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Knot Invariants and Higher Representation Theory

Ben Webster

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Abstract

We construct knot invariants categorifying the quantum knot variants for all representations of quantum groups. We show that these invariants coincide with previous invariants defined by Khovanov for $\mathfrak{sl}_2$ and $\mathfrak{sl}_3$ and by Mazorchuk-Stroppel and Sussan for $\mathfrak{sl}_n$.

Our technique is to study 2-representations of 2-quantum groups (in the sense of Rouquier and Khovanov-Lauda) categorifying tensor products of irreducible representations. These are the representation categories of certain finite dimensional algebras with an explicit diagrammatic presentation, generalizing the cyclotomic quotient of the KLR algebra. When the Lie algebra under consideration is $\mathfrak{sl}_n$, we show that these categories agree with certain subcategories of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_k$.

We also investigate the finer structure of these categories: they are standardly stratified and satisfy a double centralizer property with respect to their self-dual modules. The standard modules of the stratification play an important role as test objects for functors, as Vermas do in more classical representation theory.

The existence of these representations has consequences for the structure of previously studied categorifications. It allows us to prove the non-degeneracy of Khovanov and Lauda's 2-category (that its Hom spaces have the expected dimension) in all symmetrizable types, and that the cyclotomic quiver Hecke algebras are symmetric Frobenius.

In work of Reshetikhin and Turaev, the braiding and (co)evaluation maps between representations of quantum groups are used to define polynomial knot invariants. We show that the categorifications of tensor products are related by functors categorifying these maps, which allow the construction of bigraded knot homologies whose graded Euler characteristics are the original polynomial knot invariants.

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CHAPTER 1

Introduction

1. Quantum topology

Much of the theory of quantum topology rests on the structure of monoidal categories and their use in a variety of topological constructions. In this paper, we define a categorification of one of these constructions: the R-matrix construction of quantum knot invariants, inspired by the work of Reshetikhin and Turaev [**Tur88**, **RT90**].

Their work is in the context of the tensor category of representations of a quantized universal enveloping algebra $U_q(\mathfrak{g})$. They assign natural maps between tensor products to each ribbon tangle labeled with representations. One special case of this is a polynomial invariant of framed knots for each finite-dimensional representation of $U_q(\mathfrak{g})$. These maps are natural with respect to tangle composition; thus they can be reconstructed from a small number of constituents: the maps associated to a ribbon twist, crossing, cup and cap. The map associated to a link whose components are labeled with a representation of $\mathfrak{g}$ is thus simply a Laurent polynomial.

Particular cases of these invariants include:

- the **Jones polynomial** when $\mathfrak{g} = \mathfrak{sl}_2$ and all strands are labeled with the defining representation.
- the **colored Jones polynomials** for other representations of $\mathfrak{g} = \mathfrak{sl}_2$.
- specializations of the **HOMFLYPT polynomial** for the defining representation of $\mathfrak{g} = \mathfrak{sl}_n$.
- the **Kauffman polynomial** (not to be confused with the Kauffman bracket, a variant of the Jones polynomial) for the defining representation of $\mathfrak{so}_n$.

These special cases have been categorified to knot homologies from a number of perspectives, beginning with work of Khovanov on the Jones polynomial. However, the vast majority of representations previously had no homology theory attached to them. In this paper, we will construct such a theory for any labels; that is:

Theorem A.1 *For each simple complex Lie algebra $\mathfrak{g}$, there is a homology theory $\mathcal{K}(L, \{\lambda_i\})$ for links L whose components are labeled by finite dimensional representations of $\mathfrak{g}$ (here indicated by their highest weights λ_i), which associates to such a link a bigraded vector space whose graded Euler characteristic is the quantum invariant of this labeled link.*

Given the extensive past work on knot homology, it is natural to ask which of the homology theories mentioned above coincide with those of Theorem A.1 in special cases.

Theorem A.2 *When $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$ and the link is labeled with the defining representation of these algebras, the theory $\mathcal{K}(L, \{\lambda_i\})$ coincides up to grading shift with Khovanov's homologies for $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3$. In the case $\mathfrak{g} = \mathfrak{sl}_n$ and we use the defining representation, $\mathcal{K}(L, \{\lambda_i\})$ agrees with the Mazorchuk-Stroppel-Sussan homology.*

Previous approaches to categorifying the special Reshetikhin-Turaev invariants mentioned above have been given by Khovanov and Khovanov-Rozansky [**Kho00**, **Kho02**,**Kho04**,**Kho07**,**KR08b**,**KR07**,**KR08a**], Stroppel and Mazorchuk-Stroppel [**Str05**,**MS09**], Sussan [**Sus07**], Seidel-Smith [**SS06**], Manolescu [**Man07**], Cautis-Kamnitzer [**CK08a**,**CK08b**], Mackaay, Stošić and Vaz [**MSV09**,**MSV11**] and the author and Williamson [**WW**]. All of these approaches depend heavily on special features of minuscule representations.

There has been some progress on other representations of $\mathfrak{sl}_2$. In a paper still in preparation, Stroppel and Sussan also consider the case of the colored Jones polynomial [**SS**] (building on previous work with Frenkel [**FSS12**]); it seems likely their construction is equivalent to ours (see Chapter 4). Similarly, Cooper and Krushkal have given a categorification of the colored Jones polynomial using Bar-Natan's cobordism formalism for Khovanov homology [**CK12**]. We show in [**Webf**] that Cooper and Krushkal's theory agrees with ours for colored Jones polynomials.

On the other hand, the work of physicists suggests that categorifications for all representations exist; one schema for defining them is given by Witten [**Wit**]. The relationship between Witten's proposals and the invariants presented in this paper is completely unknown (at least to the author) and presents a very interesting question for consideration in the future.

Another question of particular interest is whether $\mathcal{K}(L, \{\lambda_i\})$ for the defining representation of $\mathfrak{sl}_n$ coincides with Khovanov-Rozansky homology [**KR08b**]; we will establish this agreement in future work with Mackaay [**MW**].

At the moment, we have not proven that $\mathcal{K}(L, \{\lambda_i\})$ is functorial, but we do have a proposal for the map associated to a cobordism when the weights λ_i are all minuscule. As usual in knot homology, this proposed functoriality map is constructed by picking a Morse function on the cobordism, and associating simple maps to the addition of handles. We have no proof that this definition is independent of Morse function and we anticipate that proving this will be quite difficult.

2. Categorification of tensor products

The program of "higher representation theory," begun (at least as an explicit program) by Chuang and Rouquier in [**CR08**] and continued by Rouquier [**Roua**] and Khovanov-Lauda [**KL10**], is aimed at studying "2-analogues" of the universal enveloping algebras of simple Lie algebras $U(\mathfrak{g})$ and their quantizations $U_q(\mathfrak{g})$. The 2-analogue for us of the quantum group $U_q(\mathfrak{g})$ is the strict 2-category $\mathcal{U}$ defined in [**CL15**, §2]. In this paper, we'll define an algebra $T^{\underline{\boldsymbol{\lambda}}}$ for each list $\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_\ell)$ of dominant weights for any symmetrizable Kac-Moody algebra $\mathfrak{g}$. Our 2-analogue of a tensor product of simple $\mathfrak{g}$-representations is the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ of finite-dimensional representations of $T^{\underline{\boldsymbol{\lambda}}}$.

Our first objective is to show that we have defined a categorification of such tensor products.

Theorem B *The category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ carries a categorical action of $\mathfrak{g}$, that is, it carries an action of the 2-category $\mathcal{U}$. The Grothendieck group of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is canonically isomorphic to the tensor product*

$$V_{\underline{\boldsymbol{\lambda}}} \cong V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_\ell}$$

of irreducible representations V_{λ_i}.

In the case where $\mathfrak{g}$ is finite-type and simply-laced, the classes of indecomposable projectives in $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ are Lusztig's canonical basis of a tensor product by [**Web15**, 6.11].

When $\underline{\boldsymbol{\lambda}} = (\lambda)$, the algebra $T^{\lambda} := T^{(\lambda)}$ is a cyclotomic KLR algebra in the sense of [**KL10**, §3.4]. Even in this case, the categorical action of Theorem B is new, and it implies that the induction and restriction functors on these categories are biadjoint. This was proved independently by Kang and Kashiwara [**KK12**] using completely different methods. This action can be used to prove that the 2-category $\mathcal{U}$ is nondegenerate in the finite type case. Earlier versions of this paper included a discussion of non-degeneracy outside of finite type, which is now proven in [**Webb**] instead.

These algebras $T^{\underline{\boldsymbol{\lambda}}}$ are also quite interesting from the perspective of pure representation theory. We'll prove that they have a number of properties which have already appeared in the literature.

Theorem C *The projective-injective objects of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ form a categorification of the subrepresentation $V_{\lambda_1+\cdots+\lambda_n} \subset V_{\underline{\boldsymbol{\lambda}}}$. In particular, if $\underline{\boldsymbol{\lambda}} = (\lambda)$, then all projectives are injective and the algebra $T^{(\lambda)}$ is Frobenius.*

The sum of all indecomposable projective-injectives has the double centralizer property. This realizes $T^{\underline{\boldsymbol{\lambda}}}$ as the endomorphisms of a natural collection of modules over the algebra $T^{(\lambda_1+\cdots+\lambda_n)}$.

The algebra $T^{\underline{\boldsymbol{\lambda}}}$ is standardly stratified. The semi-orthogonal decomposition for this stratification categorifies the decomposition of $V_{\underline{\boldsymbol{\lambda}}}$ as the sum of tensor products of weight spaces.

These algebras also have connections in the type A case to classical representation theory, as has been explored by Brundan and Kleshchev [**BK09**]. Using Theorem C, we will build on their work in Chapter 9 by showing that the algebras $T^{\underline{\boldsymbol{\lambda}}}$ are endomorphism algebras of certain projectives in parabolic category $\mathcal{O}$, while in type $\widehat{A}$, they are related to the representations of the cyclotomic q-Schur algebra. This last relationship will be explored more fully in work of the author and Stroppel [**SW**, **Webe**].

The method of proof leaves little hope for finding connections with category $\mathcal{O}$ in other types. We see no reason to think that $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ has a similar description in terms of classical representation theory when $\mathfrak{g} \not\cong \mathfrak{sl}_n, \widehat{\mathfrak{sl}}_n$, though we would be quite pleased to be proven wrong in this speculation.

3. Topology

We now turn to the construction of knot invariants. As mentioned above, the original construction of Reshetikhin-Turaev invariants is encoded in a ribbon structure on the category of $U_q(\mathfrak{g})$-representations. We can depict the structure maps of this category in terms of diagrams.

- Crossing two ribbons: the corresponding operator in representations of the quantum group is called the **braiding** or **R-matrix**[1]. More generally for any braid σ on ℓ strands, this defines a homomorphism $\Phi(\sigma)\colon V^{\underline{\boldsymbol{\lambda}}} \to V^{\sigma(\underline{\boldsymbol{\lambda}})}$.
- Creating a cup, or closing a cap: the corresponding operators in representations of the quantum group are called the **(co)evaluation** and **quantum (co)trace**.
- Adding a full twist to one of the ribbons: the corresponding operator in the quantum group is called the **ribbon element**.

In this paper, we will categorify every of these maps to a functor, and use these the define the invariants $\mathcal{K}(L, \{\lambda_i\})$. This approach was pioneered by Stroppel for the defining rep of $\mathfrak{sl}_2$ [**Str09, Str**] and was extended to $\mathfrak{sl}_n$ by Sussan [**Sus07**] and Mazorchuk-Stroppel [**MS09**]. To work in complete generality, we must use the derived categories[2] $\mathcal{V}^{\underline{\boldsymbol{\lambda}}} = D(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}})$ of finite dimensional $T^{\underline{\boldsymbol{\lambda}}}$-representations, rather than the variations of category $\mathcal{O}$ used by those authors.

Theorem D *The derived category $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ carries functors categorifying all the structure maps of the ribbon category of $U_q(\mathfrak{g})$-modules:*

(i) If σ is a braid, then we have an exact functor $\mathbb{B}_\sigma\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\sigma(\underline{\boldsymbol{\lambda}})}$ such that the induced map on Grothendieck groups $K_0(T^{\underline{\boldsymbol{\lambda}}}) \to K_0(T^{\sigma(\underline{\boldsymbol{\lambda}})})$ coincides with $\Phi(\sigma)$. Furthermore, these functors induce a strong action of the braid groupoid on the categories associated to permutations of the set $\underline{\boldsymbol{\lambda}}$.

(ii) If two consecutive elements of $\underline{\boldsymbol{\lambda}}$ label dual representations and $\underline{\boldsymbol{\lambda}}^-$ denotes the sequence with these removed, then there are functors $\mathbb{T}, \mathbb{E}\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}^{\underline{\boldsymbol{\lambda}}^-}$ which induce the quantum trace and evaluation on the Grothendieck group, and similarly functors $\mathbb{K}, \mathbb{C}\colon \mathcal{V}^{\underline{\boldsymbol{\lambda}}^-} \to \mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ for the coevaluation maps and quantum cotrace maps.

(iii) When $\mathfrak{g} = \mathfrak{sl}_n$, the structure functors above can be described in terms of twisting and Enright-Shelton functors on $\mathcal{O}$.

By definition (see [**CP95**, §4]), the quantum knot invariants are given by a composition of the decategorifications of the functors constructed in Theorem D. Combining the functors themselves in the same pattern gives the knot homology of Theorem A.1.

4. Summary

Let us now summarize the structure of the paper.

- In Chapters 2 and 3, we discuss the basics of the 2-category $\mathcal{U}$, and prove it acts on $\mathfrak{V}^\lambda$. This is accomplished by the construction of categorifications $\mathcal{U}_i^-$ for the minimal non-solvable parabolics $U(\mathfrak{p}_i)$. These categories carry a mixture of the characteristics of $U(\mathfrak{b})$ and $U(\mathfrak{sl}_2)$; an appropriate non-degeneracy result is already known for both of these algebras separately.

[1] As usual, the R-matrix is a map between tensor products of representations $V \otimes W \to V \otimes W$ intertwining the usual and opposite coproducts; we use the term **braiding** to refer to the composition of this with the usual flip map, which is thus a homomorphism of representations $V \otimes W \to W \otimes V$.

[2] We'll want to use slightly unusual finiteness conditions on $D(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}})$, so we'll leave the precise definition of these categories to the body of the paper. See Definition 4.8.

By modifying the proofs of these previous results, we can show that $\mathcal{U}_i^-$ acts on $\mathfrak{V}^\lambda$. It is an easy consequence of this that the full $\mathcal{U}$ acts. These results are of independent interest (and, in fact, some of them have been proven independently by Kang and Kashiwara [**KK12**]).

- In Chapter 4, we define the algebras $T^{\underline{\boldsymbol{\lambda}}}$, using the familiar tool of graphical calculus. This graphical calculus gives an easy description of the action of the category $\mathcal{U}$. We also study the relationship of this category to $T^{(\lambda_1+\cdots+\lambda_\ell)}$.
- In Chapter 5, we develop a special class of modules which we term **standard modules**, which define a standardly stratified structure. In the case where all λ_i are minuscule, this structure is even quasi-hereditary. These modules also serve as categorifications of pure tensors.
- In Chapter 6, we prove Theorem D(i). That is, we construct the functor lifting the braiding of the monoidal category of $U_q(\mathfrak{g})$-representations. This functor is the derived tensor product with a natural bimodule. Particularly interesting and important special cases correspond to the half-twist braid, which sends projective modules to tiltings, and the full twist braid, which gives the right Serre functor of $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$.
- In Chapter 7, we prove Theorem D(ii). The most important element of this is to identify a special simple module in the category for a pair of dual fundamental weights, which categorifies an invariant vector.
- In Chapter 8, we prove Theorem A.1 using the functors constructed in Theorem D and a small number of explicit computations. We also suggest a map for the functoriality along a cobordism between links in the minuscule case. As mentioned before, it is unknown whether this is independent of choices.
- In Chapter 9, we consider the case $\mathfrak{g} = \mathfrak{sl}_n$ or $\widehat{\mathfrak{sl}}_n$. In this case, we employ results of Brundan and Kleshchev to show that $T^{\underline{\boldsymbol{\lambda}}}$ is in fact the endomorphism algebra of a projective in a parabolic category $\mathcal{O}$ in finite type. In affine type, there is a similar description using the cyclotomic q-Schur algebra. In Chapter 4, we relate the functors appearing in Theorem D to previously defined functors on category $\mathcal{O}$. This allows us to show the portions of Theorem A.1 regarding comparisons to Khovanov homology and Mazorchuk-Stroppel-Sussan homology.

We should note that an earlier version of this paper contained a section on the connection between the algebraic material in this paper to the geometry of quiver varieties and canonical bases. In the interest of length and heaviness of machinery, that material has been moved to other papers [**Weba**, **Webg**, **Web15**].

Notation

We let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra, which we will assume is fixed for the remainder of the paper. Let Γ denote the Dynkin diagram of this algebra, considered as an unoriented graph. We'll also consider the weight lattice $Y(\mathfrak{g})$, root lattice $X(\mathfrak{g})$, simple roots α_i and coroots $\alpha_i^\vee$. Let $c_{ij} = \alpha_i^\vee(\alpha_j)$ be the entries of the Cartan matrix.

We let $\langle -,- \rangle$ denote the symmetrized inner product on $Y(\mathfrak{g})$, fixed by the fact that the shortest root has length $\sqrt{2}$ and

$$2\frac{\langle \alpha_i, \lambda \rangle}{\langle \alpha_i, \alpha_i \rangle} = \alpha_i^\vee(\lambda).$$

As usual, we let $2d_i = \langle \alpha_i, \alpha_i \rangle$, and for $\lambda \in Y(\mathfrak{g})$, we let

$$\lambda^i = \alpha_i^\vee(\lambda) = \langle \alpha_i, \lambda \rangle / d_i.$$

We note that we have $d_i c_{ij} = d_j c_{ji} = \langle \alpha_i, \alpha_j \rangle$ for all i, j.

We let ρ be the unique weight such that $\alpha_i^\vee(\rho) = 1$ for all i and $\rho^\vee$ the unique coweight such that $\rho^\vee(\alpha_i) = 1$ for all i. We note that since $\rho \in {}^1\!/\!_2 X$ and $\rho^\vee \in {}^1\!/\!_2 Y^*$, for any weight λ, the numbers $\langle \lambda, \rho \rangle$ and $\rho^\vee(\lambda)$ are not necessarily integers, but $2\langle \lambda, \rho \rangle$ and $2\rho^\vee(\lambda)$ are (not necessarily even) integers.

Throughout the paper, we will use $\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_\ell)$ to denote an ordered ℓ-tuple of dominant weights, and always use the notation $\lambda = \sum_i \lambda_i$.

We let $U_q(\mathfrak{g})$ denote the deformed universal enveloping algebra of $\mathfrak{g}$; that is, the associative $\mathbb{C}(q)$-algebra given by generators E_i, F_i, K_μ for $i \in \Gamma$ and $\mu \in Y(\mathfrak{g})$, subject to the relations:

i) $K_0 = 1$, $K_\mu K_{\mu'} = K_{\mu+\mu'}$ for all $\mu, \mu' \in Y(\mathfrak{g})$,
ii) $K_\mu E_i = q^{\alpha_i^\vee(\mu)} E_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
iii) $K_\mu F_i = q^{-\alpha_i^\vee(\mu)} F_i K_\mu$ for all $\mu \in Y(\mathfrak{g})$,
iv) $E_i F_j - F_j E_i = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}_{-i}}{q^{d_i} - q^{-d_i}}$, where $\tilde{K}_{\pm i} = K_{\pm d_i \alpha_i}$,
v) For all $i \neq j$

$$\sum_{a+b=-c_{ij}+1} (-1)^a E_i^{(a)} E_j E_i^{(b)} = 0 \qquad \text{and} \qquad \sum_{a+b=-c_{ij}+1} (-1)^a F_i^{(a)} F_j F_i^{(b)} = 0.$$

This is a Hopf algebra with coproduct on Chevalley generators given by

$$\Delta(E_i) = E_i \otimes 1 + \tilde{K}_i \otimes E_i \qquad \Delta(F_i) = F_i \otimes \tilde{K}_{-i} + 1 \otimes F_i$$

We let $U_q^{\mathbb{Z}}(\mathfrak{g})$ denote the Lusztig (divided powers) integral form generated over $\mathbb{Z}[q, q^{-1}]$ by $\frac{E_i^n}{[n]_q!}, \frac{F_i^n}{[n]_q!}$ for $n \geq 0$. We let $\dot{U}^{\mathbb{Z}}$ be the algebra obtained by adjoining idempotents 1_μ projecting to integral weight spaces to $U_q^{\mathbb{Z}}(\mathfrak{g})$. The integral form of the representation of highest weight λ over this quantum group will be denoted by $V_\lambda^{\mathbb{Z}}$. We will always think of this integral form as generated by a fixed highest weight vector v_λ. For a sequence $\underline{\boldsymbol{\lambda}}$, we will be interested in the tensor product

$$V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} = V_{\lambda_1}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \cdots \otimes_{\mathbb{Z}[q,q^{-1}]} V_{\lambda_\ell}^{\mathbb{Z}}.$$

The category of representations over $U_q(\mathfrak{g})$ is a braided monoidal category; of particular importance for us is the R-matrix. We use the opposite R_{21} of the R-matrix defined by Tingley in [**Tin**]; since Tingley uses the opposite coproduct, this will be an R-matrix for us. Thus, for two representations M, M', we have an isomorphism $\sigma_{M,M'} \colon M \otimes M' \to M' \otimes M$ sending $m \otimes m' \mapsto s(R_{12} m \otimes m')$. In the notation of [**Tin**], this map is defined by

$$\mathcal{R}_{M,M'}(m \otimes m') \in q^{\langle \mathrm{wt}(m), \mathrm{wt}(m') \rangle} m' \otimes m + \sum q^{\langle \mathrm{wt}(m)-\beta, \mathrm{wt}(m')+\beta \rangle} X_\beta m' \otimes Y_\beta m \tag{1.1}$$

where X_β has weight $\beta > 0$ and Y_β weight $-\beta > 0$. In particular, if m is lowest weight or m' highest weight, then this equation simplifies to $\mathcal{R}_{M,M'}(m \otimes m') \in q^{\langle \mathrm{wt}(m), \mathrm{wt}(m') \rangle} m' \otimes m$.

For not especially important technical reasons, it will also be helpful to consider $V_{\underline{\boldsymbol{\lambda}}}^{1/D} = V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}[q^{1/D}]$ and $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{C}} \cong V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}[\{q^z\}_{z\in\mathbb{C}}]$, this module with either the Dth roots of q (for a fixed D) or all complex powers of q adjoined. We will also consider the completion of these modules in the q-adic topology $V_{\underline{\boldsymbol{\lambda}}} = V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}((q))$.

For a graded ring R, we let R-mod denote the category of finitely generated right graded R-modules. For a graded R-module M, we let M_n denote the vectors of degree n. Let $K^0(R)$ denote the Grothendieck group of finitely generated graded projective right R-modules. This group carries an action of $\mathbb{Z}[q, q^{-1}]$ by grading shift $[A(i)] = q^i[A]$, where $A(i)_n = A_{i+n}$. The careful reader should note that this is opposite to the grading convention of Khovanov and Lauda.

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CHAPTER 2

Categorification of quantum groups

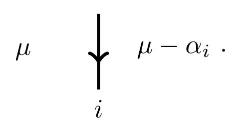
1. Khovanov-Lauda diagrams

In this paper, our notation builds on that of Khovanov and Lauda, who give a graphical version of the 2-quantum group, which we denote $\mathcal{U}$ (leaving $\mathfrak{g}$ understood). These constructions could also be rephrased in terms of Rouquier's description and we have striven to make the paper readable following either [**KL10**] or [**Roua**]; however, it is most sensible for us to use the 2-category defined by Cautis and Lauda [**CL15**], which is a variation on both of these. See the introduction of [**CL15**] for more detail on the connections between these different approaches.

The object of interest for this subsection is a strict 2-category; as described, for example, in [**Lau10**], one natural yoga for discussing strict 2-categories is planar diagrammatics. The 2-category $\mathcal{U}$ is thus most clearly described in this language.

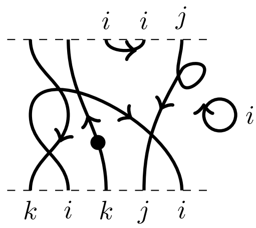
Definition 2.1 *A* **blank KL diagram** *is a collection of finitely many oriented curves in* $\mathbb{R} \times [0, 1]$ *which has no triple points or tangencies, decorated with finitely many dots. Every strand is labeled with an element of* Γ*, and any open end must meet one of the lines* $y = 0$ *or* $y = 1$ *at a distinct point from all other ends.*

A **KL diagram** *is a blank KL diagram together with a labeling of regions between strands (the components of its complement) with weights following the rule*



We identify two KL diagrams if they are isotopic via an isotopy which does not cancel any critical points of the height function or move critical points through crossings or dots. Ultimately, we will need to introduce scalar corrections for isotopies that do have these features, as shown in relations (2.1a–2.1b); particular, the biadjunctions given by cups and caps will typically not be cyclic. In the interest of simplifying diagrams, we'll often write a dot with a number beside it to indicate a group of that number of dots.

We call the lines $y = 0, 1$ the **bottom** and **top** of the diagram. Reading across the bottom and top from left to right, we obtain a sequence of elements of Γ, which we wish to record in order from left to right. Since orientations are quite important, we let $\pm\Gamma$ denote $\Gamma \times \{\pm 1\}$, and associate i to a strand labeled with i which is oriented upward and $-i$ to one oriented downward. For example, we have a blank KL diagram

$m =$ 

with top given by $(-k, k, -i, i, -j)$ and bottom given by $(-k, i, k, -j, -i)$.

We also wish to record the labeling on regions; since fixing the label on one region determines all the others, we'll typically only record $\mathcal{L}$, the weight of the region at far left and $\mathcal{R}$, the weight at far right. In addition, we will typically not draw the weights on all regions in the interest of simplifying pictures. We call the pair of a sequence $\mathbf{i} \in (\pm\Gamma)^n$ and the weight $\mathcal{L}$ a **KL pair**; let $\mathcal{R} := \mathcal{L} + \sum_{j=1}^{n} \alpha_{i_j}$ where we let $\alpha_{-i} = -\alpha_i$.

Definition 2.2 *Given KL diagrams a and b, their* **(vertical) composition** *ab is given by stacking a on top of b and attempting to join the bottom of a and top of b. If the sequences from the bottom of a and top of b don't match or $\mathcal{L}_a \neq \mathcal{L}_b$, then the composition is not defined and by convention is 0, which is not a KL diagram, just a formal symbol.*

The **horizontal composition** *$a \circ b$ of KL diagrams is the diagram which pastes together the strips where a and b live with a to the right of b. The only compatibility we require is that $\mathcal{L}_a = \mathcal{R}_b$, so that the regions of the new diagram can be labeled consistently. If $\mathcal{L}_a \neq \mathcal{R}_b$, the horizontal composition is 0 as well.*

Implicit in this definition is a rule for **horizontal composition** of KL pairs in $\pm\Gamma$, which is the reverse of concatenation

$$(i_1, \ldots, i_m) \circ (j_1, \ldots, j_n) = (j_1, \ldots, j_n, i_1, \ldots, i_m),$$

and gives 0 unless $\mathcal{L}_{\mathbf{i}} = \mathcal{R}_{\mathbf{j}}$.

We should warn the reader, this convention requires us to read our diagrams differently from the conventions of [**Lau10**, **KL10**, **CL15**]; in our diagrammatic calculus, 1-morphisms point from the left to the right, not from the right to the left as indicated in [**Lau10**, §4]. The practical implication will be that our relations are the reflection through a vertical line of Cautis and Lauda's.

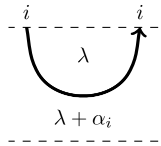
Definition 2.3 *Let $\tilde{\tilde{\mathcal{U}}}$ be the strict 2-category where*

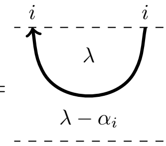
- *objects are weights in $Y(\mathfrak{g})$,*
- *1-morphisms $\mu \to \nu$ are KL pairs with $\mathcal{L} = \mu, \mathcal{R} = \nu$, and composition is given by horizontal composition as above.*
- *2-morphisms $h \to h'$ between KL pairs are $\Bbbk$-linear combinations of KL diagrams with h as bottom and h' as top, and vertical and horizontal composition of 2-morphisms is defined above.*

We'll typically use $\mathcal{E}_i$ to denote the 1-morphism (i) (leaving the weight $\mathcal{L}$ implicit) and $\mathcal{F}_i$ to denote $(-i)$. More generally, we'll let $\mathcal{E}_{\mathbf{i}} = \mathcal{F}_{-\mathbf{i}}$ denote the 1-morphism for a sequence $\mathbf{i}$.

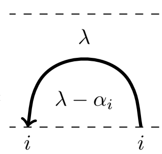
Morse theory shows that the 2-morphism spaces of $\tilde{\tilde{\mathcal{U}}}$ are generated under horizontal and vertical composition by identity morphisms and the following diagrams:

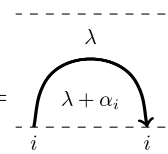
- a cup $\iota' : \emptyset \to \mathcal{E}_i\mathcal{F}_i$ or $\iota : \emptyset \to \mathcal{F}_i\mathcal{E}_i$

$\iota' =$ 

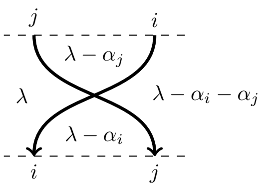
$\iota =$ 

- a cap $\epsilon : \mathcal{E}_i\mathcal{F}_i \to \emptyset$ or $\epsilon' : \mathcal{F}_i\mathcal{E}_i \to \emptyset$

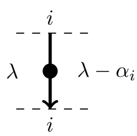
$\epsilon =$ 

$\epsilon' =$ 

- a crossing $\psi : \mathcal{F}_j\mathcal{F}_i \to \mathcal{F}_i\mathcal{F}_j$

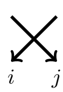
$\psi =$ 

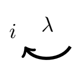
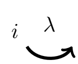
- a dot $y : \mathcal{F}_i \to \mathcal{F}_i$

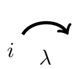
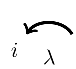
$y =$ 

In the diagrams above, we have included dashed lines to indicate the source and target of the 2-morphisms; we will not use this convention in the future in the interest of simplifying diagrams.

We can define a **degree** function on KL diagrams. The degrees are given on elementary diagrams by

deg  $= -\langle \alpha_i, \alpha_j \rangle$ deg  $= \langle \alpha_i, \alpha_i \rangle$ deg  $= -\langle \alpha_i, \alpha_j \rangle$ deg  $= \langle \alpha_i, \alpha_i \rangle$

deg  $= \langle \lambda, \alpha_i \rangle - d_i$ deg  $= -\langle \lambda, \alpha_i \rangle - d_i$

deg  $= \langle \lambda, \alpha_i \rangle - d_i$ deg  $= -\langle \lambda, \alpha_i \rangle - d_i.$

For a general diagram, we sum together the degrees of the elementary diagrams it is constructed from. This defines a grading on the 2-morphism spaces of $\tilde{\tilde{\mathcal{U}}}$.

2. The 2-category $\mathcal{U}$

Once and for all, fix a matrix of polynomials $Q_{ij}(u,v) = \sum_{k,m} Q_{ij}^{(k,m)} u^k v^m$ valued in $\Bbbk$ and indexed by $i \neq j \in \Gamma$; by convention $Q_{ii} = 0$. We assume $Q_{ij}(u,v)$ is homogeneous of degree $-\langle \alpha_i, \alpha_j \rangle = -2d_i c_{ij} = -2d_j c_{ji}$ when u is given degree $2d_i$ and v degree $2d_j$. We will always assume that the leading order of Q_{ij} in u is $-c_{ij}$, and that $Q_{ij}(u,v) = Q_{ji}(v,u)$. We let $t_{ij} = Q_{ij}^{(-c_{ij},0)} = Q_{ij}(1,0)$; by convention $t_{ii} = 1$. In [**CL15**], the coefficients of this polynomial are denoted

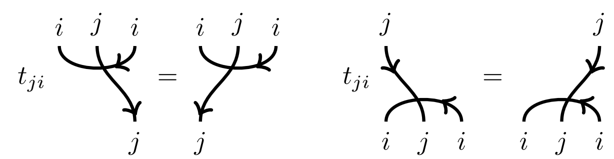
$$Q_{ij}(u,v) = t_{ij}u^{-c_{ij}} + t_{ji}v^{-c_{ji}} + \sum_{pd_i+qd_j=d_ic_{ij}} s_{ij}^{pq} u^p v^q.$$

Khovanov and Lauda's original category uses the choice $Q_{ij} = u^{-c_{ij}} + v^{-c_{ji}}$. To simplify, we'll always set the parameter r_i from [**CL15**] to be $r_i = 1$.

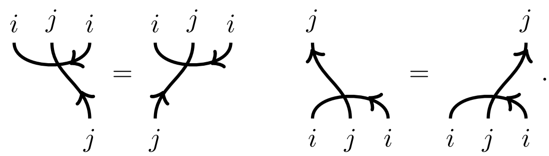
Definition 2.4 *Let $\mathcal{U}$ be the quotient of $\tilde{\tilde{\mathcal{U}}}$ by the following relations on 2-morphisms:*

- *the cups and caps are the units and counits of a biadjunction. The morphism y is cyclic. The cyclicity for crossings can be derived from the pitchfork relation:*

(2.1a)



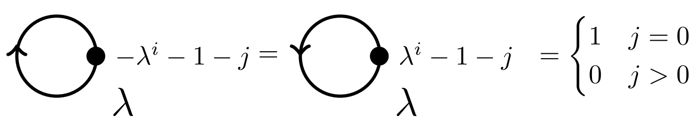
(2.1b)



The mirror images of these relations through a vertical axis also hold.

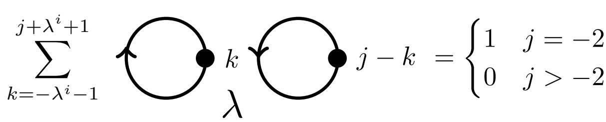
- *Recall that a* **bubble** *is a morphism given by a closed circle, endowed with some number of dots. Any bubble of negative degree is zero, any bubble of degree 0 is equal to 1. Labeling all strands with i, we have that:*

(2.2a)



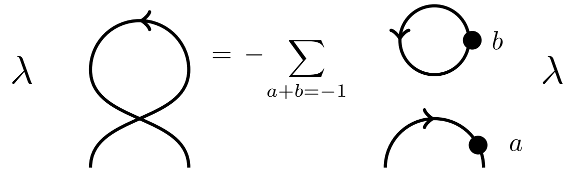
We must add formal symbols called "fake bubbles" which are bubbles labelled with a negative number of dots (these are explained in [**KL10**, §3.1.1]*); given these, we have the inversion formula for bubbles:*

(2.2b)

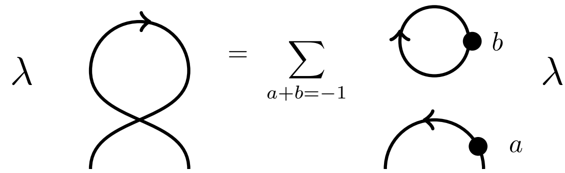


- *2 relations connecting the crossing with cups and caps, shown in* (2.3a-2.3d)*. Since these only involve one label i, we will leave it out of the* *diagrams below.*

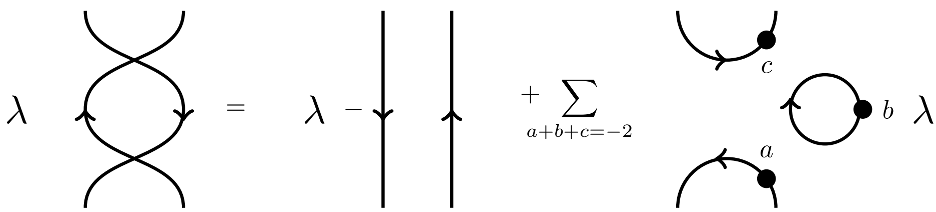
(2.3a)



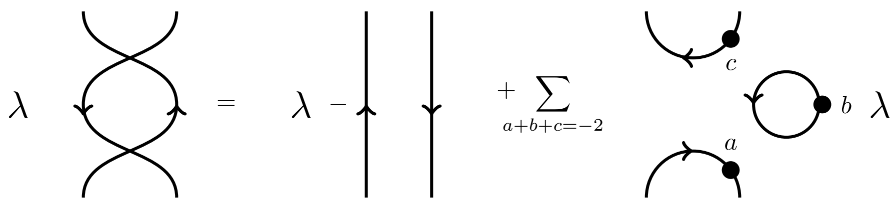
(2.3b)



(2.3c)

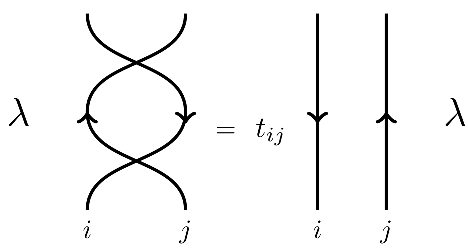


(2.3d)

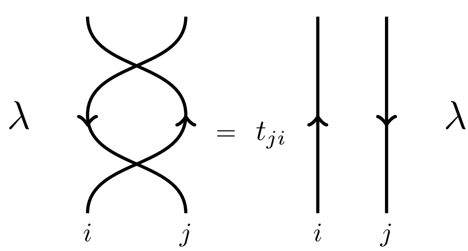


- *Oppositely oriented crossings of differently colored strands simply cancel with a scalar.*

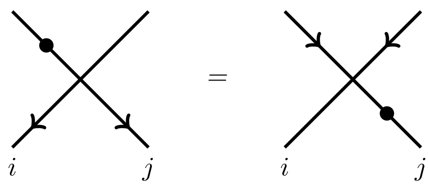
(2.4a)

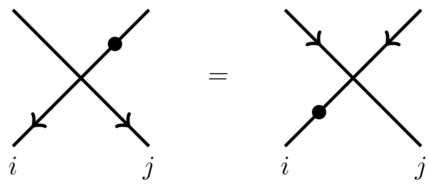


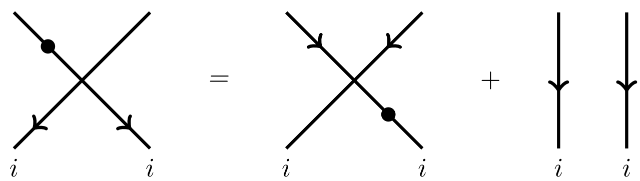
(2.4b)

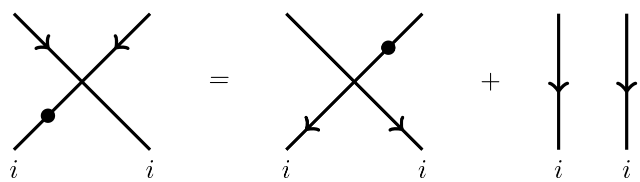


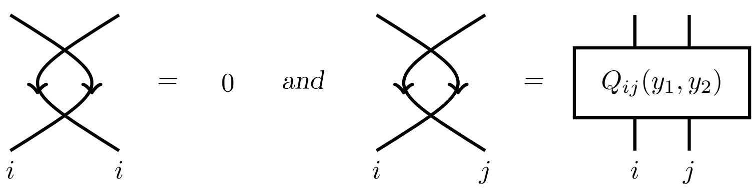
- *the endomorphisms of words only using $\mathcal{F}_i$ (or by duality only $\mathcal{E}_i$'s) satisfy the relations of the* **quiver Hecke algebra** *R.*

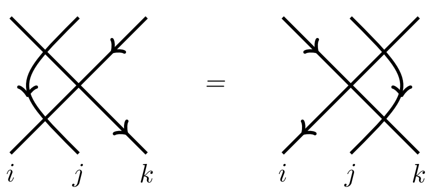
(2.5a)  *unless* $i = j$

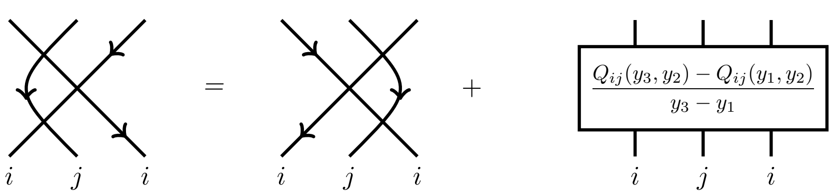
(2.5b)  *unless* $i = j$

(2.5c) 

(2.5d) 

(2.5e) 

(2.5f)  *unless* $i = k \neq j$

(2.5g) 

This completes the definition of the category $\mathcal{U}$.

There are 2-categorical analogues of the positive and negative Borels as well.

Definition 2.5 *Let $\tilde{\tilde{\mathcal{U}}}^-$ be the 2-subcategory of $\tilde{\tilde{\mathcal{U}}}$ formed by sequences and diagrams where only downward pointed strands are allowed. We let $\mathcal{U}^-$ be the quotient of $\tilde{\tilde{\mathcal{U}}}^-$ by the relations* (2.5a–2.5g) *on 2-morphisms. We let $\mathcal{U}^+$ denote the analogous 2-category where only upward pointing strands are allowed.*

Note that the relations (2.5a–2.5g) in this 2-category are insensitive to the labeling of regions (that is, to $\mathcal{L}$ and $\mathcal{R}$). Thus, we can capture all the structure of $\mathcal{U}^-$ in an algebra.

Definition 2.6 *The algebra $R = \mathrm{End}_{\mathcal{U}^-}(\oplus_{\mathbf{i}}\mathcal{F}_{\mathbf{i}})$ where we let $\mathbf{i}$ range over all sequences in $+\Gamma$ with $\mathcal{L}$ fixed is called the* **KLR algebra** *or* **quiver Hecke algebra** *(QHA), which is discussed in* [**Roua**, §4] *and an earlier paper of Khovanov and Lauda* [**KL09**]*. We can realize the same algebra (in a slightly different presentation) as $R = \mathrm{End}_{\mathcal{U}^+}(\oplus_{\mathbf{i}}\mathcal{E}_{\mathbf{i}})$.*

3. A spanning set

For the 2-category $\mathcal{U}$, there is an expected "size" of the category predicted by Khovanov and Lauda, both in terms of its Grothendieck group, and the graded dimension of Hom spaces between objects. However, in [**KL10**] this is only proven for $\mathfrak{g} = \mathfrak{sl}(n)$. In particular, they give a spanning set B in [**KL10**, 3.2.3] for the set of 2-morphisms between fixed 1-morphisms, which we will show is a basis.

For KL pairs G and H, any KL diagram with bottom G and top H induces a matching between the union of the sequences for G and H, by connecting the opposite end of each strand. The set B is indexed by the set of matchings that occur this way: they must connect entries with opposite signs within G or H, and like signs when connecting an entry in G to one in H.

For each such matching φ, we choose (arbitrarily) a diagram m_φ which realizes this matching, and which has no dots and a minimal number of crossings. We also choose (arbitrarily) a preferred location on each strand of m_φ.

Definition 2.7 *We let $B_{G,H}$ denote the set of diagrams obtained from m_φ, ranging over matchings φ, by*

- *adding any number of dots at the location we have fixed on each arc*
- *multiplying on the left by a monomial in positive degree, clockwise oriented bubbles.*

4. Bubble slides

Since these calculations are not done in Cautis and Lauda [**CL15**], let us record the form of the bubble slide relations when the bubble and the strand crossing it have different labels.

Fixing $i \neq j$, we have that the polynomial

$$t_{ji}^{-1}u^{-c_{ji}}Q_{ji}(u^{-1}, v) = \sum t_{ji}^{-1}Q_{ji}^{(-c_{ji}-k,m)}u^k v^m \in 1 + u\Bbbk[u, v] + v\Bbbk[u, v]$$

has an inverse in $\Bbbk[[u,v]]$ which is given by

$$(t_{ji}^{-1}u^{-c_{ji}}Q_{ji}(u^{-1},v))^{-1} = \sum S_{ji}^{(c_{ji}-p,q)}u^p v^q.$$

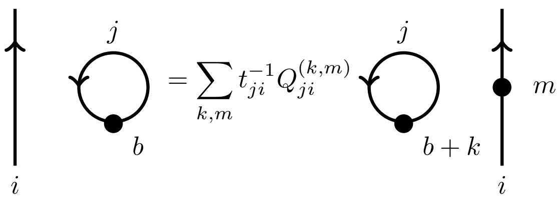
More explicitly, there's a unique collection $S_{ji}^{(p,q)} \in \Bbbk$ such that

$$\sum_{(k,m)} t_{ji}^{-1}Q_{ji}^{(k-p,m-q)}S_{ji}^{(p,q)} = \begin{cases} 1 & (k,m)=(0,0) \\ 0 & (k,m)\neq(0,0) \end{cases}$$

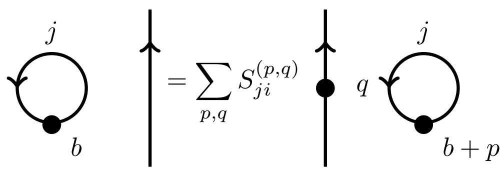
subject to the condition that $S_{ji}^{(c_{ji}-p,q)} = 0$ whenever $p<0$ or $q<0$.

Proposition 2.8 *In the category $\mathcal{U}$, the following relations and their mirror images hold:*

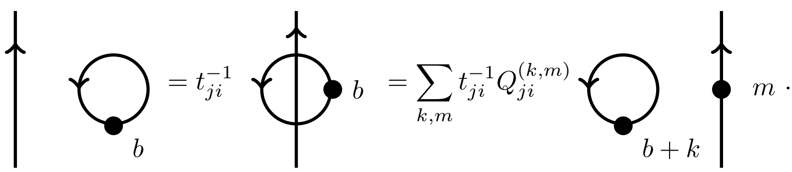
(2.6)



(2.7)



PROOF. The equations are equivalent by the definition of $S_{ij}^{(p,q)}$. Thus, we only need to prove (2.6). Thus follows from



$\square$

CHAPTER 3

Cyclotomic quotients

1. A first approach to the categorification of simples

When one presents a category by generators and relations, it can be difficult to confirm that these relations have not killed more elements then expected, or that the category is not 0. The usual technique for doing this is find a representation of the 2-category where it is easy to confirm that things are non-zero.

For $\mathcal{U}$, this representation will be a natural categorification of the simple representation $V_\lambda^{\mathbb{Z}}$. We define this by imitating the construction of $V_\lambda^{\mathbb{Z}}$ as a quotient of a Verma module.

Definition 3.1 *Let $\widetilde{DR}_\mu^\lambda$ be the algebra $\mathrm{End}_{\mathcal{U}}(\bigoplus \mathcal{F}_{\mathbf{i}})$ where $(\mathbf{i}, \lambda)$ ranges over all KL pairs with $\mathcal{L} = \lambda$ and $\mathcal{R} = \mu$. Unlike in Definition 2.6, we allow entries both from Γ and $-\Gamma$.*

Let DR_μ^λ be the quotient of $\widetilde{DR}_\mu^\lambda$ by the two-sided ideal I_μ generated by

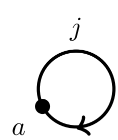
- *the identity on $\mathcal{E}_{\mathbf{i}}$ if $i_1 \in +\Gamma$ (that is, the left-most strand is upward oriented) and*
- *the horizontal composition $a \circ b$ of any map a with a positive degree bubble b.*

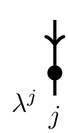
We let $DR^\lambda \cong \oplus_\mu DR_\mu^\lambda$.

We let $D\mathfrak{V}_\mu^\lambda$ denote the category of finitely generated right modules over DR_μ^λ.

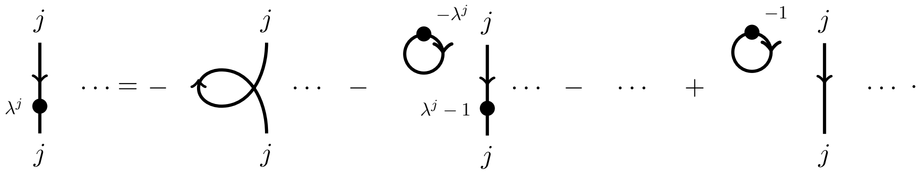
We can write these relations graphically as:

(3.1a)  $\ldots \quad = 0$

(3.1b)  $\ldots \quad = 0 \quad a \geq -\lambda^j$

(3.1c)  $\ldots \quad = 0$

where (3.1c) actually follows from (3.1a–3.1b) by (2.3b):
(3.2)



Note that if λ is not dominant, the algebra DR^λ is 0, since if $\lambda^i < 0$, the identity functor can be written as an honest clockwise bubble with label i and $-\lambda^i - 1$ dots, which is 0 by (3.1a).

We have an obvious functor $\varpi(u) = \mathrm{Hom}(\bigoplus \mathcal{F}_{\mathbf{i}}, u)/I_\mu$ from the category of 1-morphisms $\lambda \to \mu$ in $\mathcal{U}$ to $D\mathfrak{V}^\lambda_\mu$.

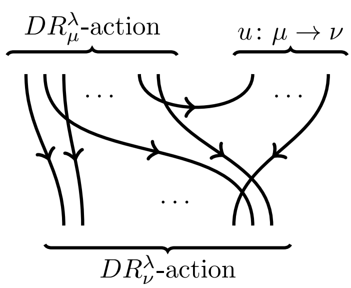
Now, we wish to define an action of $\mathcal{U}$ on the categories $D\mathfrak{V}^\lambda_\mu$ with λ fixed. That is, a 2-functor sending the weight μ to the category $D\mathfrak{V}^\lambda_\mu$, and each 1-morphism $\mu \to \nu$ to a DR^λ_μ-DR^λ_ν-bimodule.

There is only one way to do this which is compatible with ϖ. In the interest of explicitness, we will define this action as tensor product with certain bimodules. We let $\tilde{\delta}_u = \mathrm{Hom}(\bigoplus \mathcal{F}_{\mathbf{i}}, u \circ \bigoplus \mathcal{F}_{\mathbf{j}})$ for u a 1-morphism $\mu \to \nu$ in $\mathcal{U}$. We let

$$\delta_u \cong \tilde{\delta}_u/(I_\nu \cdot \tilde{\delta}_u + \tilde{\delta}_u \cdot I_\mu).$$

Pictorially, we can visualize elements of this bimodule as below.

(3.3)



The 2-morphisms of $\mathcal{U}$ act on the bimodule δ_u by attaching at the top right of the diagram (3.3). This is well-defined since the ideals I_* are closed under horizontal composition $b \mapsto a \circ b$.

Proposition 3.2 *There is a representation of the 2-category $\mathcal{U}$ which sends $\mu \mapsto D\mathfrak{V}^\lambda_\mu$ such that $u \mapsto - \otimes_{DR^\lambda} \delta_u$, with the induced action of 2-morphisms.*

Proof. We have already defined the functors associated to 1-morphisms and the action of 2-morphisms, so we need only check that the relations (2.1a-2.5g) hold. This follows immediately from the locality of the relations, since it makes no difference if we apply them before or after attaching at the top of the diagram. □

Unfortunately, it's not immediately clear that $DR^\lambda_\mu \neq 0$. From a certain perspective, the main result of this section is the fact that this ring contains a non-zero element if the μ weight space of λ is non-trivial. We'll resolve this issue by showing that this ring is Morita equivalent to a more familiar ring, the **cyclotomic quotient** of R.

