to be independent of S. (If the three-dimensional body modelled by this one-dimensional theory is homogeneous, then this requirement means that the reference configuration is prismatic. Note that (2.16), (2.17) embody both the mechanical and the geometric properties of the three-dimensional body.) We assume that n_a, m_a are continuously differentiable functions of their arguments on their domains. The variable z is restricted by (2.7); all other arguments can assume any real values.

We impose the following restrictions on the constitutive relations:

i) *Strict monotonicity.* The symmetric part of the matrix

$$\begin{pmatrix} \left(\dfrac{\partial \mathrm{n}_a}{\partial y_b}\right) & \left(\dfrac{\partial \mathrm{n}_a}{\partial u_b}\right) \\ \left(\dfrac{\partial \mathrm{m}_a}{\partial y_b}\right) & \left(\dfrac{\partial \mathrm{m}_a}{\partial u_b}\right) \end{pmatrix} \tag{2.18}$$

is positive definite, i.e., the quadratic form based on (2.18) satisfies

$$\xi_a \frac{\partial \mathrm{n}_a}{\partial y_b} \xi_b + \xi_a \frac{\partial \mathrm{n}_a}{\partial u_b} \eta_b + \eta_a \frac{\partial \mathrm{m}_a}{\partial y_b} \xi_b + \eta_a \frac{\partial \mathrm{m}_a}{\partial u_b} \eta_b > 0 \tag{2.19}$$

for $\xi_a\xi_a + \eta_a\eta_a \neq 0$. This implies the physically reasonable result that n_1, n_2, n_3, m_1, m_2, m_3 are respectively monotonically increasing functions of y_1, y_2, y_3, u_1, u_2, u_3 for fixed values of the remaining arguments. Assumptions of this form, when coupled with a coercivity condition like the following, underlie the available existence theories [2, 3, 5] and qualitative studies [4] for fully nonlinear one-dimensional problems of elasticity. Moreover, strict monotonicity conditions form the natural one-dimensional analogue of the strong ellipticity condition of three-dimensional nonlinear elasticity [5]. (See the discussion on components given above. Note that in terms of other components, this requirement would be a mess.)

ii) *Coercivity.* We require that

$$\frac{\mathrm{n}_a(y_b, u_b)y_a + \mathrm{m}_a(y_b, u_b)u_a}{(y_c y_c + u_c u_c)^{1/2}} \tag{2.20}$$

approach ∞ as $y_c y_c + u_c u_c \to \infty$ and approach $-\infty$ as $y_3 \equiv z \to 0$. This condition essentially ensures that the resultants get large as the corresponding strains get large and that an infinitely large resultant is needed to violate (2.7). In a few instances it will be necessary to supplement this condition with slightly more specific restrictions on the growth of the constitutive functions.

In the context of this paper, these two constitutive restrictions enable us to apply the

Global implicit function theorem: If the strict monotonicity and coercivity conditions hold, then the algebraic equations

$$\mathrm{n}_a(y_b, u_b) = n_a, \qquad \mathrm{m}_a(y_b, u_b) = m_a \tag{2.21}$$

have a unique solution for y_a, u_a in terms of n_a, m_a. Moreover, the subset of these equations

$$\mathrm{n}_a(y_b, u_\rho, v) = n_a, \qquad \mathrm{q}(y_b, u_\rho, v) = q \quad (q \equiv m_3) \tag{2.22}$$

can be solved uniquely for y_a, v in terms of n_a, u_ρ, q:

$$y_a = \mathrm{y}_a^*(n_b, u_\rho, q), \qquad v = \mathrm{v}^*(n_b, u_\rho, q). \tag{2.23}$$

The proof of existence requires only coercivity and the proof of uniqueness, only strict monotonicity (cf. [6, 19]). We remark that an operator-theoretic generalization of this theorem forms the basis of a general existence theory for equations of this type [5].

We also define

$$\mathrm{m}_\rho^*(n_a, u_\rho, q) \equiv \mathrm{m}_\rho(\mathrm{y}_b^*(n_c, u_\sigma, q), u_\rho, \mathrm{v}^*(n_c, u_\sigma, q)) \tag{2.24}$$

so that (2.23) and

$$m_\rho = \mathrm{m}_\rho^*(n_a, u_\sigma, q) \tag{2.25}$$

represent a set of constitutive equations equivalent to (2.16), (2.17). In Appendix 1 (Sec. 8) we show that the symmetric part of the matrix of partial derivatives of y_1^*, y_2^*, y_3^*, m_1^*, m_2^*, v^* with respect to n_1, n_2, n_3, u_1, u_2, q is positive definite.

iii) *Isotropy.* The previous conditions are universal in the sense that they are reasonable for any problem. We now adopt a special symmetry assumption that generalizes Kirchhoff's assumption of "kinetic symmetry." He assumed that $\mathbf{d}_1$, $\mathbf{d}_2$ are the principal inertia axes of the cross-section, that the moments of inertia with respect to these axes are equal, and that the constitutive equations for the bending moments have the forms

$$m_1 = EI\, u_2, \qquad m_2 = EI\, u_2, \tag{2.26}$$

where E is the elastic modulus and I is the common value of the moments of inertia. Within the geometric and mechanical structure of Kirchhoff's theory, (2.26) represents the class of rods whose mechanical response is indistinguishable from that of rods with circular cross-sections. We adopt this characterization as a criterion for symmetry in our theory. In terms of the constitutive equations (2.23), (2.25), we specifically require that y_ρ^* and m_ρ^* be isotropic two-vector functions of the two-vectors n_ρ, u_ρ and of the scalars p, q and that z^*, v^* be isotropic scalar functions of the same arguments. This means that

$$\mathrm{y}_\rho^*(n_\sigma, p, u_\sigma, q) = Q_{\rho\tau}\mathrm{y}_\tau^*(Q_{\nu\sigma}n_\sigma, p, Q_{\nu\sigma}u_\sigma, q), \tag{2.27a}$$

$$\mathrm{z}^*(n_\sigma, p, u_\sigma, q) = \mathrm{z}^*(Q_{\nu\sigma}n_\sigma, p, Q_{\nu\sigma}u_\sigma, q), \tag{2.27b}$$

$$\mathrm{m}_\sigma^*(n_\sigma, p, u_\sigma, q) = Q_{\rho\tau}\mathrm{m}_\tau^*(Q_{\nu\sigma}n_\sigma, p, Q_{\nu\sigma}u_\sigma, q), \tag{2.27c}$$

$$\mathrm{v}^*(n_\sigma, p, u_\sigma, q) = \mathrm{v}^*(Q_{\nu\sigma}n_\sigma, p, Q_{\nu\sigma}u_\sigma, q) \tag{2.27d}$$

are to be satisfied identically for all n_σ, p, u_σ, q and for all two-dimensional orthogonal tensors $Q_{\rho\sigma}$. In Appendix 2 (Sec. 9), we obtain representations for the constitutive functions under the special conditions prevailing for our problem. We note that this isotropy requires n_1, n_2, m_1, m_2 to vanish in the reference state.

Summary of equations. The geometrical variables are the components x_k of the position $\mathbf{r}$ of the axis, the Euler angles ψ, ϕ, θ giving the orientation of the sections, and the strains y_a, n_a. The mechanical variables are the components n_a of the stress resultant

n and the components m_α of stress couple **m**. These eighteen quantities must satisfy the six "strain-displacement" relations (2.6), (2.9), the six scalar equations of equilibrium of (2.13), (2.14), and the six "stress-strain" laws (2.16), (2.17) or (2.23), (2.25).

