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# Klein–Gordon oscillator in Kaluza–Klein theory

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**Abstract** In this contribution we study the Klein–Gordon oscillator on the curved background within the Kaluza–Klein theory. The problem of the interaction between particles coupled harmonically with topological defects in Kaluza–Klein theory is studied. We consider a series of topological defects, then we treat the Klein–Gordon oscillator coupled to this background, and we find the energy levels and corresponding eigenfunctions in these cases. We show that the energy levels depend on the global parameters characterizing these spacetimes. We also investigate a quantum particle described by the Klein–Gordon oscillator interacting with a cosmic dislocation in Som–Raychaudhuri spacetime in the presence of homogeneous magnetic field in a Kaluza–Klein theory. In this case, the energy spectrum is determined, and we observe that these energy levels represent themselves as the sum of the terms related with Aharonov–Bohm flux and of the parameter associated to the rotation of the spacetime.

## 1 Introduction

Harmonic interactions play an important role in physics mainly when we consider the motion of particles in the presence of molecular potentials and electromagnetic fields. Particularly, the harmonic oscillator appears as a prototype model in many areas of physics such as solid states physics, quantum statistical mechanics, and quantum field theory, and indeed it serves as an important physical example to study the concepts and mathematical tools in standard quantum physics. At the same time, within relativistic quantum mechanics, the effects introduced by the peculiar motion of the particles in physical system at the high energy can be considered. Together, quantum and relativistic effects have received a great deal of attention within the quantum treatment of dynamics of particles occurring in backgrounds produced by topological defects [1–4]. A well-known version for relativistic harmonic oscillator was proposed in Ref. [5] for a spin-1/2 particle. This oscillator was named a Dirac oscillator. In the non-relativistic limit this model has a behavior of harmonic oscillator with a very strong spin–orbit coupling term. This Dirac oscillator is characterized by a new coupling of the momentum of the particle that is linear in the coordinates. Most recently, the relativistic harmonic oscillator was studied in a commutative and noncommutative field theory [5,6] among other configurations, including magnetic fields [7–9]. The Dirac oscillator was investigated for a spin-1/2 particle in the presence of topological defects in Refs. [10–18]. However, these studies were carried out for the quantum dynamics of spin-1/2 particles, leaving a gap in the treatment of harmonic interaction for relativistic scalar particles.

Several studies have demonstrated an interest in relativistic models [19–22] where the interaction potential is similar to the one of the harmonic oscillator, such as the vibrational spectrum of diatomic molecules [23], the binding of heavy quarks [24,25], and the oscillations of atoms in crystal lattices, by mapping them as a position-dependent mass system [26–29]. The importance of these potentials arises from the presence of a strong potential field. Recently, Bahar and Yasuk [19], intending to study the quark–antiquark interaction [30,31], have investigated the problem of a relativistic spin-0 particle possessing a position-dependent mass, where the mass term acquires a contribution given by an interac-

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tion potential that consists of linear and harmonic confining potentials plus a Coulomb potential term.

The Klein–Gordon oscillator [32,33] was inspired by earlier papers on the Dirac oscillator [5] applied to half-integer spin particles. Recently Rao et al. [34] studied the spectral distribution of energy levels and eigenfunctions sets describing the state of a particle by solving the Klein–Gordon equation in one-dimensional version of Minkowski spacetimes. In a recent paper, Boumali and Messai [35] have investigated a Klein–Gordon oscillator in the background of cosmic strings in the presence of a uniform magnetic field. The Klein–Gordon oscillator was investigated in the presence of a Coulomb potential considering two ways of coupling of Coulomb potential within the Klein–Gordon equation: in the first paper [36] the Coulomb potential is incorporated in the equation by a modification mass term, in the second model [37] this potential is introduced in the Klein–Gordon equation via the minimal coupling, in the latter case the linear potential also was included in the equation. Our intention now is to extend these studies not only to other dimensions but mostly to consider this dynamics in general background spacetimes produced by topological defects using Kaluza–Klein theory [2,38–42]. These sources of gravitational fields play an important role in condensed matter physics systems [43–46], mainly due to the possibility to compensate for the elastic contribution introduced by the defect by fine tuning of the external magnetic field. Besides topological defects like cosmic strings [47] and domain walls [48], a global monopole [49] provides a tiny relation between the effects in cosmology and gravitation and those in condensed matter physics systems, where topological defects analogous to cosmic strings appear in phase transitions in liquid crystals [50,51]. Recently, the Klein–Gordon oscillator in the Som–Raychaudhuri spacetime in the presence of uniform magnetic fields was investigated by Wang et al. [52].

This contribution is organized as follows: in Sect. 2, using the Kaluza–Klein theory, we study the energy levels of particles interacting with a gravitational field produced by a cosmic string in the presence of a Klein–Gordon oscillator. In Sect. 3, we study the quantum dynamics in the presence of the magnetic cosmic string, calculating the energy spectrum as well as the corresponding eigenfunctions. In Sect. 4, we consider the case in which the background has a torsional source of a gravitational field added to the curvature source introduced by the conical defect. To consider the contribution introduced by a magnetic field to this dynamics, in Sect. 5, we consider a homogeneous magnetic field filling the space accessible to the Klein–Gordon particle. Additionally to the magnetic field we consider rotational spacetimes, whose rotation is introduced via a geometric description of Kaluza–Klein theory. In Sect. 6, some discussion of our results will be presented. Throughout the article we will consider the system of units where $\hbar = c = G = 1$.