Definition 3.3 *The* **cyclotomic quiver Hecke algebra** *(QHA) or* **cyclotomic KLR algebra** R^λ *for a weight* λ *is the quotient of* R *by the* **cyclotomic ideal**, *the 2-sided ideal generated by the elements* $y_1^{\lambda^{i_1}} e(\mathbf{i})$ *for all sequences* $\mathbf{i}$. *This is precisely the two-sided ideal generated by the relations* (3.1c).

We let $\mathfrak{V}^\lambda$ *denote the category of finite dimensional graded* R^λ*-modules.*

This algebra has attracted great interest recently in the work of Brundan-Kleshchev [**BK09**], Kleshchev-Ram [**KR11**], Hoffnung-Lauda and Lauda-Vazirani [**LV11**, **HL10**], Hill-Melvin-Mondragon [**HMM12**] and Tingley and the author [**TW**]. It has a very rich structure and representation theory, and some surprising connections to classical representation theory.

Note that we have a natural map $p' \colon R \to DR^\lambda$ thinking of a diagram in R as one in $\widetilde{DR}^\lambda$, and then applying the quotient map. The relation (3.1c) shows that the map p' factors through a map $p \colon R^\lambda \to DR^\lambda$. We will eventually show that this map induces a Morita equivalence.

Unfortunately, it is not easy to attack the question of whether this map is a Morita equivalence directly. Luckily, we can deduce this result for $\mathfrak{sl}_2$ by work of Lauda [**Lau10**]. Since a Kac-Moody algebra is essentially a bunch of $\mathfrak{sl}_2$'s with their interactions described by a Borel, we can hope that this case can lead us to the more general case.

Let us first give a rough sketch of the argument:

- First, we construct an intermediary Morita equivalence between R^λ and a ring similar to DR^λ where one only allows upward strands labeled with one of the elements of Γ.
- We can use this Morita equivalence to show that the restriction and induction functors (defined below) $\mathfrak{E}_i$ and $\mathfrak{F}_i$ define an action of $\mathcal{U}_{\mathfrak{sl}_2}$; in particular, these functors are biadjoint.
- We then need only check one extra relation to confirm that we have a categorical action of $\mathcal{U}$ on the modules over the cyclotomic quotient; this action can be used to confirm the Morita equivalence of R^λ and DR^λ, and to show non-degeneracy for $\mathcal{U}$.

2. Categorifications for minimal parabolics

2.1. The parabolic categorification. Fix $i \in \Gamma$ for the remainder of this chapter.

Definition 3.4 *We let* $\tilde{\tilde{\mathcal{U}}}_i^-$ *be the* 2*-subcategory of* $\tilde{\tilde{\mathcal{U}}}$ *where we allow downward pointing strands with all labels and upward pointing strands only with the label* i. *We let* $\mathcal{U}_i^-$ *denote the quotient of* $\tilde{\tilde{\mathcal{U}}}_i^-$ *where we impose those relations* (2.1a-2.5g) *which still make sense in this* 2*-category.*

In Rouquier's language, we would construct this category by adjoining $\mathcal{E}_i$ to $\mathcal{U}^-$ as a formal left adjoint to $\mathcal{F}_i$, and impose the relations that

- the map $\rho_{s,\lambda}$ is an isomorphism whose inverse is described by the equations (2.3c–2.3d) (in the "style" of Rouquier, one would not impose this equation, but simply adjoin an inverse to $\rho_{s,\lambda}$).

- the right adjunction between $\mathcal{F}_i$ and $\mathcal{E}_i$ is determined by the equations (2.3a–2.3b).

It seems very likely that using an argument in the style of [**Bru**] would show that this defines an equivalent category, but we have not confirmed the details of this.

There are functors $\mathcal{U}^- \to \mathcal{U}_i^- \to \mathcal{U}$, neither of which is manifestly faithful, since new relations could appear when the other objects are added. We note that the 2-morphisms in this category have a spanning set defined as in Definition 2.7:

Definition 3.5 *Let $B_{i,G,H}$ be the subset of $B_{G,H}$ given by KL diagrams which make sense in $\mathcal{U}_i^-$, that is those where any cups, caps or bubbles are labeled with i.*

This category corresponds to the parabolic $\mathfrak{p}_i \cong \mathfrak{b}^- \oplus \mathfrak{g}_{\alpha_i}$.

2.2. The quiver flag category. Lauda defines an action of his categorification of $\mathfrak{sl}_2$ on a "flag category," which gives an algebraic encapsulation of the geometry of Grassmannians. There is an appropriate generalization of Grassmannians for $\mathfrak{g}$ of higher rank with symmetric Cartan matrix. These are the quiver varieties of the graph Γ. Unfortunately, these are analogues of cotangent bundles of Grassmannians, not the Grassmannians themselves, and are typically not cotangent bundles. This makes the geometry required for defining a geometric action of $\mathcal{U}$ considerably more complex. The author has implemented one version of such an action using deformation quantization in [**Weba**] and an action has been defined on coherent sheaves on quiver varieties in [**CKL**]. However, neither of these constructions are suitable for algebras without symmetric Cartan matrices.

A standard trick to get around this issue is to work one vertex at a time. One fixes a vertex i, assumes that it is a source, and replaces the Grassmannian with the space of quiver representations where the sum of all maps out of i is injective. For example, this approach is used by Zheng [**Zhe**] in his construction of a weak categorical action on certain categories of sheaves attached to quiver varieties.

Inspired by this approach, we will develop an algebraic replacement for these quiver varieties which works even in non-symmetric types.

Let us give a brief reminder on Lauda's action on the equivariant cohomology of Grassmannians from [**Lau11**]. Over any field $\Bbbk$, we have an isomorphism

$$H^q_m := H^*_{GL_q}(\mathrm{Gr}(m,q)) \cong \Bbbk[\mathsf{x}_1,\dots,\mathsf{x}_m,\mathsf{y}_1,\dots,\mathsf{y}_{q-m}],$$

where $\mathsf{x}_g = c_g(T)$ and $\mathsf{y}_g = c_g(\mathbb{C}^q/T)$, the Chern classes of the two tautological bundles. By convention, $x_0 = y_0 = 1$. From these, we can construct another very important sequence of classes. Let

$$\mathsf{w}_g := c_g(\mathbb{C}^q) = \sum_{k=0}^{g} \mathsf{x}_k \mathsf{y}_{g-k}, \tag{3.4}$$

where the latter equality follows from the Whitney sum formula; these are the image of the Chern classes in $H_{GL_q}(*)$ under the pullback map. Note, in particular, that $\mathsf{w}_N = 0$ for $N > q$.

We can also write (3.4) as an equality of generating series $\mathsf{w}(u) = \mathsf{x}(u)\mathsf{y}(u)$ where $\mathsf{w}(u) = \sum_{i=0}^{\infty} \mathsf{w}_i u^i$ and similarly for $\mathsf{x}(u)$ and $\mathsf{y}(u)$. Note that in Lauda's construction, the clockwise bubbles correspond to the coefficients of the quotient $\mathsf{x}(-u)/\mathsf{y}(-u)$ and the counter-clockwise to $\mathsf{y}(-u)/\mathsf{x}(-u)$ by [**Lau11**, (3.18)] (keep in mind that our conventions differ from Lauda's by a reflection through the y-axis).

The ring $\Bbbk[\mathsf{w}_1, \dots, \mathsf{w}_q]$ functions as a base ring for Lauda's construction, since all bimodules and and bimodule maps he considers arise from pullback and push-forward in equivariant cohomology. This is also easily seen from the algebraic formulas he gives. In particular, for any commutative ring B and ring homomorphism $\Bbbk[\mathsf{w}_1, \dots, \mathsf{w}_q] \to B$, we can base change Lauda's action by tensoring all bimodules in the construction with B. For example, the action on the usual cohomology of Grassmannians given in [**Lau10**] is the base change via the quotient map $\Bbbk[\mathsf{w}_1, \dots, \mathsf{w}_q] \to \Bbbk$.

The construction with Grassmannians we've discussed gives a categorification of the representation $V_q = \mathrm{Sym}^q(\mathbb{C}^2)$ of $\mathfrak{sl}_2$. We can endow this vector space with a module structure over the whole Levi $\mathfrak{sl}_2 + \mathfrak{h}$ for $\mathfrak{p}_i$ by extending the highest weight to an integral weight λ such that $q = \lambda^i$.

For a fixed weight λ with $\lambda^i \geq 0$, there is essentially a universal categorical representation of $\mathcal{U}_i^-$ which satisfies a few basic properties. As usual, we let $\Lambda(\mathbf{p})$ be the algebra of symmetric polynomials on an alphabet $\mathbf{p}$, and let $e_i(\mathbf{p}), h_i(\mathbf{p})$ denote the elementary and complete symmetric polynomials of degree i. We want to find a module over $\mathcal{U}_i^-$ such that:

(1) The category attached to a weight $\mu = \lambda - \sum_{j\in\Gamma} m_j\alpha_j$ is the category of representations of an algebra $\mathbf{\Lambda}_\mu$, with $\mathbf{\Lambda}_\lambda \cong \Bbbk$. If $\mu \nleq \lambda$ then $\mathbf{\Lambda}_\mu = 0$.
(2) If one only uses $\mathcal{F}_j$ for $j \neq i$, then the representation coincides with the polynomial representation of $\mathcal{U}^-$. That is, in the special case $\nu = \lambda - \sum_{j\neq i} m_j\alpha_j$, we have that $\mathbf{\Lambda}_\nu \cong \bigotimes_{j\neq i}\Lambda(\mathbf{p}_j)$ where $\mathbf{p}_j$ is the alphabet $\{p_{j,1}, \dots, p_{j,m_j}\}$, and $\mathcal{F}_{\mathbf{i}}(\mathbf{\Lambda}_\lambda)$ is a copy of the polynomial ring $\Bbbk[\{p_{j,k}\}]$ with action given by the usual polynomial action of [**Roua**, Prop. 3.12] or [**KL11**] (also shown in equations (3.12)).
(3) On the string $\{\nu, \nu - \alpha_i, \nu - 2\alpha_i, \dots\}$ then the algebras $\mathbf{\Lambda}_{\nu - m_i\alpha}$ arise as the base change of $H^{\nu^i}_{m_i}$ by a map $\gamma_\nu \colon H^{\nu^i}_0 \cong \Bbbk[\mathsf{w}_1, \dots, \mathsf{w}_{\nu^i}] \to \mathbf{\Lambda}_\nu$.

We can determine the map γ_ν using bubble slides. Again, consider a weight of the form $\nu = \lambda - \sum_{j\neq i} m_j\alpha_j$. In this case, we have that $\mathsf{x}(u) = 1$, so $(-1)^k\mathsf{w}_k = (-1)^k\mathsf{y}_k$ corresponds to the clockwise bubble labeled i of degree k as calculated in [**Lau11**, §4.2]. Thus, we can calculate how w_k should act in $\mathbf{\Lambda}_\nu$ using the bubble slide (2.6), reflected through a horizontal axis.

Fix a sequence $(-j_1, \dots, -j_m)$ where $j \in -\Gamma$ appears m_j many times. We will use $\circlearrowright_i (k)$ as shorthand for the clockwise bubble of degree k labeled with i. We're interested in how this bubble will act in the rightmost region with label ν. We introduce a power series

$$\Theta_p(u) = (-1)^k \sum \mathrm{id}_{(-j_{p+1},\dots,-j_m)} \circ \circlearrowright_i (k) \circ \mathrm{id}_{(-j_1,\dots,-j_p)}\, u^k$$

given by the action of these bubbles when placed between the pth and $p+1$st strand. The bubble slides allow us to compute these power series inductively. By assumption $\Theta_0(u) = 1$, and the bubble slide (2.6) can be restated as the formula

$$\Theta_p(u) = \Theta_{p-1}(u)t_{ij_p}^{-1} \cdot (-u)^{-c_{ij_p}} Q_{j_p i}(y_k, -u^{-1}).$$

Thus, we have that

$$\gamma_\nu(\mathsf{w}(u)) = \Theta_m(u) = \prod_{j\neq i}\prod_{k=1}^{m_j} t_{ij}^{-1} \cdot (-u)^{-c_{ij}} Q_{ji}(p_{j,k}, -u^{-1}). \tag{3.5}$$

Since w_i has the same action on the left and right of all bimodules in Lauda's construction, this formula holds for all weights of the form $\mu = \nu - m_i\alpha_i$ as well. Thus, we can rephrase points (1-3) above as:

Definition 3.6 *The ring $\mathbf{\Lambda}_\mu$ is the base extension of $\Bbbk[\mathsf{x}_1, \dots, \mathsf{x}_{m_i}, \mathsf{y}_1, \dots, \mathsf{y}_{\mu^i+m_i}]$ via the map γ_ν. That is, if we let $\gamma_\mu \colon \Bbbk[\mathsf{x}_1, \dots, \mathsf{x}_{m_i}, \mathsf{y}_1, \dots, \mathsf{y}_{\mu^i+m_i}] \to \mathbf{\Lambda}_\mu$ denote the induced maps, then this ring is generated over $\bigotimes_{j\neq i} \Lambda(p_{j,1}, \dots, p_{j,m_j})$ by the elements $\gamma_\mu(\mathsf{x}_k)$ for $k = 1, \dots, m_i$ and $\gamma_\mu(\mathsf{y}_k)$ for $k = 1, \dots, \mu^i + m_i$, subject only to the relation*

$$\gamma_\mu(\mathsf{x}(u))\gamma_\mu(\mathsf{y}(u)) = \prod_{j\neq i}\prod_{k=1}^{m_j} t_{ij}^{-1} \cdot (-u)^{-c_{ij}} Q_{ji}(p_{j,k}, -u^{-1}). \tag{3.6}$$

We call the category $\mathbf{\Lambda}_\mu$-mod the **quiver flag category**.

We identify the images $\gamma_\mu(\mathsf{x}_i)$ with the elementary symmetric polynomials in an alphabet $p_{i,1}, \dots, p_{i,m_i}$. That is:

$$\gamma_\mu(\mathsf{x}(u)) = \sum_{m=1}^{m_i} e_m(\mathbf{p}_i) = \prod_{k=1}^{m_i}(1 + p_{i,k}u),$$

where $\mathbf{p}_i$ denotes the alphabet of variables $p_{i,*}$. We can thus write $\mathbf{\Lambda}_\mu$ as a quotient of the ring

$$\tilde{\mathbf{\Lambda}}_\mu \cong \bigotimes_{j\in\Gamma} \Lambda(p_{j,1}, \dots, p_{j,m_j})$$

of polynomials, symmetric in each of a union of alphabets, one for each node of Γ, with size given by m_j. In view of (3.6), we can identify $\gamma_\mu(\mathsf{y}(u))$ with the power series in $\tilde{\mathbf{\Lambda}}_\mu$ given by

$$\Xi_\mu(\mathbf{p}, u) := \left(\sum_{k=0}^{\infty} h_k(\mathbf{p}_i)(-u)^k\right) \prod_{j\neq i}\prod_{k=1}^{m_j} t_{ij}^{-1} \cdot (-u)^{-c_{ij}} Q_{ji}(p_{j,k}, -u^{-1}).$$

Note that if each element of $\mathbf{p}_j$ is given degree $2d_j$, and u given degree $-2d_i$, then Ξ_μ is homogeneous of degree 0; this is clear for the first term in the product, and follows from the fact that $Q_{ji}(p_{j,k}, -u^{-1})$ is homogeneous of degree $-2d_ic_{ij}$ by assumption.

We let $f(u)\{u^g\}$ denote the u^g coefficient of a polynomial or power series.

Lemma 3.7 *The map sending*

$$e_k(\mathbf{p}_i) \to \gamma_\mu(\mathsf{x}_k) \qquad\qquad \Xi_\mu\{u^g\} \mapsto \gamma_\mu(\mathsf{y}_g)$$

induces an isomorphism from the quotient of $\tilde{\mathbf{\Lambda}}_\mu$ by the relations:

$$\Xi_\mu\{u^g\} = 0 \qquad \text{for all } g > \mu^i + m_\mu^i. \tag{3.7}$$

to $\mathbf{\Lambda}_\mu$.

Proof. Let $\mathbf{\Lambda}'$ be the quotient of $\tilde{\mathbf{\Lambda}}_\mu$ by the relations (3.7). First, we note that we have a map $\mathbf{\Lambda}' \to \mathbf{\Lambda}_\mu$. The equality $\gamma_\mu(\mathsf{y}(u)) = \gamma_\mu(\mathsf{w}(u))/\gamma_\mu(\mathsf{x}(u)) = \Xi_\mu(\mathbf{p}, u)$ implies that the relations (3.7) hold as the corresponding coefficients of $\mathsf{y}(u)$ vanish as well.

The ring $\mathbf{\Lambda}_\mu$ has rank $\binom{\mu^i+2m_i}{m_i}$ as a module over $\bigotimes_{j\neq i}\Lambda(p_{j,1},\dots,p_{j,m_j})$. On the other hand, the coefficients $\Xi_\mu\{u^g\}$ have the form $h_g(\mathbf{p}_i)+\cdots$ where the remaining terms have lower order in $\mathbf{p}_i$. Thus, $\mathbf{\Lambda}'$ is spanned over $\bigotimes_{j\neq i}\Lambda(p_{j,1},\dots,p_{j,m_j})$ by any spanning set in $\Lambda(\mathbf{p}_i)/(h_g(\mathbf{p}_i)\mid g>\mu^i+m^i_\mu)$ which is isomorphic to $H^*(\mathrm{Gr}(m^i,\mu^i+2m^i))$. Therefore, the rank of $\mathbf{\Lambda}'$ is $\leq\binom{\mu^i+2m_i}{m_i}$, which is only possible if the map is an isomorphism. $\square$

2.3. The action. We wish to define a 2-functor $\mathcal{G}_\lambda$ from $\mathcal{U}_i^-$ to the 2-category of $\Bbbk$-linear categories which on the level of 0-morphisms sends $\mu\mapsto\mathbf{\Lambda}_\mu$-mod. On 1-morphisms, we need only say where we send the 1-morphisms $\mathcal{E}_i$ and $\mathcal{F}_j$ for all $j\in\Gamma$.

- The functors $\mathcal{F}_j$ for $j\neq i$ act by tensoring with the $\mathbf{\Lambda}_\mu$ - $\mathbf{\Lambda}_{\mu-\alpha_j}$ bimodule $\mathbf{\Lambda}_\mu[p_{j,m_j+1}]$. The left-module structure over $\mathbf{\Lambda}_\mu$ is the obvious one, and right-module over $\mathbf{\Lambda}_{\mu-\alpha_j}$ is a slight tweak of this: $e_k(\mathbf{p}'_j)$ acts by $e_k(\mathbf{p}_j,p_{j,m_j+1})$, $e_k(\mathbf{p}'_m)$ by $e_k(\mathbf{p}_m)$ for $m\neq j$.
- The functor $\mathcal{F}_i$ acts by an analogue of the action in Lauda's paper [**Lau11**]; tensor product with a natural $\mathbf{\Lambda}_\mu$ - $\mathbf{\Lambda}_{\mu-\alpha_i}$-bimodule $\mathbf{\Lambda}_{\mu;i}$ which is a quotient of $\mathbf{\Lambda}_\mu[p_{i,m_i+1}]$ by the relation

$$\left(\sum_{c=0}^{\infty}(-p_{i,m_i+1}u)^c\right)\Xi_\mu\{u^g\}=0\qquad\text{for all } g>\mu^i+m_i-1 \tag{3.8}$$

  with the same left and right actions as above.
- Similarly, the functor $\mathcal{E}_i$ acts by tensor product with $\dot{\mathbf{\Lambda}}_{\mu+\alpha_i;i}$, the bimodule defined above with the actions above reversed. This can also be presented as a quotient of $\mathbf{\Lambda}_\mu[p_{i,m_i}]$ by the relation

$$(1+p_{i,m_i+1}u)\,\Xi_\mu\{u^g\}=0\qquad\text{for all } g>\mu^i+m_i.$$

Now, we must specify the action of 2-morphisms.

All the morphisms only involving only the label i are inherited from the corresponding construction for $\mathfrak{sl}_2$, given by Lauda in [**Lau11**]. Let s be a 1-morphism in $\mathcal{U}_{\mathfrak{sl}_2}$ and let $\eta'(s)$ be the bimodule over equivariant cohomology rings of Grassmannians associated to s under the representation Γ_p^G defined in [**Lau11**, §4.1]; recall that we have an auto-functor $\tilde{\omega}\colon\mathcal{U}\to\mathcal{U}$ defined in [**KL10**, §3.3.2] which swaps $\mathcal{E}_i$ and $\mathcal{F}_i$. Note, we must use this functor because Lauda's construction uses a lowest weight representation, rather than a highest weight; that is, he associates the Grassmannian of 0 planes in $\mathbb{C}^N$ to the $-N$ weight space of a representation, by [**Lau11**, (4.1)].[1]

Lemma 3.8 *The base change by the map γ_μ sends the bimodule $\eta'(\tilde{\omega}s)$ to the bimodule $\mathcal{G}_\lambda(s)$.*

Proof. This follows from the definition for $\mathcal{E}_i$ and $\mathcal{F}_i$; since these bimodules are flips of each other, we need only check it that $\eta'(\mathcal{E}_i)$ is sent to $\eta(\mathcal{F}_i)$. Using the notation of [**Lau11**], the bimodule $\eta'(\mathcal{E}_i)$ is just the polynomial ring

[1] Of course, there are other differences between our conventions and Lauda's, but these cancel. In this paper, we read diagrams from left to right, rather than right to left, but because we use left modules, we have the same conventions for ordering bimodules as Lauda.

$\Bbbk[w_1,\dots,w_{m_i},\xi,z_1,\dots,z_{p-m_i-1}]$ and we can define a map of this to $\mathbf{\Lambda}_{\mu;i}$ by sending

$$w_k \mapsto e_k(\mathbf{p}_i) \qquad \xi \mapsto p_{i,m_i+1} \qquad z_\ell \mapsto \Xi_{\mu-\alpha_i}(u)\{u^\ell\}.$$

Indeed, the left and right actions match those given by Lauda in [**Lau11**, §3.1]; for the left action, this follows from

$$\mathsf{x}_k \mapsto w_k \mapsto e_k(\mathbf{p}_i) \qquad \mathsf{y}_\ell \mapsto z_\ell + \xi\cdot z_{\ell-1} = \Xi_\mu(u)\{u^\ell\}$$

where the last equality holds since $(1+up_{i,m_i+1})\Xi_{\mu-\alpha_i}(u) = \Xi_\mu(u)$. For the right action, we have that

$$\mathsf{x}_k \mapsto w_k + \xi w_{k-1} \mapsto e_k(\mathbf{p}_i, p_{i,m_i+1}) \qquad \mathsf{y}_\ell \mapsto z_\ell \mapsto \Xi_{\mu-\alpha_i}(u)\{u^\ell\},$$

as desired.

The same result holds for tensor products of these bimodules, since they are free both as left and as right modules, that is, they are **sweet**. Thus, tensor products and base change commute, and we are done. $\square$

Thus, we can define all 2-morphisms between $\mathcal{F}_i$'s and $\mathcal{E}_i$'s by simply base changing the same 2-morphisms from [**Lau11**, §4.1]. That is, in our notation, we have that

- the transformations y acts by

$$y\colon \mathcal{F}_i \to \mathcal{F}_i \mapsto (f \mapsto fp_{i,m_i+1}) \qquad y\colon \mathcal{E}_i \to \mathcal{E}_i \mapsto (f \mapsto fp_{i,m_i}) \tag{3.9}$$

- the transformation $\psi\colon \mathcal{F}_i^2 \to \mathcal{F}_i^2$ acts by multiplication by the usual Demazure operator in the last two variables:

$$f \mapsto \frac{f^s - f}{p_{i,m_i+2} - p_{i,m_i+1}} \tag{3.10}$$

  where s is the transformation switching p_{i,m_i+1} and p_{i,m_i+2}.
- the adjunctions are given by:

$$\iota = \underset{\mu}{\overset{\mu+\alpha_i}{i \curvearrowleft i}} \mapsto \Big(1 \mapsto \sum_{j=0}^{\mu^i+m_i} (-1)^j \Xi_\mu(u)\{u^j\} \otimes p_{i,m_i}^{\mu^i+m_i-j}\Big) \tag{3.11a}$$

$$\iota' = \underset{\mu}{\overset{\mu-\alpha_i}{i \curvearrowright i}} \mapsto \Big(1 \mapsto \sum_{j=0}^{m_i} (-1)^j p_{i,m_i+1}^{m_i-j} \otimes e_j(\mathbf{p}_i)\Big) \tag{3.11b}$$

(3.11c)

$$\epsilon' = \underset{\mu+\alpha_i}{\overset{\mu}{i \curvearrowright i}} \mapsto \Big(p_{i,m_i}^a \otimes p_{i,m_i}^b \mapsto h_{a+b-m_i+1}(\mathbf{p}_i)\Big)$$

(3.11d)

$$\epsilon = \underset{\mu-\alpha_i}{\overset{\mu}{i \curvearrowleft i}} \mapsto \Big(p_{i,m_i+1}^a \otimes p_{i,m_i+1}^b \mapsto (-1)^{a+b+1-m_i-\mu^i} \Xi_\mu^{-1}(u)\{u^{a+b+1-m_i-\mu^i}\}\Big)$$

Above, we use that by our assumptions on Q_{ij}, the power series $\Xi(u)$ has a non-zero constant term, and thus has a formal inverse in $\Lambda(\mathbf{p})[[t]]$, which we denote $\Xi^{-1}(u)$. By the usual Cauchy formula, we have

$$\Xi^{-1}(u) = \left(\sum_{k=0}^{\infty} e_k(\mathbf{p}_i)u^k\right) \prod_{j\neq i} \prod_{k=0}^{m_j} \frac{t_{ij} \cdot u^{c_{ij}}}{Q_{ji}(p_{j,k}, -u^{-1})}.$$

Now we turn to the question of describing the action of 2-morphisms involving labels other than i. Choose an orientation on Γ so that i is a source; this is necessary in order to pin down conventions for the action of the KLR algebra in its polynomial representation, as we see in (3.12) below. We can identify $\mathcal{F}_j\mathcal{F}_k$ with tensor product with $\mathbf{\Lambda}_{\mu;k} \otimes_{\mathbf{\Lambda}_{\mu-\alpha_k}} \mathbf{\Lambda}_{\mu-\alpha_k;j}$, and define the transformation $\psi\colon \mathcal{F}_j\mathcal{F}_k \to \mathcal{F}_k\mathcal{F}_j$ as in [**Roua**, Proposition 3.12] or [**KL11**] via the formulae

$$\psi(f) = \begin{cases} \frac{f^s - f}{p_{j,m_j+2} - p_{i,m_j+1}} & j = k \\ Q_{jk}(p_{j,m_j+1}, p_{k,m_k+1})f & j \to k \\ f & j \not\to k \end{cases} \tag{3.12}$$

Finally, we use (2.1a) as the definition of a crossing of a downward j colored strand and an upward i colored one. Note that it is not obvious that these formulae are well defined in all cases; we will check this in the course of the proof.

Theorem 3.9 *The formulas of* (3.9-3.12) *define a strict 2-functor $\mathcal{G}_\lambda$ from $\mathcal{U}_i^-$ to the 2-category of $\Bbbk$-linear categories which sends $\mu \mapsto \mathbf{\Lambda}_\mu$*-mod.

Proof. First, we must check the maps we have given above make sense, and then we must confirm that they satisfy the correct relations. For diagrams only involving the label i, this follows immediately from base change by [**Lau11**, Theorem 4.13].

On the other hand, for diagrams not involving i's, the variables p_{j,m_j+1} and p_{k,m_k+1} act freely. Thus, the formulae of (3.12) give well-defined maps. The relations (2.5a–2.5g) follow since these operators match a known representation of the KLR algebra (given, for example, in [**Roua**, Proposition 3.12]).

Thus, the only issue is the interaction between these 2 classes of functors. In particular, it remains to show the maps corresponding to elements of $R(\nu)$ are well defined (the relations between them then automatically hold, since quotienting out by relations will not cause two things to become unequal).

Now, consider the bimodules $\mathbf{\Lambda}_{\mu;j} \otimes_{\mathbf{\Lambda}_{\mu-\alpha_j}} \mathbf{\Lambda}_{\mu-\alpha_j;i}$ and $\mathbf{\Lambda}_{\mu;i} \otimes_{\mathbf{\Lambda}_{\mu-\alpha_i}} \mathbf{\Lambda}_{\mu-\alpha_i;j}$. The functors of tensor product with these are canonically isomorphic to $\mathcal{F}_i\mathcal{F}_j$ and $\mathcal{F}_j\mathcal{F}_i$, respectively (though they are not the same "on the nose"), so it suffices to define the map ψ as a map between these bimodules. The latter is just $\mathbf{\Lambda}_{\mu;i}[p_{j,m_j+1}]$, so the relations are just (3.8).

The former is a quotient of $\mathbf{\Lambda}_\mu[p_{j,m_j+1}, p_{i,m_i+1}]$ by

$$u^{-c_{ij}} Q_{ji}(p_{j,m_j+1}, -u^{-1}) \left(\sum_{c=0}^{\infty} (-p_{i,m_i+1}u)^c\right) \Xi_\mu\{u^g\} = 0 \tag{3.13}$$

for all $g > \mu^i + m_i - 1 - c_{ji}$.

Note that

$$\begin{aligned}(3.14)\qquad \frac{(-u)^{-n}}{1+p_{i,m_i+1}u} &= \frac{(-u)^{-n}-p^n_{i,m_i+1}}{1+p_{i,m_i+1}u} + \frac{p^n_{i,m_i+1}}{1+p_{i,m_i+1}u} \\ &= (-u)^{-n} + (-u)^{-n+1}p_{i,m_i+1} + \cdots - u^{-1}p^{n-1}_{i,m_i+1} + \frac{p^n_{i,m_i+1}}{1+p_{i,m_i+1}u}\end{aligned}$$

Modulo the relations (3.7) of $\mathbf{\Lambda}_\mu$, we have the equality

$$u^{-c_{ij}}Q^{(k,n)}_{ji}p^k_{j,m_j+1}((-u)^{-n} + (-u)^{-n+1}p_{i,m_i+1} + \cdots - u^{-1}p^{n-1}_{i,m_i+1})\Xi_\mu\{u^g\} = 0$$

for all $g > \mu^i + m_i - 1 - c_{ji}$, so replacing every $(-u)^n$ in (3.13) with the equality from (3.14), we have that

$$u^{-c_{ij}}Q_{ji}(p_{j,m_j+1}, p_{i,m_i+1})\left(\sum_{c=0}^{\infty}(-p_{i,m_i+1}u)^c\right)\Xi_\mu\{u^g\} = 0$$

for all $g > \mu^i + m_i - 1 - c_{ji}$. Thus, the new relations introduced are exactly $Q_{ji}(p_{j,m_j+1}, p_{i,m_i+1})$ times those of $\mathbf{\Lambda}_{\mu;i}[p_{j,m_j+1}]$. Thus, the definition of ψ given above indeed induces a map as desired.

Let us illustrate this point in the simplest case, when $\mu = \lambda$. In this case, we have that

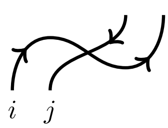
$$\begin{aligned}\mathbf{\Lambda}_\lambda &= \Bbbk, & \mathbf{\Lambda}_{\lambda-\alpha_i} &= \Bbbk[p_i]/(p_i^{\lambda^i}) \\ \mathbf{\Lambda}_{\lambda-\alpha_j} &= \Bbbk[p_j] & \mathbf{\Lambda}_{\lambda-\alpha_i-\alpha_j} &= \Bbbk[p_i,p_j]/(p_i^{\lambda^i}Q_{ji}(p_j,p_i))\end{aligned}$$

The only one of these requiring any appreciable computation is the last. In this case, we have the relation $p_i^{\lambda^i}Q_{ji}(p_j,p_i) = 0$ by taking the $u^{\lambda^i-1-c_{ij}}$ term of $(1 - p_it + \cdots)u^{-c_{ij}}Q_{ji}(p_j, -u^{-1})$.

Finally, we must prove the relations (2.4a) and (2.4b). This is simply a calculation, given that we have already defined the morphisms for all the diagrams which appear. The composition

$$(3.15)\qquad \mathcal{F}_j\mathcal{E}_i \xrightarrow{\iota_1'} \mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathcal{E}_i \xrightarrow{\psi_2} \mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathcal{E}_i \xrightarrow{\epsilon_3'} \mathcal{E}_i\mathcal{F}_j$$

pictorially



is given by

$$\begin{aligned}\epsilon_3'\psi_2\iota_1'(p^a_{i,m_i}\otimes p^b_{j,m_j+1}) &= \epsilon_3'\psi_2\left(\sum_{k=0}^{m+m_i-1}(-1)^k p^a_{i,m_i}\otimes p^b_{j,m_j+1}\otimes p^{m_i-k-1}_{i,m_i}\otimes e_k(\mathbf{p}_i^-)\right) \\ &= \epsilon_3'\left(\sum_{k=0}^{m_i-1}(-1)^k p^a_{i,m_i}\otimes p^{m_i-k-1}_{i,m_i}\otimes p^b_{j,m_j+1}\otimes e_k(\mathbf{p}_i^-)\right) \\ &= \sum_{k=0}^{a}(-1)^k p^b_{j,m_j+1}\otimes h_{a-k}(\mathbf{p}_i)e_k(\mathbf{p}_i^-) \\ &= p^b_{j,m_j+1}\otimes p^a_{i,m_i}\end{aligned}$$

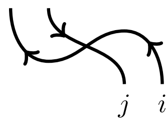
Above, we use $\mathbf{p}_i^-$ to denote the alphabet $\{p_{i,1},\dots,p_{i,m_i-1}\}$, and we use the identity

$$\sum_{k=0}^{a}(-1)^k h_{a-k}(\mathbf{p}_i)e_k(\mathbf{p}_i^-) = \frac{\prod_{k=1}^{m_i-1}(1-up_{i,k})}{\prod_{k=1}^{m_i}(1-up_{i,k})}\{u^a\} = p_{i,m_i}^a.$$

The composition

$$\mathcal{E}_i\mathcal{F}_j \xrightarrow{\iota_3} \mathcal{E}_i\mathcal{F}_j\mathcal{F}_i\mathcal{E}_i \xrightarrow{\psi_2} \mathcal{E}_i\mathcal{F}_i\mathcal{F}_j\mathcal{E}_i \xrightarrow{\epsilon_1} \mathcal{F}_j\mathcal{E}_i \tag{3.16}$$

pictorially



is given by

$$\begin{aligned}
&\epsilon_1\psi_2\iota_3(p_{j,m_j+1}^b\otimes p_{i,m_i}^a)\\
&\quad=\epsilon_1\psi_2\left(\sum_{k=0}^{m_i-1}(-1)^k\Xi_\mu(u)\{u^k\}\otimes p_{i,m_i}^{\mu^i+m_i-k}\otimes p_{j,m_j+1}^b\otimes p_{i,m_i}^a\right)\\
&\quad=\epsilon_1\left(\sum_{k=0}^{m_i-1}(-1)^k\Xi_\mu(u)\{u^k\}\otimes p_{j,m_j+1}^b\otimes p_{i,m_i}^{m_i-k+1}Q_{ij}(p_{i,m_i},p_{j,m_j+1})\otimes p_{i,m_i}^a\right)\\
&\quad=\sum_{k=0}^{a}(-1)^k\Xi_\mu(u)\{u^k\}\cdot\Xi_{\mu+\alpha_i-\alpha_j}(u)^{-1}Q_{ji}(p_{j,m_j+1},-u)\{u^{a-k-c_{ji}}\}\otimes p_{j,m_j+1}^b\\
&\quad=t_{ij}p_{i,m_i}^a\otimes p_{j,m_j+1}^b
\end{aligned}$$

Thus, composing the maps (3.15) and (3.16) in either order gives t_{ij} times the identity, confirming the relations (2.4a-2.4b). □

Recall the spanning set $B_{i,G,H}$ defined in Definition 3.5. In fact, this set is a basis:

Corollary 3.10 *Every non-trivial linear combination of elements of $B_{i,G,H}$ in $\mathcal{U}_i^-$ acts non-trivially in one of these categories. That is, $\mathcal{U}_i^-$ is non-degenerate in the sense of Khovanov-Lauda.*

Proof. If there is any pair of 1-morphisms G,H where the set $B_{i,G,H}$ is not linearly independent, then using the biadjunction of $\mathcal{F}_i$ and $\mathcal{E}_i$ and the commutation relations, we can assume that G and H only involve elements of $-\Gamma$, that is downward strands. In this case, the functor $\mathcal{F}_{\mathbf{i}}$ corresponds to adjoining new variables, followed by certain relations, where morphisms in $\mathcal{U}^-$ act on the polynomial ring by the polynomial representation we've defined.

No linear combination in B_i acts trivially before modding out by the relations (3.7). Furthermore, for each degree N, we can choose λ sufficiently large that all relations in $\mathbf{\Lambda}_\mu$ are of degree $> N$. Thus, there can be no non-trivial linear combinations in degree N. □

3. Cyclotomic quotients

Now that we understand how to add the adjoint of one of the $\mathcal{F}_i$'s to $\mathcal{U}^-$, we move towards considering all of them. Just as with $\mathcal{U}^-$ and $\mathcal{U}_i^-$, we prove non-degeneracy by constructing a family of actions which are jointly faithful. As in the previous chapter, i will denote a fixed element of Γ, and we will use j for an arbitrary index.

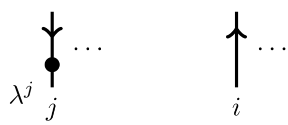
Our first step is to realize the cyclotomic quotient in terms of the category $\mathcal{U}_i^-$. Fix a dominant weight λ. Now, let Σ_i be the set of sequences in $-\Gamma \cup \{+i\}$, all considered as KL pairs with $\mathcal{L} = \lambda$. In the definition below, we'll only be interested in 1-morphisms that originate at λ, and will thus use $\mathcal{F}_{\mathbf{i}}$ to denote the 1-morphism with this name originating at λ.

Definition 3.11 *Let D^iR^λ be the quotient of the algebra* $\mathrm{End}_{\mathcal{U}_i^-}(\oplus_{\mathbf{i}\in\Sigma_i}\mathcal{F}_{\mathbf{i}})$ *by the relations*

$$y_1^{\lambda^{-i_1}} 1_{\mathbf{i}} = 0 \qquad i_1 \in -\Gamma \tag{3.17}$$

$$1_{\mathbf{i}} = 0 \qquad i_1 = +i \tag{3.18}$$

In terms of diagrams, this means that we kill all diagrams of the form



irrespective of what occurs to the right of the first strand. Note that these relations are equivalent to (3.1a–3.1c), since any positive degree counter-clockwise bubble is killed by (3.17). In particular, as before, if λ is not dominant, this algebra is 0. For purposes of reference in the proof below, we'll call the strand which is at the far left in either diagram above **violating**. We'll call a KL pair **downward** if it only uses entries from $-\Gamma$.

In D^iR^λ, we have a natural idempotent e_- which kills $\mathcal{F}_{\mathbf{i}}$ if $\mathbf{i}$ is not downward, and acts as the identity on it if it is downward. We have a natural ring map $I\colon R^\lambda \to D^iR^\lambda$ which lands in the subalgebra $e_-D^iR^\lambda e_-$.

Lemma 3.12 *The map I induces an isomorphism $R^\lambda \cong e_-D^iR^\lambda e_-$.*

Proof. The argument will be easier if we give a slightly different presentation of R^λ. Let $\check{R}$ denote the tensor product of the ring R with the ring $\mathrm{End}_{\mathcal{U}_i^-}(\mathrm{id}_\lambda)$, which is a polynomial ring generated by counter-clockwise bubbles with label i; when we draw a picture, we place this element of $\mathrm{End}_{\mathcal{U}_i^-}(\mathrm{id}_\lambda)$ at the far left of the diagram. By Corollary 3.10, we can identify

$$\check{R} \cong \mathrm{End}_{\mathcal{U}_i^-}(\oplus_{\mathbf{i}}\mathcal{F}_{\mathbf{i}}).$$

The bubble slides (2.6) allow us to interpret a bubble placed anywhere in the diagram as an element of $\check{R}$. We have a ring map $\check{R} \to R$ which just kills positive degree bubbles, and thus induces maps $\check{R} \to R^\lambda$ and $\check{R} \to D^iR^\lambda$.

First, we must show surjectivity, that any diagram $d \in e_-D^iR^\lambda e_-$ is in the image of I. We know that the set $B_{G,H,i}$ spans the corresponding 2-morphisms $G \to H$ in $\mathcal{U}_i^-$. Thus we can rewrite d as an element of the image of I, times counter-clockwise bubbles at the left, labeled i. If such a bubble has $< \lambda^i - 1$ dots, the diagram is 0 and if the bubble has at least λ^i dots, then it is 0 by the relation (3.17). We are left with the case where it has exactly $\lambda^j - 1$ dots, and can thus be deleted by (2.2a). This shows surjectivity.

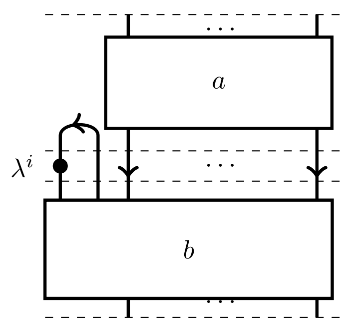
We need now to show injectivity. That is, we need to show that the maps from $\check{R} \to R^\lambda$ and $\check{R} \to D^i R^\lambda$ have the same kernel. Let J_1, J_2 be the kernels of these maps. Using the identification $\check{R} \cong \operatorname{End}_{\mathcal{U}_i^-}(\oplus_{\mathbf{i}} \mathcal{F}_{\mathbf{i}})$, we have that the ideal J_2 is spanned by KL diagrams such that the slice at $y = 1/2$ looks like the relations (3.17–3.18). We'll chop our diagram into 3 pieces: the narrow band with $y \in [1/2 - \epsilon, 1/2 + \epsilon]$ and the remainder above and below this. We'll call the leftmost strand at its point of intersection with the line $y = 1/2$ the **violating point**.

We can rewrite the pieces above $y = 1/2 + \epsilon$ and below $y = 1/2 - \epsilon$ in terms of $B_{G,H,i}$. If the KL pair obtained from the slices at $1/2 \pm \epsilon$ is downward, then we obtain an element of the cyclotomic ideal and thus we are done. Thus, we can assume there is at least one upward strand at $y = 1/2$.

There must be at least one one cap in the top half, and one cup in the bottom half. By the form of $B_{G,H,i}$, we may assume that at least one of these caps/cups has no crossings or smaller caps/cups inside it. If there is any such cap above $y = 1/2$ that does not connect to the violating point, we can use an isotopy to sink it through the band $y \in [1/2 - \epsilon, 1/2 + \epsilon]$ (which contains no crossings). We have thus reduced the number of upward strands at $y = 1/2$, and we can continue this process until the KL pair at $y = 1/2$ is downward (in which case are done), or there is a single cap in the top part and single cap in the bottom part connecting to the violating strand. This cup and cap are necessarily labeled i.

Similarly, if the remaining cap has any crossings with other strands above $y = 1/2 + \epsilon$, we can use an isotopy to sink these through the band $y \in [1/2 - \epsilon, 1/2 + \epsilon]$, so that no crossings with this cap remain above this point.

Case 1: downward orientation at the violating point. We have now reduced to the case where the diagram is as below:

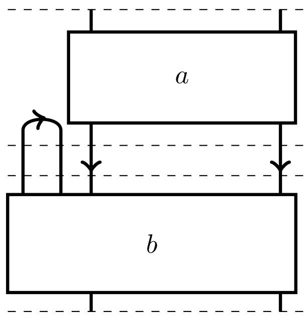


We now rewrite the top and bottom piece of the diagram in terms of $B_{G,H,i}$. Since this leaves the middle unchanged, it will still be in J_2. By our assumptions, the top half consists of a diagram where every strand points downward throughout, with one counter-clockwise cap added at the far left.

Now, consider the structure of the bottom half. It must have exactly one cup. If this cup doesn't connect to the violating point, then it must connect at $y = 1/2 - \epsilon$ to some strand further right. As argued earlier, we can isotope this cup upward through $y \in [1/2 - \epsilon, 1/2 + \epsilon]$, and arrive at the situation where the KL pair at $y = 1/2$ is downward.

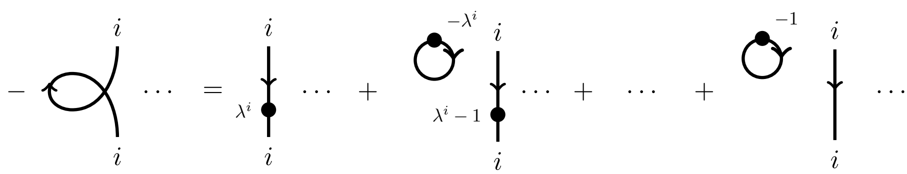
The other possibility is that the cup does connect to the violating strand, in which case we have a closed, counterclockwise oriented bubble with at least λ^i dots at the left of the diagram. This is positive degree, so the diagram is in the kernel of the map $\check{R} \to R^\lambda$.

Case 2: upward orientation at the violating point. Now, we need only consider the case where the diagram has the form:



As before, the bottom must have a single cup, but now this cup is required to meet the violating point. It may be that this cup closes up the violating strand, creating a clockwise oriented bubble at the far left. Since the region has $\lambda^i \geq 0$, this bubble has positive degree (note that it cannot be a fake bubble) so we get an element of J_1.

If this cup does not close the violating strand, it must connect to a strand at $y = 1/2 - \epsilon$ which is right of the two leftmost. Together with the top half, this must create a self-intersection in the violating strand. By our freedom of choice of basis, we can assume that it makes this crossing before either strand crosses any others. After isotopy, we see that we have obtained a diagram in which all strands point downward, except that at the violating point, we created a curl in the strand. Thus, we can apply the relation:



The first term is in J_1 because it has the requisite number of dots and the others are because they have positive degree bubbles, so the RHS is in J_1. □

Proposition 3.13 *The algebras R^λ and D^iR^λ are Morita equivalent.*

PROOF. It's a standard result that for an algebra A and idempotent e, the bimodules Ae and eA induce Morita equivalences if and only if $AeA = A$. Thus, we need only prove that $D^iR^\lambda \cdot e_- \cdot D^iR^\lambda = D^iR^\lambda$. It suffices to prove that each idempotent $e(\mathbf{i})$ for $\mathbf{i} \in \Sigma_i$ lies in $D^iR^\lambda \cdot e_- \cdot D^iR^\lambda = D^iR^\lambda$. We prove this by induction on the number of pairs $j, k \in [1, n]$ such that $i_j \in -\Gamma, i_k = i$ and $j < k$. If there is no such pair, then either i does not appear, in which case $e(\mathbf{i}) = e(\mathbf{i})e_-$, or the idempotent is 0 by (3.18). If there is any such pair, there must be one where j and k are consecutive. Thus, we can apply (2.3c) if $i_j = -i$ or (2.4a) if $i_j \neq -i$, and rewrite our idempotent has factoring through diagrams with strictly fewer such pairs. This completes the proof. □

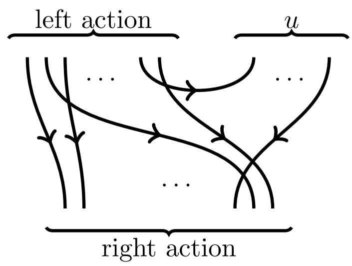
We can easily define an action of $\mathcal{U}_i^-$ on the representation category of D^iR^λ, and thus of R^λ by this Morita equivalence, using an analogous definition to that of $\mathcal{U}$ on $\mathbf{DR}^\lambda$.

If u is a 1-morphism in $\mathcal{U}_i^-$, we let e_u be the idempotent in D^iR^λ which acts by the identity on all KL pairs which end in u (that is, they are a horizontal composition $u \circ t$ for a 1-morphism t in $\mathcal{U}$) and by 0 all others.

Definition 3.14 *Let β'_u be the D^iR^λ-D^iR^λ bimodule $e_u \cdot D^iR^\lambda$. The left and right actions of $D^iR^{\underline{\boldsymbol{\lambda}}}$ on this space are by the formula $a \cdot h \cdot b = (1_u \circ a)hb$.*

Let $\beta_u = e_- \cdot \beta'_u \cdot e_-$ be the image of this bimodule under the Morita equivalence of Theorem 3.13.

Schematically, an element of the bimodule β_u looks like

(3.19) 

If we have a 2-morphism $\phi\colon u \to v$ in $\mathcal{U}_i^-$, then we have an induced bimodule map $\beta_u \to \beta_v$ where we act by by $\phi \circ 1$. In terms of the picture (3.19), the action of 2-morphisms $u \to v$ is by attaching the diagrams at the upper right. Since the relations of $\mathcal{U}_i^-$ are local, they are satisfied by the bimodule maps.

Consider the map $\nu_j\colon R^\lambda_\mu \to R^\lambda_{\mu-\alpha_i}$ that adds a strand labeled with j at the right.

Definition 3.15 *We let $\mathfrak{F}_j = - \otimes_{R^\lambda_\mu} R^\lambda_{\mu-\alpha_i}$ denote the functor of extension of scalars by this map; we will refer to this as an* **induction** *functor.*

We let $\mathfrak{E}_j = \mathrm{Hom}_{R^\lambda_{\mu-\alpha_i}}(R^\lambda_\mu, -)(\langle \mu, \alpha_j\rangle - d_i)$ denote restriction of scalars by this map (with a grading shift), the functors left adjoint to the $\mathfrak{F}_i$'s; we call these **restriction functors**.

Proposition 3.16 *There is a strict 2-functor $\mathcal{U}_i^- \to \mathsf{Cat}$ given by*

$$\begin{aligned} \mu &\mapsto R^\lambda_\mu\text{-}\mathsf{pmod} \\ u &\mapsto - \otimes_{R^\lambda} \beta_u \\ \mathcal{F}_j &\mapsto \mathfrak{F}_j \\ \mathcal{E}_i &\mapsto \mathfrak{E}_i \end{aligned}$$

In particular, the functors $\mathfrak{E}_i$ and $\mathfrak{F}_i$ are biadjoint (up to grading shift) since $\mathcal{F}_i$ and $\mathcal{E}_i$ are biadjoint in $\mathcal{U}_i^-$.

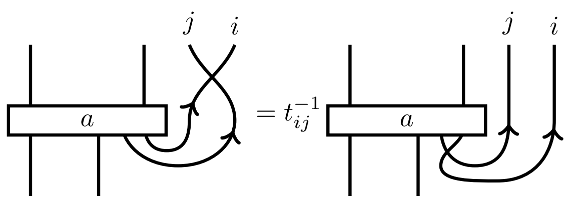
PROOF. As noted above, the fact that this is a 2-representation is immediate; one way to interpret Proposition 3.13 is that it realizes the category of modules over R^λ as an appropriate quotient of $\mathcal{U}_i^-$ which obviously carries such a representation.

Thus, we need only check the assignments $\mathcal{F}_j \mapsto \mathfrak{F}_j$ and $\mathcal{E}_i \mapsto \mathfrak{E}_i$. By adjunction, it is only necessary to check one of these. The bimodule $\beta_{\mathcal{F}_i}$ is by definition of the subspace of R^λ where the last entry of the sequence at the top of the diagram is i; this exactly the definition of the induction bimodule given in [**KL09**, §3.2]. □

4. The categorical action on cyclotomic quotients

We can upgrade the action of $\mathcal{U}_i^-$ to an action of the full category $\mathcal{U}$. This action will, of course, assign $\mathcal{E}_j \mapsto \mathfrak{E}_j, \mathcal{F}_j \mapsto \mathfrak{F}_j$. In particular, for any basic 2-morphism in $\mathcal{U}$ except the upward oriented crossing, we have a well-defined natural transformation of bimodules induced from the action of $\mathcal{U}_i^-$.