3. Geometry of the Kirchhoff solutions. The equilibrium equation (2.13) immediately yields

$$\mathbf{n} = \mathbf{const} = \mathbf{n}(L). \tag{3.1}$$

We choose Cartesian base vactors $\mathbf{e}_k$ so that

$$\mathbf{n}(L) = N\mathbf{e}_3 . \tag{3.2}$$

By taking components of (3.1) with respect to $\mathbf{d}_a$, we obtain

$$n_1 = -N \sin \theta \cos \phi, \qquad n_2 = N \sin \theta \sin \phi, \qquad n_3 \equiv p = N \cos \theta. \tag{3.3}$$

We use (3.1) to integrate (2.14):

$$\mathbf{m} + \mathbf{r} \times \mathbf{n}(L) = \mathbf{const} = \mathbf{m}(L) + N\mathbf{r}(L) \times \mathbf{e}_3 , \tag{3.4}$$

whence it follows that

$$\mathbf{m}\cdot\mathbf{e}_3 \equiv -m_1 \sin \theta \cos \phi + m_2 \sin \theta \sin \phi + q \cos \theta = \mathbf{m}(L)\cdot\mathbf{e}_3 . \tag{3.5}$$

We seek smooth solutions of the governing equations with

$$\theta = \text{const}, \qquad 0 \leq \theta \leq \pi. \tag{3.6}$$

In this case, (2.9) and (3.3) reduce to

$$u_1 = -u \cos \phi, \quad u_2 = u \sin \phi, \quad v = \phi' + \psi' \cos \theta, \quad u \equiv \psi' \sin \theta, \tag{3.7}$$

$$n_1 = -n \cos \phi, \quad n_2 = n \sin \phi, \quad p = N \cos \theta, \quad n \equiv N \sin \theta. \tag{3.8}$$

In Appendix 2 (Sec. 9), we show that (3.7), (3.8) cause the isotropic versions of the constitutive equations (2.23), (2.25) to have the reduced forms:

$$\begin{aligned} y_1 &= -\cos \phi \ \mathrm{y}(n, p, u, q), \quad y_2 = \sin \phi \ \mathrm{y}(n, p, u, q), \quad z = \mathrm{z}(n, p, u, q), \\ m_1 &= -\cos \phi \ \mathrm{m}(n, p, u, q), \quad m_2 = \sin \phi \ \mathrm{m}(n, p, u, q), \quad v = \mathrm{v}(n, p, u, q), \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} \mathrm{y}(-n, p, -u, q) &= -\mathrm{y}(n, p, u, q), \quad \mathrm{m}(-n, p, -u, q) = -\mathrm{m}(n, p, u, q), \\ \mathrm{z}(-n, p, -u, q) &= \mathrm{z}(n, p, u, q), \quad \mathrm{v}(-n, p, -u, q) = \mathrm{v}(n, p, u, q), \end{aligned} \tag{3.10}$$

and where the symmetric part of

$$\begin{pmatrix} \dfrac{\partial \mathrm{y}}{\partial n} & \dfrac{\partial \mathrm{y}}{\partial p} & \dfrac{\partial \mathrm{y}}{\partial u} & \dfrac{\partial \mathrm{y}}{\partial q} \\ \dfrac{\partial \mathrm{z}}{\partial n} & \dfrac{\partial \mathrm{z}}{\partial p} & \dfrac{\partial \mathrm{z}}{\partial u} & \dfrac{\partial \mathrm{z}}{\partial q} \\ \dfrac{\partial \mathrm{m}}{\partial n} & \dfrac{\partial \mathrm{m}}{\partial p} & \dfrac{\partial \mathrm{m}}{\partial u} & \dfrac{\partial \mathrm{m}}{\partial q} \\ \dfrac{\partial \mathrm{v}}{\partial n} & \dfrac{\partial \mathrm{v}}{\partial p} & \dfrac{\partial \mathrm{v}}{\partial u} & \dfrac{\partial \mathrm{v}}{\partial q} \end{pmatrix} \tag{3.11}$$

is positive definite. It proves convenient to define

$$y = -\cos\phi y_1 + \sin\phi y_2, \quad m = -\cos\phi m_1 + \sin\phi m_2. \tag{3.12}$$

Then y and m satisfy the constitutive relations

$$y = \mathrm{y}(n, p, u, q), \qquad m = \mathrm{m}(n, p, u, q). \tag{3.13}$$

We now substitute (3.9), (3.13) into (2.15) for $a = 3$, to get

$$q' = u_2 m_1 - u_1 m_2 + y_2 n_1 - y_1 n_2 = (-\sin\phi\cos\phi + \sin\phi\cos\phi)(um + ny) = 0. \tag{3.14}$$

Thus

$$q = \text{const} = \mathbf{m}(L)\cdot\mathbf{d}_3(L). \tag{3.15}$$

We now show that u must be a constant. Substituting (3.6), (3.9), (3.13), (3.15) into (3.5), we get

$$\sin\theta\, \mathrm{m}(n, p, u, q) = \mathbf{m}(L)\cdot[\mathbf{e}_3 - \mathbf{d}_3(L)\cos\theta]. \tag{3.16}$$

Now $\partial \mathrm{m}/\partial u > 0$ by (3.11) and $\mathrm{m}(n, p, u, q) \to \pm\infty$ as $u \to \pm\infty$ for fixed n, p, q, by coercivity. Thus if $\sin\theta \neq 0$, we can uniquely solve (3.16) for u in terms of the remaining (constant) variables. If $\sin\theta = 0$, then (3.7) implies that $u \equiv 0$. In either case, the specification of terminal load and of θ uniquely defines a real constant value for u.

If $\sin\theta \neq 0$, then (3.7) implies that ψ' is a uniquely determined constant and (3.7), (3.9) then imply that $\phi' = \mathrm{v}(n, p, u, q) - \psi'\cos\theta$, a uniquely determined constant. In this case, if there is a solution satisfying (3.6), then all the variables are uniquely determined by θ and the loads. (But θ need not be uniquely determined by the data, as we show below.) If $\sin\theta = 0$, then all we can conclude from (3.7), (3.9) is that $\phi' \pm \psi' = \mathrm{v}(0, p, 0, q)$, a constant. This ambiguity is a consequence of our use of spherical polar coordinates and leads to no ambiguity in the solution.

Let s be the arc length parameter of the deformed axis:

$$ds/dS = [\mathbf{r}'(S)\cdot\mathbf{r}'(S)]^{1/2} = (y^2 + z^2)^{1/2} \equiv \lambda \text{ (const)}. \tag{3.17}$$

Substituting this and the constitutive representations for y_a into (2.6), we obtain

$$dx_1/ds = a\cos\psi, \qquad dx_2/ds = a\sin\psi, \qquad dx_3/ds = b, \tag{3.18}$$

where

$$\psi = \lambda^{-1}\psi' s + \text{const}, \quad \lambda a = -y\cos\theta + z\sin\theta, \quad \lambda b = y\sin\theta + z\cos\theta. \tag{3.19}$$

If $\sin\theta \neq 0$, then $\psi' = $ const and (3.18) gives the equation of a circular helix (possibly degenerate) of radius $|\lambda a/\psi'|$ and pitch b. If $\sin\theta = 0$, then $n = 0$, $u = 0$, and $y = 0$ by (3.7), (3.8), (3.10), (3.13). Then $a = 0$, so that (3.14) represents a straight line along the x_3-axis. Thus if the governing equations admit solutions with $\theta = $ const, then the axes of such solutions must be helical. Note that a and b cannot vanish simultaneously since $z > 0$. We also note that

$$\lambda^{-1}\mathbf{n}\times\mathbf{r}' = Na(\sin\phi\mathbf{d}_1 + \cos\phi\mathbf{d}_2). \qquad \lambda^{-1}\mathbf{r}'\cdot\mathbf{n} = Nb, \tag{3.20}$$

so that Na is the magnitude of the component of resultant force in the plane perpendicular to the axis and Nb is the component of resultant force along the axis.