## 2 Klein–Gordon oscillator in cosmic string background

The purpose of this section is to study the Klein–Gordon oscillator in the background of the cosmic string with use of Kaluza–Klein theory [2,38–40]. A first study of a topological defect in Kaluza–Klein theory was carried out in Ref. [40], where the authors have investigated a series of cylindrically symmetric solutions of Einstein and Einstein–Gauss–Bonnet equations. In [40], one has found various solutions of a cosmic string form in five dimension, such as: the neutral cosmic string, cosmic dislocation, and superconducting cosmic and multi-cosmic string spacetimes. The metric corresponding to this geometry can be written

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2 \rho^2 d\phi^2 + dz^2 + dx^2 \tag{1}$$

where $t$ is the time coordinate, $x$ is the coordinate associated with the fifth additional dimension, and $(\rho, \phi, z)$ are cylindrical coordinates. These coordinates assume, respectively, the following ranges: $-\infty < (t, z) < \infty$, $0 \leq \rho < \infty$, $0 \leq \phi \leq 2\pi$, $0 < x < 2\pi a$, where $a$ is the radius of the compact dimension $x$. The parameter $\alpha$, characterizing the cosmic string, is given in terms of the mass density of the string $\mu$, by $\alpha = 1 - 4\mu$ [47,53]. Cosmology and gravitation imposes limits to the range of the $\alpha$ parameter, which is restricted to $\alpha < 1$ [47]. Moreover, in condensed matter physics systems, this restriction is free and the opposite case $\alpha > 1$, the well-known negative disclination case [54], can occur in several systems as those described by [51].

To couple the Klein–Gordon oscillator [32,33] to this background we use the generalization of the Mirza and Mohadesi prescription [6], in which we carry out the following change in the momentum operator:

$$p_\mu \to (p_\mu + i M w X_\mu) \tag{2}$$

where we have defined in polar coordinates $X_\mu = (0, \rho, 0, 0, 0)$, $\rho$ is the transverse distance from the particle to the defect. In this way, the general Klein–Gordon equation becomes

$$\left\{ \frac{1}{\sqrt{-g}} (\partial_\mu + M w X_\mu) \sqrt{-g} g^{\mu\nu} (\partial_\nu - M w X_\nu) - M^2 \right\} \Psi = 0 \tag{3}$$

with $g$ being the determinant of the metric tensor and $g^{\mu\nu}$ the inverse metric tensor; $\partial_{\mu,\nu}$ are partial derivatives with respect to the coordinates. The matrix $g^{\mu\nu}$ is of the form

$$g^{\mu\nu} = \text{diag}(-1, 1, 1/\alpha^2 \rho^2, 1, 1), \tag{4}$$

and in this way Eq. (3) becomes

$$\left\{ \frac{1}{\rho} \partial_\rho (\rho \partial_\rho) + \frac{\partial_\phi^2}{\alpha^2 \rho^2} - M^2 w^2 \rho^2 + \gamma \right\} \Psi = 0 \tag{5}$$

with $\gamma = -\partial_t^2 + \partial_z^2 + \partial_x^2 - 2Mw - M^2$. $x$ is the fifth spatial coordinate in Kaluza–Klein theory. Assuming a temporal independence of our background and translational symmetry along the axes of $x$ and $z$, we can choose the following *ansatz*:

$$\Psi = e^{-i(Et - kz - l\phi - \lambda x)} R(\rho). \quad (6)$$

In this way, Eq. (5) transforms into

$$\left\{ \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \left[ \frac{l^2}{\alpha^2\rho^2} + M^2w^2\rho^2 - \gamma \right] \right\} R(\rho) = 0, \quad (7)$$

so that, with the help of (6) the last term in the above equation is rewritten as $\gamma = E^2 - M^2 - 2Mw - k^2 - \lambda^2$. This equation can be transformed into another one by a change of the variable $\xi = Mw\rho^2$. The result is

$$R''(\xi) + \frac{1}{\xi}R'(\xi) - \left( \frac{l^2}{4\alpha^2\xi^2} + \frac{1}{4} - \frac{\gamma}{4Mw\xi} \right) R(\xi) = 0. \quad (8)$$

Now we proceed with a study of the asymptotic limit at the origin and at infinity. In the following, we assume

$$R(\xi) = \xi^{\frac{|l|}{2\alpha}} e^{-\frac{\xi}{2}} F(\xi). \quad (9)$$

Substituting of $R(\xi)$ in this form in Eq. (8) results in

$$\xi F''(\xi) + \left( \frac{|l|}{\alpha} + 1 - \xi \right) F'(\xi)$$
$$- \left( \frac{l}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw} \right) F(\xi) = 0 \quad (10)$$

where $\gamma$ was defined before. We can see that this equation is just the one for the confluent hypergeometric function, $xF''(x) + (c + 1 - x)F'(x) - aF(x) = 0$, whose solution is a polynomial of degree $n$. Naturally there is a convergence problem in this solution when $n$ tends to infinity. To avoid this divergence, we can choose the independent term, the last term, in Eq. (10) to be equal to a non-negative number. Mathematically we have