Thus, we must define a map for the upward crossing. If we apply $\mathfrak{E}_i\mathfrak{E}_j$ to a module M over R^λ, then the resulting diagrams look like the picture (3.3) with an element of M attached at the top left. Both the strands coming from the top right must form cups whose minimum we can assume comes to the right of all other strands. We define the morphism $\psi\colon \mathfrak{E}_j\mathfrak{E}_i \to \mathfrak{E}_i\mathfrak{E}_j$ to be given by t_{ij}^{-1} times the diagram where the crossing is done left of the cups. Pictorially:



The locality of the relations assures that this map is well-defined.

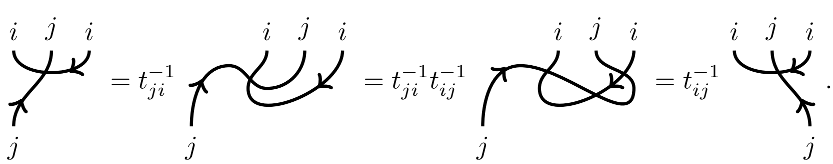
Theorem 3.17 *There is a strict 2-functor* $\mathcal{U} \to \mathsf{Cat}$ *given by*

$$\begin{aligned} \mu &\mapsto R^\lambda_\mu\text{−}\mathsf{pmod} \\ u &\mapsto - \otimes_{R^\lambda} \beta_u \\ \mathcal{F}_j &\mapsto \mathfrak{F}_j \\ \mathcal{E}_i &\mapsto \mathfrak{E}_i \end{aligned}$$

We should note that this theorem has been independently proven by Cautis and Lauda [**CL15**, 7.1] based on work of Kang and Kashiwara [**KK12**].

PROOF OF THEOREM 3.17. We also know that every relation of $\mathcal{U}$ that only involves upward strands of one color is satisfied, since these hold in one of the $\mathcal{U}_-^i$. The only remaining relations are those of (2.1b).

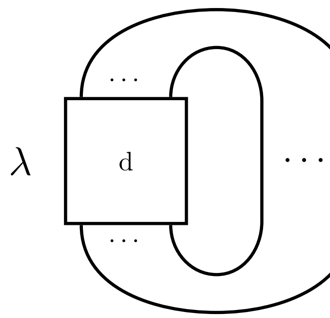
These simply involve manipulating the definition above and the relations (2.1a) and (2.4a-2.4b). The argument for the first one is that:



The second equation follows an analogous calculation. □

This completes the main goal of this chapter. However, there are several consequences of this theorem which we must draw out, including the Morita equivalence of R^λ and $\mathbf{DR}^\lambda$.

First, this equips R^λ with a map $\mathrm{tr}_\lambda : R^\lambda \to \Bbbk$ given by closing a diagram at the right (if top and bottom strands match) and considering the scalar with which this acts on $R^\lambda_\lambda \cong \Bbbk$, as shown below.



Recall that a **Frobenius** structure on a $\Bbbk$-algebra A is a linear map $\mathrm{tr}\colon A \to \Bbbk$ which kills no left ideal.

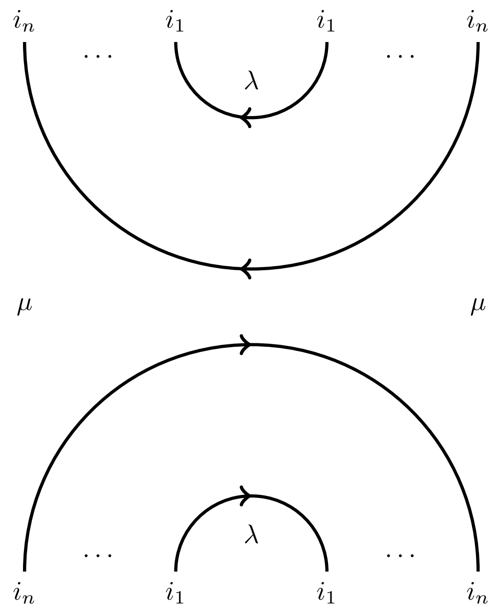
Theorem 3.18 *The map* $\mathrm{tr}_\lambda\colon R^\lambda \to \Bbbk$ *is a Frobenius trace.*

PROOF. This is essentially automatic from the fact that $\mathcal{F}_i$ and $\mathcal{E}_i$ are biadjoint, and the map of "capping off" is the counit of this adjunction; however, let us give a more concrete proof.

A trace is Frobenius if and only if the bilinear form $R^\lambda \times R^\lambda \to \Bbbk$ given by $(a,b) \mapsto \mathrm{tr}_\lambda(ab)$ is non-degenerate. This is the case if and only if there exist dual bases $\{b_1, \dots, b_m\}$ and $\{c_1, \dots, c_m\}$ such that $a = \sum_{i=1}^m \mathrm{tr}(ab_i)c_i$ for every $a \in R^\lambda$. Alternatively, it is equivalent to the existence of a canonical element $\sum b_i \otimes c_i$.

We can write this canonical element implicitly using the action of $\mathcal{U}$. Consider the morphism from $(i_n, \dots, i_1, -i_1, \dots, -i_n)$ given by the "arches":

$a =$

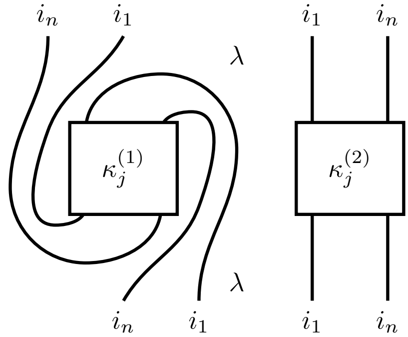


We can use the relations (2.3c) and (2.4a) to rewrite this element. This will be an inductive process, where at each step, we apply a relation to decrease the number of pairs of a cup and a cap which don't cross, by pushing the bottom arches upward and the top arches downward. The first step is to apply (2.3c) to the outermost cup and cap. This results in one term where this cup and cap cross, and possibly others with fewer cups and caps.

At each step, in the resulting diagram, if we see any pair of a cup and cap that don't intersect, then there is such a pair where nothing separates them, and we can apply (2.3c) or (2.4a) depending on the label. The resulting terms have fewer such pairs.

Thus, at the end, we only have terms where every cap intersects every cup. If there are any caps, then the region just above the minimum of the bottom of the cup is labeled with $\lambda + \alpha_i$. This diagram will act trivially on $\mathfrak{V}_\mu^\lambda$, since the object corresponding to the horizontal slice through this region is trivial. Thus, the action of a on $\mathfrak{V}_\mu^\lambda$ coincides with that of the morphism a' consisting only of the diagrams resulting from our inductive process with no arches.

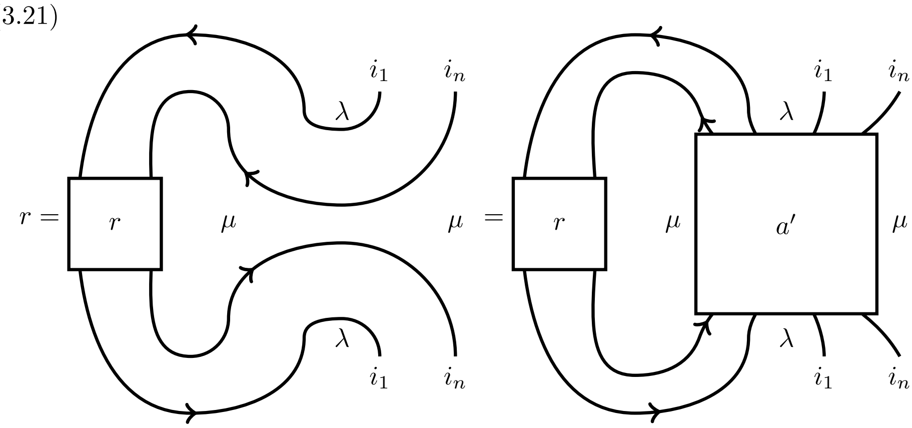
That is, we can write the natural transformation a' induced by a as a sum:

(3.20) $a' = \sum_{j=1}^{p}$ 

Note that $\kappa_j^{(q)}$ gives a well-defined element of R^λ, since in this representation of $\mathcal{U}$, the cyclotomic relation has become a local relation that holds whenever a strand separates λ and $\lambda - \alpha_i$. In particular, we can use this relation and the bubble slides to remove any positive degree bubbles. By the manifest bilinearity, we can assume without loss of generality that $\kappa_j^{(2)}$ ranges over any chosen basis of R^λ.

On the other hand, if we choose $r \in R^\lambda$, and connect the left edges of the arches using r, we simply obtain r itself by biadjunction as shown in (3.21).

(3.21)



In this case, we can interpret the equation (3.21) as saying that

$$r = \sum_j \mathrm{tr}_\lambda(r\kappa_j^{(1)})\kappa_j^{(2)}.$$

That is, $\kappa_j^{(1)}$ and $\kappa_j^{(2)}$ are dual bases, and κ is essentially the canonical element of the Frobenius pairing. This shows that the pairing is non-degenerate. $\square$

This Frobenius trace is not symmetric, since the 2-category $\mathcal{U}$ is not cyclic. For example, assume $\lambda = \omega_1 + \omega_2$ is the highest weight of the adjoint rep for $\mathfrak{sl}_3$, and $Q_{12}(u,v) = u - v$. The algebra R_0^λ has two idempotents $e_{(1,2)}, e_{(2,1)}$ corresponding to the crossingless diagrams with a strand each of label 1 and 2 in the corresponding order. We have that

$$\mathrm{tr}(\psi^2 e_{(1,2)}) = \mathrm{tr}(y_1 e_{(1,2)}) - \mathrm{tr}(y_2 e_{(1,2)}) = 0 - 1 = -1.$$

On the other hand, we have that

$$\operatorname{tr}(\psi e_{(1,2)}\psi) = \operatorname{tr}(\psi^2 e_{(2,1)}) = -\operatorname{tr}(y_1 e_{(2,1)}) + \operatorname{tr}(y_2 e_{(2,1)}) = 1 - 0 = 1.$$

In general, moving a crossing to the other side of the diagram requires multiplying by t_{ji}/t_{ij}, as is shown by the pitchfork moves (2.1a–2.1b).

Remark 3.19 *However, this trace can easily be adjusted to become symmetric. One fixes one reference sequence $\mathbf{i}_\mu$ for each weight μ; for each other sequence $\mathbf{i}$, we pick a diagram connecting it to $\mathbf{i}_\mu$ and for each crossing with and consider the scalar $t(\mathbf{i})$ which is the product over all crossings in the diagram of t_{ji}/t_{ij} where the NE/SW strand of the crossing is labeled with i and the NW/SE strand is labeled j. If we multiply the trace on $e(\mathbf{i})R^\lambda e(\mathbf{i})$ by $t(\mathbf{i})$, the result will still be Frobenius and symmetric.*

The reader may sensibly ask why we use the trace above instead; it is in large part so we may match the conventions of [**CL15**] *and use their results. That said, their choice arises very naturally from a coherent principle: that degree 0 bubbles should be 1. Trying to recover cyclicity in $\mathcal{U}$ will definitely break this condition.*

Corollary 3.20 *The map $p\colon R^\lambda \to DR^\lambda$ is a Morita equivalence.*

Proof. There is an idempotent e_- that picks out downward KL pairs in DR^λ, just as in D^iR^λ. The proof that the map R^λ surjects onto $e_-DR^\lambda e_-$ is the same the proof in Lemma 3.12. This map is injective, since any element a killed by this map must be killed by $\operatorname{tr}_\lambda(ab) = 0$ for all $b \in R^\lambda$, and there are no such elements by Theorem 3.18. The surjectivity of the map $DR^\lambda e_- DR^\lambda \to DR^\lambda$ is the same proof as Proposition 3.13. □

Proposition 3.21 *We have an isomorphism of representations $K_0(R^\lambda) \cong V_\lambda^{\mathbb{Z}}$.*

Proof. First, we note that the map $K^0(R) \to K^0(R^\lambda)$ is surjective, since every projective R^λ module is the quotient of a projective R module by the cyclotomic ideal. In particular, $K_0(R^\lambda)$ is generated over U_q^- by a single highest weight vector of weight λ.

We need only note that

- $K^0(R^\lambda)$ is thus a quotient of the Verma module of highest weight λ.
- On the other hand, $\mathfrak{V}^\lambda$ is an integrable categorification in the sense of Rouquier: acting by $\mathfrak{F}_i$ or $\mathfrak{E}_i$ a sufficiently large number of times kills any R^λ-module, so $K_0(R^\lambda)$ is integrable.
- $V_\lambda^{\mathbb{Z}}$ is the only integrable quotient of the the Verma module which is free as a $\mathbb{Z}[q, q^{-1}]$ module. □

Recall that the q**-Shapovalov form** $\langle -, - \rangle$ is the unique bilinear form on $V_\lambda^{\mathbb{Z}}$ such that

- $\langle v_h, v_h \rangle = 1$ for a fixed highest weight vector v_h.
- $\langle u \cdot v, v' \rangle = \langle v, \tau(u) \cdot v' \rangle$ for any $v, v' \in V_\lambda$ and $u \in U_q(\mathfrak{g})$, where τ is the q-antilinear antiautomorphism defined by

$$\tau(E_i) = q_i^{-1}\tilde{K}_{-i}F_i \qquad \tau(F_i) = q_i^{-1}\tilde{K}_i E_i \qquad \tau(K_\mu) = K_{-\mu} \tag{3.22}$$

- $f\langle v, v' \rangle = \langle \bar{f}v, v' \rangle = \langle v, fv' \rangle$ for any $v, v' \in V_\lambda^{\mathbb{Z}}$ and $f \in \mathbb{Z}[q, q^{-1}]$.

On the other hand, the Grothendieck group $K_0(R^\lambda)$ carries an **Euler form**, defined by:

$$\langle [P_1], [P_2]\rangle = \dim_q \operatorname{Hom}(P_1, P_2).$$

Corollary 3.22 *The isomorphism $K_0(R^\lambda) \cong V_\lambda^{\mathbb{Z}}$ intertwines the Euler form with the q-Shapovalov form described above. In particular,*

$$\dim_q e(\mathbf{i})R^\lambda e(\mathbf{j}) = \langle F_{\mathbf{i}} v_\lambda, F_{\mathbf{j}} v_\lambda\rangle$$

We let $\langle -, -\rangle_1$ denote the specialization of this form at $q = 1$, which is thus the ungraded Euler form.

5. Universal categorifications

In [**Roua**, §5.1.2], Rouquier discusses universal categorifications of simple integrable modules. Of course, to speak of universality, we must have a notion of morphisms between categorical modules. Let $\aleph_1, \aleph_2 \colon \mathcal{U} \to \mathsf{Cat}$ be two strict 2-functors.

Definition 3.23 *A* **strongly equivariant** *functor β is a collection of functors $\beta(\lambda)\colon \aleph_1(\lambda) \to \aleph_2(\lambda)$ together with natural isomorphisms of functors $c_u \colon \beta\circ\aleph_1(u) \cong \aleph_2(u) \circ \beta$ for every 1-morphism $u \in \mathcal{U}$ such that*

$$c_v \circ (\mathrm{id}_\beta \otimes \aleph_1(\alpha)) = (\aleph_2(\alpha) \otimes \mathrm{id}_\beta) \circ c_u$$

for every 2-morphism $\alpha\colon u \to v$ in $\mathcal{U}$. (Here we use $\otimes$ for horizontal composition, and $\circ$ for vertical composition of 2-morphisms).

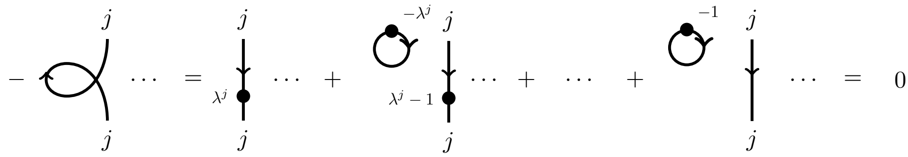
In [**Roua**, §5.1.2], it is proven that there is a unique $\mathcal{U}$-module category $\check{\mathfrak{V}}^\lambda$ (he uses the notation $\mathcal{L}(\lambda)$) with generating highest weight object P with the universal property that

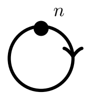
($*$) for any additive, idempotent-complete $\mathcal{U}$-module category $\mathcal{C}$ and any object $C \in \operatorname{Ob}\mathcal{C}_\lambda$ with $\mathfrak{E}_i(C) = 0$ for all i, there is a unique (up to unique isomorphism) strongly equivariant functor $\phi_C \colon \check{\mathfrak{V}}^\lambda \to \mathcal{C}$ sending $P_\emptyset$ to C.

This is a higher categorical analogue of the universal property of a Verma module, but somewhat surprisingly, $\check{\mathfrak{V}}^\lambda$ does *not* categorify a Verma module, but rather an integrable module.

Consider the algebra $\check{R} := R \otimes \mathbf{\Lambda}$ where $\mathbf{\Lambda} \cong (\otimes_{j\in\Gamma}\Lambda(\mathbf{p}_j))$ and $\mathbf{p}_j$ is an infinite alphabet attached to each node. This algebra can represented diagrammatically with R given by diagrams as usual, the clockwise bubble at the left of the diagram of degree $2n$ corresponding to $(-1)^n e_n(\mathbf{p}_j)$, and the counterclockwise one of degree $2n$ corresponding to $h_n(\mathbf{p}_i)$. Note, this is compatible with our conventions from Chapter 2, but will not require the same sort of involved calculations. This gives us a natural homomorphism $\check{R} \to \operatorname{End}_{\mathcal{U}}(\oplus_{\mathbf{i}}\mathcal{F}_{\mathbf{i}}\lambda)$ where $\mathbf{\Lambda} \cong (\otimes_{j\in\Gamma}\Lambda(\mathbf{p}_j))$ and $\mathbf{p}_j$ is an infinite alphabet attached to each node. In [**Webb**, ϵ.8], we show that this map is an isomorphism, but we do not use this fact in what follows (in fact, the results below are useful in proving the results of [**Webb**]).

Definition 3.24 *Let $\check{R}^\lambda$ be the quotient of $\check{R}$ by the relations*





$$\cdots \;= 0 \qquad (n \geq 0)$$

where in both pictures, the ellipses indicate that the portion of the diagram shown is at the far left. More algebraically, these relations can be written in the form

$$e(\mathbf{i})(y_1^{\lambda^{i_i}} - e_1(\mathbf{p}_{i_1})y_1^{\lambda_{i_1}-1} + \cdots + (-1)^{\lambda^{i_1}} e_{\lambda^{i_1}}(\mathbf{p}_{i_1})) = 0$$
$$e_n(\mathbf{p}_j) = 0 \qquad (n > \lambda^j)$$

Note that if we specialize $e_n(\mathbf{p}_j) = 0$ for every $n > 0$, then we recover the usual cyclotomic quotient R^λ.

If we extend scalars to polynomials in the $p_{*,*}$ and form the algebra $\check{R} \otimes_{\mathbf{\Lambda}} \Bbbk[p_{1,1}, \ldots,]$ then we can rewrite these equations as

$$e(\mathbf{i})(y_1 - p_{i_1,1})(y_1 - p_{i_1,2}) \cdots (y_1 - p_{i_1,\lambda^{i_1}}) = 0$$
$$p_{j,n} = 0 \qquad (n > \lambda^j)$$

In terms of the geometry of quiver varieties, $\check{R}^\lambda$ arises from considering equivariant sheaves for the action of the group $\prod \mathrm{GL}(W_i)$, and its extension to polynomials from equivariant sheaves for a maximal torus of this group.

Proposition 3.25 *The algebra $\check{R}^\lambda$ is Morita equivalent to the quotient of* $\mathrm{End}_{\mathcal{U}}(\mathcal{F}_{\mathbf{i}}\lambda)$ *by the ideal generated by all morphisms factoring through $\mathcal{F}_{\mathbf{j}}\mathcal{E}_j$ for all $\mathbf{j}, j$. The category of $\check{R}^\lambda$ modules thus carries a natural action of $\mathcal{U}$.*

The proof of this proposition is so similar to that of Lemma 3.12 and Propositions 3.13-3.16 that we leave it to the reader.

Note that this implies that:

Corollary 3.26 *The ring $\check{R}^\lambda$ is a free module over $\check{R}^\lambda_\lambda$. That is, $\check{R}^\lambda$ is a flat deformation of R^λ.*

Proof. In order to check this, we need only confirm that $\dim R^\lambda$ coincides with the generic rank of the $\check{R}^\lambda_\lambda$-module $\check{R}^\lambda$. Using the adjunction between $\mathfrak{E}_i$ and $\mathfrak{F}_i$ one can write $\dim R^\lambda$ as the sum of the dimensions of the modules $\mathfrak{E}_{\mathbf{j}}\mathfrak{F}_{\mathbf{i}}P_\emptyset$ over all sequences $\mathbf{j}$ and $\mathbf{i}$ of the same weight. Similarly, the generic rank of $\check{R}^\lambda$ is the sum of the generic rank of $\mathfrak{E}_{\mathbf{j}}\mathfrak{F}_{\mathbf{i}}\check{P}_\emptyset$ over all such sequences. Just using the fact that $\mathfrak{E}_i$ kills the highest weight space, we find that $\mathfrak{E}_{\mathbf{j}}\mathfrak{F}_{\mathbf{i}}$ acting on the λ weight space is a sum of some number of copies of the identity functor; this number is both the contribution of $\mathfrak{E}_{\mathbf{j}}\mathfrak{F}_{\mathbf{i}}P_\emptyset$ to the dimension of R^λ and of $\mathfrak{E}_{\mathbf{j}}\mathfrak{F}_{\mathbf{i}}\check{P}_\emptyset$ to the generic rank and thus these numbers coincide. □

The following corollary is essentially equivalent to [**Roub**, 4.25]; we include it mainly to spare the reader the difficulty of translating between formalisms.

Corollary 3.27 *For any additive, idempotent-complete $\mathcal{U}$-module category $\mathcal{C}$ and any object $C \in \operatorname{Ob}\mathcal{C}_\lambda$ with $\mathfrak{E}_i(C) = 0$ for all i, there is a unique strongly equivariant functor (up to unique isomorphism) $\phi_C \colon \check{R}^\lambda$-pmod $\to \mathcal{C}$ sending $P_\emptyset$ to C. The induced base change functor $\phi'_C \colon (\check{R}^\lambda \otimes_{\check{R}^\lambda_\lambda} \operatorname{End}(C))$-pmod $\to \mathcal{C}$ is fully faithful.*

PROOF. For any object C, there is a unique strongly equivariant functor $\mathcal{U}(\lambda) \to \mathcal{C}$ sending $\mathrm{id}_\lambda \mapsto C$. We wish to show that this factors through the functor from $\mathcal{U}(\lambda) \to \check{R}^\lambda$-pmod. By Proposition 3.25, it suffices to check that this map kills any 2-morphism factoring through $u\mathcal{E}_i \,\mathrm{id}_\lambda$. Indeed, this is sent to $u\mathfrak{E}_i(C) = 0$, so we kill the required 2-morphisms.

Thus, we have a base change functor ϕ'_C. We wish to show that

$$\operatorname{Hom}_{\mathcal{C}}(\phi'_C(M), \phi'_C(N)) \cong \operatorname{Hom}_{\check{R}^\lambda \otimes_{\check{R}^\lambda_\lambda} \operatorname{End}(C)}(M, N) \tag{3.23}$$

for all projectives M and N. This is clear if the weight of M is λ; in this case, we can assume that $M = \operatorname{End}(C)$ as a module over itself, and N either has the wrong weight (so both sides of the desired equation are 0), or N may also be assumed to be $\operatorname{End}(C)$, in which case (3.23) is a tautology.

Now, let us induct on the weight of M. Every indecomposable projective of weight $< \lambda$ is a summand of one of the form $\mathcal{F}_i M'$. Thus, we may assume that $M = \mathcal{F}_i M'$, so

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\phi'_C(M), \phi'_C(N)) &\cong \operatorname{Hom}_{\mathcal{C}}(\phi'_C(M'), \phi'_C(\mathcal{E}_i N)) \\ &\cong \operatorname{Hom}_{\check{R}^\lambda \otimes_{\check{R}^\lambda_\lambda} \operatorname{End}(C)}(M', \mathcal{E}_i N) \cong \operatorname{Hom}_{\check{R}^\lambda \otimes_{\check{R}^\lambda_\lambda} \operatorname{End}(C)}(M, N),\end{aligned}$$

which establishes (3.23). □

These algebras are quite interesting; though they are infinite dimensional (unlike R^λ), they seem to have finite global dimension (unlike R^λ). We will explore these algebras and their tensor product analogues in future work.

CHAPTER 4

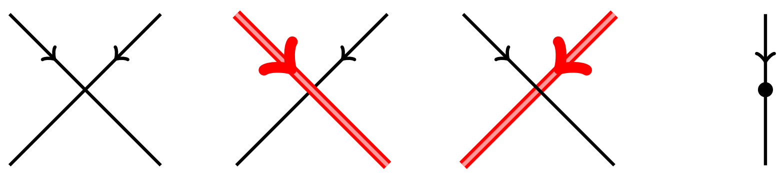
The tensor product algebras

1. Stendhal diagrams

Definition 4.1 *A* **Stendhal diagram** *is a collection of finitely many oriented curves in* $\mathbb{R} \times [0, 1]$*. Each curve is either*

- *colored red and labeled with a dominant weight of* $\mathfrak{g}$*, or*
- *colored black and labeled with* $i \in \Gamma$ *and decorated with finitely many dots.*

The diagram must be locally of the form



with each curve oriented in the negative direction. In particular, no red strands can ever cross. Each curve must meet both $y = 0$ *and* $y = 1$ *at points we call* **termini**. *No two strands should meet the same terminus.*

We'll typically only consider Stendhal diagrams up to isotopy. Since the orientation on a diagram is clear, we typically won't draw it.

We call the lines $y = 0, 1$ the **bottom** and **top** of the diagram. Reading across the bottom and top from left to right, we obtain a sequence of dominant weights and elements of Γ. We record this data as

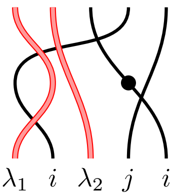
- the list $\mathbf{i} = (i_1, \dots, i_n)$ of elements of Γ, read from the left;
- the list $\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_\ell)$ of dominant weights, read from the left;
- the weakly increasing function $\kappa \colon [1, \ell] \to [0, n]$ such that there are $\kappa(m)$ black termini left of the mth red terminus. In particular, $\kappa(i) = 0$ if the ith red terminus is left of all black termini.

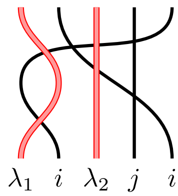
We call such a triple of data a **Stendhal triple**. We will often want to partition the sequence $\mathbf{i}$ in the groups of black strands between two consecutive reds, that is, the groups

$$\mathbf{i}_0 = (i_1, \dots, i_{\kappa(1)}), \mathbf{i}_1 = (i_{\kappa(1)+1}, \dots, i_{\kappa(2)}), \dots, \mathbf{i}_\ell = (i_{\kappa(\ell)+1}, \dots, i_n).$$

We call these **black blocks**.

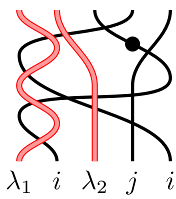
Here are two examples of Stendhal diagrams:

$a =$ 

$b =$ 

- At the top of a, we have $\mathbf{i} = (i, i, j)$, $\underline{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2)$ and $\kappa = (1 \mapsto 0, 2 \mapsto 0)$.
- At the top of b and bottom of a and b, $\mathbf{i} = (i, j, i)$, $\underline{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2)$ and $\kappa = (1 \mapsto 0, 2 \mapsto 1)$.

Definition 4.2 *Given Stendhal diagrams a and b, their* **composition** *ab is given by stacking a on top of b and attempting to join the bottom of a and top of b. If the Stendhal triples from the bottom of a and top of b don't match, then the composition is not defined and by convention is 0, which is not a Stendhal diagram, just a formal symbol.*

$ab =$ 

$ba = 0$

Fix a field $\Bbbk$ and let $\tilde{\tilde{T}}$ be the formal span over $\Bbbk$ of Stendhal diagrams (up to isotopy). The composition law induces an algebra structure on $\tilde{\tilde{T}}$.

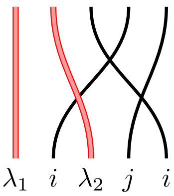
Let $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ be the unique crossingless, dotless diagram where the triple read off from both top and bottom is $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$. Composition on the left/right with $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ is an idempotent operation; it sends a diagram a to itself if the top/bottom of a matches $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ and to 0 otherwise. We'll often fix $\underline{\boldsymbol{\lambda}}$, and thus leave it out from the notation, just writing $e(\mathbf{i}, \kappa)$ for this diagram.

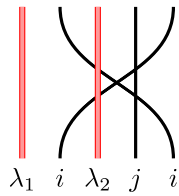
Considered as elements of $\tilde{\tilde{T}}$, the diagrams $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ are orthogonal idempotents. The algebra $\tilde{\tilde{T}}$ is not unital, but it is **locally unital**. That is, for any finite linear combination a of Stendhal diagrams, there is an idempotent such that $ea = ae = a$. This can be taken to be the sum of $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ for all triples that occur at the top or bottom of one of the diagrams in a.

Alternatively, we can organize these diagrams into a category whose objects are Stendhal triples $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ and whose morphisms are Stendhal diagrams read from bottom to top. In this perspective, the idempotents $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ are the identity morphisms of different objects.

Definition 4.3 *We call a black strand in a Stendhal diagram* **violating** *if at some horizontal slice $y = c$ for $c \in [0, 1]$, it is the leftmost strand. A Stendhal diagram which possesses a violating strand is called* **violated**.

Both the diagrams a and b above are violated. The diagrams

$c =$ 

$d =$ 

are not violated. The diagram $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ is violated if and only if $\kappa(1) > 0$.

Definition 4.4 *The* **degree** *of a Stendhal diagram is the sum over crossings and dots in the diagram of*

- $-\langle \alpha_i, \alpha_j \rangle$ *for each crossing of a black strand labeled i with one labeled j;*
- $\langle \alpha_i, \alpha_i \rangle = 2d_i$ *for each dot on a black strand labeled i;*
- $\langle \alpha_i, \lambda \rangle = d_i \lambda^i$ *for each crossing of a black strand labeled i with a red strand labeled λ.*

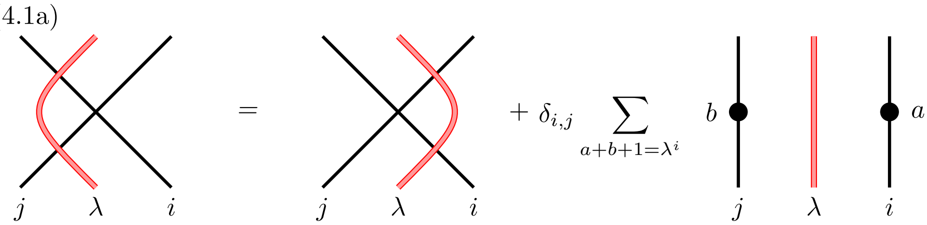
The degree of diagrams is additive under composition. Thus, the algebra $\tilde{\tilde{T}}$ inherits a grading from this degree function.

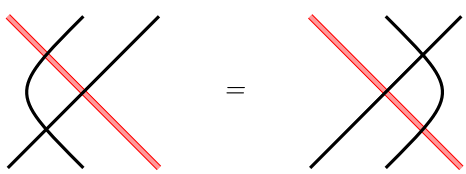
Consider the reflection through the horizontal axis of a Stendhal diagram a with its orientations reversed. This is again a Stendhal diagram, which we denote $\dot{a}$. Note that $\dot{(ab)} = \dot{b}\dot{a}$, so reflection induces an anti-automorphism of $\tilde{\tilde{T}}$.

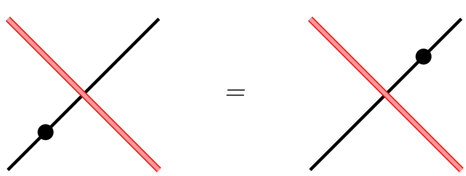
2. Definition and basic properties

Definition 4.5 *Let $\tilde{T}$ be the quotient of $\tilde{\tilde{T}}$ by the following local relations between Stendhal diagrams:*

- *the KLR relations* (2.5a–2.5g)
- *All black crossings and dots can pass through red lines. For the latter two relations* (4.1b–4.1c)*, we also include their mirror images:*

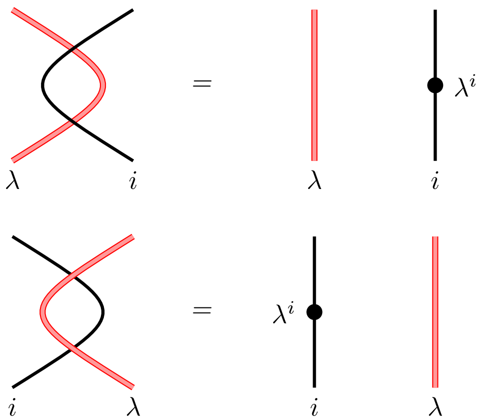
(4.1a)


(4.1b)


(4.1c)


- *The "cost" of a separating a red and a black line is adding $\lambda^i = \alpha_i^\vee(\lambda)$ dots to the black strand.*

(4.2)



The algebra $\tilde{T}$ will play a mostly auxilliary role in this paper, but it is a very natural object. For example, it has a geometric description, as we discuss in [**Webg**, §4].

Definition 4.6 *Let T be the quotient of $\tilde{T}$ by the 2-sided ideal K generated by all violated diagrams.*

Now, as before, fix a sequence of dominant weights $\underline{\boldsymbol{\lambda}} = (\lambda_1, \dots, \lambda_\ell)$ and let $\lambda = \sum_{i=1}^{\ell} \lambda_i$.

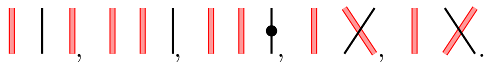
Definition 4.7 *We let $T^{\underline{\boldsymbol{\lambda}}}$ (resp. $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$) be the subalgebra of T (resp. $\tilde{T}$) where the red lines are labeled, from left to right, with the elements of $\underline{\boldsymbol{\lambda}}$. Let $T^{\underline{\boldsymbol{\lambda}}}_\alpha$ for $\alpha \in Y(\mathfrak{g})$ be the subalgebra of $T^{\underline{\boldsymbol{\lambda}}}$ where the sum of the roots associated to the black strands is $\lambda - \alpha$, and let $T^{\underline{\boldsymbol{\lambda}}}_n$ be the subalgebra of diagrams with n black strands (and similarly for $\tilde{T}^{\underline{\boldsymbol{\lambda}}}_\alpha, \tilde{T}^{\underline{\boldsymbol{\lambda}}}_n$).*

We use the notation $T^{\underline{\boldsymbol{\lambda}}}_\alpha$ because we'll show later that the Grothendieck group of this algebra is canonically isomorphic to the α-weight space of $V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$ (see Proposition 4.38). Note that every indecomposable module is killed by $T^{\underline{\boldsymbol{\lambda}}}_\alpha$ for all but one value of α, since the identities of these algebras give collection of orthogonal central idempotents summing to 1.

To give a simple illustration of the behavior of our algebra, let us consider $\mathfrak{g} = \mathfrak{sl}_2$; to avoid confusion between integers and elements of the weight lattice, we'll use α to denote the unique simple root of $\mathfrak{sl}_2$, and $\omega = \alpha/2$ the unique fundamental weight (and $0 \cdot \omega$ the trivial weight). Now consider the case $\underline{\boldsymbol{\lambda}} = (\omega, \omega)$. Thus, our diagrams have 2 red lines, both labeled with ω.

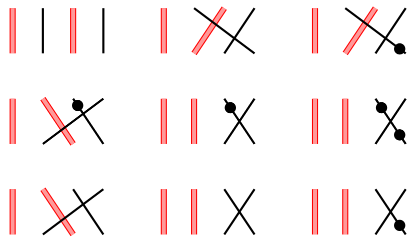
In this case, the algebras $T^{\underline{\boldsymbol{\lambda}}}_\alpha$ are easily described as follows:

- $T^{(\omega,\omega)}_{2\omega} = T^{(\omega,\omega)}_0 \cong \Bbbk$: it is spanned by the diagram  .
- $T^{(\omega,\omega)}_{0\cdot\omega} = T^{(\omega,\omega)}_1$ is spanned by

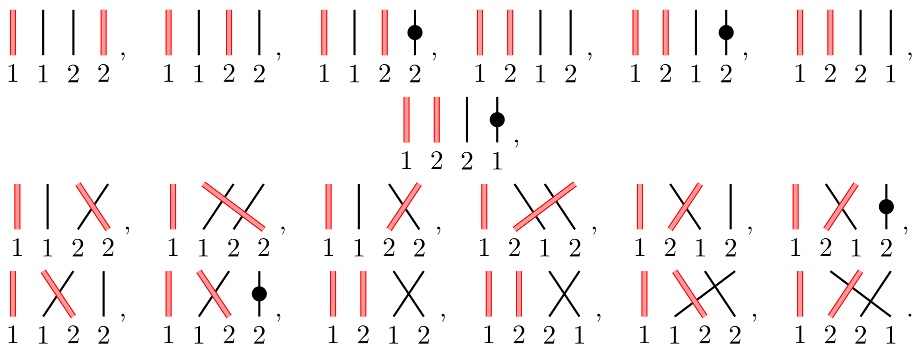


One can easily check that this is the standard presentation of a regular block of category $\mathcal{O}$ for $\mathfrak{sl}_2$ as a quotient of the path algebra of a quiver (see, for example, [**Str03**]).

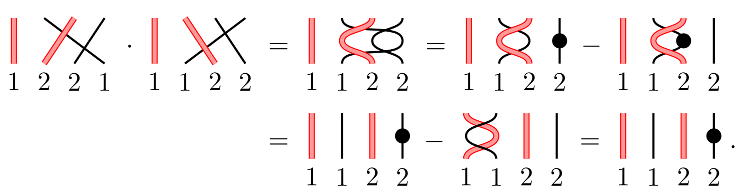
- $T^{(\omega,\omega)}_{-2\omega} = T^{(\omega,\omega)}_{2} \cong \mathrm{End}(\Bbbk^3)$: The algebra is spanned by the diagrams, which one can easily check multiply (up to sign) as the elementary generators of $\mathrm{End}(\Bbbk^3)$.



Perhaps a more interesting example is the case of $\mathfrak{g} = \mathfrak{sl}_3$ and we let $\underline{\boldsymbol{\lambda}} = (\omega_1, \omega_2)$ and $\mu = 0$. Based on the construction of a cellular basis in [**SW**], we can calculate that this algebra is 19 dimensional, with a basis given by



We leave the calculation of the multiplication in this basis to the reader; it is a useful exercise to those wishing to become more comfortable with these sorts of calculations. For example, when we multiply the last two vectors in the basis above, we get that (for $Q_{21}(u,v) = u - v$)



Definition 4.8 *Let $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\alpha}$ be the category of finite dimensional modules over $T^{\underline{\boldsymbol{\lambda}}}_{\alpha}$. Let $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}_{\alpha}$ be the derived category of complexes in $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\alpha}$ that lie in $C^{\uparrow}(T^{\underline{\boldsymbol{\lambda}}})$, the category of complexes of finite dimensional graded modules such that the degree j part of the ith homological term $C^i_j = 0$ for $i \geq N$ or $i + j \leq M$ for some constants M, N (depending on the complex).*

There are two explanations for this (somewhat unfamiliar) category. The first is that since in each graded degree, this complex is finite, any element of this category will have a well-defined class in the completion $V_{\underline{\boldsymbol{\lambda}}}$. The second is that it arises naturally from simple operations over these algebras. Note that $T_1^{2\omega} \cong \Bbbk[y]/(y^2)$. The trivial module $\Bbbk$ has a minimal projective resolution given by $\cdots \to T_1^{2\omega}(-2n) \to \cdots \to T_1^{2\omega}(-2) \to T_1^{2\omega} \to \Bbbk$. In particular, $\Bbbk \overset{L}{\otimes}_{T_1^{2\omega}} \Bbbk$ is an unbounded complex (with trivial differential), but does lie in $C^{\uparrow}(T^{\underline{\boldsymbol{\lambda}}})$.

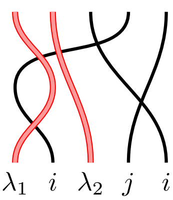
3. A basis and spanning set

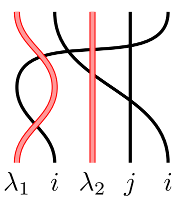
Given a Stendhal diagram d, we obtain a permutation by considering how its black strands are reordered, reading from the bottom to the top. Actually, we obtain more information than this, since the Stendhal diagram gives a factorization of this permutation into simple transpositions. As usual, we let S_n be the symmetric group on n letters, and s_m denote the simple transposition $(m, m+1)$.

Definition 4.9 *Assume d is a generic Stendhal diagram (no two crossings occur at the same value of y). Let $\mathbf{s}_d = (s_{j_1}, \dots, s_{j_m})$ be the list of simple transpositions in the symmetric group S_n obtained by reading off the crossings of black strands from bottom to top.*

Note that $\mathbf{s}_d$ is not isotopy independent, since commuting transpositions can move past each other. The list $\mathbf{s}_d$ may or may not be a reduced expression; it will be reduced if no two black strands cross twice.

For our running examples

$a =$ 

$b =$ 

we have that $\mathbf{s}_a = (s_2, s_1)$ and $\mathbf{s}_b = (s_2, s_1, s_2)$, which are both reduced.

For each permutation $w \in S_n$, and each Stendhal triple $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ and weakly increasing function $\kappa' \colon [1, \ell] \to [0, n]$, we choose a Stendhal diagram $\psi_{w,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ such that

- the bottom of $\psi_{w,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ corresponds to $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ and the top to $(w\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa')$.
- the sequence of transpositions $\mathbf{s}_{\psi_{w,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)}$ is a reduced expression for w; that is, the permutation on black strands reading bottom to top is w and no two black strands cross twice.
- no pair of red and black strands cross twice.

We should emphasize that this choice is very far from unique; there are various ways one can make it more systematically, but we see no reason to prefer one of these over any other.

Let $\mathbf{y}^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{Z}_{\geq 0}^n$ denote the monomial $y_1^{a_1} \cdots y_n^{a_n}$. Let B be the set

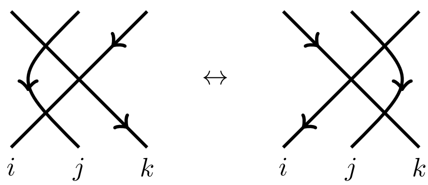
$$\{\psi_{w,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa) \mathbf{y}^{\mathbf{a}}\}$$

as $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ ranges over all Stendhal triples, κ' over weakly increasing functions, w over S_n (here $n = |\mathbf{i}|$), and $\mathbf{a}$ over $\mathbb{Z}_{\geq 0}^n$.

A basic observation, but one we will use many times through the paper is:

Lemma 4.10

(1) *Consider two Stendhal diagrams a and b with n crossings which differ by a finite number of isotopies, switches through triple points (involving all black or black and red strands) as in*



and switches of dots through crossings. The diagrams a and b agree as elements of $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ modulo the subspace spanned by diagrams with $< n$ total crossings.

(2) *If the isotopies and switches in (1) are contained in a subset U of the plane, then $a - b$ is a sum of diagrams with fewer crosses agreeing with a outside U.*

(3) *Any diagram c with n crossings containing a bigon (either all black or black/red) defines an element of $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ which lies in the span of diagrams with $< n$ crossings, which agree with c outside a neighborhood of the bigon.*

Proof. For part (1), we need only check this when a and b differ by a single triple point switch or a single dot moving through a crossing. This is clear from the relations (2.5f–2.5g,4.1a) in the first case and (2.5a–2.5d,4.1c) in the second. Part (2) follows from the locality of these relations.

Now consider part (3). We can assume that this bigon contains no smaller bigons inside it, but there may still be some number of strands which pass through, crossing each side of the bigon once. However, by doing triple point switches, we can move these strands out, and assume that our bigon is empty. Then we simply apply the relations (2.5e,4.2) to rewrite this diagram in terms of those with fewer crossings. □

Lemma 4.11 *The set B spans $\tilde{T}$.*

Proof. Given a Stendhal diagram d, we must show that modulo the relations of $\tilde{T}$, we can rewrite d as a sum of elements of B. We'll induct on the number of crossings. If there are 0 crossings, then d must be $e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ multiplied by a monomial in the dots, which is an element of B by definition.

By Lemma 4.10, we can assume that d has no bigons and that all dots are at the bottom of the diagram. Let w be the induced permutation on black strands. It only remains to show that we can rewrite a dotless diagram d with no bigons as the fixed diagram $\psi_{w,\kappa'}$ with the same top, bottom and induced permutation on black strands, plus diagrams with fewer crossings.

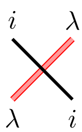
The isotopy class of d is encoded not just in the expression $\mathbf{s}_d$, but also contains encodes a reduced decomposition of the permutation induced on both red and black strands. Thus, the moves necessary to get from d to $\psi_{w,\kappa'}$ are encoded in the series of braid relations that takes one reduced expression to the other. The swapping of commuting transpositions is just an isotopy, and the braid relation corresponds to a triple point switch. Each time we apply one of these, Lemma 4.10 shows that the class modulo diagrams with fewer crossings is unchanged. After finitely many moves, we get to $\psi_{w,\kappa'}$, and the result is proven. $\square$

Fix $\underline{\boldsymbol{\lambda}}$ and $n \geq 0$. Let $\mathcal{P}_n$ be a free module over the polynomial ring $\Bbbk[Y_1, \cdots, Y_n]$ generated by elements $\varepsilon(\mathbf{i}, \kappa)$ for each Stendhal triple $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$. Choose polynomials $P_{ij}(u,v)$ such that $Q_{ij}(u,v) = P_{ij}(u,v)P_{ji}(v,u)$.

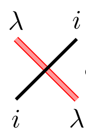
Lemma 4.12 *The algebra $\tilde{T}^{\underline{\boldsymbol{\lambda}}}_n$ acts on $\mathcal{P}_n$ by the rule that:*

- *The dots y_i act as the variables Y_i.*
- $e(\mathbf{i}, \kappa) \cdot \varepsilon(\mathbf{i}', \kappa') = \delta_{\mathbf{i},\mathbf{i}'}\delta_{\kappa,\kappa'}\varepsilon(\mathbf{i}, \kappa)$.
- *Assume $\kappa(j) = k$. The diagram crossing the kth black strand right over the jth red strand sends $\varepsilon(\mathbf{i}, \kappa) \mapsto Y_k^{\lambda_j^{i_k}}\varepsilon(\mathbf{i}, \kappa')$ where $\kappa'(m) = \kappa(m) - \delta_{j,m}$.*
- *Assume $\kappa(j) = k$. The diagram crossing the $k+1$st black strand left of the jth red sends $\varepsilon(\mathbf{i}, \kappa) \mapsto \varepsilon(\mathbf{i}, \kappa'')$ where $\kappa''(m) = \kappa(m) + \delta_{j,m}$.*
- *Crossing the mth and $m+1$st black strands (assuming there is no red between them) sends $\varepsilon(\mathbf{i}, \kappa) \mapsto 0$ if $i_m = i_{m+1}$ and $\varepsilon(\mathbf{i}, \kappa) \mapsto P_{ji}(Y_m, Y_{m+1})\varepsilon(s_m \cdot \mathbf{i}, \kappa)$ if $i_m \neq i_{m+1}$.*
- *Since the elements $\varepsilon(\mathbf{i}, \kappa)$ generate $\mathcal{P}_n$ over the polynomial ring $\mathbb{C}[Y_i]$, the action on any other element can be computed using the relations commuting elements of $T^{\underline{\boldsymbol{\lambda}}}$ past y_i's.*

More schematically, if we leave all but the two strands after the $k-1$st black out of the diagram, we can represent this action by:

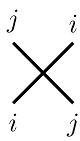


$\bullet f = f$



$\bullet f = Y_k^{\lambda^i} \cdot f$

$\bullet f = Y_k \cdot f$



$$\bullet f = \begin{cases} P_{ji}(Y_k, Y_{k+1})f^{s_k} & i \neq j \\ \dfrac{f^{s^k} - f}{Y_{k+1} - Y_k} & i = j \end{cases}$$

Proof. The KLR relations (2.5a–2.5g) follow from [**Roua**, Proposition 3.12]. Thus the only relations we need check are our additional relations (4.1a-c) and (4.2). All of these are manifest except for (4.1a) in the case where $i = j$. The LHS is

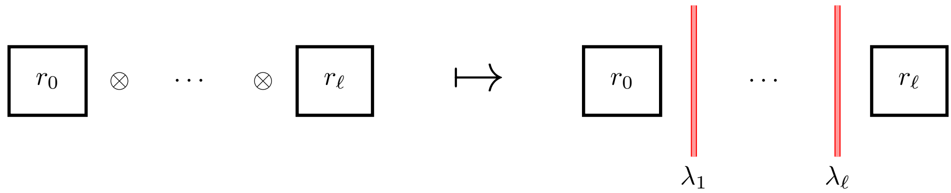
$$f \mapsto \frac{Y_{k+1}^{\lambda^i} f^{s_k} - Y_k^{\lambda^i} f}{Y_{k+1} - Y_k}$$

and the RHS is

$$f \mapsto Y_{k+1}^{\lambda^i} \frac{f^{s_k} - f}{Y_{k+1} - Y_k} + \frac{Y_{k+1}^{\lambda^i} - Y_k^{\lambda^i}}{Y_{k+1} - Y_k} f$$

so the relation is verified. $\square$

Fix any sequence of elements of the root lattice $\underline{\boldsymbol{\nu}} = (\nu_0, \dots, \nu_\ell)$. Then we have a map from the tensor product of KLR algebras $\wp_{\underline{\boldsymbol{\nu}}} \colon R_{\nu_0} \otimes \cdots \otimes R_{\nu_\ell} \to \tilde{T}^{\underline{\boldsymbol{\lambda}}}$ sending
(4.3)



In the KLR algebra, there are idempotents attached not just to sequences of elements of Γ, but to divided powers of these elements, as defined in [**KL09**, 2.5]. That is, consider $\mathbf{i} = (i_1^{(\vartheta_1)}, \dots, i_n^{(\vartheta_n)})$ for $i_j \in \Gamma$ and $\vartheta_j \in \mathbb{Z}_{>0}$ with $\sum_j \vartheta_j \alpha_{i_j} = \nu$ (in the notation of [**KL09**], this is an element of $\mathrm{Seqd}(\nu)$). We denote the idempotent attached to this sequence by $e(\mathbf{i}) \in R_\nu$ (the same idempotent is denoted $1_{\mathbf{i}}$ in [**KL09**]).

Now, consider such a sequence $\mathbf{i}$ together with $\underline{\boldsymbol{\lambda}}$ and κ as in a Stendhal triple, and let $\mathbf{i}_0, \dots, \mathbf{i}_\ell$ be the black blocks of the sequence $\mathbf{i}$ (that is, $\mathbf{i}_0$ is the first $\kappa(1)$ entries, $\mathbf{i}_1$ the next $\kappa(2) - \kappa(1)$, etc.) and $\nu_j = \mathrm{wt}(\mathbf{i}_j)$

Definition 4.13 *Let $e(\mathbf{i}, \kappa) := \wp_{\nu_0, \dots, \nu_\ell}(e(\mathbf{i}_0) \boxtimes \cdots \boxtimes e(\mathbf{i}_\ell))$. Note that if $\vartheta_j = 1$ for all j, this is agrees with the previous definition of $e(\mathbf{i}, \kappa)$*

Usually, we will not require these multiplicities, and will thus exclude them from the notation. Unless they are indicated explicitly, the reader should assume that they are 1.