To verify that there actually exist solutions of this kind, we must first show that all the equilibrium equations are satisfied. We have actually done this for the equations beginning with n_1', n_2', p', q' by our use of (3.1), (3.15). If we now substitute the results of this section into the remaining two equilibrium equations, taking special note of the constancy of y and m, we find that these equations are satisfied if and only if

$$\psi'(m \cos \theta - q \sin \theta) = N(y \cos \theta - z \sin \theta). \tag{3.21}$$

We must then check that the resulting solution is consistent with the data. This will be done in the next three sections.

Love [16, §270], in his treatment of the Kirchhoff problem, showed that a helical deformation can always be maintained by a force of magnitude N acting along the x_3-axis (which is the axis of the circular cylinder containing the helix) and by a couple parallel to $\mathbf{e}_3$. (This loading is called a *wrench.*) The force and couple are communicated to the end of the rod by some rigid connection. We show that the same is true for our problem.

We assume that the axis is not straight, so that $a\psi' \neq 0$. At the end L of the rod, the applied force is $N\mathbf{e}_3$ and the applied couple is $\mathbf{m}(L)$. This loading is equivalent to a force $N\mathbf{e}_3$ applied at $\mathbf{r} = z\mathbf{e}_3$ for any z, and a couple

$$\mathbf{m}(L) + \mathbf{r}_0(L) \times N\mathbf{e}_3 , \tag{3.22}$$

where $\mathbf{r}_0(L)$ is the projection of $\mathbf{r}(L)$ onto the (x_1 , x_2)-plane. From (3.18) we find that

$$\psi'\mathbf{r}_0 = \lambda a(\sin \psi \mathbf{e}_1 - \cos \psi \mathbf{e}_2). \tag{3.23}$$

If the force along the x_3-axis and (3.22) are to form a wrench, the components of (3.22) in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions must vanish. An evaluation of these components shows that they vanish if and only if (3.21) is satisfied. If the rod is straight, a simple computation shows that an equilibrium state can always be maintained by a wrench. Thus every deformation of a rod having the material properties described in Sec. 2 must have a helical axis and can always be maintained by a terminal wrench acting along the x_3-axis. This is not surprising since the problem becomes purely statical once the helical form is known so our demonstration is therefore effectively equivalent to Love's.

4. Deformations with a straight axis. Eq. (3.18) implies that the axis will be straight if and only if $a\psi' \equiv 0$. Now $a = 0$ if and only if

$$y \cos \theta = z \sin \theta, \tag{4.1}$$

in which case (3.21) reduces to

$$\psi'(m \cos \theta - q \sin \theta) = 0.$$

These two conditions and (2.7) restrict the six variables θ, y, z, ψ', m, q. Under these circumstances the axis is straight and parallel to the direction $\mathbf{e}_3$ of the terminal force.

If $\psi' \equiv 0$, then (3.21) implies that

$$Na = 0. \tag{4.3}$$

If $a = 0$, we revert to the previous case. If $N = 0$, then $\mathbf{e}_3$ (the direction of the terminal force) is arbitrary and we simply choose $\mathbf{e}_3$ to be along the axis by setting

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{e}_3 . \tag{4.4}$$

In this case we have from (3.7), (3.8) that

$$n = 0, \quad p = 0, \quad u = 0, \quad y = \mathrm{y}(0, 0, 0, q) = 0, \tag{4.5}$$

whence (2.5) implies

$$|\mathbf{r}'|\mathbf{e}_3 = z\mathbf{d}_3 \quad \text{or} \quad \mathbf{e}_3 = \mathbf{d}_3\,. \tag{4.6}$$

Eq. (2.3c) then implies that

$$\sin\theta = 0. \tag{4.7}$$

Thus (4.1) is satisfied by virtue of (4.5), (4.7), so that $a = 0$. Thus the axis will be straight if and only if $a = 0$.

We first examine the special case $\theta = k\pi$, $k = 0, 1$, in which the sections are perpendicular to the axis. From (3.10), we have

$$y = \mathrm{y}(0, p, 0, q) = 0, \qquad m = \mathrm{m}(0, p, 0, q) = 0, \tag{4.8}$$

so that (4.1), (4.2) are satisfied. Moreover, we have from (3.7)–(3.9) that

$$y_1 = y_2 = 0, \qquad u_1 = u_2 = 0, \qquad n_1 = n_2 = 0, \qquad m_1 = m_2 = 0. \tag{4.9}$$

Thus a solution with $\theta = k\pi$ is impossible if terminal data are inconsistent with (4.9). From (3.9), (3.8) we obtain

$$z = \mathrm{z}(0, p, 0, q), \qquad v = \mathrm{v}(0, p, 0, q), \qquad p = (-1)^k N. \tag{4.10a, b, c}$$

Thus the stretch z and the twist $v = \phi' + (-1)^k\psi'$ are determined uniquely by (4.10) if the axial force $p = (-1)^k N$ and the twisting couple q are prescribed (at the ends). Conversely, if z and v are prescribed, then p and q are uniquely determined by the unique invertibility of (4.10a, b). (The prescription of a terminal stretch is generally not a proper boundary condition for a well-set boundary value problem. This does not disturb us for two reasons: i) we are not studying boundary value problems. ii) the stretch z is constant by (4.10a) and the axis is straight, so that in the present problem the specification of stretch is equivalent to the specification of the relative longitudinal displacement of the ends, and this is a proper boundary condition.) Also, if z and q are prescribed, we can solve (4.10a) uniquely for p in terms of z and q and then substitute this representation for p into (4.10b) to obtain a unique value for v. We can similarly find z and q uniquely if p and v are prescribed. In summary, *if* $\sin\theta = 0$ *and if any one of the four pairs* (p, q), (z, v), (p, v), (z, q) *is prescribed arbitrarily, then the governing equations have a unique solution meeting these conditions. This deformation has a straight axis along* $\mathbf{e}_3$ *with sections perpendicular to it, has a constant twist* v, *and satisfies* (4.9). Note that we do not obtain ϕ' and ψ' separately, but merely get $v = \phi' + (-1)^k\psi'$. It is just this combination that determines $\mathbf{d}_1$, $\mathbf{d}_2$ uniquely from (2.3) however. In this case it is thus immaterial whether or not we take $\psi' = 0$.

Eqs. (4.10) have some interesting consequences. For simplicity let us assume that the reference state is stress-free so that $\mathrm{z}(0, 0, 0, 0) = 1$, $\mathrm{v}(0, 0, 0, 0) = 0$. If $N = 0$, but $q \neq 0$, then z need not equal 1, in which case the application of a terminal twisting moment produces not only torsion but also a change in length. Similarly, a nonzero twist v could be maintained solely by a nonzero axial force. The first possibility is the "Poynting" effect, which has been observed experimentally and has been studied in the three-dimensional theory of incompressible nonlinearly-elastic materials [21]. The second possibility is an example of a "shear instability" in which a nonzero shear deformation (namely torsion) is maintained by a shearless loading on the boundary. This case is related to the "Coulomb" effect.