$$\frac{|l|}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw} = -n, \quad (11)$$

and using the respective expressions for $\gamma$ and solving the resultant equation for $E$ we obtain the eigenvalues for this problem,

$$E^2 = M^2 + 4Mw\left( n + \frac{|l|}{2\alpha} + 1 \right) + k^2 + \lambda^2, \quad (12)$$

and the eigenfunctions sets,

$$\Psi(\vec{r}, t) = C_{n,l}\, e^{-i(Et - kz - l\phi - \lambda x)} \rho^{\frac{|l|}{\alpha}} e^{-\frac{Mw}{2}\rho^2} F$$
$$\times \left( -n, \frac{|l|}{\alpha} + 1, Mw\rho^2 \right). \quad (13)$$

Note that the dependence on nonlocal parameters of the background for energy levels as well as for the eigenfunctions is

responsible in breaking of the degeneracy of the energy levels due to the presence of the parameter $\alpha$. We observe that the present result is similar to the one obtained by Boumali and Messai [35] for the Klein–Gordon oscillator in the background of the cosmic string in Einstein gravity. Further, in the weak oscillator limit $w \to 0$, our particles behave like free particles. Moreover, if the flat spacetime limit $\alpha \longrightarrow 1$ is taken the results in [34] are reproduced.

## 3 Klein–Gordon oscillator in the background of a magnetic cosmic string in a Kaluza–Klein theory

Now, let us consider the quantum dynamics of a particle moving in the magnetic cosmic string background. In the Kaluza–Klein theory [2,38,39], the corresponding metric with a magnetic flux $\Phi$ passing along the symmetry axis of the string assumes the following form:

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\phi^2 + dz^2 + \left( dx + \frac{\Phi}{2\pi}d\phi \right)^2 \quad (14)$$

where cylindrical coordinates are used. The quantum dynamics is described by Eq. (3) with the following change in the inverse matrix tensor $g^{\mu\nu}$:

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha^2\rho^2} & 0 & -\frac{\Phi}{2\pi\alpha^2\rho^2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{\Phi}{2\pi\alpha^2\rho^2} & 0 & 1 + \frac{\Phi^2}{4\pi^2\alpha^2\rho^2} \end{pmatrix}. \quad (15)$$

In this way Eq. (3) becomes

$$\left\{ \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) + \frac{[\partial_\phi - (\Phi/2\pi)\partial_x]^2}{\alpha^2\rho^2} - M^2w^2\rho^2 + \gamma \right\} = 0 \quad (16)$$

with $\gamma = -\partial_t^2 + \partial_z^2 + \partial_x^2 - 2Mw - M^2$. Assuming a temporal independence of our background, we can choose the following ansatz:

$$\Psi = e^{-i(Et - kz - l\phi - \lambda x)} R(\rho); \quad (17)$$

thus, Eq. (16) transforms into

$$\left\{ \frac{1}{\rho}\partial_\rho(\rho\partial_\rho) - \left[ \frac{(l - \lambda\Phi/2\pi)^2}{\alpha^2\rho^2} + M^2w^2\rho^2 - \gamma \right] \right\} R(\rho) = 0, \quad (18)$$

so that with the help of (17) the last term in the above equation becomes $\gamma = E^2 - M^2 - 2Mw - k^2 - \lambda^2$. After the change of variables $x = Mw\rho^2$, this equation takes the form

$$R''(x) + \frac{1}{x}R'(x) - \left(\frac{\beta^2}{4\alpha^2 x^2} + \frac{1}{4} - \frac{\gamma}{4Mwx}\right)R(x) = 0.$$
(19)

Now, we study the behavior of our function in the small $x$ and large $x$ limits. Asymptotically at $x \to 0$ we find the following expression:

$$R''(x) + \frac{1}{x}R'(x) - \frac{\beta^2}{4\alpha^2 x^2}R(x) = 0,$$
(20)

whose solution is given by $R(x) = x^{\frac{|\beta|}{2\alpha}}$. In the other limit, $x \to \infty$, the resultant equation is

$$R''(x) - \frac{1}{4}R(x) = 0,$$
(21)

which yields $R(x) = e^{-\frac{x}{2}}$. A general solution can be determined by choosing

$$R(x) = x^{\frac{|\beta|}{2\alpha}}e^{-\frac{x}{2}}F(x).$$
(22)

Substitution of this expression into Eq. (19) results in

$$xF''(x) + \left(\frac{|\beta|}{\alpha} + 1 - x\right)F'(x)$$
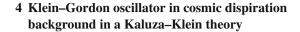$$- \left(\frac{\beta}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw}\right)F(x) = 0,$$
(23)

where $\gamma$ is as before and $\beta = l - \lambda\Phi/2\pi$. Again this equation is of the same form as the one describing the confluent hypergeometric function $xF''(x) + (c+1-x)F'(x) - aF(x) = 0$. Requiring the convergence of the corresponding series, we arrive at the following condition:

$$\frac{\beta}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw} = -n.$$
(24)

Using the respective expressions for $\beta$ and $\gamma$ and solving the resultant equation for $E$, we obtain the following eigenvalues problems:

$$E^2 = M^2 + 4Mw\left(n + \frac{|l - \lambda\Phi/2\pi|}{2\alpha} + 1\right) + k^2 + \lambda^2.$$
(25)