Recall that the KLR algebra R_ν has a faithful polynomial representation[1] Π_ν defined in [**Roua**, 3.2.2]; special cases of this are also defined in [**KL09**, **KL11**].

Lemma 4.14 *The action of $R_{\nu_0} \otimes \cdots \otimes R_{\nu_\ell}$ on $\wp_{\underline{\boldsymbol{\nu}}}(1)\mathcal{P}_n$ via $\wp_{\underline{\boldsymbol{\nu}}}$ is isomorphic to $\Pi_{\nu_0} \boxtimes \cdots \boxtimes \Pi_{\nu_\ell}$.*

Proof. Obviously, any element of the image of $\wp_{\underline{\boldsymbol{\nu}}}$ will act trivially if the weight of the black block does not match $\nu_0, \dots, \nu_\ell$. If it does, then the generated of R_{ν_i} act by the formulas given in [**Roua**, 3.2.2] which exactly match those of Lemma 4.12. $\square$

Corollary 4.15 *The map $\wp_{\underline{\boldsymbol{\nu}}}$ is injective.*

Proof. Any element of the kernel acts trivially on $\Pi_{\nu_0} \boxtimes \cdots \boxtimes \Pi_{\nu_\ell}$ and this is impossible by [**Roua**, 3.2.2]. $\square$

Proposition 4.16 *The set B is a basis of $\tilde{T}$.*

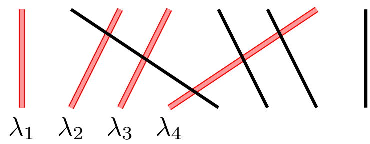
We will always refer to the process of rewriting an element in terms of this basis as "straightening" since, visually, it is akin to pulling all the strands taut until they

[1] This representation is denoted by R_n in [**Roua**]; for obvious reasons, we won't use this notation.

are straight. In the course of the proof, we'll need the element θ_κ, which is the sum over all $\mathbf{i}$ of the unique Stendhal diagram which

- has bottom triple given by $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, 0)$,
- has top triple given by $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$,
- has no dots and a minimal number of crossings.

For example, for $\kappa = (1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 3)$, we sum over all ways of adding black labels with the diagram:



The product $\dot{\theta}_{\kappa'}\psi_{w,\kappa'}\mathbf{y}^{\mathbf{a}}e(\mathbf{i},\kappa)\theta_\kappa$ is quite close to being an element of B, except that we may have created some bigons between red and black strands. Such a bigon will have been created with the strand which connects to the kth black terminus at the bottom and pth red strand if either $k < \kappa(p)$ or $w(k) < \kappa'(p)$. We define a vector $\mathbf{b} \in \mathbb{Z}^n_{\geq 0}$ whose kth entry is the sum over such p of $\lambda_p^{i_k}$.

Lemma 4.17 *The diagram $\dot{\theta}_{\kappa'}\psi_{w,\kappa'}\mathbf{y}^{\mathbf{a}}e(\mathbf{i},\kappa)\theta_\kappa$ is equal to $\psi_{w,0}\mathbf{y}^{\mathbf{a}+\mathbf{b}}e(\mathbf{i},0)$ modulo the span of diagrams with fewer crossings than $\psi_{w,0}$.*

Proof. It is easiest to see this inductively. If $\kappa \neq 0$, then we can multiply $\psi_{w,\kappa'}e(\mathbf{i},\kappa)$ on the bottom by crossing the black strand attached to the $\kappa(j)$th terminus (at the bottom) over the jth red strand, reducing κ. If this black strand had not already crossed the jth in $\psi_{w,\kappa'}e(\mathbf{i},\kappa)$, then this is still a basis vector (modulo diagrams with fewer crossings), and $\mathbf{b}$ is unchanged. On the other hand, if it had, then we can apply Lemma 4.10 and the relation (4.2) to move this strand to the right side of the jth strand and arrive at a basis vector, at the cost of multiplying it by $\lambda_j^{i_{\kappa(j)}}$ dots. However, we also must decrease $\mathbf{b}$ in order to compensate for this change, meaning that $\psi_{w,0}\mathbf{y}^{\mathbf{a}+\mathbf{b}}e(\mathbf{i},0)$ is left unchanged (modulo diagrams with fewer crossings). Applying this until $\kappa = \kappa' = 0$ shows the claim. □

Proof of Proposition 4.16. First, consider the map $\wp_{0,\dots,0,\nu}\colon R_\nu \to \tilde{T}^{\underline{\boldsymbol{\lambda}}}_n$. By Corollary 4.15, this map is injective. Furthermore, the algebra R_ν has a basis denoted S in [**Roua**, 3.1.2] which depends on a choice of reduced word for each permutation. As long as we choose these compatibly with the reduced word given by our basis vectors $\psi_{w,0}e(\mathbf{i},0)$, the basis S of R_ν is sent to the elements $\psi_{w,0}\mathbf{y}^{\mathbf{a}}e(\mathbf{i},0) \in B$ for $\mathbf{i}$ such that $\sum_j \alpha_{i_j} = \nu$. By the injectivity of $\wp_{0,\dots,0,\nu}$, these vectors are linearly independent.

Fix κ and κ', and suppose there is a non-trivial linear relation between elements of B with κ in the Stendhal triple at bottom and κ' at top. Now, multiply the relations on the left by $\dot{\theta}_{\kappa'}$ and on the right by θ_κ. As shown in Lemma 4.11, we can rewrite each term of the resulting relation in terms of the vectors $\psi_{w',0}\mathbf{y}^{\mathbf{a}'}e(\mathbf{i},0)$.

Now, choose a permutation $w \in S_n$ such that for some $\mathbf{a}$, the vector $\psi_{w,\kappa'}\mathbf{y}^{\mathbf{a}}e(\mathbf{i},\kappa)$ has nontrivial coefficient m, and such that w is maximal in Bruhat order amongst such permutations. Now, multiply by θ_κ and $\dot{\theta}_{\kappa'}$ and rewrite in terms of B. We find that $\psi_{w,0}\mathbf{y}^{\mathbf{a}+\mathbf{b}}e(\mathbf{i},0)$ also has coefficient m since no element of B other than

$\psi_{w,\kappa'}\mathbf{y}^{\mathbf{a}}e(\mathbf{i},\kappa)$ could contribute to its coefficient by Lemma 4.17. Since the elements $\{\psi_{w',0}\mathbf{y}^{\mathbf{a}'}e(\mathbf{i},0)\}$ are linearly independent, we must have $m=0$, giving a contradiction. Thus, this relation is trivial and we have a basis of $\tilde{T}^{\lambda}$. $\square$

If $\underline{\boldsymbol{\lambda}}=(\lambda)$, then we will simplify notation by writing T^{λ} for $T^{\underline{\boldsymbol{\lambda}}}$.

Theorem 4.18 $R^{\lambda}\cong T^{\lambda}$.

Proof. We have an injective map $\wp\colon R\hookrightarrow \tilde{T}^{\lambda}$ given by adding a red line at the left. Composing with the projection $\tilde{T}^{\lambda}\to T^{\lambda}$, we obtain a map $\wp'\colon R\to T^{\lambda}$. This map is a surjection since each element of the basis of Proposition 4.16 is in the image.

Thus, it only remains to show that the kernel of the map $\wp'$ is precisely the cyclotomic ideal. To show that the latter is contained in the former, we need only show that the image of $y_1^{\lambda^{i_1}}e(\mathbf{i})$ is 0 in T^{λ}; this follows immediately from (4.2).

Consider a diagram d with a violating strand in $\tilde{T}^{\lambda}$; we will prove by induction that d lies in the image of the cyclotomic ideal of R. The statistic c on which we induct on is half the number of red/black crossings in d plus the number of black/black crossings left of the red line. If $c=1$, we must have a single black strand labeled with i which crosses over and immediately crosses back, and (4.2) shows that this diagram is equal to one with no strands left of the red, but with λ^i dots on the left-most strand at some value of y, which is thus in the cyclotomic ideal.

If $c>1$, then there is either a bigon or a triangle formed with a red strand on the right side. Applying either the relation (4.2) if there is a bigon or (4.1a) if there is a triangle, every term on the RHS has c lower, but ≥ 1. Thus, applying the inductive hypothesis, we can rewrite d in $\tilde{T}$ as sum of diagrams in the cyclotomic ideal.

Thus, if $r\in R$ lies in the kernel of the map $\wp'$, its image is a sum of diagrams in the cyclotomic ideal. Thus, it can be rewritten as a sum of elements of the cyclotomic ideal. By the injectivity of $\wp$, the element r thus lies in the cyclotomic ideal. This completes the proof. $\square$

4. Splitting red strands

This leads us to an observation which will be quite useful in the future. Let e_ℓ be the idempotent given by the sum of $e(\mathbf{i},\kappa)$ where $\kappa(\ell)=n$, i.e., those where the last strand is colored red, not black. Let $\underline{\boldsymbol{\lambda}}^-=(\lambda_1,\dots,\lambda_{\ell-1})$.

Proposition 4.19 *There is an isomorphism $T^{\underline{\boldsymbol{\lambda}}^-}\to e_\ell T^{\underline{\boldsymbol{\lambda}}}e_\ell$.*

Proof. The map is induced by the map $\tilde{T}^{\underline{\boldsymbol{\lambda}}^-}\mapsto e_\ell\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_\ell$ which adds a red strand at the far right of the Stendhal diagram. Proposition 4.16 shows that this is surjective, and obviously it sends violated diagrams to violated diagrams. Thus, we have a surjective map $T^{\underline{\boldsymbol{\lambda}}^-}\to e_\ell T^{\underline{\boldsymbol{\lambda}}}e_\ell$.

Now assume, we have an element of $e_\ell\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_\ell$, which is a sum of violated diagrams. We need only to consider diagrams where top and bottom satisfy $\kappa(1)=0$, since otherwise the diagrams are automatically 0 in $e_\ell\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_\ell$. In particular, if $\ell=1$, we need only consider diagrams with no black strands, and thus obtain an isomorphism $T^{\underline{\boldsymbol{\lambda}}^-} \cong \Bbbk \cong e_\ell T^{\underline{\boldsymbol{\lambda}}} e_\ell$. Assume from now on that $\ell > 1$.

Thus, let a be a violated diagram whose top and bottom satisfy $\kappa(\ell) = n$. If at any point, there is a black strand right of the rightmost red strand, these strands must form a bigon. By Lemma 4.10, we can rewrite a as a sum of diagrams with fewer crossings without this bigon. Furthermore, in the proof, we use isotopies are relations that never change the fact that a is violating. Thus, if we write an element a of the kernel of the map $e_\ell \tilde{T}^{\underline{\boldsymbol{\lambda}}} e_\ell \to e_\ell T^{\underline{\boldsymbol{\lambda}}} e_\ell$ as a sum of violated diagrams with a minimal number of crossings, there will be no bigons involving the ℓth red strand. Thus, a is in the image of the violating ideal in $\tilde{T}^{\underline{\boldsymbol{\lambda}}^-}$ and we have the desired isomorphism. $\square$

This isomorphism induces a $T^{\underline{\boldsymbol{\lambda}}^-}$ - $T^{\underline{\boldsymbol{\lambda}}}$-bimodule structure on $e_\ell T^{\underline{\boldsymbol{\lambda}}}$.

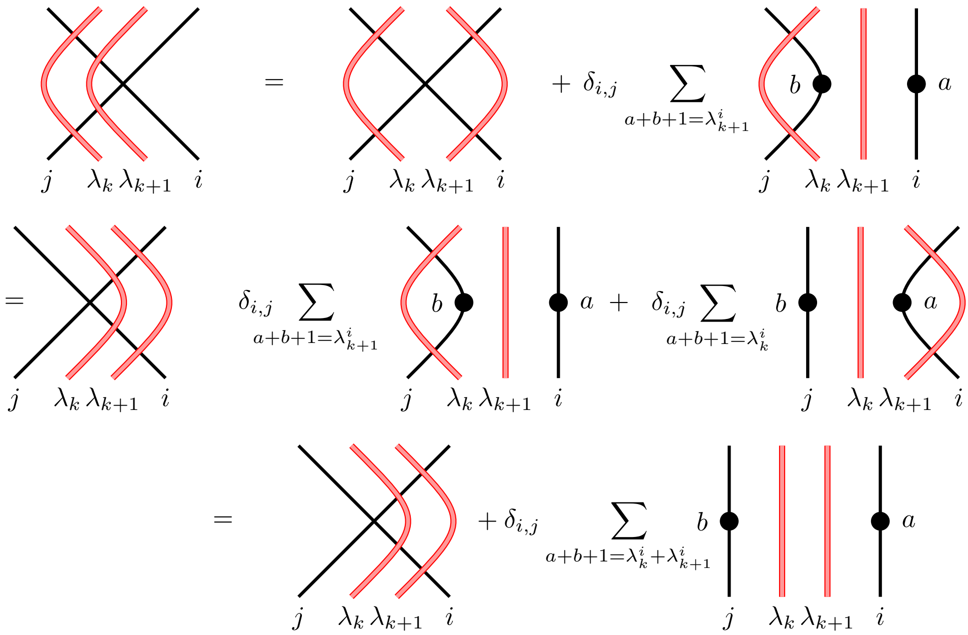
Definition 4.20 *Let $\mathfrak{I}_{\lambda_\ell}(M) := M \otimes_{T^{\underline{\boldsymbol{\lambda}}^-}} e_\ell T^{\underline{\boldsymbol{\lambda}}}$. Let $\mathfrak{I}^R_{\lambda_\ell}(N) := N e_\ell$ be its right adjoint.*

We'll often use the functor $\mathfrak{I}_\mu$ without carefully defining in relevant lists of weights first. For any sequence $\underline{\boldsymbol{\lambda}}$, by definition $\mathfrak{I}_\mu$ is a functor $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \to \mathfrak{V}^{(\lambda_1,\dots,\lambda_\ell,\mu)}$.

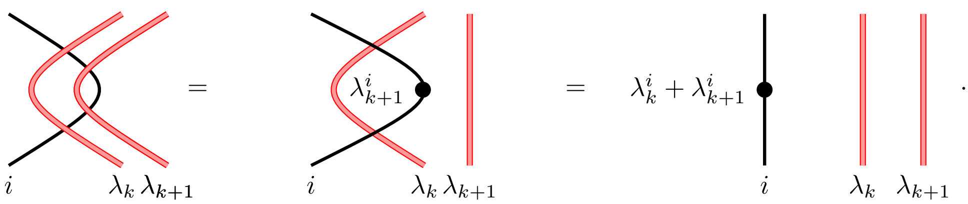
Fix $1 \leq k < \ell$, and let $\underline{\boldsymbol{\lambda}}' = (\lambda_1, \dots, \lambda_k + \lambda_{k+1}, \dots, \lambda_\ell)$. Given a Stendhal diagram with red lines labeled by $\underline{\boldsymbol{\lambda}}'$, we can obtain a new Stendhal diagram by "splitting" the kth red strand into two, labeled with λ_k and λ_{k+1}. This is compatible with composition and thus induces an algebra map $\sigma\colon \tilde{\tilde{T}}{}^{\underline{\boldsymbol{\lambda}}'}_n \to \tilde{\tilde{T}}{}^{\underline{\boldsymbol{\lambda}}}_n$. The algebra $\tilde{\tilde{T}}{}^{\underline{\boldsymbol{\lambda}}'}_n$ is unital; its unit is the sum over all Stendhal diagrams for $\underline{\boldsymbol{\lambda}}'$ with n black strands and no crossings or dots. However, this homomorphism is not unital. It sends $1 \in \tilde{\tilde{T}}{}^{\underline{\boldsymbol{\lambda}}'}_n$ to an idempotent $e_{\underline{\boldsymbol{\lambda}}'} \in \tilde{\tilde{T}}{}^{\underline{\boldsymbol{\lambda}}}_n$ consisting of the sum of $e(\mathbf{i}, \kappa)$ for all κ with $\kappa(k) = \kappa(k+1)$.

Proposition 4.21 *The map σ induces isomorphisms $\tilde{T}^{\underline{\boldsymbol{\lambda}}'} \to e_{\underline{\boldsymbol{\lambda}}'} \tilde{T}^{\underline{\boldsymbol{\lambda}}} e_{\underline{\boldsymbol{\lambda}}'}$ and $T^{\underline{\boldsymbol{\lambda}}'} \to e_{\underline{\boldsymbol{\lambda}}'} T^{\underline{\boldsymbol{\lambda}}} e_{\underline{\boldsymbol{\lambda}}'}$.*

Proof. First, we must show that σ induces a homomorphism $\tilde{T}^{\underline{\boldsymbol{\lambda}}'} \to \tilde{T}^{\underline{\boldsymbol{\lambda}}}$. Obviously, the KLR relations present no issue, nor do (4.1b) and (4.1c). Thus, we need only confirm (4.1a) and (4.2). The first follows from



and the second from



This further induces a map $T^{\underline{\boldsymbol{\lambda}}'} \to T^{\underline{\boldsymbol{\lambda}}}$ since it sends violated diagrams to violated diagrams.

That the image lies in $e_{\underline{\boldsymbol{\lambda}}'}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\underline{\boldsymbol{\lambda}}'}$ is clear from the definition. Furthermore, the map $\tilde{T}^{\underline{\boldsymbol{\lambda}}'} \to e_{\underline{\boldsymbol{\lambda}}'}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\underline{\boldsymbol{\lambda}}'}$ sends the basis B in $\tilde{T}^{\underline{\boldsymbol{\lambda}}'}$ to the intersection of the same basis with $e_{\underline{\boldsymbol{\lambda}}'}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\underline{\boldsymbol{\lambda}}'}$. Thus, it is an isomorphism.

Finally, we must show that this remains an isomorphism when we pass to the map $T^{\underline{\boldsymbol{\lambda}}'} \to e_{\underline{\boldsymbol{\lambda}}'}T^{\underline{\boldsymbol{\lambda}}}e_{\underline{\boldsymbol{\lambda}}'}$. That is, we must show that any violated diagram d is the image of a sum of violated diagrams. This is achieved by an argument very similar to Lemma 4.19: if d is not the image of a diagram under the splitting, then there must be a bigon or triangle inside the region between the k and $k+1$st red strands with one side formed by one of the strands. We can use Lemma 4.10(1) for a triangle, or Lemma 4.10(3) for a bigon in order to remove these features from between the two strands, modulo diagrams with fewer crossings Furthermore, by locality, these operations don't change whether the diagram is violated. Thus, writing d as a sum of violated diagrams with a minimal number of crossings and none between the kth and $k+1$st strands, we see that d is in the image of the violating ideal under the map $\tilde{T}^{\underline{\boldsymbol{\lambda}}'} \to e_{\underline{\boldsymbol{\lambda}}'}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\underline{\boldsymbol{\lambda}}'}$, so we have the desired isomorphism. $\square$

5. The double tensor product algebras

We'll give a presentation of a Morita equivalent algebra to $T^{\underline{\boldsymbol{\lambda}}}$. This involves a "doubled" generalization of Stendhal diagrams which roughly includes both the original Stendhal diagrams and morphisms from $\mathcal{U}$. More formally.

Definition 4.22 *A* **blank double Stendhal diagram** *is a collection of finitely many oriented curves in* $\mathbb{R} \times [0,1]$*. Each curve is either*

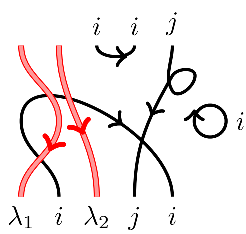
- *colored red and labeled with a dominant weight of* $\mathfrak{g}$*, or*
- *colored black and labeled with* $i \in \Gamma$ *and decorated with finitely many dots.*

and has the same local restrictions as a Stendhal diagram. However only the red strands are constrained to be oriented downwards, and the black strands are allowed to close into circles, self-intersect, etc.

Blank double Stendhal diagrams divide their complement in $\mathbb{R}^2 \times [0,1]$ *into finitely many connected components, and we define a* **double Stendhal diagram** *(DSD) to be a blank DSD together with a labeling of these regions by weights consistent with the rules*

Since this labeling is fixed as soon as one region is labeled, we will typically not draw in the weights in all regions in the interest of simplifying pictures.

Any Stendhal diagram is also a blank double Stendhal diagram, but not *vice versa*. For example,

$a' =$ 

is blank double Stendhal, but not Stendhal. Similarly, every KL diagram is a DSD. There is a unique extension of the degree function of Stendhal and KL diagrams to DSD's which is compatible with composition.

For the top and bottom of a double Stendhal diagram, we must record orientation information in addition to the labels. Thus, in the list of labels on black strands, we write $-i$ for a strand with label i oriented downward and $+i$ when it is oriented upward. Note that this means that when we consider consider a usual Stendhal diagram as a DSD, we will only have elements of $-\Gamma$ at the top and bottom; this convention saves us from negating everything, and matches better the literature on KLR algebras.

Definition 4.23 *A double Stendhal triple*[2] *(DST) is a pair of lists* $\mathbf{i} \in (\pm\Gamma)^n$*,* $\underline{\boldsymbol{\lambda}} \in X^+(\mathfrak{g})^\ell$*, a weakly increasing function* $\kappa\colon [1,\ell] \to [0,n]$*, and weights* $\mathcal{L}$ *and* $\mathcal{R}$

[2]Somewhat inaccurately named.

such that

$$\mathcal{L} + \sum_{k=1}^{\ell} \lambda_k + \sum_{m=1}^{n} \alpha_{i_m} = \mathcal{R}.$$

As usual, we employ the convention that $\alpha_{-i} = -\alpha_i$.

Thus, for the diagram a' above, the blank double Stendhal triple at the top is $\mathbf{i} = (-i, i, -j), \underline{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2)$ and $\kappa = (1 \mapsto 0, 2 \mapsto 0)$, and for the bottom it is $\mathbf{i} = (i, -j, -i), \underline{\boldsymbol{\lambda}} = (\lambda_1, \lambda_2)$ and $\kappa = (1 \mapsto 0, 2 \mapsto 1)$. We haven't chosen labelings of the regions, but if the leftmost region is labeled with $\mathcal{L}$, the rightmost must carry $\mathcal{R} = \mathcal{L} + \lambda_1 + \lambda_2 - \alpha_j$.

We can define **(vertical) composition** for double Stendhal diagrams as with usual Stendhal diagrams, though we must also require that orientations on strands and labels of regions match at bottom of a and top of b to get a non-zero result for ab.

We can also define **horizontal composition** $a \circ b$ of DSD's which pastes together the strips where a and b live with a to the *right* of b. The only compatibility we require is that $\mathcal{L}_a = \mathcal{R}_b$, so that the regions of the new diagram can be labeled consistently. Of course, this gives a notion of composition of DST's h_2h_1 where $h_m = (\mathbf{i}_m, \underline{\boldsymbol{\lambda}}_m, \kappa_m, \mathcal{L}_m, \mathcal{R}_m)$. In terms of sequences, we take the concatenations $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ and $\underline{\boldsymbol{\lambda}} = \underline{\boldsymbol{\lambda}}_1\underline{\boldsymbol{\lambda}}_2$,

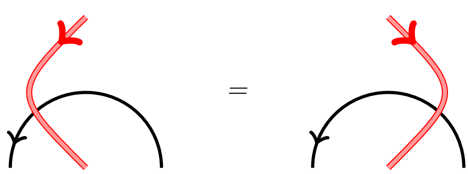
$$\kappa(j) = \begin{cases} \kappa_1(j) & j \leq \ell_1 \\ \kappa_2(j) + n_1 & j > \ell_1, \end{cases}$$

and $\mathcal{L} = \mathcal{L}_1, \mathcal{R} = \mathcal{R}_2$, with the composition being 0 if $\mathcal{L}_2 \neq \mathcal{R}_1$.

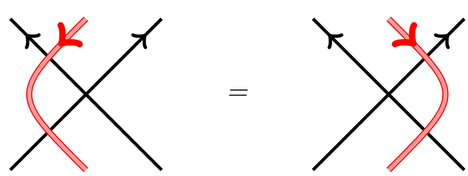
Definition 4.24 *Let $\mathcal{T}$ be the strict 2-category whose*

- *objects are weights in $X(\mathfrak{g})$,*
- *1-morphisms $\mu \to \nu$ are DST's with $\mathcal{L} = \mu, \mathcal{R} = \nu$ and composition is given by horizontal composition as above.*
- *2-morphisms $h \to h'$ between DST's are $\Bbbk$-linear combinations of DSD's with h as bottom and h' as top, modulo the relations*
  - $*$ *all the relations of Figures (2.1a–2.5g) hold for KL diagrams thought of as DSD's.*
  - $*$ *all the relations of (3.1-2) hold for Stendhal diagrams thought of DSD's (ignoring labeling of regions).*
  - $*$ *the further relations and their mirror images through a vertical line, which are again independent of labels, hold*

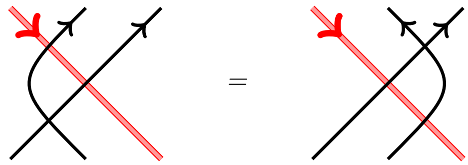
(4.4a)



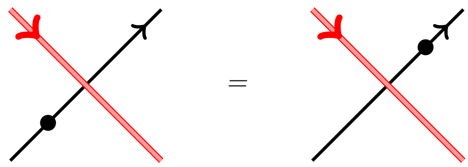
(4.4b)



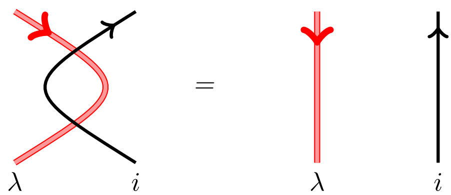
(4.4c)

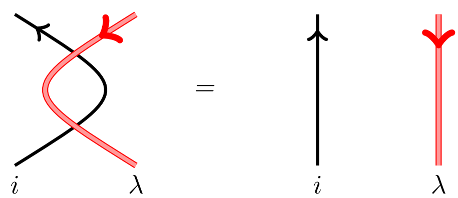


(4.4d)



(4.4e)

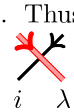




Note that if λ, ν is are dominant weights, we have natural map $R^{\nu+\lambda} \to R^{\nu}$ induced by the inclusion of cyclotomic ideals. We let $\mathfrak{I}_\lambda \colon R^{\nu}\text{-mod} \to R^{\nu+\lambda}\text{-mod}$ denote the functor of pullback by these maps.

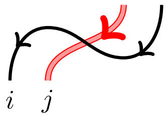
Theorem 4.25 *There is a representation of $\mathcal{T}$ in the strict 2-category of categories, sending $\mu \mapsto \oplus_\nu R^{\nu}_{\mu}$-mod, sending the image of $\mathcal{U}$ to the previously defined action of Theorem 3.17 and a single red line with label λ to $\mathfrak{I}_\lambda$.*

PROOF. The action of $\mathcal{U}$ on the same category defines how all diagrams only involving black strands act, and checks all of their relations. Thus, we need only define how the diagrams involving red strands act. Luckily, this is quite easy: the functors $\mathfrak{I}_\lambda$ of pullback and and $\mathfrak{E}_i$ of restriction obviously commute, since they are pullbacks along the two sides of a commuting square. Thus, the morphisms $a =$  and $b =$  are assigned to the identity map on the underlying vector spaces.

The relations (4.4a–4.4e) follow immediately from this assignment. Thus, we need only calculate where this sends the diagrams $b' =$  and $a' =$  and check the relations (3.1-2).

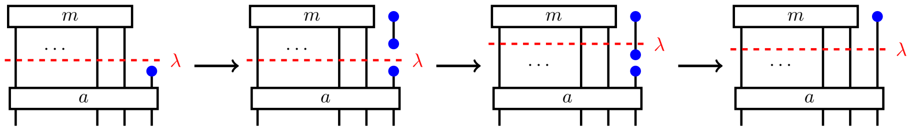
We can write a' and b' in terms of the morphisms a and b above and the adjunctions from $\mathcal{U}$. We can factor a' as the sequence $\mathfrak{F}_i\mathfrak{I}_\lambda \to \mathfrak{F}_i\mathfrak{I}_\lambda\mathfrak{E}_i\mathfrak{F}_i \to \mathfrak{F}_i\mathfrak{E}_i\mathfrak{I}_\lambda\mathfrak{F}_i \to$

$\mathfrak{I}_\lambda\mathfrak{F}_i$. Pictorially,

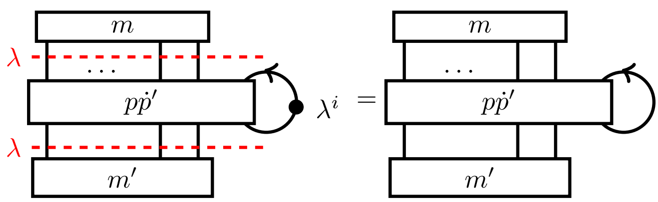
$a =$ 

Consider a $T^{\underline{\boldsymbol{\lambda}}}$-module M (which we will sometimes consider as a module over $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$). Both the modules $\mathfrak{F}_i\mathfrak{I}_\lambda M$ and $\mathfrak{I}_\lambda\mathfrak{F}_i M$ are quotients of $\tilde{\mathfrak{F}}_i M$, the induction of M considered as a module over $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$. The identity map $\tilde{\mathfrak{F}}_i M \to \tilde{\mathfrak{F}}_i M$ induces a natural projection $c\colon \mathfrak{F}_i\mathfrak{I}_\lambda \to \mathfrak{I}_\lambda\mathfrak{F}_i$. We claim that this is the map induced by a'. In order to represent the functors that appear diagrammatically, we use blue dots to represent the strands created by an $\mathfrak{F}_i$ or eaten by an $\mathfrak{E}_i$, and use a dashed line to denote the moment where we do the pullback. Since we consider right (i.e. bottom) modules, the order these functors appear is reading down the page.

(4.5)



On the other hand, the map $y^{\lambda^i}\colon \tilde{\mathfrak{F}}_i M \to \tilde{\mathfrak{F}}_i M$ induces a map $d\colon \mathfrak{I}_\lambda\mathfrak{F}_i \to \mathfrak{F}_i\mathfrak{I}_\lambda$; we claim that this coincides with b'. In order to show this, we note that the map b' is the dual of a' under the natural pairing between $\mathcal{F}_i\mathfrak{I}_\lambda M$ and $\mathcal{F}_i\mathfrak{I}_\lambda M^\star$ and similarly with the functors in the other order. From the relation (2.3a) and the bubble slides of [**KL10**, §3.1.2], we see that decreasing all labeling of regions by λ and adding λ^i dots to each bubble and any loop formed by the rightmost strand just applies the projection $R^{\nu+\lambda} \to R^\nu$. That is, given two elements $m\otimes p \in \mathcal{F}_i\mathfrak{I}_\lambda M, m'\otimes p'\mathcal{F}_i\mathfrak{I}_\lambda M^\star$, we have that



This shows that $\langle m\otimes p, d(m'\otimes p')\rangle = \langle a'(m\otimes p), m'\otimes p'\rangle$, so we must have $d = b'$.

Thus, we have that the compositions $a'b'$ and $b'a'$ are both y^{λ^i}. This confirms (4.2). The relations (4.1a–4.1c) are confirmed by same the calculations as the proof of Lemma 4.12. □

As with a usual Stendhal diagram, we call a DSD **violated** if it factors through a DST with $\kappa(0) > 1$, that is, which has a black strand (of either orientation) at the far left.

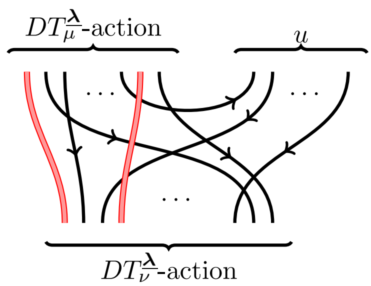
Definition 4.26 *Let the* **double tensor product algebra** $DT^{\underline{\boldsymbol{\lambda}}}$ *be the* $\Bbbk$*-algebra spanned by DSD's with* $\mathcal{L} = 0$ *and red lines labeled by* $\underline{\boldsymbol{\lambda}}$*, modulo the relations of* $\mathcal{T}$ *and all violated diagrams.*

We let $D\mathfrak{V}^{\underline{\boldsymbol{\lambda}}} = DT^{\underline{\boldsymbol{\lambda}}}$-mod be the category of finite dimensional representations of equivalently $DT^{\underline{\boldsymbol{\lambda}}}$ graded by $\mathbb{Z}$. We wish to show that this category carries a categorical $\mathfrak{g}$-action. Consider a 1-morphism u in $\mathcal{U}$. This is a word in $\mathcal{E}_i$ and $\mathcal{F}_i$, which we can consider as a DST with no red lines. Let e_u be the idempotent in $DT^{\underline{\boldsymbol{\lambda}}}$ which acts by the identity on all DST's which end in $u\colon \mu \to \nu$ (that is, they are a horizontal composition $u \circ t$ for a 1-morphism t in $\mathcal{T}$) and by 0 all others.

Definition 4.27 *Let β'_u be the $DT^{\underline{\boldsymbol{\lambda}}}_\mu$-$DT^{\underline{\boldsymbol{\lambda}}}_\nu$ bimodule $e_u \cdot DT^{\underline{\boldsymbol{\lambda}}}$. The left and right actions of $DT^{\underline{\boldsymbol{\lambda}}}$ on this space are by the formula $a \cdot h \cdot b = (1_u \circ a)hb$.*

This definition is perhaps a bit clearer from the schematic diagram

(4.6)



Any 2-morphism $\phi\colon u \to v$ in $\mathcal{U}$ can be considered as a DSD, and it defines a map of bimodules $\beta'_\phi = (\phi \circ 1)(-)\colon \mathfrak{F}_u \to \mathfrak{F}_v$; in the diagram (4.6), this action attaches the 2-morphism to the group of strands in the upper right. Since DSD's satisfy all the relations of $\mathcal{U}$, it immediately follows that:

Theorem 4.28 *There is a representation of $\mathcal{U}$ which sends*

$$\mu \mapsto D\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_\mu \qquad u \mapsto \beta'_u \qquad \phi \mapsto \beta'_\phi$$

6. A Morita equivalence

Note that we have a natural map $f\colon T^{\underline{\boldsymbol{\lambda}}} \to DT^{\underline{\boldsymbol{\lambda}}}$ given by considering a Stendhal diagram as a double Stendhal diagram; the image of the identity in $T^{\underline{\boldsymbol{\lambda}}}$ is an idempotent $e_- \in DT^{\underline{\boldsymbol{\lambda}}}$.

Lemma 4.29 *The map f induces an isomorphism $T^{\underline{\boldsymbol{\lambda}}} \cong e_- DT^{\underline{\boldsymbol{\lambda}}} e_-$.*

Proof. We first show that any diagram d of $e_- DT^{\underline{\boldsymbol{\lambda}}} e_-$ is in the image of f. As in the proof of [**KL10**, 3.9], we can apply the relations of $\mathcal{T}$ to write d as a sum of diagrams with fewer strands that intersect twice or self-intersect until neither of these occurs, and we have slid all bubbles to the far left. Any diagram with a bubble at far left is 0, so we are left with only diagrams with no bubbles, and all strands connect top to bottom. That is, we are left only with Stendhal diagrams.

Now, we need to show that the map is injective. If we use [**Webb**, 4.5], then we can apply the argument of Lemma 3.12 to show that any element of the kernel can be rewritten in terms of the cyclotomic ideal.

We can also sketch out a proof of this result that follows the path of Chapter 3 and thus keeps this paper self-contained. Since the results are, on the whole, very similar, we will spare the reader most of the details.

As before, we can define an intermediate category $\mathcal{T}^i_-$ and quotient algebra $D^iT^{\underline{\boldsymbol{\lambda}}}$ spanned by DSD's where only strands with label i can be upward. The action of Chapter 2 can be extended to $\mathcal{T}^i_-$ analogously to the action of Theorem

4.25. If we let $\mathbf{\Lambda}_\nu^\lambda$ denote the ring $\mathbf{\Lambda}_\nu$ attached to the weight ν when fix λ at the start of the construction, then the relations (3.7) show that we have a projection map $\mathbf{\Lambda}_{\nu+\mu}^{\lambda+\mu} \to \mathbf{\Lambda}_\nu^\lambda$. The action of $\mathcal{T}_-^i$ sends $\mathcal{I}_\mu$ to the pullback map under this homomorphism.

This action allows us to show the analogue of Corollary 3.10 in this case, that the diagrams given by $B_{i,G,H}$ with the red strands adding introducing a minimal number of crossings is a basis for the Hom space. Repeating the proof of Lemma 3.12 and Proposition 3.13 shows that the quotient $D^iT^{\underline{\boldsymbol{\lambda}}}$ is Morita equivalent to $T^{\underline{\boldsymbol{\lambda}}}$ via the analogue of f. Thus, by the argument of Proposition 3.16, we have an action of $\mathcal{T}_-^i$ on $\oplus_{\underline{\boldsymbol{\lambda}}}\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$, which we can extend as in Theorem 3.17 to an action of $\mathcal{T}$. Thus, any diagram in $\mathcal{T}$ can be interpreted as a natural transformation between functors from $T^{\underline{\boldsymbol{\lambda}}}$-mod to $T^{\underline{\boldsymbol{\lambda}}'}$-mod in a functorial way. In particular, the operator of left multiplication by a diagram appears this way, by thinking of that diagram in $\mathcal{T}$ and letting it act on the identity of the weight 0.

Thus if a in $T^{\underline{\boldsymbol{\lambda}}}$ is in the kernel of this map, this means that if we interpret this diagram as an 2-morphism of $\mathcal{T}$, this 2-morphism acts trivially on the identity of the weight 0. But this means that left multiplication by a is 0, that is $a = 0$. This proves injectivity. □

Theorem 4.30 *The algebras $T^{\underline{\boldsymbol{\lambda}}}$ and $DT^{\underline{\boldsymbol{\lambda}}}$ are Morita equivalent.*

PROOF. Recall again that for an algebra A and idempotent e, the bimodules Ae and eA induce Morita equivalences if and only if $AeA = A$. Thus, we need only prove that the idempotent attached to any DST in $DT^{\underline{\boldsymbol{\lambda}}}$ actually lies in $DT^{\underline{\boldsymbol{\lambda}}} \cdot e_- \cdot DT^{\underline{\boldsymbol{\lambda}}}$. In order to prove this, we fix a region near $y = 1/2$. If we see a pair of consecutive black lines where the rightward is upward oriented, and the leftward downward oriented, we use the relations (2.3c) and (2.4a) to swap them past each other. If we see an upward oriented strand immediately to the right of a red strand, we use relation (4.4e) to swap them. Thus, ultimately, we can rewrite the idempotent as a sum of DSD's factoring through DST's which have all their upward oriented strands left of all other strands, red or black. Of course, such a DST will only be non-zero in $DT^{\underline{\boldsymbol{\lambda}}}$ if it has no upward oriented strands. Thus, this central portion is in the image of f, and the whole diagram lies in $DT^{\underline{\boldsymbol{\lambda}}} \cdot e_- \cdot DT^{\underline{\boldsymbol{\lambda}}}$. The result follows. □

We can consider the image $\beta_u = e_- \cdot \beta_u' \cdot e_-$ of the action bimodules under this Morita equivalence. It immediately follows that:

Theorem 4.31 *There is a representation of $\mathcal{U}$ which sends*

$$\mu \mapsto \mathfrak{V}_\mu^{\underline{\boldsymbol{\lambda}}} \qquad u \mapsto \beta_u \qquad \phi \mapsto \beta_\phi$$

The bimodule β_u is the subspace inside β_u' such that the only upward termini are attached to u in the diagram above. In the interior of the diagram, we allow bubbles and self-intersections, and the diagram is only constrained by the rules of a DSD. However, elements like self-intersections can always be removed using the relations.

Two special cases of these functors merit special attention. When $u = \mathcal{F}_i, \mathcal{E}_i$, we denote the corresponding functors $\mathfrak{F}_i := - \otimes_{T^{\underline{\boldsymbol{\lambda}}}} \beta_{\mathcal{F}_i}$ and $\mathfrak{E}_i := - \otimes_{T^{\underline{\boldsymbol{\lambda}}}} \beta_{\mathcal{E}_i}$. In Figure 1, we show the diagrams as in (4.6) for these functors. The bimodule $\beta_{\mathcal{F}_i}$ is

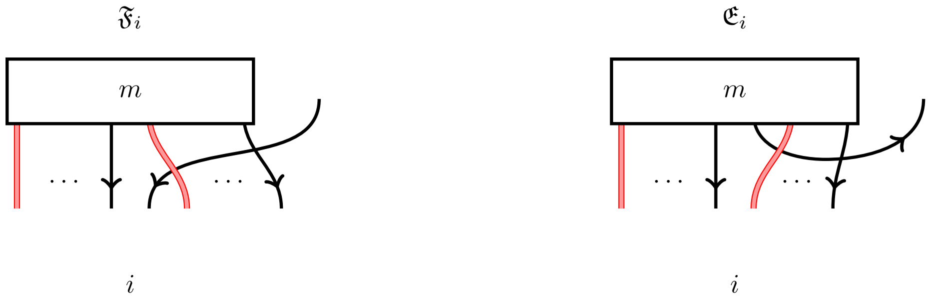


FIGURE 1. The functors $\mathfrak{E}_i$ and $\mathfrak{F}_i$

spanned by diagrams where all strands are downward, and $\beta_{\mathcal{E}_i}$ by diagrams where all but a single cup turned up at the right. As in the case of the cyclotomic quotient, we can interpret $\mathfrak{F}_i$ as an extension of scalars via the map $\nu_i \colon T^{\underline{\boldsymbol{\lambda}}} \to T^{\underline{\boldsymbol{\lambda}}}$ given by adding a i-labeled strand at the far right. We will often call this strand **new** to distinguish it from the others. In $\beta_{\mathcal{F}_i}$, this is the strand connected to the rightmost terminal at top.

Similarly, we can interpret $\mathfrak{E}_i$ as restriction under the same map ν_i (with a grading shift, due to the cup).

Proposition 4.32 *We have*

$$\mathfrak{I}_\mu^R \mathfrak{I}_\mu = \mathrm{id} \qquad \mathfrak{I}_\mu^R \mathfrak{F}_i \mathfrak{I}_\mu = \mathfrak{F}_i(-\mu^i) \qquad \mathfrak{I}_\mu^R \mathfrak{E}_i \mathfrak{I}_\mu = \mathfrak{E}_i$$

PROOF. We only need to check these equalities on the algebra $T^{\underline{\boldsymbol{\lambda}}}$ itself. The image of this algebra under the functor $\mathfrak{I}_\mu$ is $e_{\ell+1}T^{(\lambda_1,\dots,\lambda_\ell,\mu)}$. The image of this under $\mathfrak{I}_\mu^R$ is indeed $T^{\underline{\boldsymbol{\lambda}}} \cong e_{\ell+1}T^{(\lambda_1,\dots,\lambda_\ell,\mu)}e_{\ell+1}$.

Now we turn to the interaction of $\mathfrak{I}_\mu$ and $\mathfrak{F}_i$. We note that $\mathfrak{I}_\mu^R \mathfrak{F}_i \mathfrak{I}_\mu(T^{\underline{\boldsymbol{\lambda}}}) = e_\ell \beta^+_{\mathcal{F}_i} e_\ell$ where $\beta^+_{\mathcal{F}_i}$ denotes the action bimodule for $T^{(\lambda_1,\dots,\lambda_\ell,\mu)}$. This is the subspace spanned by diagrams in $\beta_{\mathcal{F}_i}$ where no strand is right of the rightmost (labeled μ) except the new strand attached to the rightmost terminal at top, corresponding to $\mathcal{F}_i$. There's a map of $\beta_{\mathcal{F}_i}$, the corresponding bimodule for $T^{\underline{\boldsymbol{\lambda}}}$, to $e_\ell \beta^+_{\mathcal{F}_i} e_\ell$ given by adding in the μ labeled strand adding just a crossing with the new strand. Since the bimodule action only works on the strands left of the red labeled μ, this will be a bimodule map, and simply deleting the red strand is its inverse. To see that these maps are well-defined, just note that they respect local relations and neither can get rid of a violating strand. Note that this map is of degree μ^i, because of the degree of the red/black crossing.

A similar argument shows the same for $\mathfrak{E}_i$. This completes the proof. $\square$

7. Decategorification

In order to understand the Grothendieck group $K_0(T^{\underline{\boldsymbol{\lambda}}})$, we need to better understand its Euler form. In particular, we need a candidate bilinear form on $V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$, which we hope will match with the Euler form under a hypothetical isomorphism $K_0(T^{\underline{\boldsymbol{\lambda}}}) \cong V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$. There is a system of non-degenerate $U_q(\mathfrak{g})$-invariant sesquilinear forms $\langle -, - \rangle$ on all tensor products $V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$ defined by $\langle v, w \rangle = \langle \Theta^{(\ell)} v, w \rangle_{\mathrm{III}}$, where $\Theta^{(\ell)}$

is the ℓ-fold **quasi-R-matrix** and $\langle -, - \rangle_{\mathrm{III}}$ is the factor-wise q-Shapovalov form. The usual quasi-R-matrix $\Theta^{(2)}$ on two tensor factors is defined in [**Lus93**, §4]; the ℓ-fold one is defined inductively by $\Theta^{(\ell)} = (\Theta^{(2)} \otimes 1^{\otimes \ell-2})(\Delta \otimes 1^{\otimes \ell-2}(\Theta^{(\ell-1)}))$. Let $\langle -, - \rangle_1$ denote the specialization of this form at $q = 1$, which is the same as the factor-wise Shapovalov form.

Proposition 4.33 *The form $\langle -, - \rangle$ is the unique system of sesquilinear forms on $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ which are*

(1) non-degenerate, and
(2) if τ is the antiautomorphism defined in (3.22), then $\langle u \cdot v, v' \rangle = \langle v, \tau(u) \cdot v' \rangle$ for any $v, v' \in V_{\underline{\boldsymbol{\lambda}}}$ and $u \in U_q(\mathfrak{g})$; that is, the form is τ-Hermitian, and
(3) the natural map tensoring with a highest weight vector $V_{\lambda_1}^{\mathbb{Z}} \otimes \cdots \otimes V_{\lambda_{\ell-1}}^{\mathbb{Z}} \otimes \{v_{\lambda_\ell}\} \hookrightarrow V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ is an isometric embedding.

Proof. The uniqueness follows by induction on the number of tensor factors. Two τ-hermitian forms on a $U_q^{\mathbb{Z}}(\mathfrak{g})$-module M agree if they agree on a generating subspace M' which is invariant under $U_q^{\geq 0}(\mathfrak{g})$. Since $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{\ell-1}} \otimes \{v_{\lambda_\ell}\} \subset V_{\underline{\boldsymbol{\lambda}}}$ is such a subspace, the uniqueness follows immediately from the inductive hypothesis.

Non-degeneracy follows from the fact that $\Theta^{(\ell)}$ is invertible and the non-degeneracy of the q-Shapovalov form for q generic.

That $\langle -, - \rangle$ is τ-Hermitian follows from the following calculation, where we use the notation $\Delta^{(\ell)}(u)v$ freely in place of $u \cdot v$ to emphasize when we are using the usual coproduct and when we are using its bar-conjugate $\bar{\Delta}^{(\ell)}(u)v$.

$$\begin{aligned}\langle u \cdot v, v' \rangle = \langle \Theta^{(\ell)} \Delta^{(\ell)}(u) v, v' \rangle_{\mathrm{III}} = \langle \bar{\Delta}^{(\ell)}(u) \Theta^{(\ell)} v, v' \rangle_{\mathrm{III}} &= \langle \Theta^{(\ell)} v, (\tau \otimes \cdots \otimes \tau) \bar{\Delta}^{(\ell)}(u) v' \rangle_{\mathrm{III}} \\ &= \langle \Theta^{(\ell)} v, \Delta^{(\ell)}(\tau(u)) v' \rangle_{\mathrm{III}} = \langle v, \tau(u) \cdot v' \rangle.\end{aligned}$$

Above, we use the fact that $\Theta^{(\ell)}$ conjugates the coproduct to the bar-coproduct, that the q-Shapovalov form on a simple is τ-Hermitian, and that τ also conjugates the bar-coproduct to the coproduct.

Statement (3) follows from the fact that $\Theta^{(n)} \in 1 \otimes \cdots \otimes 1 + \sum_i U_q(\mathfrak{g})^{\otimes \ell-1} \otimes U_q(\mathfrak{g}) E_i$, so $\Theta^{(n)}$ fixes $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_{\ell-1}} \otimes v_h$. □

Definition 4.34 *Consider a double Stendhal triple $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$, possibly with divided powers in $\mathbf{i}$. We let $P_{\mathbf{i}}^{\kappa} = e(\mathbf{i}, \kappa) D T^{\underline{\boldsymbol{\lambda}}} e_- = e(\mathbf{i}, \kappa) T^{\underline{\boldsymbol{\lambda}}}$ and $\tilde{P}_{\mathbf{i}}^{\kappa} = e(\mathbf{i}, \kappa) \tilde{T}^{\underline{\boldsymbol{\lambda}}}$.*

Fix a Stendhal triple $(\mathbf{i}, \kappa)$, and $i \in \Gamma$. We'll want to consider a DST $(\mathbf{i}^{(j)}, \kappa^{(j)})$ where we add an upward oriented i-labeled strand right of the jth black strand and a Stendhal triple $(\mathbf{i}_{(j)}, \kappa_{(j)})$ where we remove the $j+1$st strand. More precisely, we consider the (D)STs corresponding to $\mathbf{i}^{(j)} = (-i_1, \dots, -i_j, i, -i_{j+1}, \dots, -i_n)$ and $\kappa^{(j)}(m) = \kappa(m) + \delta_{\kappa(m) \leq j}$, and $\mathbf{i}_{(j)} = (-i_1, \dots, -i_j, -i_{j+2}, \dots, -i_n)$ and $\kappa_{(j)}(m) = \kappa(m) - \delta_{\kappa(m) \leq j+1}$. We let $\mu_{(j)} = \sum_{\kappa(m) < j} \lambda_m - \sum_{k=1}^{j} \alpha_{i_k}$ be the weight of the region right of the jth black strand in the original idempotent. Visually, these correspond to the diagrams

$$e(\mathbf{i}^{(j)},\kappa^{(j)}) = \cdots \underset{i_j}{\downarrow}\ \underset{i}{\uparrow}\ \underset{i_{j+1}}{\downarrow}\ \underset{i_{j+2}}{\downarrow} \cdots \qquad e(\mathbf{i}_{(j)},\kappa_{(j)}) = \cdots \underset{i_j}{\downarrow}\ \underset{i_{j+2}}{\downarrow} \cdots .$$

We've left out red strands from this diagram, but there could be some present. When a red strand separates i_j andi_{j+1}, there is ambiguity in the definition of $e(\mathbf{i}^{(j)},k^{(j)})$, based on whether the new upward strand is to the left or right of the red strand. However, the relation (4.4e) shows the corresponding projectives are isomorphic.