We now examine another special case: $a = 0$, $\sin\theta \neq 0$, $\psi' = 0$. Here (4.2) is satisfied, so we need only concern ourselves with (4.1). Now $N \neq 0$, since the vanishing of N would imply that $\sin\theta = 0$ by the argument developed in (4.4)–(4.7). Thus the substitution of (3.13) into (4.1) yields

$$\mathrm{y}(N \sin\theta, N\cos\theta, 0, q)\cos\theta = \mathrm{z}(N\sin\theta, N\cos\theta, 0, q)\sin\theta. \tag{4.11}$$

The right-hand side of (4.11) is positive by (2.7) and (3.6), so $y\cos\theta$ must be positive. Without loss of generality we assume that $\mathbf{r}' = |\mathbf{r}'|\mathbf{e}_3$. (Since $a = 0$, we know that $\mathbf{r}'$ is proportional to $\mathbf{e}_3$.) Then (2.7) gives

$$0 < z = \mathbf{r}'\cdot\mathbf{d}_3 = |\mathbf{r}'|\cos\theta,$$

so that $\cos\theta > 0$ and y itself must be positive. Since $\partial \mathrm{y}/\partial n > 0$ by (3.11), it follows that N must be positive, for if not, we should have

$$\mathrm{y}(N\sin\theta, N\cos\theta, 0, q) < \mathrm{y}(0, N\cos\theta, 0, q) = 0,$$

in contradiction to the consequences of (4.11). Thus deformations of this type can only occur in the presence of tensile forces. (Recall that (3.16) and (4.11) show that the component of resultant force perpendicular to the axis must vanish.) Beyond this fact, our constitutive restrictions cast no light on whether there actually do exist triples θ, N, q, with $\theta \in (0, \pi/2)$, $N > 0$, that satisfy (4.11). The possibility of finding such solutions becomes somewhat more remote if we were to require that some component or components of the moment vanish, for this would add more conditions to (4.11) without increasing the number of variables. It is not hard, however, to construct a set of admissible constitutive relations for which (4.11) has solutions and another set for which (4.11) does not have solutions. Such deformations have the normal to each section making a constant nonzero angle with the straight axis. This sort of shear instability models aspects of the Lüders band phenomenon in which bars of certain metals go into a state of shear under the action of purely tensile terminal loads (cf. [17]).

The remaining possibilities of solutions with straight axes are characterized by $a = 0$, $\sin\theta \neq 0$, $\psi' \neq 0$. In this case (4.1) and (4.2) require

$$\mathrm{y}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\cos\theta = \mathrm{z}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\sin\theta, \tag{4.12}$$

$$\mathrm{m}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\cos\theta = q\sin\theta. \tag{4.13}$$

Here there is one more condition to be satisfied and one more parameter at our disposal than in (4.11). As in the previous case, our constitutive restrictions say nothing about the possibility of these equations having solutions.

5. Deformations with a circular axis. Eq. (3.18) implies that the axis will be circular if and only if $b = 0$, $\psi' \neq 0$. From (3.15) it follows that $b = 0$ if and only if

$$y\sin\theta + z\cos\theta = 0. \tag{5.1}$$

Since a and b cannot vanish simultaneously, the coefficient of N in (3.21) cannot vanish. Moreover, $\sin\theta \neq 0$, since the vanishing of $\sin\theta$ would cause (5.1) to violate (2.7).

We thus have two equations (5.1), (3.21) and three inequalities (2.7), $\psi' \neq 0$, $\sin\theta \neq 0$ restricting the seven variables θ, y, z, ψ', N, m, q.

We first treat the special case $\theta = \pi/2$. Eqs. (5.1) and (3.21) then imply that

$$y = 0, \qquad q\psi' = Nz, \tag{5.2a, b}$$

so we have these two equations and two inequalities (2.7), $\psi' \neq 0$ restricting the five variables y, z, ψ', N, q. We find

$$n = N, \qquad p = 0, \qquad u = \psi', \qquad v = \phi', \tag{5.3}$$

so that

$$0 = \mathrm{y}(N, 0, \psi', q), \quad z = \mathrm{z}(N, 0, \psi', q), \quad m = \mathrm{m}(N, 0, \psi', q), \quad \phi' = \mathrm{v}(N, 0, \psi', q). \tag{5.4a, b, c, d}$$

From (2.5) we find $\mathbf{r}' = z\mathbf{d}_3$ so that the sections are perpendicular to the axis. Now if we take $N = 0$, then (5.2b) implies that $q = 0$ and conversely. For there to be a deformation in this case, the equation

$$\mathrm{y}(0, 0, \psi', 0) = 0 \tag{5.5}$$

must have a pair of nonzero solutions $\pm\psi_0'$. (This would certainly occur if y did not depend on u.) For any such solution ψ_0', we could then compute z, m, ϕ' from (5.4). From (3.9), (2.3) we have that

$$\mathbf{m} = m(-\cos\phi\mathbf{d}_1 + \sin\phi\mathbf{d}_2) = m\mathbf{e}_3\,, \tag{5.6}$$

with $m = \mathrm{m}(0, 0, \psi_0', 0)$ having the same sign as ψ_0' since $\mathrm{m}(0, 0, 0, 0) = 0$ and $\partial\mathrm{m}/\partial u > 0$. Thus for such deformations the stress couple is a pure nonzero bending moment about the normal to the plane of the deformed circular axis (just as in the classical elastica theory).

If (5.5) fails to have a nonzero solution, i.e. if

$$\mathrm{y}(0, 0, u, 0) \neq 0 \quad \text{for all} \quad u \neq 0, \tag{5.7}$$

there is a deformation with a circular axis and perpendicular sections, provided we suspend the requirement that N and q vanish and we impose some further mild constitutive restrictions. Since $\partial y/\partial n > 0$ by (3.11) and since y assumes all real values as n assumes all real values for fixed p, u, q, we can solve (5.4a) uniquely for

$$N = \mathfrak{N}(\psi', q) \tag{5.8}$$

with $\mathfrak{N}(0, q) = 0$ since $\mathrm{y}(0, 0, 0, q) = 0$ and with $\mathfrak{N}(\psi', 0) \neq 0$ for $\psi' \neq 0$ by (5.7). We can now write (5.2b) as

$$q\psi' = \mathfrak{N}(\psi', q)\mathrm{z}(\mathfrak{N}(\psi', q), 0, \psi', q). \tag{5.9}$$

Let $\psi' \neq 0$ be arbitrary. We seek a q satisfying (5.9). Note that $q = 0$ cannot be a solution since $\mathfrak{N}(\psi', 0)z \neq 0$. If we assume that the constitutive functions y and z are such that

$$|q|^{-1}\mathfrak{N}(\psi', q)\mathrm{z}(\mathfrak{N}(\psi', q), 0, \psi', q) \to 0 \quad \text{as} \quad |q| \to \infty, \tag{5.10}$$

then (5.9) has a nonzero solution q, as is immediately seen from a plot of the left- and right-hand sides of (5.9) as functions of q. (The condition (5.10) is mild because it does not restrict the dependence of y and z on their principal arguments n and p. This condition is consistent with strict monotonicity and coercivity.) We can then find the other variables from (5.4). (It is conceivable that $m = 0$). The condition (5.7) indicates that flexure and shear are coupled. In this case, our analysis implies that a deformation with a circular axis and perpendicular sections must be maintained by a nonzero shear resultant and a nonzero twisting moment. This is another example of a nonlinear effect.

We could impose a restriction on the growth of $\mathfrak{N}(z)$ as a function of ψ' for fixed q and thereby show that for any fixed q, Eq. (5.7) has a solution ψ'. But we have no assurance that the solution ψ' is not the zero solution. In the next section we develop some techniques to handle this sort of difficulty.