Compared to the case without magnetic flux string in the center of the defect, the angular quantum number is shifted by a quantity related with the magnetic field due to the presence of an Aharonov–Bohm flux in a magnetic string. Additionally, we can see that the presence of the defects breaks the degeneracy of the energy levels due to the presence of the curvature source. Besides, in the absence of magnetic fields, $\Phi = 0$, the results of the previous section are obtained. Note that the presence of a magnetic string and the Aharonov–Bohm flux modifies the energy spectrum; this effect is known as the Aharonov–Bohm effect for a bound state [41,55]. In fact, the energy levels are shifted by a quantity proportional to the Aharonov–Bohm flux.

## 4 Klein–Gordon oscillator in cosmic dispiration background in a Kaluza–Klein theory

Now we investigate the Klein–Gordon oscillator in a cosmic dispiration background in Kaluza–Klein theory [2]. Let us study the concurrency of gravitational effects due to the torsion and curvature and electromagnetic contributions due to the presence of this topological defect. In this way, we consider the magnetic cosmic string with torsion source besides curvature and electromagnetic ones. The corresponding background is described by the metric [2]

$$ds^2 = -dt^2 + d\rho^2 + \alpha^2\rho^2 d\phi^2 + (dz + Jd\phi)^2$$
$$+ \left(dx + \frac{\Phi}{2\pi}d\phi\right)^2$$
(26)

with $\alpha$, $J$, and $\Phi$, respectively, the sources of curvature, torsion, and the electromagnetic field. The coordinates $(t, \rho, \phi, z, x)$ are defined as before, and the inverse metric tensor is

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\alpha^2\rho^2} & -\frac{J^2}{\alpha^2\rho^2} & -\frac{\Phi}{2\pi\alpha^2\rho^2} \\ 0 & 0 & \frac{J}{\alpha^2\rho^2} & 1+\frac{J^2}{\alpha^2\rho^2} & \frac{\Phi J}{2\pi\alpha^2\rho^2} \\ 0 & 0 & -\frac{\Phi}{2\pi\alpha^2\rho^2} & \frac{\Phi J}{2\pi\alpha^2\rho^2} & \left(1+\frac{\Phi^2}{4\pi^2\alpha^2\rho^2}\right) \end{pmatrix}.$$
(27)

In this way Eq. (3) becomes

$$\left\{\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{(\partial_\phi - J\partial_z - \Phi\partial_y/2\pi)^2}{\alpha^2\rho^2} - M^2w^2\rho^2 + \gamma\right\}$$
$$\times \Psi(t,\vec{r}) = 0,$$
(28)

with $\gamma = -\partial_t^2 + \partial_z^2 + \partial_x^2 - 2Mw - M^2$. This partial differential equation does not involve an explicit dependence on the variables $t, \phi, z$, and $x$, and therefore, has translational symmetry around these axes. These properties allow us to suppose a general solution of Eq. (28) in the form $e^{-iEt+il\phi+ikz+i\lambda y}R(\rho)$. Thus,

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{dR(\rho)}{d\rho}\right] - \left[\frac{\Lambda^2}{\alpha^2\rho^2} - M^2w^2\rho^2 - \gamma\right]R(\rho) = 0.$$
(29)

Now $\Lambda = l - Jk - \lambda\Phi/2\pi$ and $\gamma = E^2 - M^2 - 2Mw - k^2 - \lambda^2$. Taking $\xi = Mw\rho^2$, this radial equation transforms into

$$\frac{d^2R(\xi)}{d\xi^2} + \frac{1}{\xi}\frac{dR(\xi)}{d\xi} - \left(\frac{\Lambda^2}{4\alpha^2\xi^2} + \frac{1}{4} - \frac{\gamma}{4Mw\xi}\right)R(\xi) = 0.$$
(30)

An analysis of the divergence at the origin and at infinity suggests the use of the general solution $R(\xi) = \xi^{\frac{|\Lambda|}{2\alpha}}e^{-\frac{\xi}{2}}F(\xi)$.

After some calculations, we obtain the final equation for $F(\xi)$, which should have the same form as the confluent hypergeometric equation,

$$\xi \frac{\mathrm{d}^2 F(\xi)}{\mathrm{d}\xi^2} + (c + 1 - \xi) \frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi} - aF(\xi) = 0,$$

$$\xi \frac{\mathrm{d}^2 F(\xi)}{\mathrm{d}\xi^2} + \left(\frac{|\Lambda|}{\alpha} + 1 - \xi\right) \frac{\mathrm{d}F(\xi)}{\mathrm{d}\xi}$$

$$- \left(\frac{|\Lambda|}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw}\right) F(\xi) = 0$$

$$\Lambda = l - Jk - \frac{\lambda\Phi}{2\pi}. \tag{31}$$

A solution for this equation is a polynomial of degree $n$ in $\xi$ of the form $F(\xi) = \sum a_n \xi^n$. We can see that this solution diverges for all values of $n$, which represents the degree of the hypergeometric series. This divergence is avoided by a truncation method in the coefficients $a_n$. If we assume that $a_n = 0$ for a finite numbers of terms in the polynomial series, then we guarantee normality of our solution in $\xi$ to $\xi \to 0$ and avoid the divergence to $\xi \to \infty$. With a general expression for the coefficients $a_n$, we conclude that to ensure this integrability, we must make $a = -n$. Thus,

$$\frac{|\Lambda|}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4Mw} = -n. \tag{32}$$