Lemma 4.35 *As right $DT^{\underline{\boldsymbol{\lambda}}}$-modules, we have isomorphisms:*

$$\begin{aligned}
P^{\kappa^{(j)}}_{\mathbf{i}^{(j)}} &\cong P^{\kappa^{(j+1)}}_{\mathbf{i}^{(j+1)}} && i \neq i_{j+1}\\
P^{\kappa^{(j)}}_{\mathbf{i}^{(j)}} \oplus (P^{\kappa_{(j)}}_{\mathbf{i}_{(j)}})^{\oplus[\mu^i_{(j+1)}]_q} &\cong P^{\kappa^{(j+1)}}_{\mathbf{i}^{(j+1)}} && i = i_{j+1}, \mu^i_{(j+1)} \geq 0\\
P^{\kappa^{(j+1)}}_{\mathbf{i}^{(j+1)}} \oplus (P^{\kappa_{(j)}}_{\mathbf{i}_{(j)}})^{\oplus[\mu^i_{(j+1)}]_q} &\cong P^{\kappa^{(j)}}_{\mathbf{i}^{(j)}} && i = i_{j+1}, \mu^i_{(j+1)} \leq 0
\end{aligned}$$

PROOF. This is an immediate consequence of the categorified commutation relations of $\mathcal{E}_i$ and $\mathcal{F}_i$. The DST's $(\mathbf{i}^{(j+1)},\kappa^{(j+1)})$ and $(\mathbf{i}^{(j)},\kappa^{(j)})$ differ by commuting the upward oriented strand labeled i past the $j+1$st black strand, and any red strands with $\kappa(m) = j$. Commuting past red strands is immediate from (4.4e), so we need only deal with commuting past the $j+1$st black strand, in which case the desired isomorphism follows from (2.3c–2.4b) as argued in [**KL10**, 3.25]. $\square$

For any DST $(\mathbf{i},\kappa)$, let $p^\kappa_{\mathbf{i}} \in V_{\underline{\boldsymbol{\lambda}}}$ be defined inductively by:

- if $\kappa(\ell) = n$, then $p^\kappa_{\mathbf{i}} := p^{\kappa^-}_{\mathbf{i}} \otimes v_{\lambda_\ell}$ where, as defined earlier, v_{λ_ℓ} is the highest weight vector of V_{λ_ℓ}, and κ^- is the restriction to $[1,\ell-1]$.
- If $\kappa(\ell) \neq n$, so $p^\kappa_{\mathbf{i}} := E_{i_n} p^\kappa_{\mathbf{i}^-} = F_{-i_n} p^\kappa_{\mathbf{i}^-}$, where $\mathbf{i}^- = (i_1,\dots,i_{n-1})$.

Lemma 4.36 $\dim_q \operatorname{Hom}(P^\kappa_{\mathbf{i}}, P^{\kappa'}_{\mathbf{i}'}) = \langle p^\kappa_{\mathbf{i}}, p^{\kappa'}_{\mathbf{i}'}\rangle$.

PROOF. Note that unless $(\mathbf{i},\kappa)$ and $(\mathbf{i}',\kappa')$ have the same weight $\mathcal{R}$, both sides of the equation are 0; thus we need only consider the case where they have the same weight. As is often true, it's easier to prove a slightly more general result. Thus, we will show that the formula above holds when $(\mathbf{i},\kappa)$ is allowed to be a DST with at most one upward strand. The proof will be by induction on the statement:

$(w_{\mu,j,\ell})$ Lemma 4.36 holds when there are ℓ red strands, when $\mathcal{R} \geq \mu$ and $(\mathbf{i},\kappa)$ and $(\mathbf{i}',\kappa')$ are DSTs with at most one upward strand, which is left of the jth downward strand. Lemma 4.36 also holds in all cases with $< \ell$ red strands.

If $j = 0$, then if there is an upward strand, it comes left all downward strands by definition. Thus, this DST corresponds to a trivial idempotent and $p_{\mathbf{i},\kappa} = 0$. Thus when $j = 0$, we need only consider the case of downward DSTs. In particular, $(w_{\lambda,0,1})$ is simply the fact that $\langle p_{\emptyset,0}, p_{\emptyset,0}\rangle = 1$, and $T^\lambda_\lambda \cong \Bbbk$.

First, we wish to show that $(w_{\mu,j,\ell}) \Rightarrow (w_{\mu,j+1,\ell})$. If neither $(\mathbf{i},\kappa)$ nor $(\mathbf{i}',\kappa')$ have a upward strand in the $j+1$st position, then the formula follows from $(w_{\mu,j,\ell})$. To simplify the proof, let's assume that $(\mathbf{i},\kappa)$ has such a strand and $(\mathbf{i}',\kappa')$ does not; the other cases follow from the same argument. Thus, using the notation of Lemma 4.35, we have that $(\mathbf{i},\kappa)\cong(\mathbf{k}^{(j)},\vartheta^{(j)})$ for some DST $(\mathbf{k},\vartheta)$, with i being the label of the upward strand. The reduction to $w_{\mu,j,\ell}$ follows from the match between Lemma 4.35 and the commutator relation

$$\begin{aligned} E_i(F_{k_j}p^{\kappa}_{(k_1,\dots,k_j)}) &= F_{k_j}(E_i p^{\kappa}_{(k_1,\dots,k_j)}) + [E_i,F_{k_j}]p^{\kappa}_{(k_1,\dots,k_j)} \\ &= F_{k_j}(E_i p^{\kappa}_{(k_1,\dots,k_j)}) + \delta_{i,k_j}\mu^i_{(j)}p^{\kappa}_{(k_1,\dots,k_j)}. \end{aligned}$$

For example, if $i\neq k_j$, then $P^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}}\cong P^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}}$ and $p^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}}=p^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}}$, so

$$\dim_q \operatorname{Hom}(P^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}},P^{\kappa'}_{\mathbf{i}'}) = \dim_q \operatorname{Hom}(P^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}},P^{\kappa'}_{\mathbf{i}'}) = \langle p^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}},p^{\kappa'}_{\mathbf{i}'}\rangle = \langle p^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}},p^{\kappa'}_{\mathbf{i}'}\rangle$$

and the Lemma holds in this case. Similarly, if $i=k_j$ then

$$\begin{aligned} \dim_q \operatorname{Hom}(P^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}},P^{\kappa'}_{\mathbf{i}'}) &= \dim_q \operatorname{Hom}(P^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}},P^{\kappa'}_{\mathbf{i}'}) + [\mu^i_{(j)}]_q \dim_q \operatorname{Hom}(P^{\vartheta_{(j-1)}}_{\mathbf{k}_{(j-1)}},P^{\kappa'}_{\mathbf{i}'}) \\ &= \langle p^{\vartheta^{(j-1)}}_{\mathbf{k}^{(j-1)}},p^{\kappa'}_{\mathbf{i}'}\rangle + [\mu^i_{(j)}]_q \langle p^{\vartheta_{(j-1)}}_{\mathbf{k}_{(j-1)}},p^{\kappa'}_{\mathbf{i}'}\rangle \\ &= \langle p^{\vartheta^{(j)}}_{\mathbf{k}^{(j)}},p^{\kappa'}_{\mathbf{i}'}\rangle. \end{aligned}$$

Now, we wish to establish that $(w_{\mu,0,\ell})$ is implied by $(w_{\mu+\alpha_i,j,\ell})+(w_{\mu-\lambda_\ell,j,\ell-1})$ for all i,j. Assume that in either $\mathbf{i}$ or $\mathbf{i}'$, we have that $\kappa(\ell)<n$, that is, the rightmost strand is black, not red; for simplicity, assume this is the case for $\mathbf{i}$. Then we can use adjunction to write

$$\begin{aligned} \dim_q \operatorname{Hom}(P^{\kappa}_{\mathbf{i}},P^{\kappa'}_{\mathbf{i}'}) &= \dim_q \operatorname{Hom}(P^{\kappa}_{(i_1,\dots,i_{n-1})},\mathfrak{E}_{i_n}P^{\kappa'}_{\mathbf{i}'}) \\ &= \langle p^{\kappa}_{(i_1,\dots,i_{n-1})},E_i p^{\kappa'}_{\mathbf{i}'}\rangle = \langle p^{\kappa}_{\mathbf{i}},p^{\kappa'}_{\mathbf{i}'}\rangle. \end{aligned}$$

In the middle step, we use $(w_{\mu-\alpha_{i_n},n+1,\ell})$.

Finally, we must consider the case where $\kappa(\ell)=\kappa'(\ell)=n$. In this case, we can use Proposition 4.19 to show that

$$\dim_q \operatorname{Hom}(P^{\kappa}_{\mathbf{i}},P^{\kappa'}_{\mathbf{i}'}) = \dim_q \operatorname{Hom}(P^{\kappa^-}_{\mathbf{i}},P^{(\kappa')^-}_{\mathbf{i}'}) = \langle p^{\kappa^-}_{\mathbf{i}},p^{(\kappa')^-}_{\mathbf{i}'}\rangle = \langle p^{\kappa}_{\mathbf{i}},p^{\kappa'}_{\mathbf{i}'}\rangle.$$

In the middle step, this time, we use $(w_{\mu-\lambda_\ell,n,\ell-1})$. $\square$

Lemma 4.37 *The classes $[P^{\kappa}_{\mathbf{i}}]$ span $K_0(T^{\boldsymbol{\lambda}})$ as a $\mathbb{Z}[q,q^{-1}]$ module.*

Proof. Let $K\subseteq K_0(T^{\boldsymbol{\lambda}})$ denote the span of these classes over $\mathbb{Z}[q,q^{-1}]$. We wish to show that the class of any indecomposable projective P is in the span of these classes. As usual, we induct on the number of red lines; the case of one red line follows from [**LV11**, 7.8].

Let $q(P)$ be the minimal integer such that P is a summand of $P^{\kappa}_{\mathbf{i}}$ with $\kappa(\ell)=n-q(P)$; within a fixed number of tensor factors, we will further induct based on this statistic.

If $q(P)=0$, then P is a summand of $P^{\kappa}_{\mathbf{i}}$ with $\kappa(\ell)=n$. In this case P is the image of a module over $T^{\boldsymbol{\lambda}^-}$ under the functor $-\otimes_{T^{\boldsymbol{\lambda}^-}} e_\ell T^{\boldsymbol{\lambda}}$ induced by the isomorphism of Proposition 4.19. Thus, applying the inductive hypothesis to $P^{\kappa}_{\mathbf{i}}e_\ell$ as a module over $T^{\boldsymbol{\lambda}^-}$, we obtain that $[P]\in K$. This covers the case where $q(P)=0$.

Now, we can assume that P is a summand of $u\circ P'$ for $u\in\mathcal{U}$ and P' with $q(P')=0$ which are both indecomposable. Thus, it must be that P' is the image of a primitive idempotent endomorphism e' acting on $P^{\kappa}_{\mathbf{i}'}$ with $\mathbf{i}=(i_1,\dots,i_{\kappa(\ell)})$ and

u the image of a primitive idempotent endomorphism e'' acting on $\mathcal{F}_{i_n}\cdots\mathcal{F}_{i_{\kappa(\ell)+1}}\in\mathcal{U}^-$. Inside $\operatorname{End}(u\circ P')$, there is a 2-sided ideal I of morphisms factoring through projective modules Q with $q(Q)<q(P)$. By Proposition 4.16, any Stendhal diagram with top and bottom given by $(\mathbf{i},\kappa)$ with a black strand that crosses the rightmost red strand can be written as an element of I, plus a correction term with fewer crossings. Thus, the subalgebra A in $\operatorname{End}(u\circ P')$ generated by Stendhal diagrams where no black strand crosses the rightmost red surjects onto $\operatorname{End}(u\circ P')/I$. We have an isomorphism $A\cong e'\operatorname{End}(P^{\kappa}_{\mathbf{i}'})e'\otimes e''\operatorname{End}(\mathcal{F}_{i_n}\cdots\mathcal{F}_{i_{\kappa(\ell)+1}})e''$; since e' and e'' are primitive, the latter is a graded local ring. Thus, $\operatorname{End}(u\circ P')/I$ is again graded local. This implies that $u\circ P'$ has at most one summand H with $q(H)\geq q(P)$. That is, every summand Q of $u\circ P'$ other than P has $q(Q)<q(P)$. Let Q' be the kernel of the projection $u\circ P'\to P$.

Since K is invariant under the action of $U_q^{\mathbb{Z}}(\mathfrak{g})$ by Theorem 4.31, we have that $[u\circ P']\in K$, and by induction $[Q']\in K$. Thus, $[P]=[u\circ P']-[Q']\in K$, and we are done. □

Theorem 4.38 *There is a canonical isomorphism $\eta: K_0(T^{\underline{\boldsymbol{\lambda}}})\to V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$ given by $[P^{\kappa}_{\mathbf{i}}]\mapsto p^{\kappa}_{\mathbf{i}}$ intertwining the inner product defined above with the Euler form.*

Proof. First, we note that by the non-degeneracy of $\langle -,-\rangle$, we can interpret $V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$ as the quotient of the formal span of $p^{\kappa}_{\mathbf{i}}$ over $\mathbb{Z}[q,q^{-1}]$ modulo the kernel of the induced form.

Thus, if we find any other $\mathbb{Z}[q,q^{-1}]$-module W equipped with a bilinear form $\{-,-\}$, generated by elements $q^{\kappa}_{\mathbf{i}}$ such that

$$\{q^{\kappa}_{\mathbf{i}},q^{\kappa'}_{\mathbf{i}'}\}=\langle p^{\kappa}_{\mathbf{i}},p^{\kappa'}_{\mathbf{i}'}\rangle,$$

we immediately have a map $\eta\colon W\to V$ which sends $q^{\kappa}_{\mathbf{i}}\mapsto p^{\kappa}_{\mathbf{i}}$ such that $\{-,-\}=\eta^*\langle -,-\rangle$.

By Lemma 4.36, the Grothendieck group $K_0(T^{\underline{\boldsymbol{\lambda}}})$ and the classes $[P^{\kappa}_{\mathbf{i}}]$ are exactly such a module and set of vectors. Thus, we have a map η as desired, which is surjective.

In order to prove injectivity, we need to show that the rank of $K_0(T^{\underline{\boldsymbol{\lambda}}})$ is no greater than $V^{\mathbb{Z}}_{\underline{\boldsymbol{\lambda}}}$. Again, we induct on the number of tensor factors; we have already established the case where $\ell=1$ in Proposition 3.21.

Thus, by our inductive hypothesis, we can assume that there are precisely $\prod_{j=1}^{\ell-1}\dim V_{\lambda_j}$ indecomposable projectives with $q(P)=0$. Every indecomposable projective P appears in $u\circ Q$ for $q(Q)=0$. As shown in Lemma 4.37, there are unique indecomposable u and Q such that P is the unique summand of $u\circ Q$ with $q(P)=n-\kappa(\ell)$ (that is, the number of black termini in u).

Consider a single index i. To simplify notation, let $m=\lambda^i_\ell$. Note that the algebra $T^{\lambda_\ell}_{(m+1)\alpha_i}=0$, that is, the identity of $\tilde{T}^{\lambda_\ell}_{(m+1)\alpha_i}$ can be written as a sum of violating diagrams. Applying the map $\wp$ to this sum, we can write the idempotent $e(\mathbf{i},\kappa)$ for a Stendhal triple with $\kappa(\ell)=n-m-1$ and $i_{n-m-1}=\cdots=i_n=i$ (that is, it's last black block is $m+1$ instances of i) in terms of diagrams factoring through Stendhal triples with $\kappa(\ell)\geq n-m$. That is, the corresponding projective $P^{\kappa}_{\mathbf{i}}$ is a sum of projective modules P with $q(P)\leq m$.

Now, assume u is a summand of $u'\circ\mathcal{F}_i^{m+1}$, with p as before. As argued above, every summand P of $u\circ Q$ has $q(P)<p$. That is, we may assume that

$u\colon \mu' \to \mu$ is a summand of $\mathcal{F}_{i_n} \cdots \mathcal{F}_{i_{\kappa(\ell)+1}}$ but not a summand of $u' \circ \mathcal{F}_i^{\lambda_\ell^i+1}$ for any index i. Such a 1-morphism is the image of a primitive idempotent e in the KLR algebra $R_{\mu'-\mu}$ whose corresponding simple quotient $L = R_{\mu'-\mu}e/\operatorname{rad}(R_{\mu'-\mu}e)$ satisfies $\operatorname{Hom}(Re(\mathbf{j}), L) = 0$ if $j_1 = \cdots = j_{\lambda_\ell^i+1} = i$ for all i. In the notation of [**LV11**], this is the assertion that $\epsilon_i^*(L) \leq \lambda_\ell^i$. By [**LV11**, 7.8], such simples are in bijection with the crystal of the representation V_{λ_ℓ}, so the number of them is $\dim V_{\lambda_\ell}$.

For every indecomposable projective P, there is a unique u as above and Q with $q(Q) = 0$, such that P is a summand of $u \circ Q$ and every other summand has $q < q(P)$. In particular, no pair u and Q can correspond to two indecomposable projectives, so the number of indecomposable projectives is bounded above by the number of such pairs. By induction, there are $\prod_{j=1}^{\ell-1} \dim V_{\lambda_j}$ indecomposables with $q(Q) = 0$ and $\dim V_{\lambda_\ell}$ such u. Thus, we have that there are no more than $\prod_{j=1}^{\ell} \dim V_{\lambda_j}$ indecomposable projectives, as desired. $\square$

We can easily extend this statement to the category $\tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}} = \tilde{T}^{\underline{\boldsymbol{\lambda}}}$-mod. The $\mathbb{Z}[q, q^{-1}]$-module $U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ has left and right actions of $U_q^{-,\mathbb{Z}}$ given by

$$F_i \cdot (u \otimes w_1 \otimes \cdots \otimes w_\ell) = F_i u \otimes w_1 \otimes \cdots \otimes w_\ell$$
$$(u \otimes w_1 \otimes \cdots \otimes w_\ell) \cdot F_i = uF_i \otimes K_i^{-1}(w_1 \otimes \cdots \otimes w_\ell) + u \otimes F_i(w_1 \otimes \cdots \otimes w_\ell)$$

We can define vectors $\tilde{p}_{\mathbf{i}}^{\kappa}$ defined by the same inductive rules as $p_{\mathbf{i}}^{\kappa}$, except that $p_{\emptyset}^{\emptyset}$ is by definition the generator of the trivial representation, and $\tilde{p}_{\emptyset}^{\emptyset}$ is the element 1 in $U_q^{-,\mathbb{Z}}$. Thus, if $\underline{\boldsymbol{\lambda}} = \emptyset$, then $p_{\emptyset}^{\mathbf{i}} = F_{i_n} \cdots F_{i_1} \in U_q^{-,\mathbb{Z}}$. Let $\tilde{\mathfrak{F}}_i^*, \tilde{\mathfrak{I}}_\lambda^*$ be the conjugates of $\tilde{\mathfrak{F}}_i, \tilde{\mathfrak{I}}_\lambda$ by the algebra reflecting diagrams through a horizontal line (and multiplying each crossing of strands with the same label by -1).

Proposition 4.39 *We have an isomorphism*

$$K_q^0(\tilde{T}^{\underline{\boldsymbol{\lambda}}}) \cong U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}.$$

This isomorphism sends

$$[\tilde{\mathfrak{F}}_i^*](u \otimes w) \mapsto F_i \cdot (u \otimes w) \quad [\tilde{\mathfrak{F}}_i](u \otimes w) \mapsto (u \otimes w) \cdot F_i \tag{4.7a}$$

$$[\tilde{\mathfrak{I}}_\lambda^*](u \otimes w) \mapsto u_{(1)} \otimes (u_{(2)} v_\lambda \otimes w) \quad [\tilde{\mathfrak{I}}_\lambda](u \otimes w) \mapsto u \otimes w \otimes v_\lambda \tag{4.7b}$$

Proof. We hope to find an isomorphism $K_q^0(\tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}}) \to U_q^{-,\mathbb{Z}} \otimes V_{\lambda_1}^{\mathbb{Z}} \otimes \cdots \otimes V_{\lambda_\ell}^{\mathbb{Z}}$ which sends $[P_{\mathbf{i}}^{\kappa}] \mapsto p_{\mathbf{i}}^{\kappa}$. In order to check that such a map exists, we use the fact that both groups have non-degenerate forms which match. For any fixed dominant weight λ_0 with $\lambda_0^i \geq 1$ for all i, we have a functor $\mathfrak{r}_N\colon \tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}} \to \mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}$ given by applying $\tilde{\mathfrak{I}}_\lambda^*$ and then adding the violating relation. This functor is full, and for each degree d and fixed weight space μ, there is a bound $N(d, \mu)$ such that if $N \geq N(d, \mu)$, then this functor is also faithful is degree d. In particular, no projective in $\tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}}$ is killed for all N. This shows that the classes $[P_\kappa^{\mathbf{i}}]$ span the Grothendieck group $K_q^0(\tilde{T}^{\underline{\boldsymbol{\lambda}}})$, since the same is true of $K_q^0(T^{N\lambda_0, \underline{\boldsymbol{\lambda}}})$.

Furthermore, on the level of Euler forms, we have

$$\langle [M], [M'] \rangle_{\tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}}} = \lim_{N \to \infty} \langle [\mathfrak{r}_N M], [\mathfrak{r}_N M'] \rangle_{\mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}}$$

where the convergence is in power series with the q-adic topology. For any weight vector $m \in K_q^0(\tilde{T}^{\underline{\boldsymbol{\lambda}}})$, we can consider the minimal degree of a non-vanishing term of $\langle \mathfrak{r}_N m, [\mathfrak{r}_N M'] \rangle_{\mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}}$ for any fixed M'. This valuation is is bounded above as N varies, since each weight space is finite rank over $\mathbb{Z}[q, q^{-1}]$. Since the classes $[P_{\mathbf{i}}^{\kappa}]$ span $K_q^0(\tilde{T}^{\underline{\boldsymbol{\lambda}}})$, we must have that $\langle \mathfrak{r}_N m, [\mathfrak{r}_N P_{\mathbf{i}}^{\kappa}] \rangle_{\mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}} \neq 0$ for some $\mathbf{i}, \kappa$ for each N. While $\mathbf{i}, \kappa$ might depend on N, since there are finitely many options, there is at least one that gives a non-zero answer for infinitely many N. The upper bound on valuation shows that the limit $\lim_{N\to\infty} \langle \mathfrak{r}_N m, [\mathfrak{r}_N P_{\mathbf{i}}^{\kappa}] \rangle_{\mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}} \neq 0$ as well. This form is thus non-degenerate.

Similarly, $U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ is endowed with a form defined a similar limit. Let $\mathfrak{q}_N : U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} \to V_{N\lambda_0, \underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ such that $\mathfrak{q}_N(u \otimes w) = uv_{N\lambda_0} \otimes w$. We define a form by

$$\langle u \otimes w, u' \otimes w' \rangle_{U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}} = \lim_{N\to\infty} \langle \mathfrak{q}_N(u \otimes w), \mathfrak{q}_N(u' \otimes w') \rangle_{V_{N\lambda_0, \underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}}$$

where the form on $V_{N\lambda_0, \underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ is that given in Theorem 4.33. A similar argument gives the non-degeneracy of this form.

By Theorem 4.38, we have an isomorphism $V_{N\lambda_0, \underline{\boldsymbol{\lambda}}}^{\mathbb{Z}} \cong K_q^0(T^{N\lambda_0, \underline{\boldsymbol{\lambda}}})$ of free $\mathbb{Z}[q, q^{-1}]$ modules endowed with sesquilinear forms such that $[\mathfrak{r}_N P_{\mathbf{i}}^{\kappa}] \mapsto \mathfrak{q}_N p_{\mathbf{i}}^{\kappa}$. Thus, we have that:

$$\begin{aligned}(4.8)\quad \langle [P_{\mathbf{i}}^{\kappa}], [P_{\mathbf{i}'}^{\kappa'}] \rangle_{\tilde{\mathfrak{V}}^{\underline{\boldsymbol{\lambda}}}} &= \lim_{N\to\infty} \langle [\mathfrak{r}_N P_{\mathbf{i}}^{\kappa}], [\mathfrak{r}_N P_{\mathbf{i}'}^{\kappa'}] \rangle_{\mathfrak{V}^{N\lambda_0, \underline{\boldsymbol{\lambda}}}} \\ &= \lim_{N\to\infty} \langle \mathfrak{q}_N p_{\mathbf{i}}^{\kappa}, \mathfrak{q}_N p_{\mathbf{i}'}^{\kappa'} \rangle_{V_{N\lambda_0, \underline{\boldsymbol{\lambda}}}} = \langle p_{\mathbf{i}}^{\kappa}, p_{\mathbf{i}'}^{\kappa'} \rangle_{U_q^- \otimes V_{\underline{\boldsymbol{\lambda}}}}\end{aligned}$$

As in the proof of Theorem 4.38, we can view $K_q^0(\tilde{T}^{\underline{\boldsymbol{\lambda}}})$ and $U_q^{-,\mathbb{Z}} \otimes V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ as quotients of the free span of $[P_{\mathbf{i}}^{\kappa}]$ and $p_{\mathbf{i}}^{\kappa}$ by the kernel of these forms, so (4.8) shows that we have the desired isomorphism. Compatibility with $\tilde{\mathfrak{F}}_i$ and $\tilde{\mathfrak{I}}_\lambda$ is obvious. The functors $\tilde{\mathfrak{F}}_i^*$ and $\tilde{\mathfrak{I}}_\lambda^*$ commute with $\tilde{\mathfrak{F}}_i$ and $\tilde{\mathfrak{I}}_\lambda$, and similarly for the maps we intend to match them with in equations (4.7a–4.7b). Thus, need only check that they give the right answer when acting on $P_\emptyset$, which is clear. $\square$

CHAPTER 5

Standard modules

1. Standard modules defined

When analyzing the structure of representation-theoretic categories, such as the categories $\mathcal{O}$ appearing in Stroppel's construction of Khovanov homology [**Str09**], a crucial role is played by the Verma modules and their analogues. The property of "having objects like Verma modules" was formalized by Cline-Parshall-Scott as the property of being **quasi-hereditary** [**CPS88**]. Unfortunately, this is too strong of an assumption for us; as we noted earlier, the cyclotomic QHA is Frobenius, and thus very far from being quasi-hereditary (any ring which is both Frobenius and quasi-hereditary is semi-simple).

Luckily, our categories satisfy a weaker condition: they are **standardly stratified**, as defined by the same authors [**CPS96**]. To show this, we must construct a collection of modules which are called **standard**, and show that projectives have a filtration by these modules compatible with a preorder.

From another perspective, given the isomorphism between $K^0(T^{\underline{\boldsymbol{\lambda}}}) \cong V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$, it is natural to expect that pure tensors in $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$ correspond to modules, and that things like the definition of the coproducts

$$\begin{aligned}(5.1)\quad \Delta^{(\ell)}(E_i) = E_i \otimes 1 \otimes \cdots \otimes 1 + \tilde{K}_i \otimes E_i \otimes 1 \otimes \cdots \otimes 1 + \cdots + \\ \tilde{K}_i \otimes \cdots \otimes \tilde{K}_i \otimes E_i \otimes 1 + \tilde{K}_i \otimes \cdots \otimes \tilde{K}_i \otimes E_i.\end{aligned}$$

$$\begin{aligned}(5.2)\quad \Delta^{(\ell)}(F_i) = F_i \otimes \tilde{K}_{-i} \otimes \cdots \otimes \tilde{K}_{-i} + 1 \otimes F_i \otimes \tilde{K}_{-i} \otimes \cdots \otimes \tilde{K}_{-i} + \cdots + \\ 1 \otimes \cdots \otimes 1 \otimes F_i \otimes \tilde{K}_{-i} + 1 \otimes \cdots \otimes 1 \otimes F_i.\end{aligned}$$

will have a categorical interpretation. Standard modules are the key to both these questions.

We define a preorder on Stendhal triples $(\mathbf{i}, \kappa)$'s by $(\mathbf{i}, \kappa) \leq (\mathbf{i}', \kappa')$ if

$$\sum_{k \leq \kappa(j)} \alpha_{i_k} \leq \sum_{k \leq \kappa'(j)} \alpha_{i'_k} \qquad \text{for all } j \in [1, \ell].$$

Since there is a danger of sign confusion, let us emphasize that we are summing *positive* roots here, since we are using the sign conventions of a Stendhal triple. Put more informally, one gets higher in this order as black strands move left and red strands move right.

This preorder can be packaged as the dominance order for a function $\boldsymbol{\alpha}_{\mathbf{i},\kappa} \colon [0, \ell] \to X(\mathfrak{g})$ which we call a **root function** given by

$$\boldsymbol{\alpha}_{\mathbf{i},\kappa}(k) = \sum_{\kappa(k-1) < j \leq \kappa(k)} \alpha_{i_j}.$$

Note that this preorder is entirely insensitive to permutations of the black strands which do not cross any red strands.

Definition 5.1 *Let $U_{\mathbf{i}}^{\kappa} \subset \tilde{P}_{\mathbf{i}}^{\kappa}$ be the submodule generated by the image of all maps $\tilde{P}_{\mathbf{i}'}^{\kappa'} \to \tilde{P}_{\mathbf{i}}^{\kappa}$ with $(\mathbf{i}', \kappa') \geq (\mathbf{i}, \kappa)$. We define $S_{\mathbf{i}}^{\kappa} = \tilde{P}_{\mathbf{i}}^{\kappa}/U_{\mathbf{i}}^{\kappa}$ to be the* **standard module** *for κ and $\mathbf{i}$.*

If $\kappa(1) = 0$, then the action of $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ on $S_{\mathbf{i}}^{\kappa}$ factors through the natural map $\tilde{T}^{\underline{\boldsymbol{\lambda}}} \to T^{\underline{\boldsymbol{\lambda}}}$, and we will typically consider $S_{\mathbf{i}}^{\kappa}$ as a module over this smaller algebra.

Recall that according to our conventions, elements of the algebra $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ act at the bottom of the diagram. Thus, the submodule $U_{\mathbf{i}}^{\kappa}$ is the span of all diagrams where the slice at the top is given by $(\mathbf{i}, \kappa)$ and somewhere in the middle of the diagram is given by $(\mathbf{i}', \kappa') \geq (\mathbf{i}, \kappa)$.

By convention, we call a red/black crossing where black strands go from NW to SE **left** and the mirror image of such a crossing **right**. Note that this terminology does not apply to black/black crossings; if we call a crossing left or right we are implicitly assuming it is black/red.

a "left" crossing

a "right" crossing

(5.3)

We can alternatively define $U_{\mathbf{i}}^{\kappa}$ as the submodule generated by all diagrams with at least one right crossing and no left crossings.

Definition 5.2 *We will call a black strand that makes a right crossing above all left crossings* **standardly violating***, and a diagram containing such a strand* **standardly violated**.

Let $e_{\boldsymbol{\alpha}}$ be the idempotent which is 1 on projectives $P_{\mathbf{i}}^{\kappa}$ with root function $\boldsymbol{\alpha}_{\mathbf{i},\kappa} = \boldsymbol{\alpha}$. We let $S_{\boldsymbol{\alpha}}$ be the standard quotient of the projective $e_{\boldsymbol{\alpha}} T^{\underline{\boldsymbol{\lambda}}}$, that is, its quotient by the submodule generated by the image of all maps $P_{\mathbf{i}'}^{\kappa'} \to e_{\boldsymbol{\alpha}} T^{\underline{\boldsymbol{\lambda}}}$ with $\boldsymbol{\alpha}_{\mathbf{i}',\kappa'} > \boldsymbol{\alpha}$. Recall that we have a map $\wp_{\boldsymbol{\alpha}} \colon R_{\boldsymbol{\alpha}(1)} \otimes \cdots \otimes R_{\boldsymbol{\alpha}(\ell)} \to e_{\boldsymbol{\alpha}} \tilde{T}^{\underline{\boldsymbol{\lambda}}} e_{\boldsymbol{\alpha}}$ defined in Chapter 3. Let $\mu_i = \lambda_i - \boldsymbol{\alpha}(i)$.

Proposition 5.3 *The map $\wp_{\boldsymbol{\alpha}}$ induces an algebra map*

$$R_{\boldsymbol{\alpha}(0)} \otimes T_{\mu_1}^{\lambda_1} \otimes \cdots \otimes T_{\mu_\ell}^{\lambda_\ell} \to \operatorname{End}_{\tilde{T}^{\underline{\boldsymbol{\lambda}}}}(S_{\boldsymbol{\alpha}}).$$

Proof. First, we note that left (top) multiplication by $\wp_{\boldsymbol{\alpha}}$ induces an action of $R_{\boldsymbol{\alpha}(0)} \otimes R_{\boldsymbol{\alpha}(1)} \otimes \cdots \otimes R_{\boldsymbol{\alpha}(\ell)}$ on $e_{\boldsymbol{\alpha}} T^{\underline{\boldsymbol{\lambda}}}$. This further induces an action on $S_{\boldsymbol{\alpha}}$, since the elements of $\wp_{\boldsymbol{\alpha}}$ only rearrange strands within black blocks. Let $U_{\boldsymbol{\alpha}}$ be the sum of the submodules $U_{\mathbf{i}}^{\kappa}$ with $\boldsymbol{\alpha}(\mathbf{i}, \kappa) = \boldsymbol{\alpha}$. The map $\wp_{\boldsymbol{\alpha}}(r)$ must send $U_{\boldsymbol{\alpha}}$ to itself, since a map from $P_{\mathbf{i}'}^{\kappa'}$ composed with $\wp_{\boldsymbol{\alpha}}(r)$ is still a map from a higher projective and thus in $U_{\boldsymbol{\alpha}}$.

It follows that we have a map $R_{\boldsymbol{\alpha}(0)} \otimes R_{\boldsymbol{\alpha}(1)} \otimes \cdots \otimes R_{\boldsymbol{\alpha}(\ell)} \to \operatorname{End}_{T^{\underline{\boldsymbol{\lambda}}}}(S_{\boldsymbol{\alpha}})$. Furthermore, consider r in the span of $R_{\boldsymbol{\alpha}(0)} \otimes R_{\boldsymbol{\alpha}(1)} \otimes \cdots \otimes I_{\lambda_i} \otimes \cdots \otimes R_{\boldsymbol{\alpha}(\ell)}$ for $i = 1, \ldots, \ell$, where $I_{\lambda_i} \subset R_{\boldsymbol{\alpha}(i)}$ is the cyclotomic ideal of corresponding to λ_i. In this case, r can be written in $T^{\underline{\boldsymbol{\lambda}}}$ as elements factoring through a higher projective by Theorem 4.18. In this case r will send the entirety of $P_{\boldsymbol{\alpha}}$ to $U_{\boldsymbol{\alpha}}$, and thus acts trivially on $S_{\boldsymbol{\alpha}}$. It follows that we have the desired induced action. □

Thus, we can think of $S_{\boldsymbol{\alpha}}$ as a $R_{\boldsymbol{\alpha}(0)} \otimes T^{\lambda_1}_{\mu_1} \otimes \cdots \otimes T^{\lambda_\ell}_{\mu_\ell} - \tilde{T}^{\underline{\boldsymbol{\lambda}}}_{\alpha}$-bimodule, and $S = \oplus_{\boldsymbol{\alpha}} S_{\boldsymbol{\alpha}}$ as a $R \otimes T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell} - \tilde{T}^{\underline{\boldsymbol{\lambda}}}$-bimodule. Let

$$\mathfrak{V}^{\infty;\lambda_1;\dots;\lambda_\ell} := R \otimes T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell}\text{-mod} \qquad \mathfrak{V}^{\lambda_1;\dots;\lambda_\ell} := T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell}\text{-mod}$$

Definition 5.4 *The* **standardization functor** *is the tensor product with this bimodule:*

$$\mathbb{S}^{\underline{\boldsymbol{\lambda}}} \colon \mathfrak{V}^{\infty;\lambda_1;\dots;\lambda_\ell} \to \tilde{T}^{\underline{\boldsymbol{\lambda}}}\text{-mod} \qquad \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(-) = - \otimes_{R \otimes T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_\ell}} S$$

Note that if we restrict to sequences where $\boldsymbol{\alpha}(0) = 0$, then we can view this as a functor $\mathbb{S}^{\underline{\boldsymbol{\lambda}}} \colon \mathfrak{V}^{\lambda_1;\dots;\lambda_\ell} \to \mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$.

More generally, we can construct partial standardization modules, where we only kill the right crossings for some of the red strands. This will give us a standardization functor

$$\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m} : \mathfrak{V}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_\ell} \to \mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$$

for any list of sequences $\underline{\boldsymbol{\lambda}}_1, \dots, \underline{\boldsymbol{\lambda}}_m$ such that the concatenation $\underline{\boldsymbol{\lambda}}_1 \cdots \underline{\boldsymbol{\lambda}}_m$ is equal to $\underline{\boldsymbol{\lambda}}$.

We've already seen one example of these functors. For any dominant weight μ, we can rewrite the functor $\mathfrak{I}_\mu$ defined in Definition 4.20 as the standardization functor $\mathfrak{I}_\mu(M) = \mathbb{S}^{\underline{\boldsymbol{\lambda}};(\mu)}(M \boxtimes P_\emptyset)$. This categorifies the inclusion of $V_{\underline{\boldsymbol{\lambda}}} \otimes \{v_{high}\} \hookrightarrow V_{\underline{\boldsymbol{\lambda}}} \otimes V_\mu$. This map is not a map of $\mathfrak{g}$-representations, though we will discuss the interaction of standardization functors with the categorical $\mathfrak{g}$-action below.

The category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}$ has a categorical action of $\mathfrak{g}^{\oplus m}$ by functors we denote ${}_k\mathfrak{E}_i$ and ${}_k\mathfrak{F}_i$ which act only on the kth factor. That is:

$${}_k\mathfrak{E}_i(\cdots \boxtimes M_{k-1} \boxtimes M_k \boxtimes M_{k+1} \boxtimes \cdots) \cong \cdots \boxtimes M_{k-1} \boxtimes \mathfrak{E}_i M_k \boxtimes M_{k+1} \boxtimes \cdots .$$

These actions are compatible with the action via $\mathfrak{E}_i, \mathfrak{F}_i$ on $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ as follows:

Proposition 5.5 *For any $T^{\underline{\boldsymbol{\lambda}}_1}_{\mu_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_m}_{\mu_\ell}$-module M, the module $\mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}(M)$ has a natural filtration $Q_1 \supset Q_2 \supset \cdots$ such that*

$$Q_k/Q_{k+1} \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}({}_k\mathfrak{E}_i M)\left(\sum_{j=1}^{k-1} \langle \alpha_i, \lambda_j - \boldsymbol{\alpha}(j)\rangle\right).$$

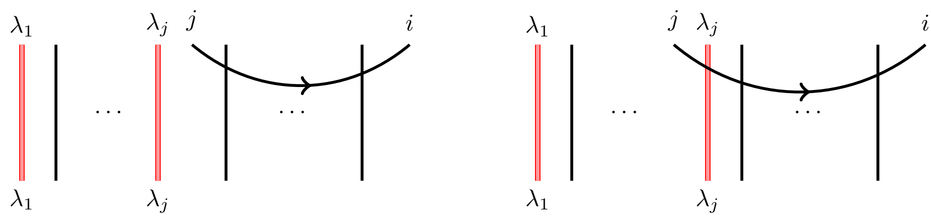
The module $\mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}(M)$ has a natural filtration $O_m \supset O_{m-1} \supset \cdots$ such that

$$O_k/O_{k-1} \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}({}_k\mathfrak{F}_i M)\left(-\sum_{j=k+1}^{k} \langle \alpha_i, \lambda_j - \boldsymbol{\alpha}(j)\rangle\right).$$

These filtrations are precisely the categorification of the coproducts (5.1) and (5.2).

Proof. We can easily reduce from the general case to the case where there are two tensor factors. For any sequence $(\underline{\boldsymbol{\lambda}}_1; \dots; \underline{\boldsymbol{\lambda}}_m)$, we first apply the two term result for $(\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2 \cdots \underline{\boldsymbol{\lambda}}_m)$ and then on $(\underline{\boldsymbol{\lambda}}_2; \underline{\boldsymbol{\lambda}}_3 \cdots \underline{\boldsymbol{\lambda}}_m)$, and so on. Thus, throughout, we'll assume that $m = 2$ and $\underline{\boldsymbol{\lambda}}_1 = (\lambda_1, \dots, \lambda_{j-1}), \underline{\boldsymbol{\lambda}}_2 = (\lambda_j, \dots, \lambda_\ell)$.

First, consider $\mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}(M)$. Let $Q(M)$ be the submodule of diagrams where the strand forming the unique cup stays to the right of the jth red strand. One can easily check that this is a subfunctor of $\mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}$. In the diagrams below, the left-hand diagram is in $Q(M)$ and the right-hand is not (or at least this is not clear from how it is written).



For any $M \in \mathfrak{V}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}$, we have a natural transformation $\gamma_2 \colon \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}({}_2\mathfrak{E}_i M) \to Q(M)$ where we take a diagram in the former module and think of it in the latter.

One can think of this as a map from of bimodules $\gamma_2 \colon (T^{\underline{\boldsymbol{\lambda}}_1} \boxtimes \beta_{\mathcal{E}_i}) \otimes_{T^{\underline{\boldsymbol{\lambda}}_1} \otimes T^{\underline{\boldsymbol{\lambda}}_2}} S \to S \otimes_{T^{\underline{\boldsymbol{\lambda}}}} \beta_{\mathcal{E}_i}$, where again, the inclusion is just isotopy of diagrams. Let c_i be the diagram just making a cup between the only upward terminal, and the ith downward terminal from the right. Every element of $(T^{\underline{\boldsymbol{\lambda}}_1} \boxtimes \beta_{\mathcal{E}_i}) \otimes S$ can be written as a sum of diagrams of the form $c_i \otimes a$ where a is an element of S; in this case, $\gamma_1((1 \boxtimes c_i) \otimes a) = 1 \otimes c_i a$. This is well-defined by the usual locality of relations, but not obviously injective.

Note, however, that this map is not grading preserving. The degree of the cup will increase by $\langle \alpha_i, \lambda_1 - \boldsymbol{\alpha}(1) \rangle$, since we must change the labeling of regions in the diagram.

Dually, we have a natural transformation $\gamma_1 : \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}({}_1\mathfrak{E}_i M) \to \mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}(M)/Q$. One can think of this as a map of bimodules $(\beta_{\mathcal{E}_i} \boxtimes T^{\underline{\boldsymbol{\lambda}}_2}) \otimes_{T^{\underline{\boldsymbol{\lambda}}_1} \otimes T^{\underline{\boldsymbol{\lambda}}_2}} S \to (S \otimes_{T^{\underline{\boldsymbol{\lambda}}}} \beta_{\mathcal{E}_i}) / \operatorname{im}(\gamma_1)$. This maps

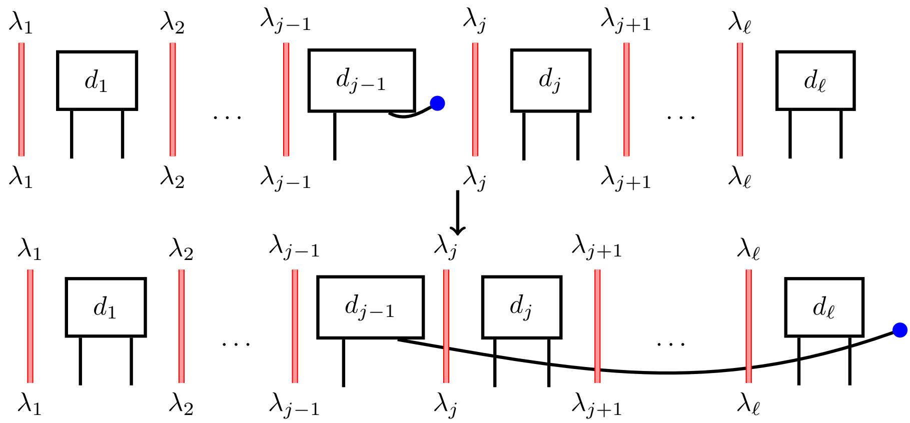
$$\gamma_1((c_i \boxtimes 1) \otimes a) = 1 \otimes c_{i + \rho^\vee(\boldsymbol{\alpha}(2))} a.$$

This map is only well-defined modulo the image $\operatorname{im}(\gamma_\rho^\vee(\boldsymbol{\alpha}(2)))$ since when need to move a dot or crossing past the cup c_{i+n}, the equations (2.5c,2.5g) show that two representations of the same element will differ by diagrams of the form $\gamma_1((1 \boxtimes c_b) \otimes a)$ for $b < n$.

Note that this map is surjective, since the module $\mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1; \underline{\boldsymbol{\lambda}}_2}(M)/Q$ is spanned by elements of the form $(1 \boxtimes e(\mathbf{j}, \kappa)) \otimes c_{i+n} a$, which is in the image of γ_1.

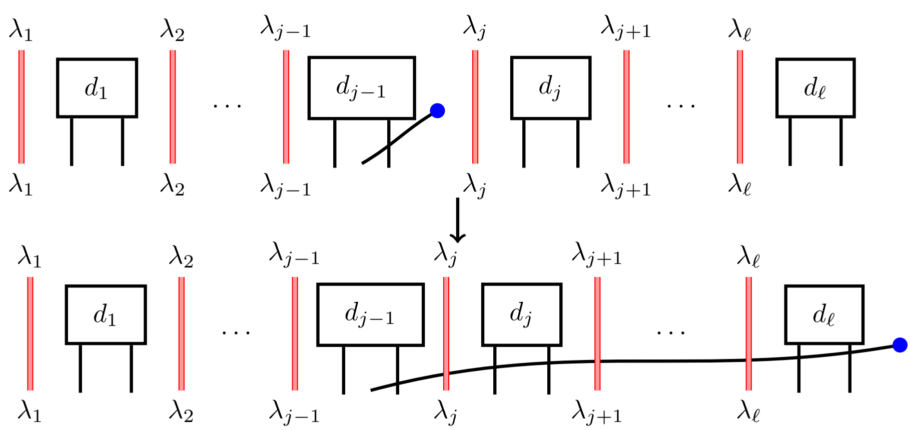
The map γ_1 is shown in (5.4). In each, case, the diagram we have shown would be acting on an element of M as in Figure 1. For the legibility of the pictures, we have not shown these elements.

(5.4)



We turn to the module $\mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}(M)$. This has a submodule O generated by the diagram g where the "new" strand at the far right is pulled to the spot left of the jth red strand with no other crossings or dots. Much like the case of $\mathfrak{E}_i$, we have a map $\delta_1 \colon \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_1\mathfrak{F}_i M) \to O$ of degree $-\langle \alpha_i, \lambda_2 - \boldsymbol{\alpha}(2)\rangle$. As in the case of $\mathcal{E}$, this can be described as a bimodule map $(\beta_{\mathcal{F}_i} \boxtimes T^{\underline{\boldsymbol{\lambda}}_2}) \otimes S \to S \otimes \beta_{\mathcal{F}_i}$ which sends a diagram $1 \otimes a \to 1 \otimes ga$. This map is shown in (5.5). Note that in the course of this proof, we will draw several diagrams representing elements of functors applied to M.

(5.5)



Dually, we have a map $\delta_2 \colon \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_2\mathfrak{F}_i M) \to \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}(M)/O$. This sends a diagram to the same underlying diagram. As with γ_1, this isn't well-defined as a map to $\mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}(M)$ since diagrams where the new strand ${}_2\mathfrak{F}_i$ adds is violating aren't sent to elements of the violating ideal. However, such a diagram does land in O, so the map to the quotient is well-defined.

Thus, in order to finish the proof, we must prove that the maps γ_k, δ_k are isomorphisms. Since the maps γ_k and δ_k are surjections, suffices to check that the dimensions of the source and target coincide. That is, it suffices to prove for any projective that

$$\dim \operatorname{Hom}(P, \mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}(M)) = \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_1\mathfrak{E}_i M)) + \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_2\mathfrak{E}_i M)) \tag{5.6}$$

$$\dim \operatorname{Hom}(P, \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}(M)) = \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_1\mathfrak{F}_i M)) + \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2}({}_2\mathfrak{F}_i M)). \tag{5.7}$$

Surjectivity implies that in both (5.6) and (5.7), the LHS must be $\leq$ the RHS.

We'll induct on ℓ, and on the weight of the module P. More precisely, our inductive hypothesis will be that

$f_{(\mu,\ell)}$ For all i, The equation (5.6) holds for any P projective over $T^{\underline{\boldsymbol{\lambda}}}_{\mu+\alpha_i}$, and the equation (5.7) holds for any P projective over $T^{\underline{\boldsymbol{\lambda}}}_{\mu}$.

For our induction, we prove $f_{(\mu,\ell)}$ assuming $f_{(\nu,k)}$ holds if $k < \ell$ or $\ell = k$ and $\nu > \mu$. When $\ell = 1$ the equations (5.6) and (5.7) are tautological, and for $\mu = \lambda$, the module P in (5.6) can only be the trivial module, and similarly for the module M in (5.7), so this establishes the base case.

Obviously, if either (5.6) or (5.7) fails for a projective P, it will still fail if P is replaced by its sum with any other projective module, and it must fail for some indecomposable summand of P. Similarly, since Hom with a projective is exact, the formulas (5.6) or (5.7) hold for M if and only if they hold for all its composition factors. Thus, we can assume that either $P = \mathfrak{I}_{\lambda_\ell} P'$, or that $P = \mathfrak{F}_j P'$ for some j and some other projective P'.

In the former case, we can assume without loss of generality that $M = {}_m\mathfrak{I}_{\lambda_\ell} M'$ for some M', since any simple which is not a composition factor of such a module has $\operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\underline{\boldsymbol{\lambda}}_2} L) = 0$. By Proposition 4.32, we have that:

$$\begin{aligned}
\dim \operatorname{Hom}(P, \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}(M)) &= \sum \dim \operatorname{Hom}(\mathfrak{I}_{\lambda_\ell} P', \mathfrak{F}_i \mathfrak{I}_{\lambda_\ell} \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m^-}(M')) \\
&= \sum \dim \operatorname{Hom}(P', \mathfrak{I}^R_{\lambda_\ell} \mathfrak{F}_i \mathfrak{I}_{\lambda_\ell} \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m^-}(M')) \\
&= \sum \dim \operatorname{Hom}(P', \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m^-}(M')) \\
&= \sum \dim \operatorname{Hom}(P', \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m^-}({}_k\mathfrak{F}_i M')) \\
&= \sum \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}({}_k\mathfrak{F}_i M))
\end{aligned}$$

applying the inductive hypothesis $f_{(\mu-\lambda_\ell,\ell-1)}$. This establishes (5.7) and (5.6) follows by a similar argument.

Thus, we may assume that $P = \mathfrak{F}_j P'$ for some j. In this case, we can apply the adjunction to show that

$$\dim \operatorname{Hom}(\mathfrak{F}_j P', \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}(M)) = \dim \operatorname{Hom}(P', \mathfrak{E}_j \mathfrak{F}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}(M)) \tag{5.8}$$

$$\leq \sum \dim \operatorname{Hom}(P, \mathfrak{E}_i \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}({}_k\mathfrak{F}_i M) \tag{5.9}$$

$$\leq \sum \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}({}_p\mathfrak{E}_j \circ {}_k\mathfrak{F}_i M) \tag{5.10}$$

where (5.9) and (5.10) follow from the inequality LHS $\leq$ RHS in (5.7) and (5.6) respectively. Applying the commutation relations in $\mathcal{U}$, we find that this implies that

$$\operatorname{Hom}(P', \mathfrak{F}_i \mathfrak{E}_j \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}(M)) \leq \sum \dim \operatorname{Hom}(P, \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}({}_k\mathfrak{F}_i \circ {}_p\mathfrak{E}_j M) \tag{5.11}$$

with equality if and only if both steps (5.9) and (5.10) are equalities. On the other hand, the inductive hypothesis $f_{(\mu+\alpha_j,\ell)}$ implies that (5.11) is an equality, by applying (5.6) to $\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}(M)$ and then (5.7) to $\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m}({}_p\mathfrak{E}_j M)$.