We summarize our results: *For* $\theta = \pi/2$ *there exists a deformation with a circular axis and perpendicular sections if either* i) *Eq.* (5.5) *has nonzero solutions* $\pm\psi_0'$, *in which case the deformation is maintained by zero force resultant, zero twisting moment, and a nonzero bending moment acting about the normal to the plane of the circular axis, or* ii) *the constitutive restrictions* (5.7), (5.10) *hold, in which case for every* $\psi' \neq 0$, *there is a deformation maintained by a nonzero shear force, a nonzero twisting moment, a bending moment, and a zero axial force.*

For the general case, we merely note that when the constitutive expressions for y and z are substituted into (5.1), the resulting equation can be solved uniquely for N as a function of θ, ψ', q, since the derivative of the left-hand side of this equation with respect to N is just

$$\frac{\partial \mathrm{y}}{\partial n}\sin^2\theta + \left(\frac{\partial \mathrm{y}}{\partial p} + \frac{\partial \mathrm{z}}{\partial n}\right)\sin\theta\cos\theta + \frac{\partial \mathrm{z}}{\partial p}\cos^2\theta,$$

which is strictly positive by (3.11), and since the left-hand side of the equation assumes all real values as N is varied. Whether or not there are solutions with $\theta \neq \pi/2$ cannot be determined by the strict monotonicity and coercivity alone. This situation is entirely analogous to that for the straight-axis case and will not be pursued further.

6. Deformations with a helical axis. We now obtain conditions ensuring that there are deformations having a nondegenerate helical axis, i.e., deformations for which $ab\psi' \neq 0$. Rather than furnishing an exhaustive treatment of all possible cases, we limit our investigation to cases analogous to those treated by Love [16, §270]. Let

$$\mathfrak{K}(\theta, \psi', N, q) \equiv \psi'[\mathrm{m}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\cos\theta - q\sin\theta]$$
$$- N[\mathrm{y}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\cos\theta - \mathrm{z}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\sin\theta]. \quad (6.1)$$

Then (3.18), (3.21) imply that there will be nondegenerate helical solutions if there are numbers θ, ψ', N, q such that

$$\mathfrak{K}(\theta, \psi', N, q) = 0 \quad (6.2)$$

with $ab\psi' \neq 0$. We note that $\sin\theta > 0$ by (3.6) and by our proof in Sec. 4 that the vanishing of $\sin\theta$ implies that of a.

We first examine the possibility of obtaining a helical solution for prescribed θ, N, q with $N < 0$, $\cos\theta > 0$. We assume that the constitutive functions y and z satisfy the mild growth conditions:

$$\frac{\mathrm{y}(n, p, u, q)}{|u|} \to 0, \ \frac{\mathrm{z}(n, p, u, q)}{|u|} \to 0 \quad \text{as} \quad |u| \to \infty. \quad (6.3)$$

These sublinear growth conditions and the coercivity imply that

$$\mathcal{H}(\theta, \psi', N, q) \to \infty \quad \text{as} \quad |\psi'| \to \infty \tag{6.4}$$

uniformly in θ, N, q for θ in any closed subset of $[0, \pi/2)$ and for N, q in any compact set. Since $\partial \mathrm{y}/\partial n > 0$ by (3.11), since $\mathrm{y}(0, p, 0, q) = 0$ by (3.10), and since $z > 0$ by (2.7), we find that

$$\mathcal{H}(\theta, 0, N, q) \equiv -N[\mathrm{y}(N \sin\theta, N\cos\theta, 0, q)\cos\theta - \mathrm{z}(N\sin\theta, N\cos\theta, 0, q)\sin\theta] < 0 \tag{6.5}$$

for $N < 0$, $\cos\theta > 0$. From (6.4), (6.5) it follows that (6.2) has at least two roots $\psi_1' < 0$, $\psi_2' > 0$ for arbitrary fixed $N < 0$, q, $\theta \in (0, \pi/2)$. Now if a were to vanish for one of these roots, say ψ_1', then (6.2) would reduce to

$$\mathrm{m}(N\sin\theta, N\cos\theta, \psi_1'\sin\theta, q)\cos\theta - q\sin\theta = 0 \tag{6.6}$$

by (6.1), (3.19). Since $\partial \mathrm{m}/\partial u > 0$ by (3.11), we have

$$\mathrm{m}(N\sin\theta, N\cos\theta, \psi_2'\sin\theta, q)\cos\theta - q\sin\theta \neq 0, \tag{6.7}$$

so a could not vanish for ψ_2'. Since a and b cannot vanish simultaneously, we conclude

i) *Let* (6.3) *hold. Let* θ, N, q *be arbitrary with* $N\cos\theta < 0$, $\sin\theta \neq 0$. *Then there is a deformation with a helical axis that is not straight and another deformation with a helical axis that is not circular.*

The generalization to include the case $N > 0$, $\cos\theta < 0$ is an immediate consequence of the invariance of (6.2) under the transformation $(\theta, \psi', N, q) \to (\pi - \theta, -\psi', -N, -q)$. In other words, the two deformations differ only by a rigid displacement.

Now let $N < 0$ and q both be fixed. If θ is restricted to an interval of the form $[0, \theta_0]$ with $0 < \theta_0 < \pi/2$, then the roots of (6.2) fall in a bounded interval by virtue of (6.4), (6.5). This result, the continuity of y and z, and the inequality (2.7) imply that y is uniformly bounded as a function of θ for $\theta \in [0, \theta_0]$ and that z is uniformly bounded away from zero as a function of θ for $\theta \in [0, \theta_0]$. Thus if we take θ small enough we can ensure that

$$\lambda b = \mathrm{y}(N\sin\theta, N\cos\theta, \psi'\sin\theta, q)\sin\theta + \mathrm{z}(N\sin\theta, N\cos\theta\cdot\psi'\sin\phi, q)\cos\theta > 0 \tag{6.8}$$

when $\psi' = \psi_1', \psi_2'$. (Note that ψ_1', ψ_2' depend on θ as well as on N, q.) Since $b = dx_3/ds$ is positive and N is negative, this state is maintained by a terminal force that acts compressively. Hence

ii) *Let* (6.3) *hold. Let* $N \neq 0$ *and* q *be arbitrary. Then there is a number* $\delta > 0$ *depending on* N *and* q *such that there is a deformation with a nondegenerate helical axis for any prescribed* θ *satisfying* $0 < \sin\theta < \delta$.

We now study whether there can be a solution with a helical axis when $N = 0$. Let us prescribe $\theta \in (0, \pi/2)$ and $q \neq 0$. Then (6.1), (6.2) imply that

$$\mathcal{H}(\theta, \psi', 0, q) = \psi'[\mathrm{m}(0, 0, \psi'\sin\theta, q)\cos\theta - q\sin\theta] = 0. \tag{6.9}$$

Since $\partial \mathrm{m}/\partial u > 0$ and since m assumes all real values as u varies over all real numbers, the equation

$$\mathrm{m}(0, 0, \psi'\sin\theta, q) = q\tan\theta \tag{6.10}$$

has a unique nonzero solution ψ_0' with the same sign as q. Moreover, $\psi_0' \to 0$ as $\theta \to 0$.