This condition gives us the energy levels and eigenfunctions for our scalar particle in the form

$$E^2 = M^2 + 4M\left(n + \frac{|l - Jk - \lambda\Phi/2\pi|}{2\alpha} + 1\right)w + k^2 + \lambda^2 \tag{33}$$

and the eigenfunctions sets

$$\Psi(\vec{r}, t) = C_{n,l} \, e^{-i(Et - kz - l\phi - \lambda x)} \rho^{\frac{|\Lambda|}{\alpha}} e^{-\frac{Mw}{2}\rho^2} F$$

$$\times \left(-n, \frac{|\Lambda|}{\alpha} + 1, Mw\rho^2\right). \tag{34}$$

We have now an important result: all global parameters of the background appear in the energy levels of the Klein–Gordon oscillator; however, this background can be locally flat. The energy degeneracy is absent due to the presence of curvature and torsion sources. As pointed out previously, this result allows us to compensate for the torsion contribution by adjusting the magnetic field appropriately, and the medium becomes torsion-free. Note that the term $\lambda\Phi/2\pi$ in (33) is responsible for the electromagnetic Aharonov–Bohm effect [41,55] for a bound state due to the presence of magnetic flux $\Phi$ due to a solenoid field in the extra dimension. The term $Jk$ in (33) is present due to a torsion of the topological defect and $J$ is associated with the Burgers vector of cosmic dispiration. The quantum effect associated to this term in the energy spectrum of Klein–Gordon oscillator is responsible for the shift in this level known as the gravitational Aharonov–Bohm effect

[2,3] due to the torsion of this spacetime. In this way, we have three different contributions which modify the energy levels of the Klein–Gordon oscillator. The first one is due to the conical nature of spacetime, represented by the deficit angle $\alpha$. The second is due to the contribution of the torsion represented by $J$ and the third is due to the electromagnetic field represented by $\Phi$, in this form $E(\alpha, J, \Phi)$.

## 5 KG oscillator in cosmic dislocation in Som–Raychaudhuri spacetime in Kaluza–Klein theory

Recent observational data indicate rotation as well as expansion of our universe. This dynamics has called a great attention to the establishment of a theory aimed to describe these scenarios. One of these theories was developed by Gödel in the 1950s for an universe with rigid rotation characterized by a term $\Omega$ in the metric and with a curvature source known as a Weyssenhoff–Raabe fluid [56,57]. Some studies of the quantum dynamics in this spacetime were carried out for $(3 + 1)$-dimensional spacetimes [58–60]. In this section we consider a cosmic dispiration in a flat Gödel solution or Som–Raychaudhuri solution in Kaluza–Klein theory. We assume that the charged scalar particle is exposed to a uniform magnetic field. This field is also introduced using the Kaluza–Klein theory via the geometry of the spacetime. Due to the importance of the rotation in actual scenarios, we can use the Kaluza–Klein theory to describe the quantum dynamics of a Klein–Gordon particle in this spinning background. Now, we present a new solution for the Som–Raychaudhuri spacetime with a topological defect of a cosmic dispiration type, localized parallel to the rotation axis. We have considered this solution in Kaluza–Klein theory and have introduced this by extra dimensions, an Aharonov–Bohm flux and a uniform magnetic field. In contrast with the previous section where we have studied quantum dynamics in a topological defect background, here we consider the influence of the introduction of a cosmic string in a Gödel-type universe. Let us consider the Som–Raychaudhuri solution of the Einstein field equation [57] with a cosmic dispiration, described in a Kaluza–Klein theory with a spinning torsion source along the symmetry axis of background spacetimes,

$$\mathrm{d}s^2 = -(\mathrm{d}t + \alpha\Omega\rho^2 \mathrm{d}\phi)^2 + \mathrm{d}\rho^2 + \alpha^2\rho^2 \mathrm{d}\phi^2 + (\mathrm{d}z + J\mathrm{d}\phi)^2$$

$$+ [\mathrm{d}x + (\Phi/2\pi + eB\rho^2/2)\mathrm{d}\phi]^2. \tag{35}$$

Here $\Omega$ is associated with the rotational source of the space. This solution is a backbone of a cosmology occurring at a large scale. The rotation parameter $\Omega$ characterizes this scale; this parameter also can viewed as the scale of a magnetic-type, or twist, gravitational field [59,61]. This spacetime is characterized by the causality safe region $0 < \rho < 1/\Omega$