Thus, we must have that (5.9) is an equality, which shows (5.7) for $P = \mathfrak{F}_j P'$ and M arbitrary. This establishes the second half of $f_{(\mu,\ell)}$ in complete generality. On the other hand, we also know that (5.10) is an equality. This establishes (5.6) for P arbitrary, and M any composition factor of ${}_k\mathfrak{F}_i M'$ with M' arbitrary.

Thus it only remains to establish (5.6) when M is a simple module which receives no maps from ${}_k\mathfrak{F}_i M'$ for any i. In this case, adjunction shows that ${}_k\mathfrak{E}_i M = 0$ for every k and i. This shows that the right hand side of (5.6) is 0, so the equation must hold. Thus, the result is proven. $\square$

We let $s_{\mathbf{i}}^{\kappa} = F_{i_{\kappa(2)}} \cdots F_{i_1} p_1 \otimes \cdots \otimes F_{i_n} \cdots F_{i_{\kappa(\ell)+1}} p_\ell$.

Proposition 5.6 $\eta([S_{\mathbf{i}}^{\kappa}]) = s_{\mathbf{i}}^{\kappa}$.

Proof. We'll induct on ℓ and on the height of $(\mathbf{i}, \kappa)$ in our preorder. For $\ell = 1$, this is simply the statement of Proposition 4.38. This establishes the base case.

Now, assume that $\kappa(\ell) = n$; in this case, we can assume that $\eta([S_{\mathbf{i}}^{\kappa^-}]) = s_{\mathbf{i}}^{\kappa^-}$ by the inductive hypothesis. The class of $S_{\mathbf{i}}^{\kappa} = \mathfrak{I}_{\lambda_\ell}(S_{\mathbf{i}}^{\kappa^-})$ is thus $s_{\mathbf{i}}^{\kappa^-} \otimes p_\ell = s_{\mathbf{i}}^{\kappa}$ by definition.

Thus, we may assume that $\kappa(\ell) < n$. We let $\mathbf{i}_k$ and κ_k be the sequence $\mathbf{i}$ with i_n moved from the end of the sequence to the end of the kth black block (so $\mathbf{i}_\ell = \mathbf{i}$), and the function κ changed appropriately, that is, with 1 added to its values above k. By Proposition 5.5 we see that the kernel of the surjection $\mathfrak{F}_{i_n} S_{\mathbf{i}^-}^{\kappa} \to S_{\mathbf{i}}^{\kappa}$ has a filtration by the standard modules $S_{\mathbf{i}_k}^{\kappa_k}$ for $k = 1, \ldots, \ell - 1$. Thus, we have that

$$\begin{aligned}
[S_{\mathbf{i}}^{\kappa}] &= [\mathfrak{F}_{i_n} S_{\mathbf{i}^-}^{\kappa}] - \sum_{k=1}^{\ell-1} q^{\alpha_i^\vee(\lambda_{k+1}+\cdots+\lambda_\ell)} [S_{\mathbf{i}_k}^{\kappa_k}] \\
&= \Delta^{(n)}(F_i) s_{\mathbf{i}^-}^{\kappa} - \sum_{k=1}^{\ell-1} q^{\alpha_i^\vee(\lambda_{k+1}+\cdots+\lambda_\ell)} s_{\mathbf{i}_k}^{\kappa_k} \\
&= \Delta^{(n)}(F_i) s_{\mathbf{i}^-}^{\kappa} - \sum_{k=1}^{\ell-1} (1 \otimes \cdots \otimes 1 \otimes F_i \otimes \tilde{K}_{-i} \otimes \cdots \otimes \tilde{K}_{-i}) s_{\mathbf{i}^-}^{\kappa} \\
&= (1 \otimes \cdots \otimes 1 \otimes F_i) s_{\mathbf{i}^-}^{\kappa} \\
&= s_{\mathbf{i}}^{\kappa}
\end{aligned}$$

$\square$

This result also shows the exactness of the standardization functor:

Proposition 5.7 *The standardization functor* $\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_m} : \mathfrak{V}^{\underline{\lambda}_1;\cdots;\underline{\lambda}_\ell} \to \mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ *is exact.*

Proof. Note that we need only consider the case where $m = \ell$ and $\underline{\boldsymbol{\lambda}}_i = (\lambda_i)$. We induct as in the proof of Theorem 4.38 on n and ℓ. It suffices to prove that $\mathrm{Hom}(P_{\mathbf{i}}^{\kappa}, \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(-))$ is always exact since every indecomposable projective is a summand of $P_{\mathbf{i}}^{\kappa}$.

Unless $n = \kappa(\ell)$, the projective $P_{\mathbf{i}}^{\kappa}$ is a sum of summands of modules of the form $\mathfrak{F}_i(P')$. Thus, we can use the adjunction

$$\operatorname{Hom}(\mathfrak{F}_i(P'), \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(-)) \cong \operatorname{Hom}(P', \mathfrak{E}_i\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(-)).$$

By Proposition 5.5, $\mathfrak{E}_i\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(M)$ is filtered by the modules $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}({}_j\mathfrak{E}_i M)$ where ${}_j\mathfrak{E}_i$ is the categorification functor applied in the jth tensor factor. By induction, we have that $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}({}_j\mathfrak{E}_i(-))$ is exact, so this establishes this induction step.

If $n = \kappa(\ell)$, then $\operatorname{Hom}(P_{\mathbf{i}}^{\kappa}, \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(M))$ is the same as $\operatorname{Hom}(P_{\mathbf{i}}^{\kappa^-}, \mathbb{S}^{(\lambda_1,\dots,\lambda_{\ell-1})}(M^+))$ where M^+ is the $T^{\lambda_1} \otimes \cdots \otimes T^{\lambda_{\ell-1}}$ submodule in M where the weight for T^{λ_ℓ} is λ_ℓ. Since $M \mapsto M^+$ is exact (it is the projection of a sum of idempotents), by induction $M \mapsto \mathbb{S}^{(\lambda_1,\dots,\lambda_{\ell-1})}(M^+)$ is exact as well. This completes the induction step, and thus the proof. □

2. Simple modules and crystals

Lauda and Vazirani show that there is a natural crystal structure on simple representations of $R^\lambda = T^\lambda$, which is isomorphic to the usual highest weight crystal $\mathcal{B}(\lambda)$. A similar crystal structure exists for simples of $T^{\underline{\boldsymbol{\lambda}}}$; we denote the set of isomorphism classes of simple modules by $\mathcal{B}^{\underline{\boldsymbol{\lambda}}}$.

Recall that the **cosocle** or **head** $\operatorname{hd}(M)$ of a representation M is its maximal semi-simple quotient. As many examples in representation theory show, it is often easiest to construct simple modules by first considering other modules that they are cosocle of. For example, this is done for KLR algebras in [**KR11**].

Theorem 5.8 *For L_i a simple T^{λ_i} module, the module $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell)$ has a unique simple quotient. This defines a bijection*

$$h\colon \mathcal{B}^{\lambda_1} \times \cdots \times \mathcal{B}^{\lambda_\ell} \to \mathcal{B}^{\underline{\boldsymbol{\lambda}}},$$

$$h(L_1, \dots, L_\ell) \mapsto \operatorname{hd}\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell).$$

We'll use the following standard lemma:

Lemma 5.9 *Let A be an algebra and M a right A-module, and $e \in A$ an idempotent. If*

(1) Me is simple as an eAe-module and
(2) Me generates M as an A module,

then M has a unique simple quotient.

Proof. Any proper submodule is killed by the idempotent e, since any nonzero vector in Me generates M. Thus, the sum of two proper submodules is killed by e, and is again proper. Therefore, we have a unique maximal submodule. □

Proof of Theorem 5.8. Since L_i is indecomposable, it makes sense to speak of its weight. Thus we have a root function $\boldsymbol{\alpha}$ of $L_1 \boxtimes \cdots \boxtimes L_\ell$, and the corresponding idempotent $e_{\boldsymbol{\alpha}}$ as defined earlier. Note that the functor $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}$ restricted to $T^{\lambda_1}_{\mu_1} \otimes \cdots \otimes T^{\lambda_\ell}_{\mu_\ell}$-modules has a right adjoint given by $\operatorname{Hom}(S_{\boldsymbol{\alpha}}, -)$. For a fixed module $T^{\underline{\boldsymbol{\lambda}}}$ module M, if $Me_{\boldsymbol{\alpha}} \neq 0$ and $\boldsymbol{\alpha}$ is maximal amongst $\boldsymbol{\alpha}'$ with this property, then we have that $\operatorname{Hom}(S_{\boldsymbol{\alpha}}, M) \cong Me_{\boldsymbol{\alpha}}$.

The unit of the adjunction gives an inclusion of $T^{\lambda_1}_{\mu_1} \otimes \cdots \otimes T^{\lambda_\ell}_{\mu_\ell}$-modules

$$L_1 \boxtimes \cdots \boxtimes L_\ell \hookrightarrow \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell)e_{\boldsymbol{\alpha}}.$$

This map is actually an isomorphism since by Proposition 4.16, we can rewrite all elements of $e_{\boldsymbol{\alpha}} T^{\underline{\boldsymbol{\lambda}}} e_{\boldsymbol{\alpha}}$ as a sum of diagrams in the image of $\wp$, which preserve $L_1 \boxtimes \cdots \boxtimes L_\ell$, and of diagrams with standardly violating strands, which act trivially.

We apply Lemma 5.9 to the idempotent $e_{\boldsymbol{\alpha}}$ and the $T^{\underline{\boldsymbol{\lambda}}}$-module $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell)$; condition (1) follows from the simplicity of L_i, and condition (2) from the definition of standard modules (these are quotients of projectives generated by the same subspace). Thus h is indeed well-defined.

Now we wish to show bijectivity by constructing an inverse. Fix a simple L and let $\boldsymbol{\alpha}$ be a maximal root function such that $Le_{\boldsymbol{\alpha}} \neq 0$. Let $L_1 \boxtimes \cdots \boxtimes L_\ell$ be a simple submodule of $Le_{\boldsymbol{\alpha}}$. Since $\boldsymbol{\alpha}$ is maximal, the counit of the adjunction between $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}$ and $\cdot e_{\boldsymbol{\alpha}}$ induces a map $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell) \to L$, which is non-zero, and thus surjective. This shows that the map h is surjective.

Now, assume there is another set of simples $L_1', \ldots, L_\ell'$ with a possibly different root function $\boldsymbol{\alpha}'$ such that L is also a quotient of $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1' \boxtimes \cdots \boxtimes L_\ell')$. Since $Le_{\boldsymbol{\alpha}} \neq 0$, we must have $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1' \boxtimes \cdots \boxtimes L_\ell')e_{\boldsymbol{\alpha}} \neq 0$. This only possible if $\boldsymbol{\alpha} \leq \boldsymbol{\alpha}'$. By symmetry, this also implies that $\boldsymbol{\alpha}' \leq \boldsymbol{\alpha}$, so we must have $\boldsymbol{\alpha}' = \boldsymbol{\alpha}$.

Furthermore, we have that

$$L_1 \boxtimes \cdots \boxtimes L_\ell \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell)e_{\underline{\boldsymbol{\lambda}}} \cong Le_{\boldsymbol{\alpha}} \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1' \boxtimes \cdots \boxtimes L_\ell')e_{\underline{\boldsymbol{\lambda}}} \cong L_1' \boxtimes \cdots \boxtimes L_\ell'.$$

This shows that the map h is also injective. $\square$

If M is a right module over $T^{\underline{\boldsymbol{\lambda}}}$, we let $\dot{M}$ be the left module given by twisting the action by the anti-automorphism $a \mapsto \dot{a}$ flipping diagrams through the vertical axis.

Definition 5.10 *For a finite-dimensional right module M, we define the* **dual module** *by $M^\star = \dot{M}^*$, where $(\cdot)^*$ denotes usual vector space duality interchanging left and right modules.*

This is a right module since both vector space dual and the anti-automorphism interchange left and right modules.

Proposition 5.11 *Any simple module $L \in \mathcal{B}^{\underline{\boldsymbol{\lambda}}}$ is isomorphic to its dual: $L \cong L^\star$.*

Proof. From Theorem 5.8, we have that two simple modules $L, L'$are isomorphic if there is a root function $\boldsymbol{\alpha}$ such that $Le_{\boldsymbol{\alpha}'} = L'e_{\boldsymbol{\alpha}'} = 0$ for all $\boldsymbol{\alpha}' \not\leq \boldsymbol{\alpha}$, and $Le_{\boldsymbol{\alpha}}$ and $L'e_{\boldsymbol{\alpha}}$ are non-zero and isomorphic as modules over $T^{\lambda_1}_{\mu_1} \otimes \cdots \otimes T^{\lambda_\ell}_{\mu_\ell}$. Since $\dot{e}_{\boldsymbol{\alpha}} = e_{\boldsymbol{\alpha}}$, we have that $Le_{\boldsymbol{\alpha}'} = 0$ if and only if $L^\star e_{\boldsymbol{\alpha}'} = 0$. Thus, the criterion above shows that $L \cong L^\star$ if and only if $Le_{\boldsymbol{\alpha}} \cong L^\star e_{\boldsymbol{\alpha}}$. Khovanov and Lauda have shown [**KL09**, §3.2] that every simple module over $T^{\lambda_i}_{\mu_i}$, and thus over the tensor products of these algebras, is self-dual. Applying this to $Le_{\boldsymbol{\alpha}}$ gives the result. $\square$

Now, we wish to understand how the simple modules of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ are related by categorification functors. In particular, it follows from [**CR08**, 5.20] that:

Proposition 5.12 *For a simple module L, the modules $\tilde{f}_i(L) := \operatorname{hd}(\mathfrak{F}_i L)$, and $\tilde{e}_i(L) := \operatorname{hd}(\mathfrak{E}_i L)$ are simple.*

Remark 5.13 *It would be more in the spirit of earlier work on crystals of representations, such as* [**LV11**], *to let $\tilde{e}_i(L)$ be the socle of the kernel of the action of y on $\mathfrak{E}_i L$; however, $\mathfrak{E}_i L$ is self-dual, so this is the same up to isomorphism.*

Theorem 5.14 *These operators make the classes of the simple modules a perfect basis of $K_0(T^{\underline{\boldsymbol{\lambda}}})$ in the sense of Berenstein and Kazhdan* [**BK07**, Definition 5.30]. *In particular, they define a crystal structure on simple modules.*

Proof. By [**CR08**, Prop. 5.20], if a is the largest integer such that $\tilde{e}_i^a(L) \neq 0$, then $\mathfrak{E}_i^a(L)$ is semi-simple; in fact, it is a sum of copies of $\tilde{e}_i^a(L)$ (since $\mathfrak{F}_i^{(a)}(\tilde{e}_i^a(L))$ surjects onto L). In particular, any other simple constituent of $\mathfrak{E}_i(L)$ is killed by $\tilde{e}_i^{a-1}$. This is the definition of a perfect basis. $\square$

Since $K_0(T^{\underline{\boldsymbol{\lambda}}}) \cong V_{\underline{\boldsymbol{\lambda}}}$, this implies that an isomorphism of crystals exists between $\mathcal{B}^{\underline{\boldsymbol{\lambda}}}$ and the tensor product $\mathcal{B}^{\lambda_1} \times \cdots \times \mathcal{B}^{\lambda_\ell}$ without actually determining what it is. In [**LW**, 7.2], the author and Losev prove that:

Theorem 5.15 *The crystal structure induced on $\mathcal{B}^{\underline{\boldsymbol{\lambda}}}$ by h has Kashiwara operators given by $\tilde{f}_i$ and $\tilde{e}_i$, where $\mathcal{B}^{\lambda_1} \times \cdots \times \mathcal{B}^{\lambda_\ell}$ is endowed with the tensor product crystal structure.*

3. Stringy triples

Our system of projectives $P_{\mathbf{i}}^{\kappa}$ is quite redundant; there are many more of them than there are simple modules, as Proposition 5.8 shows. We can produce a smaller projective generator by using string parametrizations.

Choose any infinite sequence $\{i_1, i_2, \dots\} \in \Gamma$ of simple roots such that each element of Γ appears infinitely often. For any element v of a highest weight crystal $\mathcal{B}^{\lambda}$, there are unique integers $\{a_1, \dots\}$ such that $\cdots \tilde{e}_{i_2}^{a_2} \tilde{e}_{i_1}^{a_1} v = v_{high}$ and $\tilde{e}_k^{a_k+1} \cdots \tilde{e}_{i_1}^{a_1} v = 0$. The parametrization of the elements of the crystal by this tuple is called the **string parametrization.** We can associate this to a sequence with multiplicities $(\dots, i_2^{(a_2)}, i_1^{(a_1)})$. While this is *a priori* infinite, $a_j = 0$ for all but finitely many j, so deleting entries with multiplicity 0, we obtain a finite sequence, which we'll call the **string parametrization** of the corresponding simple.

Definition 5.16 *We call a Stendhal triple $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$* **stringy** *if the jth black block, that is, the sequence of i's between the jth and $j+1$st red lines, is the string parametrization of a crystal basis vector in V_{λ_j}.*

We will implicitly use the canonical identification between stringy triples and $\mathcal{B}^{\underline{\boldsymbol{\lambda}}}$ via h.

As in Khovanov and Lauda [**KL09**, §3.2], we order the elements of the crystal $\mathcal{B}^{\underline{\boldsymbol{\lambda}}}$ by first decreasing weight (so that the smallest element is the highest weight vector) and then lexicographically by the string parametrization.

For the tensor product crystal, we use the dominance order on $\boldsymbol{\alpha}$'s, with the order discussed above in the factors used to break ties.

Proposition 5.17 *The projective cover of any simple appears as a summand of $P_{\mathbf{i}}^{\kappa}$ where $(\mathbf{i}, \kappa)$ is the corresponding stringy triple. This cover is, in fact, the* *unique indecomposable summand which doesn't appear in $P_{\mathbf{i}'}^{\kappa'}$ for $(\mathbf{i}',\kappa') > (\mathbf{i},\kappa)$. If $(\mathbf{i},\kappa)$ is not stringy, then every indecomposable summand of $P_{\mathbf{i}}^{\kappa}$ appears in $P_{\mathbf{i}'}^{\kappa'}$ for $(\mathbf{i}',\kappa') > (\mathbf{i},\kappa)$.*

As a matter of convention, we call the root function of the stringy triple where an indecomposable projective first appears the root function of that projective.

PROOF. Obviously,

$$P_{\mathbf{i}}^{\kappa} \twoheadrightarrow S_{\mathbf{i}}^{\kappa} = \mathbb{S}^{\boldsymbol{\lambda}}(\mathfrak{F}_{i_{\kappa(2)}}^{(a_{\kappa(2)})} \cdots \mathfrak{F}_{i_1}^{(a_1)} P_{\emptyset} \boxtimes \cdots \boxtimes \mathfrak{F}_{i_n}^{(a_n)} \cdots \mathfrak{F}_{i_{\kappa(\ell)+1}}^{(a_{\kappa(\ell)+1})} P_{\emptyset})$$

which in turn surjects to the corresponding simple, by the definition of Kashiwara operators on simple modules, and of the map h. Thus, the indecomposable projective cover of the simple with this string parametrization is a summand of $P_{\mathbf{i}}^{\kappa}$.

The other indecomposable projective summands of $P_{\mathbf{i}}^{\kappa}$ are precisely the projective covers of simples such that $\mathrm{Hom}(P_{\mathbf{i}}^{\kappa}, L) \neq 0$, which is the same as requiring that $Le_{\mathbf{i},\kappa} \neq 0$. This can only hold if the simple L has an associated root function (under the bijection h) which greater than or equal to that for $(\mathbf{i},\kappa)$. If it is strictly greater, then L must be a quotient of $P_{\mathbf{i}'}^{\kappa'}$ with $(\mathbf{i}',\kappa')$ having greater root function than $(\mathbf{i},\kappa)$. Thus, in this case, we must have $(\mathbf{i}',\kappa') > (\mathbf{i},\kappa)$.

On the other hand, if the root functions are equal, any map of $P_{\mathbf{i}}^{\kappa}$ to L must factor through $S_{\mathbf{i}}^{\kappa}$. In this case, $Le_{\boldsymbol{\alpha}}$ will be a quotient of $S_{\mathbf{i}}^{\kappa} e_{\boldsymbol{\alpha}} \cong \mathfrak{F}_{i_{\kappa(2)}}^{(a_{\kappa(2)})} \cdots \mathfrak{F}_{i_1}^{(a_1)} P_{\emptyset} \boxtimes \cdots \boxtimes \mathfrak{F}_{i_n}^{(a_n)} \cdots \mathfrak{F}_{i_{\kappa(\ell)+1}}^{(a_{\kappa(\ell)+1})} P_{\emptyset}$.

By [**KL09**, Lemma 3.7], this module has a unique simple quotient that doesn't appear as a quotient associated to a word higher in lexicographic order if each of the black blocks is a string parametrization, and none if any one of them is not. This completes the proof. □

For an indecomposable projective P, its **standard quotient** is its quotient under the sum of all images of maps from projectives with higher root sequences. This coincides with its image in $S_{\mathbf{i}}^{\kappa}$, the standard quotient for its associated stringy triple. This standard quotient is indecomposable, since it is a quotient of an indecomposable projective.

Proposition 5.18 *Consider $(\mathbf{i},\kappa)$ with the associated root function $\boldsymbol{\alpha}$. Then the sum of indecomposable summands of $P_{\mathbf{i}}^{\kappa}$ that have the same root function surject to $S_{\mathbf{i}}^{\kappa}$, which is a direct sum of the standard quotients of those projectives.*

PROOF. If an indecomposable summand of $P_{\mathbf{i}}^{\kappa}$ has a different root function, it must be higher, so this summand is in the image of a higher stringy projective and thus in $U_{\mathbf{i}}^{\kappa}$. Thus, the other summands must surject.

Similarly, it is clear that the intersection of any indecomposable with the same root function with $U_{\mathbf{i}}^{\kappa}$ is exactly the trace of the projectives with higher root functions. □

4. Standard stratification

Now, we proceed to showing that the algebra $T^{\boldsymbol{\lambda}}$ is standardly stratified. Fix a Stendhal triple $(\mathbf{i},\kappa)$. Any Stendhal diagram with top $(\mathbf{i},\kappa)$ thus has its black strands divided in black blocks divided by the red strands at the top of the diagram.

Consider the set $\tilde{\Phi}$ of permutations of the terminals at the top of the diagram which do not move black strands into blocks to their right and are minimal coset representatives for the permutations within blocks at the bottom of the diagram. We let Φ be the subset of $\tilde{\Phi}$ where the bottom of the diagram is not violating.

Lemma 5.19 $v_{\mathbf{i}}^{\kappa} = \sum_{\phi\in\Phi} q^{-\deg x_\phi} s_{\mathbf{i}_\phi}^{\kappa_\phi}$

Proof. As usual, we prove this by induction on the number of red and black strands. If $\kappa(\ell) = n$, then Φ is unchanged by removing the red strand, and we have that

$$v_{\mathbf{i}}^{\kappa} = v_{\mathbf{i}}^{\kappa^-} \otimes v_{\lambda_\ell} = \sum_{\phi\in\Phi} q^{-\deg x_\phi} s_{\mathbf{i}_\phi}^{\kappa_\phi^-} \otimes v_{\lambda_\ell} = \sum_{\phi\in\Phi} q^{-\deg x_\phi} s_{\mathbf{i}_\phi}^{\kappa_\phi}.$$

Thus, we may assume that $\kappa(\ell) < n$. We let Φ^- be the set of permutations associated to the Stendhal triple $(\kappa^-, \mathbf{i}^-)$ where we remove the rightmost black strand. Each element of Φ^- contributes ℓ elements to Φ given by moving the far right element to the far right of the ℓ different black blocks (it can only be at the far right since we must have a shortest coset representative). As computed in Proposition 5.5, the grading shifts of these elements match those in the coproduct formula for F_{i_n} acting on $s_{\mathbf{i}_\phi^-}^{\kappa_\phi}$. Thus, we have

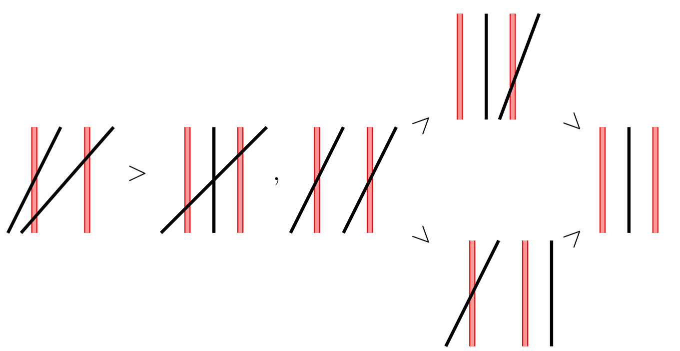
$$v_{\mathbf{i}}^{\kappa} = F_{i_n} v_{\mathbf{i}^-}^{\kappa} = \sum_{\phi\in\Phi^-} q^{-\deg x_\phi} F_{i_n} s_{\mathbf{i}_\phi^-}^{\kappa_\phi} = \sum_{\phi\in\Phi} q^{-\deg x_\phi} s_{\mathbf{i}_\phi}^{\kappa_\phi}.$$

This completes the proof. □

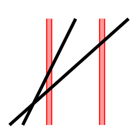
We preorder $\tilde{\Phi}$ according to the preorder on the idempotent $(\mathbf{i}_\phi, \kappa_\phi)$ which appears at the bottom of the diagram.

Let x_ϕ be a Stendhal diagram where we permute the strands exactly according to a chosen reduced word of $\phi \in \tilde{\Phi}$.

Example 5.20 *So, for example, in the case of $\mathfrak{sl}_2$, if we have $\lambda = (1,1)$, $\mathbf{i} = (1,1)$ and $\kappa(1,2) = 0,1$, then the elements in $\tilde{\Phi}$ are given (with their ordering) by:*



Only the rightmost and topmost diagrams lie in Φ. The others have a violating strand. Note that



is not one of the diagrams we consider, since it is not a shortest coset representative.

Consider the submodules

$$P_{>\phi} = \sum_{\phi' > \phi} x_{\phi'} T^{\underline{\boldsymbol{\lambda}}} \subset P_{\mathbf{i}}^{\kappa} \qquad P_{\geq\phi} = P_{>\phi} + x_\phi T^{\underline{\boldsymbol{\lambda}}}.$$

Proposition 5.21 *For any $\phi \in \Phi$, we have $P_{\geq\phi}/P_{>\phi} \cong S_{\mathbf{i}_\phi}^{\kappa_\phi}$.*

We note that some of these subquotients are trivial, but in this case the corresponding standard module is trivial as well.

PROOF. The multiplying by the element x_ϕ induces a map $P_{\mathbf{i}_\phi}^{\kappa_\phi} \to P_{\geq\phi}$. This map sends $U_{\mathbf{i}_\phi}^{\kappa_\phi}$ to $P_{>\phi}$, and thus induces a surjective map $\gamma_\phi\colon S_{\mathbf{i}_\phi}^{\kappa_\phi} \to P_{\geq\phi}/P_{>\phi}$. Since this map is surjective, we have

$$\dim P_{\geq\phi}/P_{>\phi} \leq \dim S_{\mathbf{i}_\phi}^{\kappa_\phi}. \tag{5.12}$$

On the other hand, we have $v_{\mathbf{i}}^{\kappa} = \sum_{\phi\in\Phi} q^{-\deg x_\phi} s_{\mathbf{i}_\phi}^{\kappa_\phi}$ by Lemma 5.19, so taking inner product with $[T^{\underline{\boldsymbol{\lambda}}}]$, we obtain $\dim P_{\mathbf{i}}^{\kappa} = \sum_{\phi\in\Phi} \dim S_{\mathbf{i}_\phi}^{\kappa_\phi}$.

Thus we must have equality in (5.12), and the map γ_ϕ is an isomorphism for dimension reasons. $\square$

Corollary 5.22 *The algebra $T^{\underline{\boldsymbol{\lambda}}}$ is standardly stratified with standard modules given by the standard quotients of indecomposable projectives, and the preorder on simples/standards/projectives given by the dominance order on root functions $\boldsymbol{\alpha}$.*

Note that we can easily bootstrap this to prove that

Corollary 5.23 *The algebra $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ is standardly stratified with standard modules given by the standard quotients of indecomposable projectives, and the preorder on simples/standards/projectives given by the dominance order on root functions $\boldsymbol{\alpha}$.*

PROOF. The indexing set for the standard filtration on a projective is now $\tilde{\Phi}$ instead of Φ. As before, the map of $S_{\mathbf{i}_\phi}^{\kappa_\phi}$ to the successive quotient is clear. In order to check that the dimensions are correct, the easiest thing to note is that we can add a new red strand at the left and impose the violating relation in both the projective to be filtered and the standard modules. In this case, Proposition 5.21 shows that the dimensions match in each degree.

For each fixed degree, we can choose the label on the new red strand to be sufficiently dominant so that in both the projective and standard modules, adding the red strand and imposing the violating relation kills no elements of that degree in either the projective or standard modules. Thus, Proposition 5.21 shows the same result for $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$, and the standard stratification follows. $\square$

Corollary 5.24 *Every standard module has a finite length projective resolution.*

This is a standard fact about finite dimensional standardly stratified algebras; in particular, any module with a standard filtration has a well-defined class in $K_0(T^{\underline{\boldsymbol{\lambda}}})$.

Proof. We induct on the preorder order $\leq$. If a standard is maximal in this order, it is projective. For an arbitrary standard, there is a map $P^{\mathbf{i}}_{\kappa} \to S^{\mathbf{i}}_{\kappa}$ with kernel filtered by standards higher in the partial order. Since each of these has a finite length projective resolution, we can glue these to form one of $S^{\kappa}_{\mathbf{i}}$ by Lemma 5.29. $\square$

We note that $e(\mathbf{i},\kappa)T^{\underline{\boldsymbol{\lambda}}}e(\mathbf{i},0)$ has a unique element consisting of a diagram with no dots and no crossings between black strands which simply pulls red strands to the left and black to the right. As before, we call this element θ_κ (leaving $\mathbf{i}$ implicit).

Lemma 5.25 *The map from $P^{\kappa}_{\mathbf{i}} \to P^{0}_{\mathbf{i}}$ given by the action of $\dot{\theta}_\kappa$ is injective.*

Proof. Obviously, this map is filtered, where we include $\Phi_{\mathbf{i},\kappa} \subset \Phi_{\mathbf{i},0}$ by precomposing with $\dot{\theta}_\kappa$. Furthermore, it induces an isomorphism on each successive quotient in this image. Thus, it is injective. $\square$

Let $\mathcal{C}^{\boldsymbol{\alpha}}$ be the subcategory of $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ generated by standard modules with root function $\boldsymbol{\alpha}$.

Proposition 5.26 *We have a natural isomorphism*

$$\mathrm{End}_{\tilde{T}^{\underline{\boldsymbol{\lambda}}}}(S_{\boldsymbol{\alpha}}) \cong R_{\boldsymbol{\alpha}(0)} \otimes T^{\lambda_1}_{\mu_1} \otimes \cdots \otimes T^{\lambda_\ell}_{\mu_\ell}.$$

The triangulated subcategories generated by $\mathcal{C}^{\boldsymbol{\alpha}}$ form a semi-orthogonal decomposition of $\tilde{\mathcal{V}}^{\underline{\boldsymbol{\lambda}}}$ with respect to dominance order.

For more general standardizations, this implies that for modules M and N over $R_{\mu_0} \otimes T^{\underline{\boldsymbol{\lambda}}_1}_{\mu_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_m}_{\mu_\ell}$ that

$$\mathrm{Hom}_{\tilde{T}^{\underline{\boldsymbol{\lambda}}}}(\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}(M), \mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_m}(N)) \cong \mathrm{Hom}_{R_{\mu_0}\otimes T^{\underline{\boldsymbol{\lambda}}_1}_{\mu_1}\otimes\cdots\otimes T^{\underline{\boldsymbol{\lambda}}_m}_{\mu_\ell}}(M,N),$$

that is, that standardization is fully faithful

Proof. Since standardization is exact, it's enough to check full faithfulness on standards, which is the first part of the theorem.

Since the map $\nu\colon e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}} \to S_{\boldsymbol{\alpha}}$ is surjective, the projective lifting property shows that every endomorphism of $S_{\boldsymbol{\alpha}}$ is induced by an endomorphism of $e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}}$. Thus $\mathrm{End}^{op}(S_{\boldsymbol{\alpha}})$ is a subquotient of $e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\boldsymbol{\alpha}}$: it is the quotient of the subalgebra in $e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\boldsymbol{\alpha}}$ which preserves the kernel of the map ν modulo the ideal of endomorphisms whose composition with ν is 0.

Now let us use Proposition 4.16 to better understand how elements of $e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\boldsymbol{\alpha}}$ act. We choose a reduced word for each permutation. First we split the strands, both red and black, into groups consisting of a black block at $y = 1$ and the red strand immediately to its left. For each permutation, we choose a reduced word so that so that all crossings that occur within such a group are above $y = 1/2$ and all crossings that occur between different groups are below. This implies that the diagram for any permutation which has a left crossing has at least one above any right crossings. By the definition of the standard quotient such a diagram acts trivially (assuming it preserves the kernel of ν). On the other hand, an element of $e_{\boldsymbol{\alpha}}\tilde{T}^{\underline{\boldsymbol{\lambda}}}e_{\boldsymbol{\alpha}}$ must have equal numbers of the two types of crossings, so our element acts in the same way as one that has been "straightened" so that no red and black strands ever cross. Thus, the map $R_{\boldsymbol{\alpha}(0)}\otimes\cdots\otimes R_{\boldsymbol{\alpha}(\ell)}\to \mathrm{End}_{\tilde{T}^{\underline{\boldsymbol{\lambda}}}}(S_{\boldsymbol{\alpha}})$ of Proposition 5.3 is surjective.

By definition of a standard quotient, the cyclotomic ideal of this tensor product is killed by the map to $\mathrm{End}^{op}(S_{\boldsymbol{\alpha}})$, so we have a surjective map $R_{\boldsymbol{\alpha}(0)}\otimes T^{\lambda_1}_{\boldsymbol{\alpha}(1)}\otimes\cdots\otimes T^{\lambda_\ell}_{\boldsymbol{\alpha}(\ell)}\to \mathrm{End}^{op}(S_{\boldsymbol{\alpha}})$, which we need only show is also injective. Since $\mathrm{Ext}^{>0}(S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}})=0$, this is equivalent to showing that

$$\dim_q \mathrm{End}(S_{\boldsymbol{\alpha}},S_{\boldsymbol{\alpha}})=\langle[S_{\boldsymbol{\alpha}}],[S_{\boldsymbol{\alpha}}]\rangle=\dim_q R_{\boldsymbol{\alpha}(0)}\otimes T^{\lambda_1}_{\boldsymbol{\alpha}(1)}\otimes\cdots\otimes T^{\lambda_\ell}_{\boldsymbol{\alpha}(\ell)}.$$

The second equality follows from the equality $\langle a\otimes b,a'\otimes b'\rangle=\langle a,b\rangle\langle a',b'\rangle$ if a,a' and b,b' are weight vectors with each pair having the same weight, which follows, in turn, from the upper-triangularity of $\Theta^{(2)}$.

Finally, we establish the semi-orthogonal decomposition: by Proposition 5.21, the subcategory generated by $\mathcal{C}^{\boldsymbol{\alpha}'}$ for $\boldsymbol{\alpha}'>\boldsymbol{\alpha}$ in the dominance order is the same as that generated by $P^{\kappa}_{\mathbf{i}}$ such that $\boldsymbol{\alpha}_{\mathbf{i},\kappa}>\boldsymbol{\alpha}$. Since all the simple modules in $S^{\kappa}_{\mathbf{i}}$ are given by idempotents $e_{\mathbf{i},\kappa}$ such that $\boldsymbol{\alpha}_{\mathbf{i},\kappa}\leq\boldsymbol{\alpha}$, we have

$$\mathrm{Ext}^{\bullet}(S^{\kappa'}_{\mathbf{i}'},S^{\kappa}_{\mathbf{i}})=0$$

whenever $\boldsymbol{\alpha}_{\mathbf{i},\kappa}<\boldsymbol{\alpha}_{\mathbf{i}',\kappa'}$, and higher Ext's vanish when equality holds. □

Together, the results above show that the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is a **tensor product categorification** in the sense introduced by the author and Losev in [**LW**, §3.2].

Corollary 5.27 *The $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ with its standardly stratified structure from Corollary* 5.22 *and categorical $\mathfrak{g}$-action from Theorem* 4.31 *forms a tensor product categorification of $V_{\underline{\boldsymbol{\lambda}}}$.*

Proof. We consider the axioms of a tensor product categorification in turn, and confirm them:

(TPC1) We must have that the poset underlying the stratification is that of n-tuples $\underline{\boldsymbol{\mu}}=(\mu_1,\dots,\mu_n)$, where μ_i is a weight of V_i. The poset structure is given by "inverse dominance order": we have

$$\underline{\boldsymbol{\mu}}=(\mu_1,\dots,\mu_n)\geqslant\underline{\boldsymbol{\mu}}'=(\mu'_1,\dots,\mu'_n)$$

if and only if $\sum_{i=1}^n\mu_i=\sum_{i=1}^n\mu'_i$ and for all $1\leqslant j<n$, we have

$$\sum_{i=1}^j\mu_i\leqslant\sum_{i=1}^j\mu'_i.$$

This precisely matches the definition of the order on root functions from Chapter 1, since $\mu_i=\lambda_i-\boldsymbol{\alpha}(i)$. Proposition 5.7 shows that the standardization functors are exact, as required in [**LW**].

(TPC2) Proposition 5.26 shows that the subquotients of this standardly stratified structure are equivalent to $\mathfrak{V}^{\lambda_1;\dots;\lambda_\ell}$ and thus carry the expected categorical $\mathfrak{g}^{\oplus\ell}$ action on these subquotients.

(TPC3) Proposition 5.5 shows that $\mathfrak{E}_i$ and $\mathfrak{F}_i$ acting on a standard module have the desired filtrations. □

Finally, we prove a result which, while somewhat technical in nature, is very important for understanding how to decategorify our construction. As in [**BGS96**, §2.12], we let $C^\uparrow(T^{\underline{\boldsymbol{\lambda}}})$ denote the category of complexes of graded modules such that the degree j part of the ith homological term $C^i_j = 0$ for $i \gg 0$ or $i + j \ll 0$.

Theorem 5.28 *Every simple module over $T^{\underline{\boldsymbol{\lambda}}}_\alpha$ has a projective resolution in $C^\uparrow(T^{\underline{\boldsymbol{\lambda}}})$. In particular, each simple module L has a well-defined class in $K_0(T^{\underline{\boldsymbol{\lambda}}}) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}((q)) \cong V_{\underline{\boldsymbol{\lambda}}}$.*

This observation would be clear if $T^{\underline{\boldsymbol{\lambda}}}$ were Morita equivalent to a positively graded algebra. This case is called **mixed** by Achar and Stroppel [**AS13**], and is carefully worked out in their paper. As shown in [**Webg**, 4.6], this is true when $\Bbbk$ is characteristic 0, the Cartan matrix of $\mathfrak{g}$ is symmetric, and polynomials Q_{ij} are carefully chosen, but as the example [**Web15**, 5.6] shows, outside these cases there may simply be no such Morita equivalence.

Lemma 5.29 *If a module M is filtered by modules which have finite length projective resolutions, these resolutions can be glued to give a finite length resolution of the entire module.*

Proof. This is a standard lemma in homological algebra, but let us include a proof. By induction, we need only prove this for a short exact sequence $0 \to M' \to M \to M'' \to 0$, with $M' \leftarrow P'_0 \leftarrow \cdots$ and $M'' \leftarrow P''_0 \leftarrow \cdots$ projective resolutions. If we delete M'' and P''_0 from the second resolution, we obtain a resolution of K'', the kernel of the map $P''_0 \to M''$.

By the universal property of projectives, we have a map $P''_0 \to M$ which lifts the projection $P''_0 \to M''$ and thus induces a map $K'' \to M'$. Let $\nu_i \colon P''_i \to P'_{i-1}$ be a lift of this map to the projective resolutions, and let $\nu_0 = 0$. The cone of this chain map is a new complex of projectives, necessarily exact except in degree 0. In degree 0, the homology is the cokernel of the map $P'_1 \oplus P''_1 \to P'_0 \oplus P''_1$ given by the matrix

$$\begin{bmatrix} \partial' & \nu_1 \\ 0 & \partial'' \end{bmatrix}$$

which is easily checked to be M. Thus we have found a finite length resolution of this module. □

Proof of Theorem 5.28. The proof is by induction on our order above. First, we do the base case of $\underline{\boldsymbol{\lambda}} = (\lambda)$ and $\lambda - \beta = k\alpha_i$. This case, $T^{\underline{\boldsymbol{\lambda}}}_\beta$ is Morita equivalent to its center, which is the cohomology ring on a Grassmannian of k-planes in λ^i-dimensional space. In particular, it is positively graded, so such a resolution exists.

Now, we bootstrap to the case where $\underline{\boldsymbol{\lambda}} = (\lambda)$ but β is arbitrary. In this case, we may assume that $L' = \tilde{e}^{a_1}_{i_1} L$ has this type of resolution. Now, we consider

$$M = \operatorname{Ind}_{\beta + a_1\alpha_{i_1}, a_i\alpha_{i_1}} L' \boxtimes L(i_1^{a_1}),$$

where here we use the notation of [**KL09**, §3.2]. The module M has a projective resolution of the prescribed type, by inducing the outer tensors of the resolutions on the two factors. Furthermore, there is a surjection $M \twoheadrightarrow L$ whose kernel has composition factors smaller in the order given above on simples, by [**KL09**, Theorem 3.7]. Since each of these has an appropriate resolution by induction, we may lift the inclusion of each composition factor to a map of projective resolutions, and take the cone to obtain a resolution of L in $C^{\uparrow}(T^{\underline{\boldsymbol{\lambda}}})$.

Finally, we deal with the general case using standardization; let $L = h(\{L_i\})$. By standardizing the resolutions of L_i, we obtain a standard resolution of $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(L_1 \boxtimes \cdots \boxtimes L_\ell)$. Replacing each standard with its finite projective resolution, we obtain a projective resolution of the same module. As before, the kernel of the surjection of this module to L has composition factors all smaller in the partial order, so we may attach projective resolutions of each composition factor to obtain a projective resolution of L in $C^{\uparrow}(T^{\underline{\boldsymbol{\lambda}}})$. □

5. Self-dual projectives

One interesting consequence of the module structure over $\mathcal{U}$ and standard stratification is the understanding it gives us of the self-dual projectives of our category. Self-dual projectives have played a very important role in understanding the structure of representation theoretic categories like $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$. For example, the unique self-dual projective in BGG category $\mathcal{O}$ for $\mathfrak{g}$ was key in Soergel's analysis of that category [**Soe90**, **Soe92**], and the self-dual projectives in category $\mathcal{O}$ for a rational Cherednik algebra provide an important perspective on the Knizhnik-Zamolodchikov functor defined by Ginzburg, Guay, Opdam and Rouquier [**GGOR03**]. In particular, following Mazorchuk and Stroppel [**MS08**], we use these modules to identify the Serre functor in Chapter 2.

Consider the projectives where $\kappa(i) = 0$ for all i, in which case we will simply denote the projective for κ by $P^0_{\mathbf{i}}$. We note that $P^0_{\mathbf{i}}$ carries an obvious action of R by composition on the bottom. We let $P^0 = \oplus_{\mathbf{i}} P^0_{\mathbf{i}}$ be the sum of all such projectives with $\kappa(i) = 0$.

Theorem 5.30 *If P is an indecomposable projective $T^{\underline{\boldsymbol{\lambda}}}$-module, then the following are equivalent:*

(1) P is injective.
(2) P is a summand of the injective hull of an indecomposable standard module.
(3) P is isomorphic (up to grading shift) to a summand of P^0.

Proof. (3) → (1): To establish this, we show that P^0 is self-dual; that is, there is a non-degenerate pairing $P^0_{\mathbf{i}} \otimes P^0_{\mathbf{i}} \to \Bbbk$. This is given by $(a, b) = \operatorname{tr}_\lambda(a\dot{b})$, where $\operatorname{tr}_\lambda$ is the Frobenius trace on $\operatorname{End}(P^0) \cong T^\lambda$ given in Chapter 3. Thus P^0 is both projective and injective, so any summand of it is as well.

(1) → (2): Since P is indecomposable and injective, it is the injective hull of any submodule of P. Since P has a standard stratification, it has a submodule which is standard.

(2) → (3): We have already established that P^0 is injective, so we need only establish that any simple in the socle of $S^\kappa_{\mathbf{i}}$ is a summand of the cosocle of P^0 (since the injective hull of $S^\kappa_{\mathbf{i}}$ coincides with that of its socle). It suffices to show that there is no non-trivial submodule of $S^\kappa_{\mathbf{i}}$ killed by $e_{0,\mathbf{j}}$ for all $\mathbf{j}$. If such a submodule M existed, then we would have $M\dot{\theta}_\kappa = 0$. Thus, its preimage M' in $P^\kappa_{\mathbf{i}}$ satisfies

$M'\dot{\theta}_\kappa \subset U^0_{\mathbf{i}}$. But the injectivity of Lemma 5.25 and the fact that $L^\kappa_{\mathbf{i}}\dot{\theta}_\kappa = U^0_{\mathbf{i}} \cap P^\kappa_{\mathbf{i}}\dot{\theta}_\kappa$, this implies that $M = 0$. □

For two rings A and B, we say an A-B bimodule M has the **double centralizer property** if $\mathrm{End}_B(M) = A$ and $\mathrm{End}_A(M) = B$. In particular, this implies that if M is projective as a B-module, the functor

$$\mathrm{Hom}(M, -) : B\text{-mod} \to A\text{-mod}$$

is fully faithful on projectives (it could be quite far from being a Morita equivalence, as the theorem below shows).

Proposition 5.31 $\mathrm{End}_{T^{\underline{\boldsymbol{\lambda}}}}(P^0) \cong T^\lambda \cong R^\lambda$.

Proof. The first isomorphism follows from repeated application of Corollary 4.21. The second is just a restatement of Proposition 4.18 □

Corollary 5.32 *The projective-injective P^0 has the double centralizer property for the actions of T^λ and $T^{\underline{\boldsymbol{\lambda}}}$ on the left and right.*

Proof. By [**MS08**, Corollary 2.6], this follows immediately from the fact that the injective hull of an indecomposable standard is also a summand of P^0. □

Thus, in this case, our algebra can be realized as the endomorphisms of a collection of modules over R^λ, in a way analogous to the realization of a regular block of category $\mathcal{O}$ as the modules over endomorphisms of a particular module over the coinvariant algebra, or of the cyclotomic q-Schur algebra as the endomorphisms of a module over the Hecke algebra.

In fact, these modules are easy to identify. Given $(\mathbf{i}, \kappa)$, we consider the element $y_{\mathbf{i},\kappa}$ of $P^0_{\mathbf{i}}$ given by

$$y_{\mathbf{i},\kappa} = e_{\mathbf{i}} \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} y_k^{\lambda_j^{i_k}}.$$

Pictorially this is given by multiplying the element θ_κ with no black/black crossings going from $(\mathbf{i}, 0)$ to $(\mathbf{i}, \kappa)$ by its horizontal reflection $\dot{\theta}_\kappa$, and then straightening the strands.

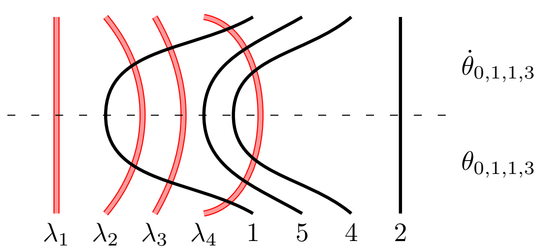


Figure 1. The element $y_{(1,5,4,2),(0,1,1,3)}$.

Proposition 5.33 *The algebra $T^{\underline{\boldsymbol{\lambda}}}$ is isomorphic to the algebra $\mathrm{End}_{T^\lambda}(\bigoplus_\kappa y_{\mathbf{i},\kappa} T^\lambda)$.*

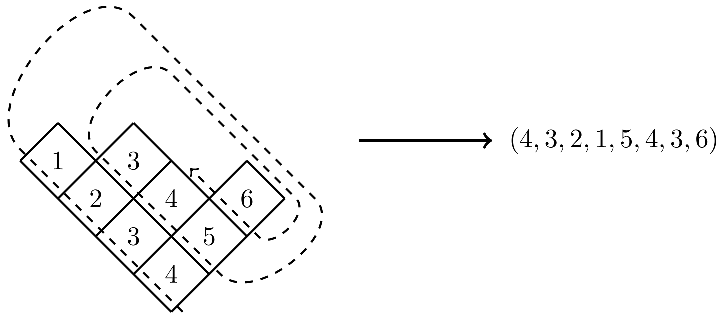


FIGURE 2. The stringy triple attached to a partition for $n = 7$ and $i = 4$.

PROOF. Based on Corollary 5.32, all we need to show is that $\mathrm{Hom}_{T^{\underline{\lambda}}}(P^0, P_{\mathbf{i}}^{\kappa}) \cong y_{\mathbf{i},\kappa}P_{\mathbf{i}}^0$ as a T^λ representation. A map m from $P_{\mathbf{i}'}^0$ to $P_{\mathbf{i}}^{\kappa}$ is simply a linear combination of diagrams starting at $\mathbf{i}$ with the correct placement of red strands and ending at $\mathbf{i}'$ with all red strands to the right. By Proposition 4.16, we can assure that all red/black crossings occur above all black/black ones, so $m = \theta_\kappa m'$, where $m \in T^\lambda$.

Thus, we have maps

$$\mathrm{Hom}_{T^{\underline{\lambda}}}(P^0, P_{\mathbf{i}}^0) \xrightarrow{\theta_\kappa} \mathrm{Hom}_{T^{\underline{\lambda}}}(P^0, P_{\mathbf{i}}^{\kappa}) \xrightarrow{\dot{\theta}_\kappa} \mathrm{Hom}_{T^{\underline{\lambda}}}(P^0, P_{\mathbf{i}}^0)$$

given by composition. The first of these is surjective, as we argued above. Furthermore, the latter is injective, by Proposition 5.25. Thus, $\mathrm{Hom}_{T^{\underline{\lambda}}}(P^0, P_{\mathbf{i}}^{\kappa})$ is isomorphic to the image of the composition of these maps, which is $y_{\mathbf{i},\kappa}T^\lambda$. □

For some choices of $\mathbf{i}$ and κ, the element $y_{\mathbf{i},\kappa}$ has already appeared in work of Hu and Mathas [**HM10**]. Assume that $\mathfrak{g} = \mathfrak{sl}_n$ and specialize to the case where for all j, we have $\lambda_j = \omega_{\pi_j}$ for some π_j. As suggested by the notation, we will later want to think of π_j as the numbers in a composition, not just arbitrary symbols indexing the nodes of the Dynkin diagram. We can define stringy triples for this algebra using the reduced decomposition of the longest element of the Weyl group $w_0 = s_{n-1}(s_{n-2}s_{n-1})\cdots(s_1\cdots s_{n-1})$.

As illustrated in Figure 2, the stringy triples for the fundamental representation V_{ω_i} are gotten by

- taking a partition diagram which fits in an $i \times (n-i)$ box,
- filling the box at (k, m) with its content $m - k + i$,
- taking the row-reading word.

For a multipartition $\xi = (\xi^{(1)}, \dots, \xi^{(\ell)})$, with $\xi^{(i)}$ fitting in a $\pi_i \times (n - \pi_i)$ box, we can thus define $(\mathbf{i}_\xi, \kappa_\xi)$ where $\mathbf{i}_\xi$ is the concatenation of these row-reading words, and $\kappa_\xi(k)$ is the number of the boxes in the first $k-1$ partitions. The element $y_{\mathbf{i}_\xi,\kappa_\xi}$ is exactly that denoted $\psi_{\mathfrak{t}^\xi\mathfrak{t}^\xi}$ in [**HM10, HM**].