Thus (6.9) has a unique nonzero solution ψ_0'. By choosing θ small enough, we can use the same uniformity arguments as above to ensure that

$$\lambda b = \mathrm{y}(0, 0, \psi_0' \sin \theta, q) \sin \theta + \mathrm{z}(0, 0, \psi_0' \sin \theta, q) \cos \theta > 0 \qquad (6.11)$$

so that the axis is not circular. The vanishing of a for this solution is equivalent to

$$\frac{\psi_0' \mathrm{y}(0, 0, \psi_0' \sin \theta, q)}{\psi_0' \sin \theta} = \frac{\mathrm{z}(0, 0, \psi_0' \sin \theta, q)}{\cos \theta}. \qquad (6.12)$$

The limit of the left-hand side of (6.12) as $\theta \to 0$ is

$$\partial \mathrm{y}/\partial u(0, 0, 0, q) \lim_{\theta \to 0} \psi_0' = 0,$$

whereas the limit of the right-hand side is $\mathrm{z}(0, 0, 0, q) \neq 0$. Thus for θ sufficiently small, (6.12) is violated and the axis is not straight either. In summary, we have

iii) *Let $q \neq 0$ be arbitrary. Then there is a number $\delta > 0$ depending on q such that there is a deformation with a nondegenerate helical axis maintained by zero terminal force for any prescribed θ satisfying $0 < \sin \theta < \delta$.*

We have the following related result:

iv) *Let $\psi' \neq 0$ be arbitrary and let* m *be such that*

$$q^{-1}\mathrm{m}(0, 0, \psi' \sin \theta, q) \to 0 \quad \text{as} \quad |q| \to \infty. \qquad (6.13)$$

Then there is a number $\delta > 0$ depending on ψ' such that there is a deformation with a nondegenerate helical axis maintained by zero terminal force for any prescribed θ satisfying $0 < \sin \theta < \delta$.

The proof of this relies on the fact that (6.13) implies that (6.10) has a solution $q_0 \neq 0$ for fixed $\psi' \neq 0$ and for fixed $\theta \in (0, \pi/2)$. Moreover (6.10) implies that

$$\bar{q}_0 - \psi' \frac{\partial \mathrm{m}}{\partial u}(0, 0, 0, \bar{q}_0) = 0, \qquad \bar{q}_0 \equiv \lim_{\theta \to 0} q_0. \qquad (6.14)$$

In Sec. 3 we showed the classical result that a deformation with a helical axis could always be maintained by a wrench applied along the x_3-axis. We now examine whether this deformation can be maintained solely by a force along the x_3-axis. This means that the couple (3.22) must vanish. We have already shown that the vanishing of the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of (3.22) is equivalent to (3.21) or (6.2). The vanishing of the $\mathbf{e}_3$ component implies that

$$\mathrm{m}(N \sin \theta, N \cos \theta, \psi' \sin \theta, q) \sin \theta + q \cos \theta = 0. \qquad (6.15)$$

Let $\theta \in (0, \pi/2)$, $\psi' \neq 0$, $N < 0$ be arbitrary. If (6.13) holds, we can solve (6.15) for q as a function of θ, ψ', N:

$$q = \mathrm{q}(\theta, \psi', N). \qquad (6.16)$$

Since $\partial \mathrm{m}/\partial u > 0$ and since $\mathrm{m} \to \pm\infty$ as $u \to \pm\infty$, we have that

$$\mathrm{q}(\theta, \psi', N) \to \pm\infty \quad \text{as} \quad \psi' \to \pm\infty. \qquad (6.17)$$

Substituting (6.15), (6.16) into (6.2), we obtain

$$\mathrm{J}(\theta, \psi', N) \equiv \psi'\mathrm{m} - N \cos \theta[\mathrm{y} \cos \theta - \mathrm{z} \sin \theta] = 0, \qquad (6.18)$$

where the arguments of m, y, z are

$$N \sin\theta, \qquad N\cos\theta, \qquad \psi'\sin\theta, \qquad \mathrm{q}(\theta, \psi', N).$$

We strengthen the coercivity condition on m and the conditions (6.3) by requiring that

$$\frac{u\mathrm{m}(n, p, u, q)}{|u|} \to \infty, \frac{\mathrm{y}(n, p, u, q)}{|u|} \to 0, \frac{\mathrm{z}(n, p, u, q)}{|u|} \to 0 \tag{6.19}$$

as $|u| \to \infty$ uniformly for all real q. It then follows that

$$\mathrm{J}(\theta, \psi', N) \to \infty \quad \text{as} \quad |\psi'| \to \infty. \tag{6.20}$$

Moreover, by following the argument leading to (6.5), we see that

$$\mathrm{J}(\theta, 0, N) < 0 \tag{6.21}$$

for $N < 0$, $\theta \in (0, \pi/2)$. We can now reproduce the arguments leading to (i), (ii), to obtain

v) *Let* (6.13), (6.19) *hold. Let* θ, N *be arbitrary with* $N \cos\theta < 0$, $\sin\theta \neq 0$. *Then there is a deformation with a helical axis that is not straight and another with a helical axis that is not circular. These deformations are maintained by the sole action of a terminal force applied along the* x_3*-axis. Moreover, there is a number* $\delta > 0$ *depending on* N *such that the first deformation does not have a circular axis either provided* θ *is prescribed such that* $0 < \sin\theta < \delta$.

One can obtain a corresponding result by solving (6.15) for ψ' as a function of θ, N, q under suitable growth restrictions and then proceeding in an analogous fashion.

7. Conclusion. In our work we used an exact fully nonlinear theory of one-dimensional elasticity. Our reward for suffering the complexity engendered by this generality was a wealth of nonlinear effects including those associated with the names of Poynting, Coulomb, Lüders, as well as nonuniqueness and nonexistence. A number of the effects were of the sort that the three-dimensional theory is not yet equipped to handle.

We note that a theory embodying the requirement (2.7) and the corresponding coercivity restriction of (2.20) must have constitutive relations depending nonlinearly on z. Moreover, on numerous occasions our analysis relied critically on (2.7), so that there were significant analytic advantages attending the use of a nonlinear theory. The simplest consistent nonlinear theory has constitutive functions of the form

$$\mathrm{y}(n, p, u, q) = An, \quad \mathrm{z}(n, p, u, q) = \mathfrak{z}(p), \quad \mathrm{m}(n, p, u, q) = Bu, \quad \mathrm{v}(n, p, u, q) = Cq, \tag{7.1}$$

where A, B, C are constants and the function $\mathfrak{z}$ satisfies

$$d\mathfrak{z}/dp > 0, \lim_{p\to-\infty} \mathfrak{z}(p) = 0, \lim_{p\to\infty} \mathfrak{z}(p) = \infty, \mathfrak{z}(0) = 1. \tag{7.2}$$

For this particular material, all the special problems treated in Secs. 4, 5, 6 become especially simple and the nonlinear effects evaporate.

We did not seek to make an exhaustive study of all possible conditions that would produce deformations with helical axes. Rather, we limited ourselves to giving the simplest sufficient conditions for such deformations. We accordingly used constitutive equations in the form (3.9), which seemed most natural for this goal. On the other hand, we did not postulate constitutive equations in the form (2.23), (2.24) because these are not the common way in which such equations are presented; we began with (2.16), (2.17), which are standard forms of "stress-strain" relations and which are closely related to the corresponding versions of three-dimensional nonlinear elasticity [5].

There are two obvious directions in which our results can be extended. The first is the study of helical deformations of initially helical rods, which is connected with the study of spiral springs [16, §271]. This would present no analytic novelties. The second is the study of helical deformations of an initially straight rod within the geometrically-richer director theory. (This would furnish the existence results for [12].) The requirement that director stretches and shear be constant should reduce such problems to tractable, though somewhat more complicated, versions of those we have treated here.

The helical solutions whose existence we have demonstrated may serve as "trivial" solutions in the study of the behavior of all solutions of boundary-value problems for these rods. In particular, they form a useful starting point for treating questions of multiplicity, stability, and qualitative behavior.