[62,63]. In this region of this spacetime we have no closed timelike curves [62,63]. Recently the quantum dynamics of scalar and spinorial particles have been studied in this spacetime and similarities with quantum dynamics in the presence of an external magnetic field were observed in the four-dimensional solution of Som–Raychaudhuri [59,61] and in M-Theory in Ref. [64]. Based on this similarity, we have obtained the solution (35) in Kaluza–Klein theory where we have considered a Som–Raychaudhuri solution with cosmic dispiration and an inclusion of a uniform magnetic field via extra dimension It is easy to write the matrix $g_{\mu\nu}(\vec{r})$ and from it obtain $g^{\mu\nu}(\vec{r})$,

$$
g^{\mu\nu} = \begin{pmatrix}
\Omega^2\rho^2 - 1 & 0 & -\frac{\Omega}{\alpha} & \frac{\Omega J}{\alpha} & \left(\frac{\Omega \Phi}{2\pi\alpha} + \frac{eB\Omega}{2\alpha}\rho^2\right) \\
0 & 1 & 0 & 0 & 0 \\
-\frac{\Omega}{\alpha} & 0 & \frac{1}{\alpha^2\rho^2} & -\frac{J^2}{\alpha^2\rho^2} & -\left(\frac{\Phi}{2\pi\alpha^2\rho^2} + \frac{eB}{2\alpha^2}\right) \\
\frac{\Omega J}{\alpha} & 0 & \frac{J}{\alpha^2}\rho^2 & 1 + \frac{J^2}{\alpha^2\rho^2} & \left(\frac{\Phi J}{2\pi\alpha^2\rho^2} + \frac{eBJ}{2\alpha^2}\right) \\
\left(\frac{\Omega \Phi}{2\pi\alpha} + \frac{eB\Omega}{2\alpha}\rho^2\right) & 0 & -\left(\frac{\Phi}{2\pi\alpha^2\rho^2} + \frac{eB}{2\alpha^2}\right) & \left(\frac{\Phi J}{2\pi\alpha^2\rho^2} + \frac{eBJ}{2\alpha^2}\right) & \left(1 + \frac{\Phi^2}{4\pi^2\alpha^2\rho^2} + \frac{e\Phi B}{2\pi\alpha^2} + \frac{e^2B^2\rho^2}{4\alpha^2}\right)
\end{pmatrix}.
\tag{36}
$$

In this way the square root of the determinant of this matrix is given by $\sqrt{-g} = \alpha\rho$. Therefore we can write the Klein–Gordon equation (3) in this background as

$$
\left\{ \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{(\partial_\phi - J\partial_z - \Phi\partial_y/2\pi)^2}{\alpha^2\rho^2} \right.
$$
$$
\left. + \left[ -M^2w^2 + \left(\Omega\partial_t + \frac{eB\partial_y}{2\alpha}\right)^2 \right]\rho^2 + \gamma \right\} \Psi(t,\vec{r}) = 0,
$$
$$
\gamma = -\partial_t^2 - \left(\partial_\phi - J\partial_z - \frac{\Phi\partial_y}{2\pi}\right)\left(\frac{2\Omega}{\alpha}\partial_t + \frac{eB}{\alpha^2}\partial_y\right)
$$
$$
+ \partial_z^2 + \partial_y^2 - 2Mw - M^2.
\tag{37}
$$

This equation is independent of the variables $t$, $\phi$, $z$, and $y$ and allows us to use the ansatz in the general form $\Psi \propto e^{-iEt+il\phi+ikz+i\lambda y}R(\rho)$. Therefore,

$$
\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}R(\rho)\right] - \left\{ \frac{(l - Jk - \lambda\Phi/2\pi)^2}{\alpha^2\rho^2} \right.
$$
$$
\left. + \left[ M^2w^2 + \left(\frac{eB\lambda}{2\alpha} - \Omega E\right)^2 \right]\rho^2 - \gamma \right\} R(\rho) = 0;
$$
$$
\gamma = E^2 + \left(l - Jk - \frac{\lambda\Phi}{2\pi}\right)\left(\frac{eB\lambda}{\alpha^2} - \frac{2\Omega E}{\alpha}\right)
$$
$$
- M^2 - 2Mw - k^2 - \lambda^2.
\tag{38}
$$

It is not difficult to show that this equation is the same as a confluent hypergeometric equation looking like

$$
x\frac{d^2R(x)}{dx^2} + \left(\frac{|\beta|}{\alpha} + 1 - x\right)\frac{dR(x)}{dx}
$$
$$
- \left(\frac{|\beta|}{2\alpha} + \frac{1}{2} - \frac{\gamma}{4\delta}\right)R(x) = 0,
$$
$$
\beta = l - Jk - \frac{\lambda\Phi}{2\pi},
$$
$$
\delta = \sqrt{M^2w^2 + \left(\frac{eB\lambda}{2\alpha} - \Omega E\right)^2}.
\tag{39}
$$

The solution of this equation is a polynomial of order $x^n$. For all limits, this function diverges. This blow-up is avoided

assuming $a = -n$ in the general expression of the coefficients of the hypergeometric series. With this condition the energy levels of our particle are

$$
E^2 = M^2 - \left(l - Jk - \frac{\lambda\Phi}{2\pi}\right)\left(\frac{eB\lambda}{\alpha^2} - \frac{2\Omega E}{\alpha}\right)
$$
$$
+ 4\sqrt{M^2w^2 + \left(\Omega E - \frac{eB\lambda}{2\alpha}\right)^2}
$$
$$
\times \left(n + \frac{|l - Jk - \frac{\lambda\Phi}{2\pi}|}{2\alpha} + \frac{1}{2}\right) + 2Mw + k^2 + \lambda^2.
\tag{40}
$$

Apparently we can see that in the absence of a homogeneous magnetic field as well as of rotation sources, the results of the previous section are reproduced. We can also see that the external sources have an important role in the dynamics of our particle due to the explicit dependence on these parameters. Assuming that the physical laws are true in any temporal scales, we believe that the rotation, magnetic fields, curvature, and torsion sources had played an important role in the dynamics of our universe in the early evolution epoch and, therefore, our contribution has an interesting application.