Mathas and Hu have defined another algebra, which they call a **quiver Schur algebra**[1] $\mathcal{S}_m^\lambda$.

Theorem 5.34 *For $\mathfrak{g} = \mathfrak{sl}_n$, the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is equivalent (as a graded category) to a sum of blocks of graded representations of $\mathcal{S}_m^\lambda$ for the charges $(\pi_1, \dots, \pi_\ell)$.*

If we considered the case where $\mathfrak{g} = \mathfrak{sl}_\infty$ (thought of as the Kac-Moody algebra of the A_∞-quiver), then we could say that $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is simply equivalent to $\oplus_m \mathcal{S}_m^\lambda$-mod.

Proof. By [**HM**, 4.35], the graded category of projectives in a block of Hu and Mathas's algebra is equivalent to an additive subcategory of T^λ-mod. By Proposition 5.33, the graded category of projectives in each weight space of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is also equivalent to such a subcategory. Thus, we need only show that these subcategories coincide.

Each block of $\mathcal{S}_m^\lambda$ is the sum of images of the idempotents $e(\mathbf{i})$ where $\mathbf{i}$ ranges over all integer sequences with a fixed number m_i of occurrences of i. As long as m_i is only non-zero for $1 \leq i \leq n-1$, we can associate to this multiplicity data a weight $\mu = \lambda - \sum_i m_i \alpha_i$. We wish to show that this block is equivalent to $\mathfrak{V}_\mu^{\underline{\boldsymbol{\lambda}}}$. Let $m = \sum m_i$.

The image of projective modules over $\mathcal{S}_m^\lambda$ is the subcategory additively generated by $\psi_{\mathfrak{t}^\xi \mathfrak{t}^\xi} T^\lambda = y_{\mathbf{i}_\xi, \kappa_\xi} T^\lambda$ as we range over all multipartitions with m boxes fitting inside the correct $\pi_i \times (n - \pi_i)$ boxes. These are the same as the images of the projectives $P_{\mathbf{i}_\xi}^{\kappa_\xi}$ under the functor $\operatorname{Hom}(P^0, -)$. By Proposition 5.17, every indecomposable projective over $T_\mu^{\underline{\boldsymbol{\lambda}}}$ is a summand of a unique one of these modules, so those which have weight μ already additively generate the image of the $T_\mu^{\underline{\boldsymbol{\lambda}}}$-pmod in T_μ^λ-mod. Thus, that image coincides with the corresponding image for the quiver Schur algebra. $\square$

[1]This is an unfortunate terminological clash with [**SW**], where a non-equivalent, but graded Morita equivalent algebra is given the same name; after forgetting the grading, this is the difference between defining Schur algebras using all permutation modules attached to partitions or to compositions.

CHAPTER 6

Braiding functors

1. Braiding

Recall that the category of integrable $U_q(\mathfrak{g})$ modules (of type I) is a **braided category**; that is, for every pair of representations V, W, there is a natural isomorphism $\sigma_{V,W}\colon V \otimes W \to W \otimes V$ satisfying various commutative diagrams (see, for example, [**CP95**, 5.2B], where the name "quasi-tensor category" is used instead). This braiding is described in terms of an R-matrix $R \in U(\widehat{\mathfrak{g}) \otimes U}(\mathfrak{g})$; here the hat denotes the completion of the tensor square with respect to the kernels of finite dimensional representations, as usual.

As we mentioned earlier, we were left at times with difficult decisions in terms of reconciling the different conventions which have appeared in previous work. One which we seem to be forced into is to use the opposite R-matrix from that usually chosen (for example in [**CP95**]), which would usually be denoted R^{21}. Thus, we must be quite careful about matching formulas with references such as [**CP95**].

Our first task is to describe the braiding in terms of an explicit bimodule $\mathfrak{B}_\sigma$ attached to each braid. We will now define bimodules which we can use as building blocks for these.

Fix a permutation $w \in S_\ell$.

Definition 6.1 *A w-Stendhal diagram is a collection of curves which form a Stendhal diagram except that the red strands read from* top *to* bottom *trace out a reduced string diagram of the permutation w (that is, one where no two strands cross twice).*

We'll draw these with the crossing of red strands given by an over-crossing to remind the reader that ultimately these will define the bimodules for positive braids.

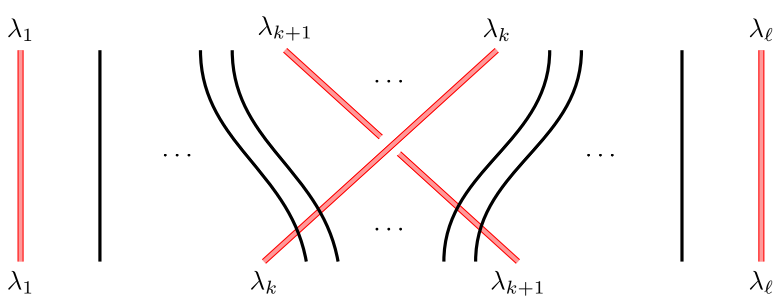


FIGURE 1. An example of a w-Stendhal diagram for $w = (k, k+1)$.

We can compose w-Stendhal diagrams with usual Stendhal diagrams. Unlike in the usual case, the triples at the top and bottom of the diagram needn't have the same sequence of red strands; instead, the sequences $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}'$ from the top and bottom must differ by the permutation $\underline{\boldsymbol{\lambda}}' = w \cdot \underline{\boldsymbol{\lambda}}$.

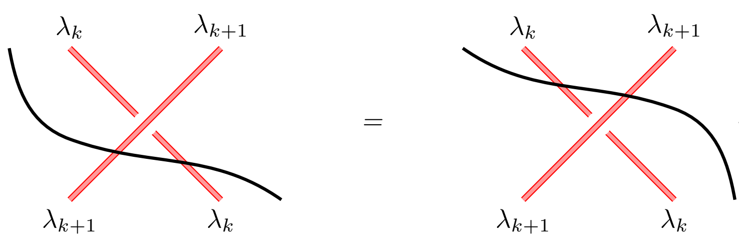
Definition 6.2 *Let $\tilde{\tilde{B}}_w$ denote the formal span of w-Stendhal diagrams over $\Bbbk$. We can consider this as a bimodule over $\tilde{\tilde{T}}$ using composition on the left and right (that is, on the top and bottom of the diagram).*

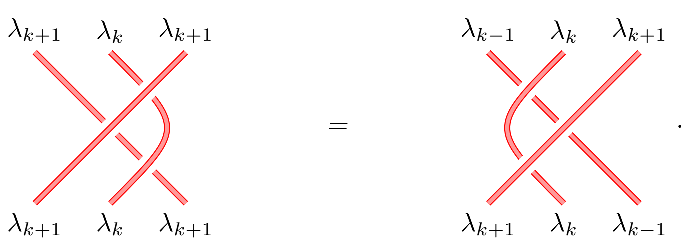
We grade w-Stendhal diagrams much the same as usual Stendhal diagrams, but with a red/red crossing with labels λ and λ' given degree $-\langle \lambda, \lambda' \rangle$. Annoyingly, this is typically not an integer. If the Cartan matrix of $\mathfrak{g}$ is invertible, then this will be an integer divided by its determinant. If the Cartan matrix is not invertible, then this can be any complex number. To avoid trouble from now on, we'll consider the categories $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}}$ and $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}}$ of modules graded by $\mathbb{C}$, not by $\mathbb{Z}$.

Definition 6.3 *Let $\tilde{\mathfrak{B}}_w$ be the quotient of $\tilde{\tilde{B}}_w$ by:*

- *All local relations of $\tilde{T}$, including planar isotopy. That is, we impose the relations of (2.5a–2.5g) and from equations (4.1a-4.2), but not the relations killing violating strands.*
- *The relations (along with their mirror images):*

(6.1a)


(6.1b)


(6.1c)


As in the usual case, we call a w-Stendhal diagram **violated** *if for some y-value, the leftmost strand is black. Let $\mathfrak{B}_w$ be the quotient of $\tilde{\mathfrak{B}}_w$ by the sub-bimodule spanned by violated w-Stendhal diagrams.*

We can define a basis of $\tilde{\mathfrak{B}}_w$ much like that of $\tilde{T}$. We fix a reduced expression of w, and only consider diagrams where the string diagram formed by the red strands follows this expression. For fixed bottom triple $(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$, we range over permutations $v \in S_n$ and top triples of the form $(v \cdot \mathbf{i}, w^{-1} \cdot \underline{\boldsymbol{\lambda}}, \kappa')$; for each such v and κ', we let $\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ be an arbitrarily chosen diagram with no dots such that the black strands give a string diagram for a reduced decomposition of v.

Proposition 6.4 *The set*

$$B_w = \{\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa) y^{\mathbf{a}} | v \in S_n, \mathbf{i} \in \Gamma^n, \kappa' \colon [1, \ell] \to [0, n], \mathbf{a} \in \mathbb{Z}^n_{\geq 0}\}$$

is a basis of $\tilde{\mathfrak{B}}_w$.

Proof. This is essentially the same as the proof of Proposition 4.16. The proof that the diagrams $\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa) y^{\mathbf{a}}$ span all diagrams that have the same reduced decomposition of w is exactly the same, using the relations (6.1a) and (6.1b) to slide through red crossings. So, now we must check that the span of the vectors $\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa) y^{\mathbf{a}}$ will not change if we change the reduced word for w.

Any two reduced words are related by switching commuting crossings and braid moves. Obviously, if two red crossings commute, then we can isotope one past the other with no problem. Thus, we need only to check that the vectors where the red strands trace out one reduced word also span the space where they trace out one that differs by a braid move. However, if 3 red strands form a triangle, we can always choose $\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ so that its strands avoid this triangle. In this case, we can apply (6.1c) to get all diagrams where the crossings are in the opposite order.

The proof of linear independence is also similar. Applying the product $\boldsymbol{\theta}(a) = \dot{\theta}_{\kappa'} a \theta_\kappa$ to an sum of diagrams that is zero in $\tilde{\mathfrak{B}}_w$ results in a relation in R, as is easily checked on a case-by-case basis. Thus, the map of $R \to \tilde{\mathfrak{B}}_w$ placing a KL diagram to the right of red strands tracing out w is injective. One can also check this by defining a "polynomial" representation of $\oplus_{w \in S_\ell} \tilde{\mathfrak{B}}_w$, where a red/red crossing acts by the identity on the underlying polynomial rings.

Therefore, $\boldsymbol{\theta}$ defines a linear map $\tilde{\mathfrak{B}}_w \to R$. Applying $\boldsymbol{\theta}$ to any element of B_w gives an element of Khovanov and Lauda's basis of R modulo terms with fewer crossings. Furthermore, if κ and κ' are fixed, no two elements of B_w yield the same one. Thus, the linear independence of Khovanov and Lauda's basis shows the linear independence of B_w as well. $\square$

Lemma 6.5 *If $\ell(ww') = \ell(w) + \ell(w')$, then $\tilde{\mathfrak{B}}_{ww'} \cong \tilde{\mathfrak{B}}_w \otimes_{\tilde{T}} \tilde{\mathfrak{B}}_{w'}$ and $\mathfrak{B}_{ww'} \cong \mathfrak{B}_w \otimes_T \mathfrak{B}_{w'}$.*

Proof. We have a map $\tilde{\mathfrak{B}}_w \otimes_{\tilde{T}} \tilde{\mathfrak{B}}_{w'} \to \tilde{\mathfrak{B}}_{ww'}$ given by composition; thus we wish to show that this map is an isomorphism.

First, we note that this map is a surjection. We can choose reduced expressions of w and w' and concatenate these to get one for ww'. Thus, we can choose the element $\psi_{v,\kappa'} e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ so that the diagram above $y = 1/2$ has red strands permuted by w' and below $y = 1/2$ has red strands permuted by w. This writes $\psi_{v,\kappa'}e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ as the image of the tensor of the 2 halves of the diagram.

Now assume that we have an element p of the tensor product which is sent to 0 in $\tilde{\mathfrak{B}}_{ww'}$. We can think of each pure tensor of w- and w'-Stendhal diagrams as a ww'-Stendhal diagram by composition. The relations in $\tilde{\mathfrak{B}}_{ww'}$ are the usual local relations of Stendhal diagrams, and the relations in $\tilde{\mathfrak{B}}_w \otimes_{\tilde{T}} \tilde{\mathfrak{B}}_{w'}$ are those relations applied to pictures in one of the two halves of the tensor (i.e. above or below $y = 1/2$) and the fact that one can isotope diagrams from one side of $y = 1/2$ to the other. The image of the element p is thus a sum of ww'-Stendhal diagrams which can be sent to 0 using our relations. In fact, as argued in Proposition 6.4, we never need to use the relation (6.1c) to write this sum as a sum of basis vectors (and thus to show that it is 0). Any other relation can be pushed above or below the line $y = 1/2$ so that it is a relation in the tensor product. Thus, the map is injective.

Now we turn to considering the tensor product $\mathfrak{B}_w \otimes_T \mathfrak{B}_{w'}$; this obviously receives a map from $\tilde{\mathfrak{B}}_w \otimes_{\tilde{T}} \tilde{\mathfrak{B}}_{w'} \cong \tilde{\mathfrak{B}}_{ww'}$, and this map sends violated diagrams to tensors of diagrams where one is violated. Thus, it induces a map $\mathfrak{B}_{ww'} \to \mathfrak{B}_w \otimes_T \mathfrak{B}_{w'}$ which is inverse to the obvious composition map. This shows that $\mathfrak{B}_{ww'} \cong \mathfrak{B}_w \otimes_T \mathfrak{B}_{w'}$ □

Definition 6.6 *Let $\mathbb{B}_w$ be the functor $- \overset{L}{\otimes} \mathfrak{B}_w : D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}}) \to D^-(\mathfrak{V}^{w\cdot\underline{\boldsymbol{\lambda}}}_{\mathbb{C}})$. We'll use $\mathbb{B}_j$ to denote $\mathbb{B}_{s_j}$.*

Here, $D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}})$ refers to the bounded above derived category of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}}$; *a priori*, the functor $\mathbb{B}_k$ does not obviously preserve the subcategory $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}} \subset D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}})$. In order to show this, and certain other important properties of this functor, we require some technical results. Of course, the image of a projective $P^{\kappa}_{\underline{\boldsymbol{\lambda}}}$ is easy to understand in diagrammatic terms: $\mathbb{B}_w(P^{\kappa}_{\underline{\boldsymbol{\lambda}}}) = e(\mathbf{i}, \kappa)\mathfrak{B}_w$ is given by the span of w-Stendhal diagrams with the top fixed to be the idempotent $e(\mathbf{i}, \kappa)$, and with $T^{w\cdot\underline{\boldsymbol{\lambda}}}$ acting by attaching at the bottom (thought of as a 1-term complex).

Proposition 6.7 *The functors $\mathbb{B}_k$ commute with all 1-morphisms in $\mathcal{U}$; in fact, $\mathbb{B}_k$ is a strongly equivariant functor $D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}_{\mathbb{C}}) \to D^-(\mathfrak{V}^{s_k\underline{\boldsymbol{\lambda}}}_{\mathbb{C}})$.*

Proof. Both $\mathfrak{F}_i$ and $\mathfrak{E}_i$ can be written as tensor product with bimodules $\beta_{\mathcal{F}_i}$ and $\beta_{\mathcal{E}_i}$; since both these functors are exact and preserve projectives, these bimodules are projective as left and right modules. Thus, the desired isomorphism of functors will be yielded by an isomorphism $\beta_{\mathcal{E}_i} \otimes \mathfrak{B}_{s_k} \cong \mathfrak{B}_{s_k} \otimes \beta_{\mathcal{E}_i}$; the same result for $\mathcal{F}_i$ will follow by biadjunction. The bimodule $\mathfrak{B}_{s_k} \otimes \beta_{\mathcal{E}_i}$ can be identified with a subspace of $\mathfrak{B}_{s_k}$ where we require that the rightmost strand at the bottom is black and colored i; the right (that is, bottom) action of $T^{s_i\underline{\boldsymbol{\lambda}}}$ ignores this strand and acts on the others.

The tensor product $\beta_{\mathcal{E}_i} \otimes \mathfrak{B}_{s_k}$ maps injectively into this space; its image is that of s_k-Stendhal diagrams where the strand at far right at the bottom (i.e. that which makes the cup) cannot pass below the red/red crossing, since we must have pulled this to the side before adding the red/red crossing. But this map is easily seen to be surjective, since the vector $\psi_{v,\kappa'}e(\mathbf{i}, \underline{\boldsymbol{\lambda}}, \kappa)$ can be chosen to never pass a black strand under the red/red crossing, using the relation 6.1b.

This proves that for any 1-morphism u, we have an isomorphism $\beta_u \otimes \mathfrak{B}_{s_k} \cong \mathfrak{B}_{s_k} \otimes \beta_u$; if we picture β_u as in (4.6), the former module comes from putting the red/red crossing at the bottom of the diagram, and the latter from putting it at the top left. The isomorphism is simply using the relation (6.1b) to slide the crossing from the top to the bottom or *vice versa*. Since the action of 2-morphisms is by attachment at top right, it does not matter whether we do this before or after we slide the crossing. This shows the strong equivariance of this functor. □

Note that $P^0_{\mathbf{i}}$ is the image of $P^0_{\emptyset} = T^{\boldsymbol{\lambda}}_{\lambda}$ under the 1-morphism in $\mathcal{U}$ given $-\mathbf{i}$. Abusing notation to let $P^0_{\mathbf{i}}$ denote the corresponding module over both $T^{\underline{\boldsymbol{\lambda}}}$ and $T^{s_i \cdot \underline{\boldsymbol{\lambda}}}$, this shows that:

Corollary 6.8 $\mathbb{B}_i P^0_{\mathbf{i}} \cong P^0_{\mathbf{i}}(\langle \lambda_i, \lambda_{i+1} \rangle)$.

Let $\underline{\boldsymbol{\lambda}}^{(j)}$ be the sequence of sequences $(\lambda_1; \dots; \lambda_{j-1}; \lambda_j, \lambda_{j+1}; \lambda_{j+2}; \dots; \lambda_\ell)$; that is almost all weights are singletons, but λ_j and λ_{j+1} are together in a block. We consider the category $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}^{(j)}}_{\mathbb{C}} := T^{\lambda_1} \otimes \cdots \otimes T^{(\lambda_j, \lambda_{j+1})} \otimes \cdots \otimes T^{\lambda_\ell}$-mod. We can define a functor $\mathbb{B}_j \colon D^-(\mathfrak{V}^{\underline{\boldsymbol{\lambda}}^{(j)}}_{\mathbb{C}}) \to D^-(\mathfrak{V}^{s_j \underline{\boldsymbol{\lambda}}^{(j)}}_{\mathbb{C}})$ by derived tensor product with the bimodule

$$T^{\lambda_1} \boxtimes \cdots \boxtimes T^{\lambda_{j-1}} \boxtimes \mathfrak{B} \boxtimes T^{\lambda_{j+2}} \boxtimes \cdots \boxtimes T^{\lambda_\ell}$$

where $\mathfrak{B}$ is the bimodule where we switch the two strands labeled with λ_j and λ_{j+1}.

Lemma 6.9 *The functor $\mathbb{B}_j$ commutes with the standardization functor $\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}$.*

Proof. As with the commutation of $\mathbb{B}_j$ with $\mathcal{U}$, the proof in terms of naive tensor products is simply identifying the pictures that describe elements of the tensor product. So, having fixed a Stendhal triple at the top, elements of the 0th cohomology of $\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}$ are just s_j-Stendhal diagrams modulo

- the usual local relations,
- the standardly violating relations that kill a right crossing above all left crossings on the red strands other than the λ_j and λ_{j+1}, and
- the same relation on the λ_j strand above the red/red crossing.

For $\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}} \circ \mathbb{B}_j$, we impose this last relation on the λ_{j+1} strand *below* the red/red crossing (so at horizontal slices where the λ_{j+1} strand is *left* of the λ_j).

Now, assume we have a strand which originates between the λ_j and λ_{j+1} red strands and has a violation below the crossing. If at the y-value of the red/red crossing, this strand is left of the crossing, it also has a violation above the crossing; if it is right of the crossing, it must have crossed both red strands below the crossing, and (6.1b) gives us a violation above it.

On the other hand, if we have a strand which originates right of both strands, it can only violate below the crossing if it crosses both red strands, and we can use (6.1b) again.

Thus, it suffices to check that the higher cohomology of both functors applied to projectives vanishes. This is clear for $\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}} \circ \mathbb{B}_j$ by the exactness of $\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}$, so we need only show it for $\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}$. We'll prove this by induction on the preorder on Stendhal triples. If $\boldsymbol{\alpha}$ is maximal, then any induction of a projective is again projective and we are done. If not, then there is a map from a projective Q to

$\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P)$ such that the kernel K is filtered with higher standardizations. Consider the usual long exact sequence:
(6.2)

$$\cdots H^{k+1}(\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P)) \to H^k(\mathbb{B}_j(K)) \to H^k(\mathbb{B}_j(Q)) \to H^k(\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P)) \to H^{k-1}(\mathbb{B}_j(K)) \to \cdots$$

Since the higher cohomology for both Q and K vanish, it immediately follows that $H^k(\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P)) = 0$ for $k > 1$, and $H^1(\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P))$ is the kernel of the map $H^0(\mathbb{B}_j(K)) \to H^0(\mathbb{B}_j(Q))$. Thus, it remains to show that this map is injective.

We can assume that Q is of the form $P^{\kappa}_{\mathbf{i}}$ and $\mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}(P)$ is its standard quotient for the sequence $\underline{\boldsymbol{\lambda}}^{(j)}$. This is not the module we denote $S^{\kappa}_{\mathbf{i}}$, since we are not imposing the standardly violating relation on the $j+1$st strand. In this case, $K \subset P^{\kappa}_{\mathbf{i}}$ is the set of all diagrams with a standardly violating strand on a red line other than the $j+1$st.

In order to show that $\mathbb{B}_j(K) \to \mathbb{B}_j(Q)$ is injective, it is enough to show that any sum of s_k-Stendhal diagrams which is 0 in $\mathbb{B}_j(Q)$ is a sum of arbitrary s_k-Stendhal diagrams composed with ones that are 0 in K. If one has a violated s_k-Stendhal diagram, then one can always push the red crossing below one of the violation points. This is possible by isotopy as long as the violating strand does not cross below the red/red crossing; if it does pass below, we can use the relations, in particular (6.1b), to push the violating strand above the crossing. Thus, if we isotope so that the red/red crossing is below $y = 1/2$ and one of the violations above it, we can cut along $y = 1/2$, and obtain the desired composition.

This allows us to write any element of the kernel of $\mathbb{B}_j(K) \to \mathbb{B}_j(Q)$ in terms of elements that are 0 in K, so the map is injective. Thus, substituting into (6.2), we see that $\mathbb{B}_j \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}^{(j)}}$ is exact. □

Proposition 6.10 $\mathbb{B}_j\left(\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\dots;\mathbf{i}_j;\emptyset;\dots})\right) \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\dots;\emptyset;\mathbf{i}_j;\dots})\Big(\big\langle \lambda_j - \boldsymbol{\alpha}(j), \lambda_{j+1}\big\rangle\Big)$

Proof. By Lemma 6.9, we can immediately reduce to the case where $\ell = 2$. In this case, $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\mathbf{i}_j;\emptyset})$ is projective, so $\mathbb{B}_j\left(\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\dots;\mathbf{i}_j;\emptyset;\dots})\right)$ is the naive tensor product of these modules. The isomorphism to $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\dots;\emptyset;\mathbf{i}_j;\dots})\Big(\big\langle \lambda_j - \boldsymbol{\alpha}(j), \lambda_{j+1}\big\rangle\Big)$ is the single diagram shown in Figure 2. □

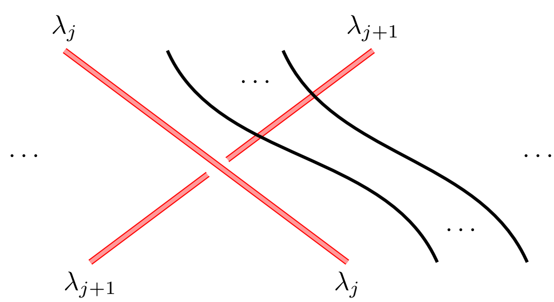


Figure 2. The generator of $\mathbb{B}_j\left(\mathbb{S}^{\boldsymbol{\lambda}}(P_{\dots;\mathbf{i}_j;\emptyset;\dots})\right)$.

Corollary 6.11 *The action of* $\mathbb{B}_k$ *categorifies the action of the braiding* $V^{\mathbb{C}}_{\underline{\boldsymbol{\lambda}}} \to V^{\mathbb{C}}_{s_k\underline{\boldsymbol{\lambda}}}$ *switching the* k *and* $k+1$*st representations with a positive crossing.*

Proof. By Proposition 6.7, the induced action on $V_{\underline{\boldsymbol{\lambda}}}$, which we denote by $\mathcal{R}_\sigma$, commutes with the action of $U_q^-(\mathfrak{g})$. Thus we need only calculate the action of R_σ on a pure tensor of weight vectors with a *highest* weight vector v_h in the $j+1$st place, since these generate $V_{\underline{\boldsymbol{\lambda}}}$ as a $U_q^-(\mathfrak{g})$ -representation.

The space of such vectors is spanned by the classes of the form $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_{\dots;\mathbf{i}_j;\emptyset;\dots})$. Thus, Proposition 6.10 implies that

$$\mathcal{R}_\sigma(v_1 \otimes \cdots \otimes v_j \otimes v_h \otimes \cdots \otimes v_\ell) = q^{\langle \mathrm{wt}(v_j), \lambda_{j+1}\rangle} v_1 \otimes \cdots \otimes v_h \otimes v_j \otimes \cdots \otimes v_\ell$$

which is exactly what the braiding (1.1) does to vectors of this form. Since vectors of this form generate the representation over $U_q(\mathfrak{g})$, there is a unique endomorphism with this behavior, and R_σ is the braiding. $\square$

Lemma 6.12 *For any projective* $P^\kappa_{\mathbf{i}}$, *the module* $\mathbb{B}_w(P^\kappa_{\mathbf{i}})$ *has a standard filtration. If* $w > ws_i$ *then* $\mathbb{B}_w \cong \mathbb{B}_{ws_i} \otimes \mathbb{B}_i$.

Proof. We will prove this by induction on the length of w. This induction is slightly subtle, so rather than attempt each step in one go, we break the theorem into 3 statements, and induct around a triangle. Consider the three statements (for each positive integer n):

- p_n : For all w with $\ell(w) = n$, if $w > ws_i$ then $\mathbb{B}_w \cong \mathbb{B}_{ws_i} \otimes \mathbb{B}_i$.
- f_n : For all w with $\ell(w) = n$, $\mathbb{B}_w$ sends projectives to objects with standard filtrations.
- s_n : For all w with $\ell(w) = n$, $\mathbb{B}_w$ sends standards to modules; that is, $\mathrm{Tor}^k(S^\kappa_{\mathbf{i}}, \mathfrak{B}_w) = 0$ for all $k > 0$.

Our induction proceeds by showing

$$\cdots \to p_n \to f_n \to s_n \to p_{n+1} \to \cdots$$

These are all obviously true for $w = 1$, so this covers the base of our induction.

$f_n \to s_n$: Consider the groups $\mathrm{Tor}^k(S^\kappa_{\mathbf{i}}, \dot{S}^{\kappa'}_{\mathbf{i}'})$. By symmetry, we may assume that $(\kappa, \mathbf{i}) \not< (\kappa', \mathbf{i}')$ in which case $S^\kappa_{\mathbf{i}}$ has a projective resolution where all higher terms are killed by tensor product with $\dot{S}^{\kappa'}_{\mathbf{i}'}$, since they are projective covers of simples which do not appear as composition factors in $S^{\kappa'}_{\mathbf{i}'}$. Thus, we have $\mathrm{Tor}^k(S^\kappa_{\mathbf{i}}, \dot{S}^{\kappa'}_{\mathbf{i}'}) = 0$ for $k > 0$, and for $k = 0$ if $(\kappa, \mathbf{i}) \neq (\kappa', \mathbf{i}')$.

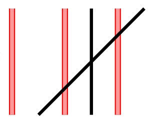
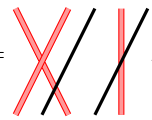
If we let $\dot{\mathfrak{B}}_w$ be $\mathfrak{B}_w$ with the left and right actions reversed by the dot-anti-automorphism, then $\dot{\mathfrak{B}}_w \cong \mathfrak{B}_{w^{-1}}$. By f_n, the bimodule $\mathfrak{B}_{w^{-1}}$ has a standard filtration as a right module, so $\mathfrak{B}_w$ has a standard filtration as a left module. Thus, we have $\mathrm{Tor}^k(S^\kappa_{\mathbf{i}}, \mathfrak{B}_w)$ for $k > 0$ and the same holds for any module with a standard filtration.

$s_n + f_n \to p_{n+1}$: By Lemma 6.5, we have that $\mathfrak{B}_w \cong \mathfrak{B}_{ws_i} \otimes \mathfrak{B}_i$, so we need only show that the higher Tor's of this tensor product vanish. By f_n, as a right module $\mathfrak{B}_{ws_i}$ has a standard filtration, as does $\mathfrak{B}_i$ as a left module by Lemma 6.9 (note that this follows from f_n if $n \geq 1$, but one needs to use Lemma 6.9 when $n = 0$). Thus, as we argued above, the higher Tor's vanish, and we are done.

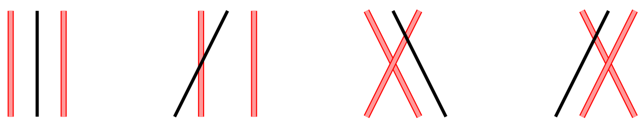
$p_n \to f_n$: Now, we construct the standard filtration on $D = \mathbb{B}_w P^\kappa_{\mathbf{i}}$. Let Φ be the parameter set of the standard filtration on the projective as defined on page

78. We let $\breve{w}$ be the permutation of terminals which keeps together a red strand and the black block to its right, and permutes these groups according to w. We let Φ^w be the permutations obtained by composing $\phi \in \Phi$ with $\breve{w}$ on the bottom.

We let y_ϕ be a choice of w-Stendhal diagram which realizes the permutation $\phi \in \Phi^w$ with a minimal number of crossings; if $w = 1$, then these satisfy the same conditions as the elements x_ϕ defined earlier. The diagrams y_ϕ are representatives of the isotopy classes of diagrams where we cannot factor off a black/black crossing, or a left crossing (as depicted in (5.3)) at the bottom of the diagram. Note that y_ϕ does not actually have to arise from composing x_ϕ with an element of the bimodule $\mathfrak{B}_w$; there may be strands that cross in x_ϕ which do not in y_ϕ. We might have a situation like:

$x_\phi =$  $y_\phi =$  .

If $\mathfrak{g} = \mathfrak{sl}_2, \lambda = (1,1), \mathbf{i} = (1)$ and we consider $\kappa(1,2) = 0, 1$ then both Φ and Φ^s have two elements with associated diagrams given by



As before, we can preorder these elements according the preorder on the idempotents found at their bottom. Note that the bijection between Φ and Φ^w is not order preserving.

We wish to show that the elements y_ϕ generate $\mathfrak{B}_w$ as a right module. For ease, let us isotope the diagram so that all red/red crossings occur above $y = 1/2$. Now we wish to apply the relations to write an arbitrary element as a sum of diagrams where the top half is of the form y_ϕ. As usual, it is enough to start with an arbitrary diagram, and rewrite as a sum of diagrams with top half given by y_ϕ, plus elements with fewer crossings, and then use induction.

As explained above, if the diagram above $y = 1/2$ is not isotopic to a y_ϕ, then we can perform an isotopy to move a dot, a black/black crossing or a left red/black crossing to lie directly above $y = 1/2$; then we can isotope this offending element through $y = 1/2$. Since this reduces the number of crossings or dots above $y = 1/2$, eventually this process will terminate. This shows that the elements y_ϕ generate.

This allows us to construct a filtration

$$D_{\leq \phi} = \sum_{\phi' \leq \phi} y_\phi T^{w \cdot \underline{\boldsymbol{\lambda}}} \qquad D_{< \phi} = \sum_{\phi' < \phi} y_\phi T^{w \cdot \underline{\boldsymbol{\lambda}}}$$

out of these elements and partial order; while the element y_ϕ involves a choice of reduced word, this filtration is independent of it. Multiplication by y_ϕ gives a surjection $d : S^{\kappa_\phi}_{\mathbf{i}_\phi} \twoheadrightarrow D_{\leq \phi}/D_{<\phi}$, which we aim to show is an isomorphism.

Since $\mathbb{B}_w$ categorifies the braiding attached to the positive lift of w to a braid, when q is specialized to 1, it categorifies the permutation map $V_{\underline{\boldsymbol{\lambda}}} \to V_{w \cdot \underline{\boldsymbol{\lambda}}}$, and is thus an isometry for $\langle -, - \rangle_1$. In particular,

$$\dim \mathbb{B}_w = \langle [T^{w \cdot \underline{\boldsymbol{\lambda}}}], w \cdot [T^{\underline{\boldsymbol{\lambda}}}] \rangle_1 = \sum_{\phi \in \Phi} \langle [T^{w \cdot \underline{\boldsymbol{\lambda}}}], [S^{\kappa_\phi}_{\mathbf{i}_\phi}] \rangle_1 = \sum_{\phi \in \Phi} \dim S^{\kappa_\phi}_{\mathbf{i}_\phi}$$

which shows that all the maps $S^{\kappa_\phi}_{\mathbf{i}_\phi} \twoheadrightarrow D_{\leq \phi}/D_{<\phi}$ must be isomorphisms. □

Lemma 6.13 *The functor $\mathbb{B}_w$ sends $\mathcal{V}_{\mathbb{C}}^{\boldsymbol{\lambda}}$ to $\mathcal{V}_{\mathbb{C}}^{w\cdot\boldsymbol{\lambda}}$.*

PROOF. From Lemma 6.12, we find that $\mathfrak{B}_w$ considered as a left module (which is the same as $\dot{\mathfrak{B}}_w$) has a standard filtration. By Corollary 5.24, standard modules have finite length projective resolutions. So any projective module M is sent to a finite length complex; since there are only finitely many indecomposable projectives, the amount which this can decrease the lowest degree is bounded below. Thus, a complex of projectives in $C^{\uparrow}(\mathfrak{V}_{\mathbb{C}}^{\boldsymbol{\lambda}})$ is sent to another collection of projectives in $C^{\uparrow}(\mathfrak{V}_{\mathbb{C}}^{w\cdot\boldsymbol{\lambda}})$. □

Consider the half twist τ. Note that according to our conventions, it is drawn with the blackboard framing, not the one with ribbon half-twists as well. Recall that a module M over a standardly stratified algebra is called **tilting** if

- M has a filtration by standards, that is, modules of the form $\mathbb{S}^{\lambda_1;\dots;\lambda_\ell}(P)$ for P projective and
- $M^\star$ has a filtration by standardizations, that is, modules of the form $\mathbb{S}^{\lambda_1;\dots;\lambda_\ell}(Q)$ for Q arbitrary.

Theorem 6.14 *The modules $\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ are tilting, and every indecomposable tilting module is a summand of these tiltings.*

PROOF. We show first that $\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ is self-dual. The pairing that achieves this duality is a simple variant on that described in Chapter 3, where as before, we form a closed diagram and evaluate its constant term.

The non-degeneracy of this pairing follows from that on $P_{\mathbf{i}}^{0}$. In Lemma 5.25, we have shown that $P_{\mathbf{i}}^{\kappa}$ has an embedding into $P_{\mathbf{i}}^{\kappa}$ into $P_{\mathbf{i}}^{0}$ consistent with the standard filtration, given by left multiplication by the element θ_κ. The quotient $P_{\mathbf{i}}^{0}/P_{\mathbf{i}}^{\kappa}$ is again filtered by standard modules, and this is sent to a module by $\mathbb{B}_\tau$ by Lemma 6.12. Thus, the usual long exact sequence shows that the induced map $\iota\colon \mathbb{B}_\tau P_{\mathbf{i}}^{\kappa} \to \mathbb{B}_\tau P_{\mathbf{i}}^{0} \cong P_{\mathbf{i}}^{0}((\langle\lambda,\lambda\rangle - \sum_{i=1}^{\ell}\langle\lambda_i,\lambda_i\rangle)/2)$ is again an injection (the last isomorphism follows by Corollary 6.8).

By Proposition 6.4, any non-zero diagram in $\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ can be drawn with a section in the middle where all black strands are right of all red strands. Thus, the map $P_{\mathbf{i}}^{0} \to P_{\mathbf{i}}^{\kappa}$ given by multiplication by $\dot{\theta}_\kappa$ is not surjective, but the induced map $\pi\colon P_{\mathbf{i}}^{0}((\langle\lambda,\lambda\rangle - \sum_{i=1}^{\ell}\langle\lambda_i,\lambda_i\rangle)/2) \cong \mathbb{B}_\tau P_{\mathbf{i}}^{0} \to \mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ is.

Note that $\iota\pi = y_{\mathbf{i},\kappa}\cdot$. Thus, the pairing we desire is defined by:

$$\langle \pi(a), \pi(b)\rangle_{\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}} = \operatorname{tr}(\dot{a}by_{\mathbf{i},\kappa}) = \operatorname{tr}(y_{\mathbf{i},\kappa}\dot{a}b).$$

This is well defined since if $\pi(a) = 0$, then $y_{\mathbf{i},\kappa}\dot{a} = 0$ and similarly for b.

We can alternatively define this as the unique pairing such that the maps π and ι are adjoint with respect to the Frobenius pairing on $P_{\mathbf{i}}^{0}$. This shows immediately that the perpendicular to the image of the inclusion contains the kernel of the surjection. Since these have the same dimension, they coincide and the pairing is non-degenerate. Thus, $\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ is self-dual.

By Lemma 6.12, $\mathbb{B}_\tau P_{\mathbf{i}}^{\kappa}$ has a filtration by standards. Since the element τ reverses the pre-order on standards, every standard which appears is below $(\kappa', \mathbf{i}')$, the sequence obtained from reversing the blocks of $(\kappa, \mathbf{i})$. So if $(\kappa, \mathbf{i})$ (and thus $(\kappa', \mathbf{i})$) is stringy, the indecomposable tilting whose highest composition factor is the head of $S^{\kappa'}_{\mathbf{i}'}$ is a summand of $\mathbb{B}_\tau P^\kappa_{\mathbf{i}}$. Thus, any tilting is a summand of $\mathbb{B}_\tau$ applied to a projective. □

Theorem 6.15 *The functor $\mathbb{B}_w$ is an equivalence $\mathcal{V}^{\boldsymbol{\lambda}}_{\mathbb{C}} \to \mathcal{V}^{w\boldsymbol{\lambda}}_{\mathbb{C}}$ for every $w \in W$.*

PROOF. We will first show $\mathbb{B}_\tau$ is a derived equivalence. The higher Ext's between tilting modules always vanish so we always have that $\operatorname{Ext}^{>0}(\mathbb{B}_\tau P^\kappa_{\mathbf{i}}, \mathbb{B}_\tau P^{\kappa'}_{\mathbf{i}'}) = 0$; thus we need only show that induced map between endomorphisms of these modules is an isomorphism.

It follows from Corollary 6.11 that

$$\dim \operatorname{Hom}(\mathbb{B}_\tau P^\kappa_{\mathbf{i}}, \mathbb{B}_\tau P^\kappa_{\mathbf{i}'}) = \langle [\mathbb{B}_\tau P^\kappa_{\mathbf{i}}], [\mathbb{B}_\tau P^\kappa_{\mathbf{i}'}]\rangle_1 = \langle [P^\kappa_{\mathbf{i}}], [P^{\kappa'}_{\mathbf{i}'}]\rangle_1 = \dim \operatorname{Hom}(P^\kappa_{\mathbf{i}}, P^{\kappa'}_{\mathbf{i}'}).$$

The functor $\mathbb{B}_\tau$ induces a map

$$\operatorname{Hom}(P^\kappa_{\mathbf{i}}, P^\kappa_{\mathbf{i}'}) \longrightarrow \operatorname{Hom}(\mathbb{B}_\tau P^\kappa_{\mathbf{i}}, \mathbb{B}_\tau P^\kappa_{\mathbf{i}'}).$$

This is injective, since no element of the image kills the element which pulls all black strands to the right of all red strands below all crossings, by Lemma 5.25. Thus, it is surjective by the dimension calculation above.

It follows that $\mathbb{B}_\tau$ is an equivalence. Since it factors through any $\mathbb{B}_k$ on the left and right, the functor $\mathbb{B}_k$ is an equivalence as well. Since all other $\mathbb{B}_\sigma$ is a composition of these functors and their adjoints, these are equivalences, finishing the proof. □

If σ is a braid, recall that σ has a canonical factorization called **Garside decomposition** $\sigma = \tau^p \xi_1 \cdots \xi_m$ into minimal lifts of non-longest permutations $w_1, \dots, w_m$, with τ a positive lift of the longest element of S_ℓ, and $p \in \mathbb{Z}$. First, p is is the lowest integer such that $\tau^{-p}\sigma$ is a positive braid. Then, the first factor ξ_1 is by definition the longest positive lift of a permutation such that $\xi_1^{-1}\tau^{-p}\sigma$ is still positive, and the rest of the decomposition is constructed inductively.

Definition 6.16 *Let $\mathbb{B}_\sigma := \mathbb{B}^p_\tau \mathbb{B}_{w_1} \cdots \mathbb{B}_{w_n}$.*

Corollary 6.17 *If $\sigma = \tau^{p'} \xi'_1 \cdots \xi'_q$ is any other factorization of σ into a power of τ and minimal positive lifts of $w'_1, \dots, w'_q$, then we have an isomorphism of functors $\mathbb{B}_\sigma \cong \mathbb{B}^{p'}_\tau \mathbb{B}_{w'_1} \cdots \mathbb{B}_{w'_q}$.*

PROOF. By multiplying by a high power of τ, we can assume that the braid is positive. Let us induct on the length $\ell(\sigma)$ of the braid. Pick a reduced expression for each w'_j; by induction, $\mathbb{B}_{w'_j}$ is isomorphic to the composition of the functors corresponding to these simple reflections. This allows us to reduce to the case where each w'_j has length 1.

The result is true when the Garside decomposition has length 1, since we can apply the statement p_n proven in the proof of Lemma 6.12 to write $\mathbb{B}_w \cong \mathbb{B}_{ws}\mathbb{B}_s$.

In the general case, this shows that the resulting functor will not change when one refines any factorization. This establishes the general case, since any two reduced expressions for the braid are related by a finite chain of Reidemeister III moves, i.e. a chain where each consecutive pair are two different refinements of a single factorization. Thus, starting with any factorization, we can refine to a product of simple twists, and then apply Reidemeister III moves until we arrive at a refinement of the Garside decomposition. ☐

Corollary 6.18 *The functors $\mathbb{B}_\sigma$ induce a strong action of the braid group on the categories $\bigoplus_{w\in S_\ell}\mathcal{V}_{\mathbb{C}}^{w\cdot\underline{\boldsymbol{\lambda}}}$.*

Proof. By work of Elias and Williamson [**EW**, 1.18], it suffices to show that we have isomorphisms lifting the braid relations which satisfy the Zamolodchikov tetrahedral equations. This will hold since we have defined a canonical functor not just for braid generators, but for all positive lifts of permutations.

By Lemma 6.12, the composition $\mathbb{B}_{\sigma_i}\circ\mathbb{B}_{\sigma_{i+1}}\circ\mathbb{B}_{\sigma_i}$ is the derived tensor product with $\mathfrak{B}_{\sigma_i}\otimes\mathfrak{B}_{\sigma_{i+1}}\otimes\mathfrak{B}_{\sigma_i}\cong\mathfrak{B}_{\sigma_i\sigma_{i+1}\sigma_i}$. By Lemma 6.5, we have a canonical isomorphism of this functor with $\mathbb{B}_{\sigma_{i+1}}\circ\mathbb{B}_{\sigma_i}\circ\mathbb{B}_{\sigma_{i+1}}$.

Given any reduced expression for the longest permutation of 4 consecutive strands, we can apply these isomorphisms to go around the loop of the Zamolodchikov tetrahedral equation, collapsing empty red triangles in the desired sequence. Since can use the relations to pull all black strands out of all the polygons created by the red strands in the permutation of 4 strands, going around this loop sends a diagram to itself.

This checks the necessary relation in terms of maps between modules. Thus the induced natural transformations on projective resolutions satisfy the Zamolodchikov tetrahedron equations up to homotopy, so the natural transformations between derived functors satisfy the same equations on the nose. ☐

Recall that the **Ringel dual** of a standardly stratified category is the category of modules over the endomorphism ring of a tilting generator, that is, the opposite category to the heart of the t-structure in which the tiltings are projective.

Corollary 6.19 *The Ringel dual of $\mathfrak{V}_{\mathbb{C}}^{\underline{\boldsymbol{\lambda}}}$ is equivalent to $\mathfrak{V}_{\mathbb{C}}^{\tau\cdot\underline{\boldsymbol{\lambda}}}$.*

If C_i and C_i' are semi-orthogonal decompositions indexed by $i\in[1,n]$ then C_i' is the **mutation** of C_i by a permutation σ if, for each $j\in[1,n]$, the category generated by C_i for $i\geq j$ is the same as that generated by $C'_{\sigma(i)}$ for $i\geq j$.

Proposition 6.20 *For any braid σ, $\mathbb{B}_\sigma$ sends the semi-orthogonal decomposition of Proposition* 5.26 *to its mutation by σ.*

Proof. First, note that we need only show this for σ_k. Of course, an equivalence sends one semi-orthogonal decomposition to another. Thus, the only point that remains to show is that $\mathbb{B}_{\sigma_k}(S_{\boldsymbol{\alpha}})$ for $\boldsymbol{\alpha}\geq\boldsymbol{\beta}$ generates the same subcategory as $S'_{\boldsymbol{\alpha}}$ for $\sigma_k^{-1}(\boldsymbol{\alpha})\leq\boldsymbol{\beta}$, where $S'_{\boldsymbol{\alpha}}$ denotes the appropriate standard module in $\mathfrak{V}_{\mathbb{C}}^{\sigma_k\cdot\underline{\boldsymbol{\lambda}}}$. Call these subcategories C_1 and C_2. Now let $P_{\boldsymbol{\alpha}}$ be the projective cover of $S_{\boldsymbol{\alpha}}$. First, note that the category C_1 is the same that generated by $\mathbb{B}_{\sigma_k}(P_{\boldsymbol{\alpha}})$ for $\boldsymbol{\alpha}\geq\boldsymbol{\beta}$, since the kernel of the map $P_{\boldsymbol{\alpha}}\to S_{\boldsymbol{\alpha}}$ is filtered by summands of $S_{\boldsymbol{\alpha}'}$ for $\boldsymbol{\alpha}'>\boldsymbol{\alpha}$ by Corollary 5.22. On the other hand, $\mathbb{B}_{\sigma_k}(P_{\boldsymbol{\alpha}})$ also has a filtration in which $S_{\sigma_k(\boldsymbol{\beta})}$ appears with multiplicity 1, and all other constituents are summands of $S'_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha}>\sigma_k(\boldsymbol{\beta})$ by Lemma 6.12. This completes the proof. ☐

2. Serre functors

It is a well-supported principle (see, for example, Beilinson, Bezrukavnikov and Mirković [**BBM04**] or Mazorchuk and Stroppel [**MS08**]) that for any suitable braid group action on a category, the Serre functor will be given by the full twist. Here the same is true, up to grading shift. Let $\mathfrak{R} = \mathbb{B}_\tau^2$ be the functor given by a full positive twist of the red strands. Let $\mathfrak{S}'$ be the functor sending $M \in \mathcal{V}^{\boldsymbol{\lambda}}_{\mathbb{C},\alpha}$ to $M\big(-\langle\alpha,\alpha\rangle + \sum_{i=1}^{\ell}\langle\lambda_i,\lambda_i\rangle\big)$. Let $\mathcal{V}^{\boldsymbol{\lambda}}_{\mathsf{per}}$ be the full subcategory of $\mathcal{V}^{\boldsymbol{\lambda}}_{\mathbb{C}}$ given by bounded perfect complexes, that is, objects which have finite projective dimension. We note that in general, this subcategory does not contain many of the important objects in $\mathcal{V}^{\boldsymbol{\lambda}}_{\mathbb{C}}$; for example, it will contain all simple modules if and only if all $\underline{\boldsymbol{\lambda}}$ are minuscule.

Proposition 6.21 *The right Serre functor of $\mathcal{V}^{\boldsymbol{\lambda}}_{\mathsf{per}}$ is given by $\mathfrak{S} = \mathfrak{R}\mathfrak{S}'$.*

Proof. First consider the action of $\mathfrak{S}$ on projective-injectives. This is the same as to say on $P^0_{\mathbf{i}}$, since these modules generate the additive category of projective-injectives. The twists of red strands are irrelevant to black strands that begin to the right of all of them, so

$$\mathfrak{R} \cong \mathrm{Id}\left(\langle\lambda,\lambda\rangle - \sum_{i=1}^{\ell}\langle\lambda_i,\lambda_i\rangle\right)$$

as functors on the projective-injective category. We let $I^\kappa_{\mathbf{i}}$ be the injective hull of the cosocle of $P^\kappa_{\mathbf{i}}$. Since $I^0_{\mathbf{i}} \cong P^0_{\mathbf{i}}(\langle\lambda,\lambda\rangle - \langle\alpha,\alpha\rangle)$, on this subcategory $\mathfrak{S}P^0_{\mathbf{i}} = P^0_{\mathbf{i}}(\langle\lambda,\lambda\rangle - \langle\alpha,\alpha\rangle) \cong I^0_{\mathbf{i}}$ and so $\mathfrak{S}$ is the graded Serre functor.

On general grounds, we know that the modules $\mathbb{B}_\tau^{-1}I^\kappa_{\mathbf{i}}$ and $\mathbb{B}_\tau P^\kappa_{\mathbf{i}}$ are dual. However, we proved in Theorem 6.14 that $\mathbb{B}_\tau P^\kappa_{\mathbf{i}}$ is a self-dual tilting module and so $\mathbb{B}_\tau^{-1}I^\kappa_{\mathbf{i}} \cong \mathbb{B}_\tau P^\kappa_{\mathbf{i}}$ (ignoring grading for the moment). Thus, $\mathfrak{R}P^\kappa_{\mathbf{i}} \cong I^\kappa_{\mathbf{i}}$ (again, ignoring the grading). In particular, $\mathfrak{R}$ sends projectives to injectives, and is an equivalence by Theorem 6.15. By [**MS08**, Theorem 3.4], the result follows. □

CHAPTER 7

Rigidity structures

Throughout this chapter and the next, $\mathfrak{g}$ is assumed to be finite-dimensional. Let D be the determinant of the Cartan matrix. For technical reasons, most convenient to use $V_{\underline{\boldsymbol{\lambda}}}^{1/D} = V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}[q^{1/D}]$. To categorify this, we consider the categories $\mathfrak{V}^{\boldsymbol{\lambda}}_{1/D}$ and $\mathcal{V}^{\boldsymbol{\lambda}}_{1/D}$ where we allow gradings in $\frac{1}{D}\mathbb{Z}$ rather than just $\mathbb{Z}$.