8. Appendix 1. Monotonicity conditions. We prove that the symmetric part of the matrix of partial derivatives of y_1^*, y_2^*, y_3^*, m_1^*, m_2^*, v^* with respect to n_1, n_2, n_3, u_1, u_2, q is positive definite. We actually prove this as a consequence of a general result of matrix theory, which, though simple, does not seem to be accessible in the literature.

Let **A**, **C**, **F**, **Y** be K-dimensional vectors and **B**, **G**, **U** be L-dimensional vectors. Let $\mathfrak{F}$ be a K-dimensional and $\mathfrak{G}$ an L-dimensional vector function of **Y**, **U**. Let $\partial\mathfrak{F}/\partial\mathbf{Y}$ denote the matrix of partial derivatives of the components of $\mathfrak{F}$ with respect to the components of **Y**, etc. We assume that the symmetric part of the matrix

$$\begin{pmatrix} \dfrac{\partial\mathfrak{F}}{\partial\mathbf{Y}} & \dfrac{\partial\mathfrak{F}}{\partial\mathbf{U}} \\[2ex] \dfrac{\partial\mathfrak{G}}{\partial\mathbf{Y}} & \dfrac{\partial\mathfrak{G}}{\partial\mathbf{U}} \end{pmatrix} \tag{8.1}$$

is positive definite, i.e., the quadratic form

$$\mathbf{A}\cdot\frac{\partial\mathfrak{F}}{\partial\mathbf{Y}}\mathbf{A} + \mathbf{A}\cdot\frac{\partial\mathfrak{F}}{\partial\mathbf{U}}\mathbf{B} + \mathbf{B}\cdot\frac{\partial\mathfrak{G}}{\partial\mathbf{Y}}\mathbf{A} + \mathbf{B}\cdot\frac{\partial\mathfrak{G}}{\partial\mathbf{U}}\mathbf{B} > 0 \tag{8.2}$$

for $\mathbf{A}\cdot\mathbf{A} + \mathbf{B}\cdot\mathbf{B} \neq 0$. Under these conditions the equations

$$\mathbf{F} = \mathfrak{F}(\mathbf{Y}, \mathbf{U}), \tag{8.3}$$

$$\mathbf{G} = \mathfrak{G}(\mathbf{Y}, \mathbf{U}), \tag{8.4}$$

can be solved locally for **Y**, **U** as functions of **F**, **G** (for **F**, **G** in the range of $\mathfrak{F}, \mathfrak{G}$) by the classical implicit function theorem. Moreover, (8.3) can be solved for **Y** as a function of **F**, **U**:

$$\mathbf{Y} = \mathbf{Y}^*(\mathbf{F}, \mathbf{U}). \tag{8.5}$$

Then we have

$$\mathbf{G} = \mathfrak{G}^*(\mathbf{F}, \mathbf{U}) \equiv \mathfrak{G}(\mathbf{Y}^*(\mathbf{F}, \mathbf{U}), \mathbf{U}). \tag{8.6}$$

THEOREM: Under these conditions, the symmetric part of

$$\begin{pmatrix} \dfrac{\partial \mathbf{Y}^*}{\partial \mathbf{F}} & \dfrac{\partial \mathbf{Y}^*}{\partial \mathbf{U}} \\ \dfrac{\partial \mathcal{G}^*}{\partial \mathbf{F}} & \dfrac{\partial \mathcal{G}^*}{\partial \mathbf{U}} \end{pmatrix} \tag{8.7}$$

is positive definite.

Proof. Solve (8.4) for $\mathbf{U}$ as a function of $\mathbf{Y}$, $\mathbf{G}$:

$$\mathbf{U} = \mathbf{U}^*(\mathbf{Y}, \mathbf{G}). \tag{8.8}$$

From the identities

$$\mathbf{F} = \mathcal{F}(\mathbf{Y}^*(\mathbf{F}, \mathbf{U}), \mathbf{U}), \qquad \mathbf{G} = \mathcal{G}(\mathbf{Y}, \mathbf{U}(\mathbf{Y}, \mathbf{G})) \tag{8.9}$$

it follows that

$$\mathbf{I} = \frac{\partial \mathcal{F}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}}, \quad \mathbf{0} = \frac{\partial \mathcal{F}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}^*}{\partial \mathbf{U}} + \frac{\partial \mathcal{F}}{\partial \mathbf{U}}, \tag{8.10a, b}$$

$$\mathbf{0} = \frac{\partial \mathcal{G}}{\partial \mathbf{Y}} + \frac{\partial \mathcal{G}}{\partial \mathbf{U}} \frac{\partial \mathbf{Y}}{\partial \mathbf{Y}}, \quad \mathbf{I} = \frac{\partial \mathcal{G}}{\partial \mathbf{U}} \frac{\partial \mathbf{Y}}{\partial \mathbf{G}}, \tag{8.10c, d}$$

where $\mathbf{I}$ represents the identity matrix and $\mathbf{0}$ the zero matrix of the right dimension. If we now take

$$\mathbf{A} = -\left(\frac{\partial \mathcal{G}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}}\right)^T \mathbf{B} - \left(\frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}}\right)^T \mathbf{C}, \tag{8.11}$$

where T denotes the transpose, then by virtue of (8.10), the quadratic form of (8.2) becomes

$$\mathbf{C} \cdot \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}} \mathbf{C} + \mathbf{C} \cdot \frac{\partial \mathbf{Y}^*}{\partial \mathbf{U}} \mathbf{B} + \mathbf{B} \cdot \frac{\partial \mathcal{G}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}} \mathbf{C} + \mathbf{B} \cdot \left[\frac{\partial \mathcal{G}}{\partial \mathbf{U}} - \frac{\partial \mathcal{G}}{\partial \mathbf{Y}} \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}} \frac{\partial \mathcal{F}}{\partial \mathbf{U}}\right] \mathbf{B}. \tag{8.12}$$

Applying the chain rule to (8.6) and substituting the resulting expression together with (8.10b) into (8.12), we reduce (8.12) to

$$\mathbf{C} \cdot \frac{\partial \mathbf{Y}^*}{\partial \mathbf{F}} \mathbf{C} + \mathbf{C} \cdot \frac{\partial \mathbf{Y}^*}{\partial \mathbf{U}} \mathbf{B} + \mathbf{B} \cdot \frac{\partial \mathcal{G}^*}{\partial \mathbf{F}} \mathbf{C} + \mathbf{B} \cdot \frac{\partial \mathcal{G}^*}{\partial \mathbf{U}} \mathbf{B}, \tag{8.13}$$

which is the quadratic form for (8.7) and is positive for $\mathbf{C} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{B} \neq 0$ by (8.2). Q.E.D.

We remark that if we set $\mathbf{C} = \mathbf{0}$ in (8.12), then the matrix

$$(\partial \mathcal{G}/\partial \mathbf{U}) - (\partial \mathcal{G}/\partial \mathbf{Y})(\partial \mathbf{Y}^*/\partial \mathbf{F})(\partial \mathcal{F}/\partial \mathbf{U})$$

has a positive-definite symmetric part. We may regard this matrix as essentially a matrix-valued determinant of (8.1).