It is important to study some limits. Let us consider the weak oscillator as well as rotation-free limit of spacetimes, which is equivalent to the assumption $(w, \Omega) \to 0$, in Eq. (40); in this limit we have no influence of the Klein–Gordon oscillator and the Som–Raychaudhuri geometry. After some calculations, we obtain the following result:

$$E^2 = M^2 + \frac{2eB\lambda}{\alpha}\left[n + \frac{|l - kJ - \frac{\lambda\Phi}{2\pi}|}{2\alpha}\right.$$
$$\left. - \frac{l - kJ - \frac{\lambda\Phi}{2\pi}}{2\alpha} + \frac{1}{2}\right] + k^2 + \lambda^2, \tag{41}$$

which is exactly the result of Eq. (35) of Ref. [2], where one of us has obtained the relativistic Landau levels for a scalar particle in Kaluza–Klein theory. Now we consider the limit where we do not have a Klein–Gordon oscillator, and we obtain from (40) the following energy eigenvalues:

$$E = \Omega\left(2n + \frac{|l - kJ - \frac{\lambda\Phi}{2\pi}|}{\alpha} - \frac{l - kJ - \frac{\lambda\Phi}{2\pi}}{\alpha} + 1\right)$$
$$\pm \left\{\left(\Omega^2\left(2n + \frac{|l-kJ-\frac{\lambda\Phi}{2\pi}|}{\alpha} - \frac{l-kJ-\frac{\lambda\Phi}{2\pi}}{\alpha} + \frac{1}{2}\right)\right.\right.$$
$$\left. - \frac{eB\lambda}{\alpha}\right) \times \left[2n + \frac{|l-kJ-\frac{\lambda\Phi}{2\pi}|}{\alpha} - \frac{l-kJ-\frac{\lambda\Phi}{2\pi}}{\alpha} + 1\right]$$
$$\left. + k^2 + M^2 + \lambda^2\right\}^{1/2}. \tag{42}$$

The eigenvalues (42) represent the energy levels for a free scalar particle in a Som–Raychaudhuri spacetime pierced by a cosmic dispiration and a uniform magnetic field introduced in a geometric way by Kaluza–Klein theory. Note the influence of topological defects in the eigenvalues: in the limit where we have $B \to 0$ in (42) we obtain the eigenvalues of the quantum dynamics of a scalar quantum particle in $(4+1)$-dimensional Som–Raychaudhuri spacetime, given by

$$E = \Omega\left(2n + \frac{|l - kJ - \frac{\lambda\Phi}{2\pi}|}{\alpha} - \frac{l - kJ - \frac{\lambda\Phi}{2\pi}}{\alpha} + 1\right)$$
$$\pm \sqrt{\Omega^2\left(2n + \frac{|l-kJ-\frac{\lambda\Phi}{2\pi}|}{\alpha} - \frac{l-kJ-\frac{\lambda\Phi}{2\pi}}{\alpha} + \frac{1}{2}\right)^2 + k^2 + M^2 + \lambda^2}. \tag{43}$$

This result is the generalization of the results obtained previously in Refs. [59,60] for a $(4 + 1)$-dimensional Gödel scenario in the presence of cosmic dispiration. We can observe in (43) the dependence of the parameter $J$, which is related with the Burgers vector of the topological defect associated with the torsion of the spacetime. In the presence of rotation and curvature sources, one can consider the limit $(J, \Phi, B, w) \to 0$ in (40). After a some algebra we obtain the result

$$E = \left(2n + \frac{|l|}{\alpha} + \frac{l}{\alpha} + 1\right)\Omega$$
$$\pm \sqrt{\left(2n + \frac{|l|}{\alpha} + \frac{l}{\alpha} + 1\right)^2\Omega^2 + M^2 + k^2}. \tag{44}$$

This is equivalent to the energy levels in the Gödel-type spacetimes studied by us [59–61,64]. Although Eq. (40) has a cumbersome form, some limits show important results in describing other simple physical systems with topological/electromagnetic interactions through simple manipulations with that equation. Finally, we can see that the degeneracy of the energy levels, in this case, is strongly broken due to the presence of curvature, torsion, magnetic fields, and the rotation source. It is important that we can observe a Landau structure in all previously studied cases.

## 6 Summary

The aim of this paper was to investigate the quantum dynamics of a scalar particle interacting harmonically with a gravitational background of topological defects, via a Klein–Gordon oscillator description, in the presence of a class of spacetimes in Kaluza–Klein theory. We determine the manner in which the non-trivial topology due to the topological defect, electromagnetic field, and rotation of this background modifies the energy spectrum and wave function of the Klein–Gordon oscillator. This perturbation in the eigenvalues is compared with the flat spacetime case, and these results can be used to investigate the presence of these defects in the cosmos. Here we investigate a harmonic interaction that can be used for a simulation of a series of physical systems, such as the vibrational spectrum of a diatomic molecule [23], the binding of heavy quarks [24,25], and the quark–antiquark interaction [30]. The possibility to use the modification in the spectra of the KG oscillator to probe the existence of these topological defects was noticed in the results obtained. In fact it is clear, from the observational point of view, that to have an observable modification in the energy eigenvalues, we need a huge number of particles in the states, otherwise the magnitude of the effect to a real spectrum may not be strong enough to be observed.