To obtain results for our problem, we merely identify

$$\mathbf{Y} = (y_1, y_2, y_3, v), \quad \mathbf{U} = (u_1, u_2), \quad \mathbf{F} = (n_1, n_2, n_3, q), \quad \mathbf{G} = (m_1, m_2). \tag{8.14}$$

In this case (8.13) implies that

$$\alpha_a \left(\frac{\partial y_a^*}{\partial n_b} \alpha_b + \frac{\partial y_a^*}{\partial u_\rho} \beta_\rho + \frac{\partial y_a^*}{\partial q} \gamma\right) + \beta_\sigma \left(\frac{\partial m_\sigma^*}{\partial n_b} \alpha_b + \frac{\partial m_\sigma^*}{\partial u_\rho} \beta_\rho + \frac{\partial m_\sigma^*}{\partial q} \gamma\right) + \gamma \left(\frac{\partial v^*}{\partial n_b} \alpha_b + \frac{\partial v^*}{\partial u_\rho} \beta_\rho + \frac{\partial v^*}{\partial q} \gamma\right) > 0 \tag{8.15}$$

for $a_a\alpha_a + \beta_\rho\beta_\rho + \gamma^2 \neq 0$. We note that there are appropriate implicit function theorems for these new mappings.

9. Appendix 2. Representation of the isotropic constitutive functions. We require the constitutive functions of (2.23), (2.25) to satisfy the isotropy conditions (2.27). Our special representations rely on relations (3.7), (3.8) which prevail for the Kirchhoff solution. We rewrite these here:

$$n_1 = -n \cos \phi, \quad n_2 = n \sin \phi, \quad u_1 = -u \cos \phi, \quad u_2 = u \sin \phi. \tag{9.1}$$

We choose

$$(Q_{\rho\sigma}) = \begin{pmatrix} -\cos \phi & \sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}. \tag{9.2}$$

Then (2.27a) yields

$$y_1^*(-n \cos \phi, n \sin \phi, p, -u \cos \phi, u \sin \phi, q) = -\cos \phi\, y_1^*(n, 0, p, u, 0, q) + \sin \phi\, y_2^*(n, 0, p, u, 0, q), \tag{9.3}$$

$$y_2^*(-n \cos \phi, n \sin \phi, p, -u \cos \phi, u \sin \phi, q) = \sin \phi\, y_1^*(n, 0, p, u, 0, p) + \cos \phi\, y_2^*(n, 0, p, u, 0, p). \tag{9.4}$$

Setting $\phi = \pi$ in (8.4) and $\phi = 0$ in (8.3), we obtain

$$y_2^*(n, 0, p, u, 0, q) = -y_2^*(n, 0, p, u, 0, q) = 0, \tag{9.5}$$

$$y_1^*(-n, 0, p, -u, 0, q) = -y_1^*(n, 0, p, u, 0, q). \tag{9.6}$$

We define

$$y(n, p, u, q) \equiv y_1^*(n, 0, p, u, 0, q). \tag{9.7}$$

Thus the first two equations of (2.23) have the reduced forms

$$y_1 = -\cos \phi\, y(n, p, u, q), \qquad y_2 = \sin \phi\, y(n, p, u, q), \qquad y(-n, p, -u, q) = -y(n, p, u, q). \tag{9.8}$$

Similarly we obtain reduced forms for the remaining equations of (2.23), (2.25):

$$z = z(n, p, u, q), \qquad z(-n, p, -u, q) = z(n, p, u, q), \tag{9.9}$$

$$m_1 = -\cos \phi\, m(n, p, u, q), \qquad m_2 = \sin \phi\, m(n, p, u, q), \qquad m(-n, p, -u, q) = -m(n, p, u, q), \tag{9.10}$$

$$v = v(n, p, u, q), \qquad v(-n, p, -u, q) = v(n, p, u, q). \tag{9.11}$$

We next observe that y, z, m, v constitute a monotone mapping of n, p, u, q, i.e., the symmetric part of the matrix of partial derivatives of y, z, m, v with respect to n, p, u, q is positive definite. To show this we substitute

$$\alpha_1 = -\alpha \cos \phi, \qquad \alpha_2 = \alpha \sin \phi, \qquad \beta_1 = -\beta \cos \phi, \qquad \beta_2 = \beta \sin \phi \tag{9.12}$$

into (8.15). Since

$$\begin{gathered} \mathrm{y} = -\cos \phi \mathrm{y}_1^* + \sin \phi \mathrm{y}_2^*, \ \mathrm{m} = -\cos \phi \mathrm{m}_1^* + \sin \phi \mathrm{m}_2^*, \\ \frac{\partial}{\partial n} = -\cos \phi \frac{\partial}{\partial n_1} + \sin \phi \frac{\partial}{\partial n_2}, \ \frac{\partial}{\partial u} = -\cos \phi \frac{\partial}{\partial u_1} + \sin \phi \frac{\partial}{\partial u_2}, \end{gathered} \tag{9.13}$$

the resulting form of (8.15) reduces to the quadratic form for the matrix of derivatives of y, z, m, v with respect to n, p, u, q.

References

[1] S. S. Antman, *Existence of solutions of the equilibrium equations for nonlinearly elastic elastic rings and arches*, Indiana Univ. Math. J. **20**, 281–302 (1970)

[2] ———, *Existence and nonuniqueness of axisymmetric equilibrium states of nonlinearly elastic shells*, Arch. Rat. Mech. Anal. **40**, 329–372 (1971)

[3] ———, *The theory of rods*, Handbuch der Physik VIa/2, Springer-Verlag, Berlin, 1972

[4] ———, *Qualitative theory of the ordinary differential equations of nonlinear elasticity*, Mechanics Today **1** (1972), Pergamon, Oxford

[5] ———, *One-dimensional boundary value problems of nonlinear elasticity*, in preparation

[6] F. E. Browder, *Existence theorems for nonlinear partial differential equations*, Global Analysis III, American Math. Soc., Providence, 1970

[7] G. F. Carrier, *On the buckling of elastic rings*, J. Math. Phys. **26**, 94–103 (1947)

[8] A Clebsch, *Theorie der Elasticität fester Körper*, Leipzig, 1862

[9] H. Cohen, *A non-linear theory of elastic directed curves*, Int. J. Eng. Sci. **4**, 511–524 (1966)

[10] E. and F. Cosserat, *Théorie des corps déformables*, Hermann, Paris, 1909

[11] C. N. DeSilva and A. B. Whitman, *A thermodynamic theory of directed curves*, J. Math Phys. **12**, 1603–1609 (1971)

[12] J. L. Ericksen, *Simpler static problems in nonlinear theories of rods*, Int. J. Solids Structures **6**, 371–377 (1970)

[13] L. Euler, *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes*, Lausanne, 1744

[14] A. E. Green and N. Laws, *A general theory of rods*, Proc. Roy. Soc. (London) **A293**, 145–155 (1966)

[15] G. Kirchhoff, *Über des Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes*, J. reine angew. Math. (Crelle) **56**, 285–313 (1859)

[16] A. E. H. Love, *A treatise on the mathematical theory of elasticity*, 4th ed., Dover, New York, 1944

[17] A. Nadai, *Theory of flow and fracture of solids*, Vol. I, 2nd ed., McGraw-Hill, New York, 1950

[18] B. de St. Venant, *Mémoire sur la torsion des prismes, etc.*, Mém. de Savants étrangers **14**, 233–560 (1855)

[19] J. T. Schwartz, *Lectures on nonlinear functional analysis*, New York Univ., 1964

[20] C. Truesdell, *A new chapter in the theory of the elastica*, Proc. First Midwestern Conf. on Solid Mechanics 1953, 52–55, 1954

[21] C. Truesdell and W. Noll, *The nonlinear field theories of mechanics*, Handbuch der Physik III/3, Springer-Verlag, Berlin, 1965

[22] A. B. Whitman and C. N. DeSilva, *An exact solution in a nonlinear theory of rods*, to appear