We have studied the quantum dynamics of a Klein–Gordon particle interacting with external field sources, by using the five-dimensional version of general relativity. The quantum dynamics in the usual as well as magnetic cosmic string cases allow us to obtain the energy levels and the eigenfunctions depending on the external parameters characterizing the background spacetimes, a result known by the gravitational analog of the well-studied Aharonov–Bohm effect.

We have investigated the Klein–Gordon oscillator in the cosmic string background in a Kaluza–Klein theory and obtained the energy eigenvalues and eigenfunctions, which turn out to depend on the parameter $\alpha$, which characterizes the cosmic string. Note that in the four-dimensional limit we recover the results found by Boumali et al. [35]. For the case of a Klein–Gordon oscillator in the presence of a magnetic flux string in Kaluza–Klein theory, we obtain the result that

the energy levels depend on the Aharonov–Bohm flux and the parameter $\alpha$. The Klein–Gordon equation for the KG oscillator in cosmic dispiration was investigated, and the energy levels turn out to depend on the parameter $\alpha$, the dislocation parameter (Burgers vector modulus) $J$, and the Aharonov–Bohm flux. The inclusion of torsion has an important role to play in this dynamics. By the results in this background, it becomes possible to compensate for the elastic contribution introduced by the topological defect, by a fine tuning of the external magnetic field strength. The degeneracy of the energy levels is strongly broken due to the presence of curvature and torsional sources in these expressions.

Note that in Sects. 2, 3, and 4 we have studied the influence of topological defects in Kaluza–Klein theory on the energy levels of a Klein–Gordon oscillator in order to observe the influence of this structure on the energy levels and wave functions. In Sect. 5, we have studied the Klein–Gordon oscillator for a $(4+1)$-dimensional Som–Raychaudhuri solution with a topological defect; this study can be employed to investigate other quantum systems.

We have obtained the spectrum and wavefunction for Klein–Gordon oscillator in the background of the cosmic dispiration in a Som–Raychaudhuri spacetime in a Kaluza–Klein theory in the presence of a uniform magnetic field and a magnetic flux. We introduced a uniform magnetic field and Aharonov–Bohm flux via Kaluza–Klein theory. The energy levels and eigenfunctions for the Klein–Gordon oscillator in this geometry were obtained, and we demonstrated their dependence on the parameters characterizing the spacetime in $(4 + 1)$ dimensions, such as $(\Omega, \alpha, J, )$ associated to the rotation, deficit angle, and torsion of spacetime, the external magnetic field $B$, and the magnetic flux $\Phi$ introduced by Kaluza–Klein theory. Note that in the appropriate limit we obtain the results of the previous section, $\Omega \longmapsto 0$, $B \longmapsto 0$, that is, the cosmic dispiration case. In the case where $\alpha = 1$, $\Phi = 0$, we obtain the results of Wang et al. [52]. In the limit of $\Omega = 0$, a spectrum similar to the Landau levels is recovered. Note that this dynamics in the presence of the confining potential due to the Klein–Gordon oscillator reinforces the characteristics of the quantum dynamics observed in this Gödel-type spacetime [59–61], which are characterized by the similarity with the Landau problem for a charged particle on a surface exposed to a uniform magnetic field. Based on this analogy, Druker et al. [59] have suggested the picture of a holographic description for a single chronologically safe region. In the same paper they have conjectured that this discussion can be extended for a $4 + 1$-dimensional Gödel solution. In this article we demonstrated that this similarity with Landau levels occurs in a $(4 + 1)$-dimensional Gödel-type solution as well and we have considered a richer structure of the spacetime: including a topological defect. If we consider the analogy between the quantum dynamics in a chronologically safe region in a Som–Raychaudhuri space-

time and Landau levels, we can consider applications of these results to the quantum dynamics in Hall droplets of finite size [65]. Because of this analogy, we can think of applications in the Hall effect in a droplet of finite size in systems of condensed matter. We can also use the results found here in our harmonic confinement via the Klein–Gordon oscillator in the Hall effect in droplets of finite size with harmonic confinement of electrons, as was done in Ref. [66]. We claim that these results can be used in a generalization of the quantum Hall effect in $(4 + 1)$ dimensions [67–69], and they can be related with quantum dynamics in gravity-based systems in higher dimensions due to the already mentioned analogy of quantum dynamics in Gödel-type solutions with Landau levels on curved surfaces [70,71]. In the limit $w \to 0$ we have obtained for the first time the spectrum of a scalar particle in a Som–Raychaudhuri spacetime with a topological defect, which combines the curvature and the torsion into a dispiration, in the presence of a uniform field and magnetic flux introduced via Kaluza–Klein theory. These energy levels (42) have several contributions due to the rotation of spacetime $\Omega$ of the parameter $J$, which is related with torsion.

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