1. Coevaluation and evaluation for a pair of representations

Now, we must consider the cups and caps in our theory. The most basic case of this is $\underline{\boldsymbol{\lambda}} = (\lambda, \lambda^*)$, where we use $\lambda^* = -w_0\lambda$ to denote the highest weight of the dual representation to V_λ. It is important to note that $V_\lambda \cong V_{\lambda^*}^*$, but this isomorphism is not canonical.

In fact, the representation $K_0(T^{\lambda})$ comes with more structure, since it is an integral form $V_{\underline{\boldsymbol{\lambda}}}^{\mathbb{Z}}$. In particular, it comes with a distinguished highest weight vector v_h, the class of the unique simple module over $T_\lambda^\lambda \cong \Bbbk$ which is 1-dimensional and concentrated in degree 0. Thus, in order to fix the isomorphism above, we need only fix a lowest weight vector v_l of V_{λ^*}, and take the unique invariant pairing such that $\langle v_h, v_l\rangle = 1$.

Our first step is to better understand the lowest weight category $T^\lambda_{w_0\lambda}$-mod. This is most efficiently done not by considering it in isolation, but in the context of the other extremal weight spaces. Consider a reduced expression $\mathbf{w} = (s_{i_1}, \dots, s_{i_k})$ of $w \in W$ in the Weyl group of $\mathfrak{g}$, and let w_j be the product of the first j reflections in this word.

Definition 7.1 *Consider the sequence*

$$\mathbf{i}_{\mathbf{w}}^{\lambda} = \big(i_1^{(\lambda^{i_1})}, i_2^{\left((w_1\lambda)^{i_2}\right)}, \dots, i_k^{\left((w_{k-1}\lambda)^{i_k}\right)}\big)$$

For example, if $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = a\omega_1 + b\omega_2$ and $\mathbf{w} = (1,2,1)$, then $\mathbf{i}_{(1,2,1)}^{\lambda} = (1^{(a)}, 2^{(a+b)}, 1^{(b)})$. Note that the number of black strands for a reduced expression of w_0 is given by $2\rho^\vee(\lambda)$.

Proposition 7.2 *The projective $P^0_{\mathbf{i}_{\mathbf{w}}^{\lambda}}$ over T^λ is irreducible, and only depends on w.*

Proof. Since the corresponding weight space is one dimensional, there can only be a single irreducible up to isomorphism, which shows that independence of expression will follow from simplicity.

The irreducibility is easily proven by induction: $P^0_{\mathbf{i}^\lambda_\emptyset}$ is obviously irreducible, and if we assume that $P^0_{\mathbf{i}^\lambda_{(s_{i_1},\dots,s_{i_{k-1}})}}$ is irreducible, [**CR08**, 5.20(a)] proves the simplicity of $P^0_{\mathbf{i}^\lambda_{\mathbf{w}}}$ applied to $\mathcal{F}_{i_k}$ (in place of E). $\square$

Fix an expression $\mathbf{w}_0$ for the longest element w_0 and consider this construction for $\mathbf{i}^\lambda = \mathbf{i}^\lambda_{\mathbf{w}_0}$. We fix $v_l = [P^0_{\mathbf{i}^\lambda}]$. Since this is a non-zero lowest weight vectors, we can use this to fix an isomorphism $V_\lambda \cong V^*_{\lambda^*}$ which we use freely throughout the rest of the paper.

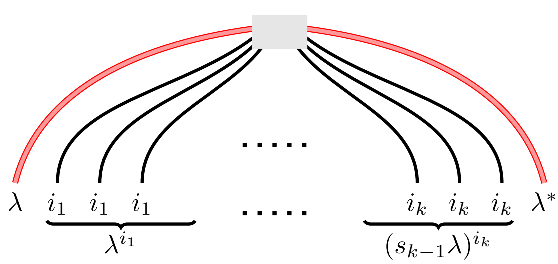
We can now consider the standardization of $P^0_{\mathbf{i}^\lambda} \boxtimes P_\emptyset$ obtaining the standard and projective module $S^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda} = P^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}$.

Lemma 7.3 *The module $S^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}$ has a unique simple quotient L_λ. The kernel of the projection map $S^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda} \to L_\lambda$ is the sum of images of every map from a projective $P^\kappa_{\mathbf{i}} \to S^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}$ with $\kappa(2) < 2\rho^\vee(\lambda)$.*

Proof. The existence of a unique simple quotient follows from Theorem 5.8 and Proposition 7.2. First, we must show that if $\kappa(2) < 2\rho^\vee(\lambda)$, then $\mathrm{Hom}(P^\kappa_{\mathbf{i}}, L_\lambda) = 0$. By adjunction, this is the same as proving that $\mathfrak{E}_i L_\lambda = 0$ for all i. By Theorem 5.14, there is a crystal isomorphism between the set of simples over T^{λ,λ^*} and the tensor product crystal. Thus, there is exactly one simple module over T^{λ,λ^*}_0 killed by all $\mathfrak{E}_i$. Every simple module other than L_λ is the image under the map h of simples (L_1, L_2) with the weight $\mathrm{wt}(L_1) > w_0\lambda$ and $\mathrm{wt}(L_2) < \lambda^*$; none of these are killed by all $\mathfrak{E}_i$. Thus, by the pigeonhole principle, L_λ just be the unique simple killed by these functors. This completes the proof. $\square$

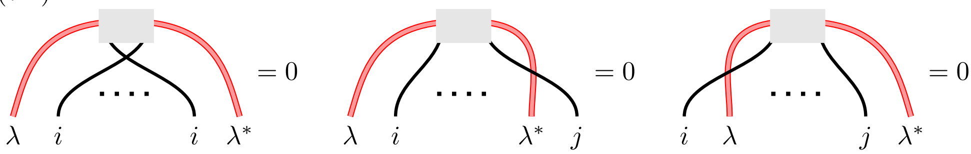
This theorem suggests a pictorial representation of L_λ which will be helpful for us in the future. We represent the image of the generating vector of $P^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}$ by a small grey box, with the red and black lines we act on springing out, as shown below:

(7.1)



The elements of $P^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}$ are given by attaching Stendhal diagrams to these inputs and imposing the relations of $T^{(\lambda,\lambda^*)}$. Recall that since we have multiplied on the left by Khovanov and Lauda's idempotent in the nilHecke algebra on each group of like-colored strands, any crossing of like-colored consecutive strands springing from the box is trivial. Also, any black strand crossing the left red strand is trivial by the violating relation.

Passing to L_λ means that we also mod out by any crossing of a black strand across the right red strand. Pictorially, we express these relations as:
(7.2)



At the moment, the reader can consider this graphical representation a convenient mnemonic, but in the next section, this will help define a generalization of this simple module.

Recall that the **coevaluation** $\mathbb{C}[q^{1/D}, q^{-1/D}] \to V_{\lambda,\lambda^*}$ is the map sending 1 to the canonical element of the pairing we have fixed, and **evaluation** is the map induced by the pairing $V_{\lambda^*,\lambda} \to \mathbb{C}[q^{1/D}, q^{-1/D}]$.

Definition 7.4 *Let*

$$\mathbb{K}_\emptyset^{\lambda,\lambda^*} : \mathcal{V}^\emptyset_{1/D} \to \mathcal{V}^{\lambda,\lambda^*}_{1/D} \text{ be the functor } \mathrm{RHom}_\Bbbk(\dot{L}_\lambda, -)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]$$

and

$$\mathbb{E}^\emptyset_{\lambda^*,\lambda} : \mathcal{V}^{\lambda^*,\lambda}_{1/D} \to \mathcal{V}^\emptyset_{1/D} \text{ be the functor } - \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}}} \dot{L}_{\lambda^*}.$$

These functors preserve the appropriate categories since by Theorem 5.28, the module L_λ has a projective resolution in $\mathcal{V}^{\boldsymbol{\lambda}}_{1/D}$.

Proposition 7.5 *The functor* $\mathbb{K}_\emptyset^{\lambda,\lambda^*}$ *categorifies the coevaluation, and* $\mathbb{E}^\emptyset_{\lambda^*,\lambda}$ *the evaluation.*

Proof. Since L_λ is self-dual, we must first check that $[L_\lambda]$ is invariant. Of course, the invariants are the space of vectors of weight 0 such that $\{v|E_i v = 0\}$ for any i. To show this, its enough to see that $\mathrm{Hom}(P, \mathfrak{E}_i L_\lambda) = \mathrm{Hom}(\mathfrak{F}_i P, L_\lambda) = 0$ for all projectives p and $i \in \Gamma$. This follows immediately from Lemma 7.3. Thus $[L_\lambda]$ is invariant. In fact, L_λ is the only invariant simple representation, since the $-\lambda^*$-weight space of V_λ is 1 dimensional.

Now, we need just check the normalization is correct. Of course, $[L_\lambda]$'s projection to $(V_\lambda)_{low} \otimes (V_{\lambda^*})_{high}$ is

$$[P^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_\lambda}] = [P^0_{\mathbf{i}^\lambda}] \otimes [P^0_\emptyset] = F_{\mathbf{i}^\lambda} v_h \otimes v_{h^*}.$$

Thus, by invariance, the projection to $(V_\lambda)_{high} \otimes (V_{\lambda^*})_{low}$ is

$$v_h \otimes S(F_{\mathbf{i}^\lambda}) v_{h^*} = (-1)^{2\rho^\vee(\lambda)} q^{-2\langle\lambda,\rho\rangle} v_h \otimes v_l.$$

On the other hand, Lemma 7.3 also implies that $- \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}}} \dot{L}_{\lambda^*}$ kills all modules of the form $\mathfrak{F}_i M$, so it gives an invariant map, whose normalization we, again, just need to check on one element. For example, $P^{(0,2\rho^\vee(\lambda))}_{\mathbf{i}_{\lambda^*}} \otimes L_{\lambda^*} \cong \Bbbk$, so we get 1 on $v_l \otimes v_h$, which is the correct normalization for the evaluation. $\square$

We represent these functors as leftward oriented cups as is done for the coevaluation and evaluation in the usual diagrammatic approach to quantum groups, as shown in Figure 1.

FIGURE 1. Pictures for the coevaluation and evaluation maps.

In order to analyze the structure of L_λ, we must understand some projective resolutions of standards. This can be done with surprising precision in the case where $\ell = 2$.

Fix a sequence $\mathbf{i} = (i_1, \dots i_n)$. Define a map $\kappa_j : [1,2] \to [0,n]$ by $\kappa_j(2) = j$ and $\kappa_j(1) = 0$. Given a subset $T \subset [j+1,n]$, we let $\mathbf{i}_T$ be the sequence given by $i_1, \dots, i_j$ followed by T in reversed order, and then $[j+1,n] \setminus T$ in sequence and let $\kappa_T(2) = j + \#T$. Let

$$\chi_T = \sum_{k \in T} \left\langle \alpha_{i_k}, -\lambda_2 + \sum_{j<m<k} \alpha_{i_m} \right\rangle .$$

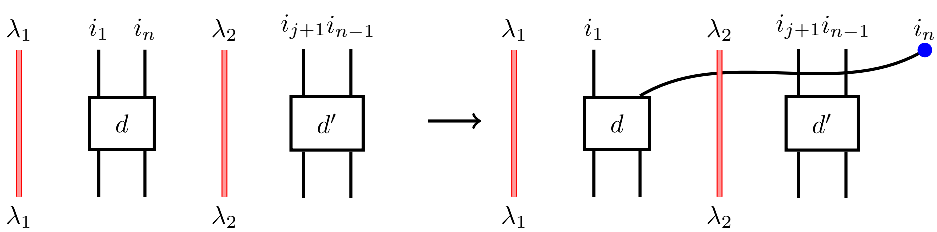
Proposition 7.6 *The standard $S_{\mathbf{i}}^{\kappa_j}$ has a projective resolution of the form*

$$\cdots \longrightarrow \bigoplus_{|T|=k} P_{\mathbf{i}_T}^{\kappa_T}(\chi_T) \longrightarrow \cdots \longrightarrow P_{\mathbf{i}}^{\kappa_j} \longrightarrow S_{\mathbf{i}}^{\kappa_j}$$

PROOF. We induct on $n - j$. If $j = n$, then $S_{\mathbf{i}}^{\kappa_j}$ is itself projective, so we may take the trivial resolution. Let $\mathbf{i}'$ be $\mathbf{i}$ with its last entry removed, and $\mathbf{i}''$ be $\mathbf{i}$ with its last entry moved to the $j+1$st position. As shown in Proposition 5.5, we have an exact sequence

$$0 \longrightarrow S_{\mathbf{i}''}^{\kappa_{j+1}}\Big(\big\langle \alpha_{i_n}, -\lambda_2 + \sum_{j<\ell<n} \alpha_{i_\ell} \big\rangle\Big) \longrightarrow \mathfrak{F}_{i_n} S_{\mathbf{i}'}^{\kappa_j} \longrightarrow S_{\mathbf{i}}^{\kappa_j} \longrightarrow 0. \tag{7.3}$$

The right hand map is the obvious projection, which imposes the standardly violating condition on the strand added by $\mathfrak{F}_{i_n}$. The kernel of this map is thus spanned by diagrams where at the top, the strand added by $\mathfrak{F}_{i_n}$ crosses the second red strand and all black ones to its right. Thus the left hand map is given by attaching a diagram in $S_{\mathbf{i}''}^{\kappa_{j+1}}$ to this diagram:



Applying the inductive hypothesis, we obtain projective resolutions of the left two factors. In the terms that appear in $\mathfrak{F}_{i_n} S_{\mathbf{i}'}^{\kappa_j}$, we have taken a subset $T' \subset [j+1, n-1]$ and moved these to the left of the second red strand (reversing their order). Clearly, we have $\mathbf{i}'_{T'} = \mathbf{i}_T$ and $\kappa_{T'} = \kappa_T$ where we take $T := T'$. In $S_{\mathbf{i}''}^{\kappa_{j+1}}$, we have now taken a subset of $[j+2,n]$, and moved these left of the red line, and right of the $j+1$st strand, which has label i_n. Thus, if we take $T = T' - 1 \cup \{n\}$, then $\mathbf{i}''_{T'} = \mathbf{i}_T$ and $(\kappa_{j+1})_{T'} = \kappa_T$, and $\langle \alpha_{i_n}, -\lambda_2 + \sum_{j<\ell<n} \alpha_{i_\ell} \rangle + \chi_{T'} = \chi_T$. This shows why we must reverse the order of T: in $\mathbf{i}''_{T'}$, all the strands for T' are right of $j+1$st, reversing the order from $\mathbf{i}$. Thus, between these two resolutions we have all the terms that appear in our expected resolution, in the correct degree shifts.

Now, we can lift the leftmost map of (7.3) to a map between projective resolutions. The cone of this map is the desired projective resolution of $S_{\mathbf{i}}^{\kappa_j}$. □

The same principle can be used for any value of ℓ to construct an explicit description of a projective resolution for any standard, but carefully writing this down is a bit more subtle and difficult than the $\ell = 2$ case, so we will not do so here.

This provides a resolution of the standard module $M_\lambda = S_{\mathbf{i}_{\lambda^*}}^{\kappa_0}$. In particular, it shows that

Corollary 7.7 $\operatorname{Ext}^i(M_\lambda, L_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \Bbbk(2\langle\lambda,\rho\rangle) & i = 2\rho^\vee(\lambda) \end{cases}.$

Proof. All of the projectives which appear in the resolution of M_λ have no maps to L_λ except the last term where $T = [1, 2\rho^\vee(\lambda)]$. We can break up the grading shift $\chi_{[1,2\rho^\vee(\lambda)]}$ of this term into the pieces corresponding to simple reflections in a reduced expression for a longest word of W, which are in turn in canonical bijection with the set of positive roots R^+. Thus, we have

$$\sum_{i=1}^{n} \left\langle \alpha_{i_k}, -\lambda^* + \sum_{m<k} \alpha_{i_m} \right\rangle = \sum_{\alpha \in R^+} \langle \alpha, -\lambda^* \rangle = -2\langle \lambda^*, \rho\rangle = -2\langle\lambda,\rho\rangle.$$

Thus, the last term in the resolution is $P_{\mathbf{i}_\lambda}^{\kappa_{2\rho^\vee(\lambda)}}(-2\langle\lambda,\rho\rangle)$. Thus we have

$$\operatorname{Ext}^i(M_\lambda, L_\lambda) \cong \operatorname{Ext}^{i-2\rho^\vee(\lambda)}(P_{\mathbf{i}_\lambda}^{\kappa_{2\rho^\vee(\lambda)}}(-2\langle\lambda,\rho\rangle), L_\lambda)$$

and the result follows. □

It also shows more indirectly that L_λ has a beautiful, if more complicated resolution.

Proposition 7.8 *There is a resolution*

$$\cdots \longrightarrow M_j \longrightarrow \cdots \longrightarrow M_1 \longrightarrow M_0 \longrightarrow L_\lambda \longrightarrow 0$$

of L_λ with the property that

- *$M_{2\rho^\vee(\lambda)-j}$ lies in the subcategory generated by $S_{\mathbf{i}}^{\kappa_j}$ for all different choices of $\mathbf{i}$. In particular, if $j > 2\rho^\vee(\lambda)$, then $M_j = 0$.*
- *$M_{2\rho^\vee(\lambda)} \cong M_\lambda(-2\langle\lambda,\rho\rangle)$.*

Proof. We prove this statement by induction on j. We take M_0 to be the standard $S_{\mathbf{i}_\lambda}^{\kappa_{2\rho^\vee(\lambda)}}$; by definition, we have a surjective map $M_0 \to L_\lambda$. Let M_1' be the kernel of this map. We wish to show that we have a surjective map from a sum of standards of the form $S_{\mathbf{i}}^{\kappa_{2\rho^\vee(\lambda)-1}}$. By the upper-triangularity of multiplicities in standards, this will follow if we show that all the simples that receive a nonzero map from M_1' are quotients of $S_{\mathbf{i}}^{\kappa_{2\rho^\vee(\lambda)-1}}$ for some $\mathbf{i}$, and not of $S_{\mathbf{i}}^{\kappa_k}$ for $k <$

$2\rho^\vee(\lambda)-1$. The simple quotients of $S_{\mathbf{i}}^{\kappa_k}$ are the same as the submodules of $(S_{\mathbf{i}}^{\kappa_k})^\star$. Thus, we wish to show that $\mathrm{Hom}\big(M_1', (S_{\mathbf{i}}^{\kappa_k})^\star\big) = 0$ for $k < 2\rho^\vee(\lambda) - 1$. Since $\mathrm{Ext}^i\left(S_{\mathbf{i}_\lambda}^{\kappa_{2\rho^\vee(\lambda)}}, (S_{\mathbf{i}}^{\kappa_k})^\star\right) = 0$, the long exact sequence shows that

$$\mathrm{Hom}\big(M_1', (S_{\mathbf{i}}^{\kappa_k})^\star\big) \cong \mathrm{Ext}^1\left(L_\lambda, (S_{\mathbf{i}}^{\kappa_k})^\star\right).$$

Dualizing the projective resolution of Proposition 7.6, we see that this can only be non-zero if $k = 2\rho^\vee(\lambda) - 1$. Thus, there exists the module M_1 as desired.

Now, we let M_2' be the kernel of the map $M_1 \to M_1'$. The composition factors of this module are quotients of $S_{\mathbf{i}}^{\kappa_k}$ for $k \le 2\rho^\vee(\lambda) - 2$. We now wish to show that that this inequality is sharp for any simple quotient as before. The long exact sequence applied again shows that

$$\mathrm{Hom}\big(M_2', (S_{\mathbf{i}}^{\kappa_k})^\star\big) \cong \mathrm{Ext}^2\left(L_\lambda, (S_{\mathbf{i}}^{\kappa_k})^\star\right).$$

Applying the projective resolution of Proposition 7.6 again, we see that this can only be non-zero if $k = 2\rho^\vee(\lambda) - 2$.

Applying this argument inductively, we see that we can construct M_i as desired.

Now we wish to analyze $M_{2\rho^\vee(\lambda)}$. This is in the subcategory generated by M_λ. Since $\mathrm{Ext}^i(M_\lambda, M_\lambda)$ vanishes for $i > 0$, we must have that $M_{2\rho^\vee(\lambda)}$ is a sum of grading shifts of M_λ. By our projective resolution, we have

$$\mathrm{Hom}(M_{2\rho^\vee(\lambda)}, \big(S_{\mathbf{i}_\lambda}^{\kappa_0}\big)^\star) \cong \mathrm{Ext}^{2\rho^\vee(\lambda)}\left(L_\lambda, \big(S_{\mathbf{i}_\lambda}^{\kappa_0}\big)^\star\right) \cong \Bbbk(-2\langle\lambda,\rho\rangle).$$

This can only be the case if $M_{2\rho^\vee(\lambda)} \cong M_\lambda(-2\langle\lambda,\rho\rangle)$, since $\mathrm{Hom}(M_\lambda, \big(S_{\mathbf{i}_\lambda}^{\kappa_0}\big)^\star) \cong \Bbbk$. ☐

Corollary 7.9 $\mathrm{Ext}^i(L_\lambda, M_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \Bbbk(2\langle\lambda,\rho\rangle) & i = 2\rho^\vee(\lambda) \end{cases}.$

Corollary 7.10 $\mathrm{Tor}^i(M_\lambda, \dot{L}_\lambda) = \begin{cases} 0 & i \neq 2\rho^\vee(\lambda) \\ \Bbbk(-2\langle\lambda,\rho\rangle) & i = 2\rho^\vee(\lambda) \end{cases}.$

2. Ribbon structure

This calculation is also important for showing how L_λ behaves under braiding:

Proposition 7.11 $\mathbb{B}_{\sigma_1} L_\lambda \cong L_{\lambda^*}[-2\rho^\vee(\lambda)](-2\langle\lambda,\rho\rangle - \langle\lambda,\lambda\rangle)$.

Proof. Note that L_λ is the unique simple module such that for all $j < 2\rho^\vee(\lambda)$

$$L_\lambda e(\mathbf{i}, \kappa_j) \cong L_\lambda \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j} \cong 0. \tag{7.4}$$

Thus we wish to check that $\mathbb{B}_{\sigma_1} L_\lambda$ has the same property. Assume $\mathbf{i}$ is a sequence of length $2\rho^\vee(\lambda)$. If $j < 2\rho^\vee(\lambda)$, then $\mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j} \cong \mathfrak{F}_i(\mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}'}^{\kappa_j})$ for a shorter sequence $\mathbf{i}'$. Thus, $\mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j}$ has a projective resolution in which $P_{\mathbf{i}}^{\kappa_{2\rho^\vee(\lambda)}}$ never appears, and

$$\mathbb{B}_{\sigma_1} L_\lambda e(\mathbf{i}, \kappa_j) \cong L_\lambda \overset{L}{\otimes} \mathfrak{B} \overset{L}{\otimes} \dot{P}_{\mathbf{i}}^{\kappa_j} \cong 0.$$

The property shows that the only composition factor which can occur in the cohomology $\mathbb{B}L_\lambda$ is L_{λ^*}. Now we need only show that it only appears with multiplicity 1 in the correct degree.

In order to see this, we note that Proposition 6.10 implies that $\mathfrak{B} \overset{L}{\otimes} \dot{P}^{\kappa_{2\rho^\vee(\lambda)}}_{\mathbf{i}_\lambda} \cong \dot{M}_\lambda(-\langle\lambda,\lambda\rangle)$. Thus, by Corollary 7.10, we have an isomorphism of vector spaces

$$\mathbb{B}L_\lambda e(\mathbf{i}_\lambda) \cong L_\lambda \overset{L}{\otimes} \mathfrak{B} \overset{L}{\otimes} \dot{P}^{\kappa_{2\rho^\vee(\lambda)}}_{\mathbf{i}_\lambda} \cong L_\lambda \overset{L}{\otimes} \dot{M}_\lambda(-\langle\lambda,\lambda\rangle) \cong \Bbbk[-2\rho^\vee(\lambda)](-2\langle\lambda,\rho\rangle - \langle\lambda,\lambda\rangle).$$

By the exactness of tensoring with a projective, we see that as a $T^{\lambda^*,\lambda}$ representation, the cohomology of $\mathbb{B}L_\lambda$ must be simple, and thus

$$\mathbb{B}L_\lambda \cong L_{\lambda^*}[-2\rho^\vee(\lambda)](-\langle\lambda,\lambda\rangle - 2\langle\lambda,\rho\rangle). \qquad \square$$

Now, in order to define quantum knot invariants, we must also have have quantum trace and cotrace maps, which can only be defined after one has chosen a ribbon structure. The Hopf algebra $U_q(\mathfrak{g})$ does not have a unique ribbon structure; in fact topological ribbon elements form a torsor over the characters $Y/X \to \{\pm 1\}$. Essentially, this action is by multiplying quantum dimension by the value of the character.

The standard convention is to choose the ribbon element so that all quantum dimensions are Laurent polynomials in q with positive coefficients; however, the calculation above shows that this choice is not compatible with our categorification! Instead we define:

Definition 7.12 *The* **ribbon functor** $\mathbb{R}_i$ *is defined by*

$$\mathbb{R}_i M = M[2\rho^\vee(\lambda_i)](2\langle\lambda_i,\rho\rangle + \langle\lambda_i,\lambda_i\rangle).$$

By Proposition 7.11, we have

$$\mathbb{B}^2 L_\lambda = L_\lambda[-4\rho^\vee(\lambda)](-4\langle\lambda,\rho\rangle - 2\langle\lambda,\lambda\rangle).$$

Thus, our ribbon functor $\mathbb{R}$ satisfies the equations

$$\mathbb{B}^2 L_\lambda \cong \mathbb{R}_1^{-2} L_\lambda = \mathbb{R}_2^{-2} L_\lambda = \mathbb{R}_1^{-1}\mathbb{R}_2^{-1} L_\lambda,$$

which are necessary for topological invariance (as we depict in Figure 2).

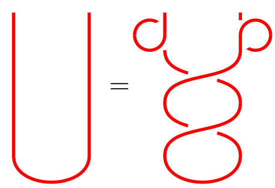


FIGURE 2. The compatibility of double twist and the ribbon element.

Taking Grothendieck group, we see that we obtain the ribbon element in $U_q(\mathfrak{g})$ uniquely determined by the fact that it acts on the simple representation of highest weight λ by $(-1)^{2\rho^\vee(\lambda)} q^{\langle\lambda,\lambda\rangle + 2\langle\lambda,\rho\rangle}$. This is the inverse of the ribbon element constructed by Snyder and Tingley in [**ST09**]; we must take inverse because Snyder and Tingley use the opposite choice of coproduct from ours. See Theorem 4.6 of that paper for a proof that this is a ribbon element.

From now on, we will term this the **ST ribbon element**. It may seem strange that this element appears more naturally from the perspective of categorification than the standard ribbon element, but it is perhaps not so surprising; the ST ribbon element is closely connected to the braid group action on the quantum group, which also played an important role in Chuang and Rouquier's early investigations on categorifying $\mathfrak{sl}_2$ in [**CR08**]. It is not surprising at all that we are forced into a choice, since ribbon structures depend on the ambiguity of taking a square root; while numbers always have 2 or 0 square roots in any given field (of characteristic $\neq 2$), a functor will often only have one.

Due to the extra trouble of drawing ribbons, we will draw all pictures in the blackboard framing.

This different choice of ribbon element will not seriously affect our topological invariants; we simply multiply the invariants from the standard ribbon structure by a sign depending on the framing of our link and the Frobenius-Schur indicator of the label, as we describe precisely in Proposition 8.8.

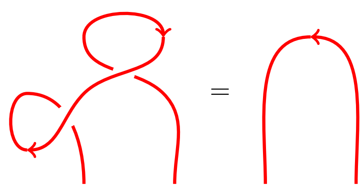


FIGURE 3. Changing the orientation of a cap

Proposition 7.13 *The quantum trace and cotrace for the ST ribbon structure are categorified by the functors*

$$\mathbb{C}_{\emptyset}^{\lambda^*,\lambda}\colon \mathcal{V}_{1/D}^{\emptyset} \to \mathcal{V}_{1/D}^{\lambda^*,\lambda} \text{ given by } \mathrm{RHom}(\dot{L}_{\lambda^*}, -)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]$$

and

$$\mathbb{T}_{\lambda,\lambda^*}^{\emptyset}\colon \mathcal{V}_{1/D}^{\lambda,\lambda^*} \to \mathcal{V}_{1/D}^{\emptyset} \text{ given by } -\otimes_{T^{\underline{\lambda}}}\dot{L}_{\lambda}.$$

PROOF. As the picture Figure 3 suggests, by definition the quantum trace is given by applying a negative ribbon twist of one strand, and then applying a positive braiding, followed by the evaluation; that is, it is categorified by

$$(\mathbb{B}\mathbb{R}_1-)\otimes\dot{L}_\lambda \cong -\otimes(\mathbb{B}\mathbb{R}_1\dot{L}_\lambda)\cong -\otimes\dot{L}_\lambda.$$

The result thus immediately follows from Proposition 7.11, and our definition of $\mathbb{R}$. The same relation between evaluation and quantum trace follows from adjunction. $\square$

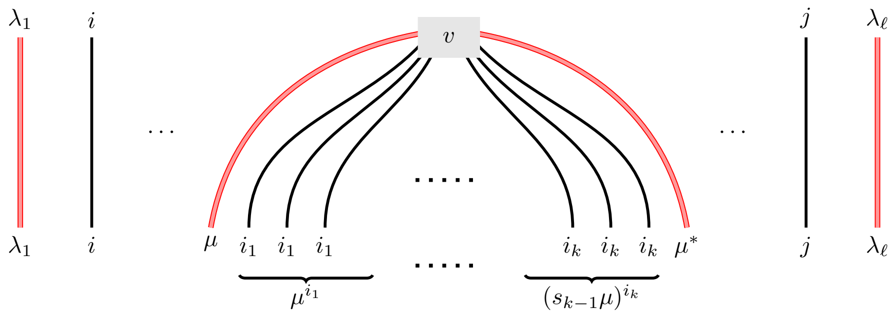
FIGURE 4. Pictures for the quantum (co)trace.

3. Coevaluation and quantum trace in general

More generally, whenever we are presented with a sequence $\underline{\boldsymbol{\lambda}}$ and a dominant weight μ, we wish to have a functor relating the categories $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}^+ = (\lambda_1, \dots, \lambda_{j-1}, \mu, \mu^*, \lambda_j, \dots, \lambda_\ell)$. This will be given by left tensor product with a particular bimodule.

Definition 7.14 *We let a $(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}^+)$-Stendhal diagram be a collection of curves like a Stendhal diagram, except that we allow a single cap given by a red strand connecting the bottom to itself; like in* (7.1)*, we insert an element of L_μ at the maximum of this cup, with appropriate inputs exploding out of its bottom.*

The $(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}^+)$-Stendhal diagrams are obtained by attaching normal Stendhal diagrams to the top and bottom of diagrams of the form:



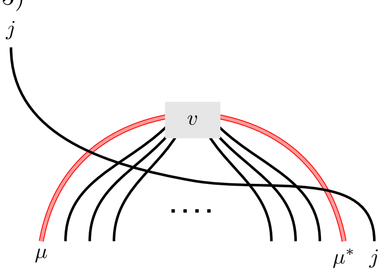
where v is an element of L_λ.

Let g_i be the number of times i appears in $\mathbf{i}^\lambda_{\mathbf{w}}$ for any reduced expression for the longest element w_0. These numbers can also be defined as the unique integers so that $\lambda - w_0(\lambda) = \sum_i g_i\alpha_i$. In particular, the sum $\sum g_i$ is precisely the quantity $2\rho^\vee(\lambda)$, which we have considered extensively

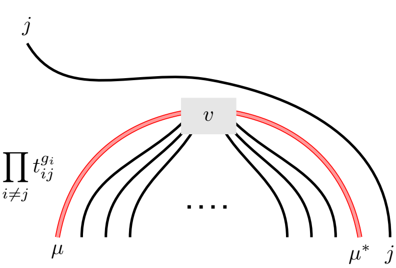
Definition 7.15 *We let $\tilde{\mathfrak{K}}^{\underline{\boldsymbol{\lambda}}^+}_{\underline{\boldsymbol{\lambda}}}$ be the quotient of the $\Bbbk$-span of all $(\underline{\boldsymbol{\lambda}}, \underline{\boldsymbol{\lambda}}^+)$-Stendhal diagrams by:*

- *We impose all local relations of $\tilde{T}$, including planar isotopy. That is, we impose the relations of* (2.5a–2.5g) *and* (4.1a-4.2)*, but not the relations killing violating strands.*
- *diagrams only involving strands that hit the maximum of the cup can act on elements of L_λ as expected.*
- *The relations:*

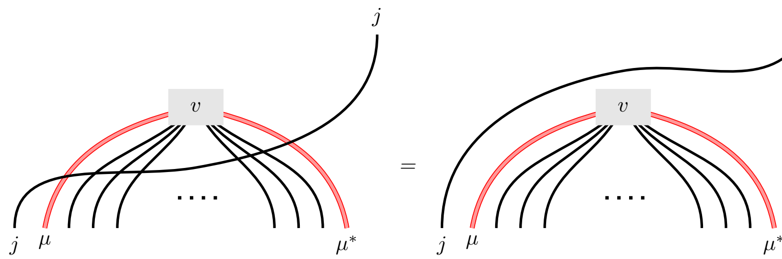
(7.5)



$$= (-1)^{g_j} \prod_{i \neq j} t_{ij}^{g_i}$$



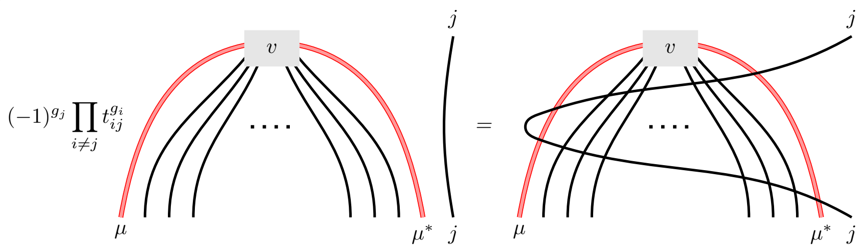
(7.6)



One can think of the relation above as categorifying the equality $(F_i v) \otimes K = F_i(v \otimes K)$ *for any invariant element* K.

In order to check the coherence of these relations, we will need to check that we can pull a strand which passes over the cup and back either off the bottom or off using the usual relations, and obtain the same answer. That is:

Lemma 7.16
(7.7)



PROOF. We note, this is equivalent to checking a relation in $\tilde{T}^{\mu,\mu^*}$: if we remove the box from the top of the diagrams, we must obtain that the RHS of (7.7) is equal to the LHS plus a sum of diagrams that give zero when they act on the cap. Unfortunately, this is quite a difficult computation and it would not be straightforward to present it cogently here. It will be greatly simplified if we can also use upward strands and assume that the weight labeling the region outside the cup is 0.

In order to do this, it is enough to check that our relation holds in $T^{(\nu,\mu,\mu^*)}$ for ν sufficiently large, after adding a red strand at the left. Finally, given a element d in $T^{\underline{\boldsymbol{\lambda}}}$, let $\gamma(d)$ be the same diagram with the sequence $\mathbf{i}^{\nu^*}$ added and then a red strand at far left with weight ν^*. This is a non-unital homomorphism, so $e = \gamma(1)$ is an idempotent. We claim that:

$$eT^{(\nu^*,\underline{\boldsymbol{\lambda}})} \cong \mathbb{S}^{(\nu^*);\underline{\boldsymbol{\lambda}}}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes T^{\underline{\boldsymbol{\lambda}}}\right). \tag{7.8}$$

This is clear if $\underline{\boldsymbol{\lambda}} = \emptyset$. As usual, we can prove this by induction on the number of red and black strands. If we add a new red strand turning $\underline{\boldsymbol{\lambda}}$ to $(\underline{\boldsymbol{\lambda}}, \lambda_{\ell+1})$, this is clear, since

$$\begin{aligned}\mathfrak{I}_{\lambda_{\ell+1}}(eT^{(\nu^*,\underline{\boldsymbol{\lambda}})}) &\cong \mathfrak{I}_{\lambda_{\ell+1}}\mathbb{S}^{(\nu^*);\underline{\boldsymbol{\lambda}}}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes T^{\underline{\boldsymbol{\lambda}}}\right)\\ &\cong \mathbb{S}^{(\nu^*);(\underline{\boldsymbol{\lambda}},\lambda_{\ell+1})}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes \mathfrak{I}_{\lambda_{\ell+1}}(T^{\underline{\boldsymbol{\lambda}}})\right)\end{aligned}$$

by the associativity of standardization. If we add a black strand with label i_{n+1}, then we have that

$$\mathfrak{F}_{i_{n+1}}(eT^{(\nu^*,\underline{\boldsymbol{\lambda}})}) \cong \mathfrak{F}_{i_{n+1}}\mathbb{S}^{(\nu^*);\underline{\boldsymbol{\lambda}}}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes T^{\underline{\boldsymbol{\lambda}}}\right) \cong \mathbb{S}^{(\nu^*);\underline{\boldsymbol{\lambda}}}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes \mathfrak{F}_{i_{n+1}}T^{\underline{\boldsymbol{\lambda}}}\right)$$

by Proposition 5.5, since $\mathfrak{F}_{i_{n+1}}(e(\mathbf{i}_{\nu^*})T^{\nu_*}) = 0$. This establishes (7.8).

Proposition 5.26 shows that standardization is fully-faithful, so

$$\text{(7.9)} \quad \operatorname{End}_{T^{(\nu^*,\nu,\mu,\mu^*)}}\left(\mathbb{S}^{(\nu^*);(\nu,\mu,\mu^*)}\left(e(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes T^{(\nu,\mu,\mu^*)}\right)\right) \cong$$

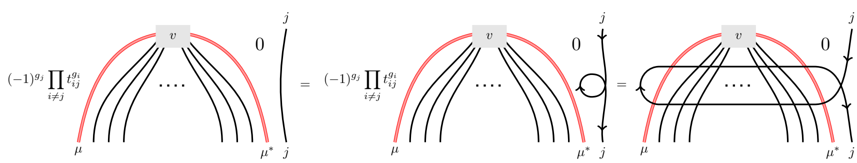
$$\operatorname{End}_{T^{\nu^*}\otimes T^{(\nu,\mu,\mu^*)}}(e\left(\mathbf{i}_{\nu^*})T^{\nu_*} \boxtimes T^{(\nu,\mu,\mu^*)}\right) \cong e(\mathbf{i}_{\nu^*})T^{\nu_*}e(\mathbf{i}_{\nu^*})\otimes T^{(\nu,\mu,\mu^*)} \cong T^{(\nu,\mu,\mu^*)}$$

where we apply the standard observation for any algebra A and idempotent e, we have $\operatorname{End}_A(eA) \cong eAe$. This also shows that

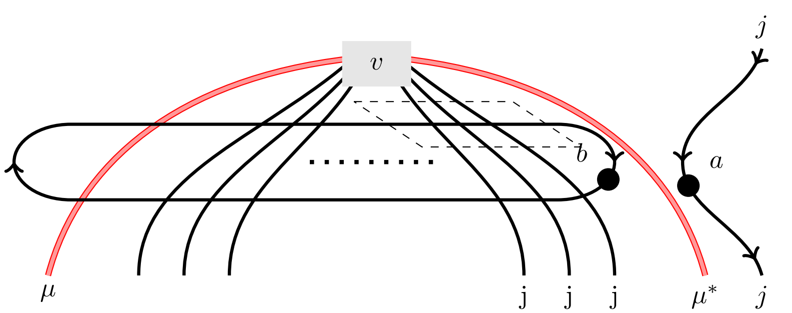
$$\text{(7.10)} \qquad \operatorname{End}_{T^{(\nu^*,\nu,\mu,\mu^*)}}(eT^{(\nu^*,\nu,\mu,\mu^*)}) \cong eT^{(\nu^*,\nu,\mu,\mu^*)}e.$$

Thus (7.8) applied with $\underline{\boldsymbol{\lambda}} = (\nu,\mu,\mu^*)$ together with (7.9–7.10) shows that the map γ induces an isomorphism $T^{(\nu,\mu,\mu^*)} \to eT^{(\nu^*,\nu,\mu,\mu^*)}e$. After doing this, we see that the label on the region above the cup is 0. Theorem 4.30 now shows that we can perform our calculation in $DT^{(\nu^*,\nu,\mu,\mu^*)}$, for sufficiently large ν.

We begin with the left-hand picture, and add a curl. Push the left side of the curl through the strands. The primary term that we arrive at has a curl wrapped over all strands; all the correction terms have a strand pulled right out of the cap, and thus are 0. By the relations (2.3c) and (2.4a) of $\mathcal{U}$, this term is multiplied by t_{ij}^{-1} each time we cross a strand labeled i for $i \neq j$, and by -1 when we cross one labeled j. Thus we obtain the equality:

(7.11) 

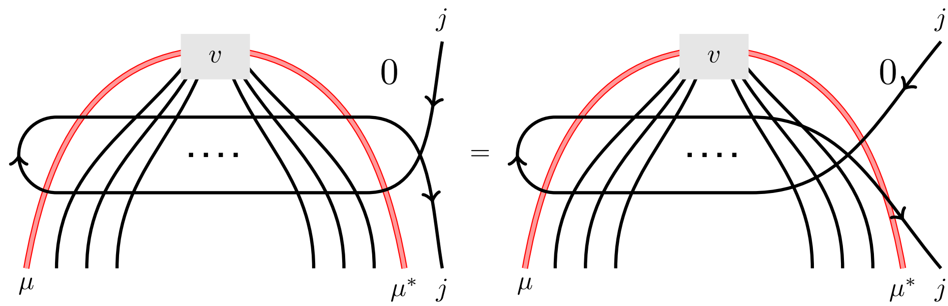
Next we move the crossing in the RHS of (7.11) left over the red strand using (4.1a). There is one term in the result where we simply isotope the crossing to the left side, and then there are others where the crossing is broken, and on the resulting strands there are $m = (\mu^*)^j - 1$ total dots. If we choose the reduced word for w_0 used to define $\mathbf{i}_\mu$ so that the last reflection appearing is s_j, then we can assume the $m+1$ rightmost black strands inside the cup are labelled j, and are multiplied by the divided power idempotent e_{m+1}. That is, fixing $a+b=m$, these have the form:



Since we have multiplied by the divided power idempotent where this group of strand with label j meet the gray box, we can write this element as an element of

L_λ times the half twist on these m strands, that is, the element D_m in the notation of [**KLMS12**, §2.2]. Taking the top row of crossings of the right most strand with these (in the dashed parallelogram above), we actually have D_{m+1} on these $m+1$ strands with label j inside the dashed parallelogram. Applying [**KLMS12**, (2.28)] to the $m+1$ black strands, we see this element is 0, since $b < m$. Thus, we have

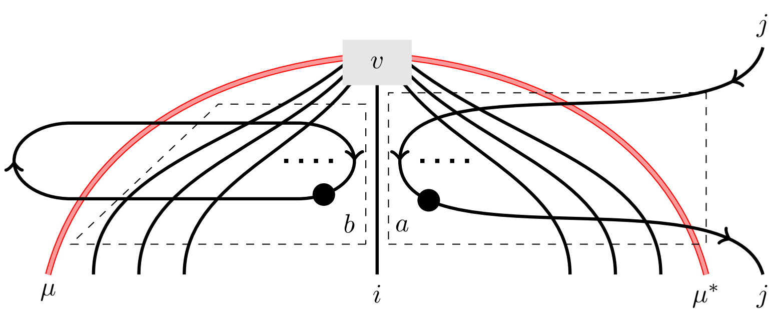
(7.12)



Now, we move the crossing in the RHS of (7.12) left through all the black strands, using (2.5g). There is a "dominant" term where the crossing simply isotopes through. There are also correction terms coming from the leftmost term in the triple point relation (2.5g), when crossing a strand of label i with $c_{ji} < 0$. In these,

- the outside strand makes bigons with all the rightmost $2\rho^\vee(\mu) - k$ black strands, and the rightward red, and carries some number of dots $a \geq 0$
- a bubble is laid over the leftmost $k-1$ black strands and the leftward red, and carries some number of dots $b \geq 0$ with $a + b + 1 = -c_{ji}$.
- there is a single strand between these which is black with label i.

Schematically, these look like:

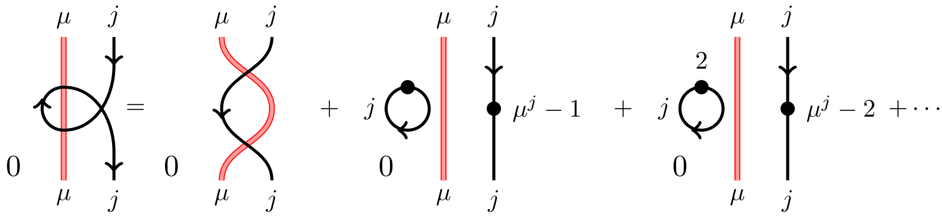


We intend to show that all these correction terms kill the cap.

We do this by applying Theorem 4.16 to simplify the diagrams inside the dashed boxes (which only involve downward strands). First, in the righthand box, we use a reduced expression for each permutation where the rightmost transposition only occurs once. In each diagram, if the rightmost terminal at the top and the bottom are connected by a single strand, then this strand will not cross any other strands. Otherwise, the strands connected to these terminals cross to the right of the red strands. In this case, the resulting diagram acts trivially, by Lemma 7.3 (as expressed in (7.2)). Thus, we can assume the rightmost strand never enters the cap.

Now consider the lefthand box, and use a reduced expression for each permutation where the leftmost transposition only occurs once. We leave unchanged the upward oriented segment of a strand left of the red strands. As above, we divide these diagrams into those where a single strand connects the lefthand terminals, and those the strands from these terminals cross immediately right of the red strand. In the former case, the upward segment closes up to a bubble just to the right of the red strand, without intersecting any black strand, and we can pull this to the left resulting in a positive degree bubble at the far left. In the latter, it has a self-intersection before crossing any others, and we can apply the relation

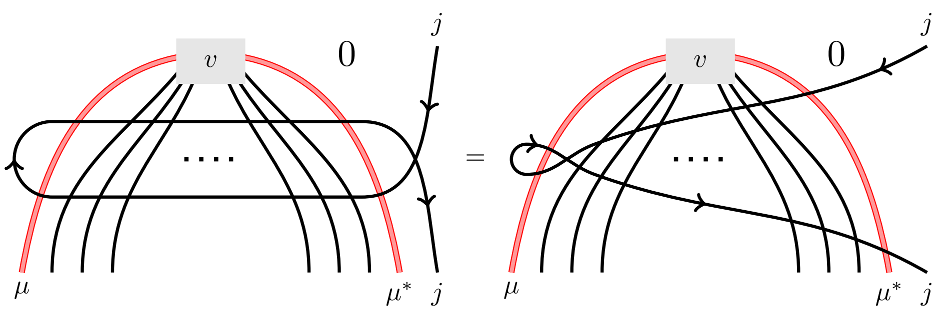
(7.13)



The leftmost term kills the cap by Lemma 7.3 and (7.2) again, so all the remaining terms have a positive degree bubble left of the cap.

Thus, the result is that the only correction terms that matter are those where there is a positive degree bubble at the far left and a rightmost strand that does not cross any of the reds. Since the total diagram has degree 0, the diagram acting between the red strands must have *negative* degree. This means that it must act trivially on L_μ, so all correction terms act trivially.

Therefore, we have that:



In order to finish, we apply the relation (7.13) again; as argued for the correction terms, all terms but the first on the RHS of (7.13) acts trivially, since each has a positive degree bubble. Thus we are just left with the first term which is precisely the RHS of the statement (7.7). □

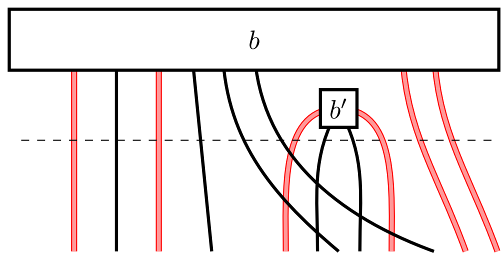
Like its analogues, the module $\tilde{\mathfrak{K}}^{\boldsymbol{\lambda}^+}_{\underline{\boldsymbol{\lambda}}}$ has a basis. First one considers the basis B for $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ and chooses a basis B' of L_λ given by Stendhal diagrams; we'll define a spanning set B'' for $\tilde{\mathfrak{K}}^{\boldsymbol{\lambda}^+}_{\underline{\boldsymbol{\lambda}}}$ is indexed by triples consisting of

(1) an element of $b \in B$,
(2) an element of $b' \in B'$,

(3) a shuffle of the bottom of b' (a sequence in Γ) and the $j-1$st black block of the bottom of b; that is, an order on the union of these sequences that coincides with the usual sequence order on each of them.

The elements of B'' are obtained by inserting the maximum of the cup after the $j-1$st black block at the top of the diagram, and using a minimal number of crossings to attain the shuffle; in particular, we never pass any black strands above the minimum of the cup, always going under it instead. A schematic representation of one of these basis vectors looks like:

(7.14)



As usual, there are choices involved in this definition, and we arbitrarily fix one for each triple.

Lemma 7.17 *The set B'' is a basis of $\tilde{\mathfrak{K}}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$.*

Proof. Let K be the formal span of the elements of B''. One can define a bimodule structure on K as follows:

- When one acts at the top, one uses the usual action of elements of $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ on the formal span of the elements B from the top (i.e. the left), and leaves the element b' unchanged; that is, one simply does simplifications above the maximum of the cap.
- If one acts at the bottom with a crossing or dot on strands which are *not* between the left edge of the cap and the jth red strand from the top (the next one to the right of the cap), one simply isotopes the diagram up to the top and lets it act on the formal span of B by the usual multiplication on the bottom (i.e. the right).
- If at the bottom, we cross the left edge of the cap with a black strand to its left, that is a new basis vector where we have only changed the shuffle.
- If we apply a crossing or dot to the strands between the left edge of the cap and the jth red strand from the top, then we apply Theorem 4.16 to rewrite the portion of the diagram below b' using basis diagrams that put all dots and all crossings between pairs of strands both from b or both from b' occur above those between strands coming from b and b'. Once we have fixed basis diagrams, as we can using a fixed longest reduced word, this expansion is unique.

  That is, our diagrams look like the one above, with a shuffle between strands from b and b' at the bottom, and the elements b and b' at the top, but possibly with some crossings and dots on the strands coming out of b and on those coming out of b' at the y-value where there is a dashed line. We let these crossings and dots act on the span of B and B' in the usual way, by thinking of them as bases of $\tilde{T}^{\underline{\boldsymbol{\lambda}}}$ and L_{λ}.

- If at the bottom, we cross a black strand from the left over the jth red strand (from the top), then the strand must have come from b, and passed under b'. We simply pull the strand through the top of the cap, multiplying by a scalar as in (7.5). Similarly, when the black strand comes from the right, we must do this operation in the opposite direction.

Now, we wish to define a map $\tilde{\mathfrak{K}}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \to K$. The local relations (2.5a–2.5g) and (4.1a-4.2) are immediate, so we need only confirm (7.5–7.6). The relation (7.5) holds by the definition of the action, and (7.6) follows from Lemma 7.16. This map is surjective since every element of B'' is in the image.

We need only show that $\tilde{\mathfrak{K}}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ is spanned by B''. This is easily shown using techniques analogous to Lemma 4.11; the only new trick needed is to show that we don't need diagrams where a strand starts left of the jth red strand, but passes right of the maximum of the cap. This is avoided using (7.5). □

As usual, we let $\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ be the quotient of $\tilde{\mathfrak{K}}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ by the submodule spanned by violated diagrams.

Definition 7.18 *The coevaluation functor is*

$$\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} = - \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}^+}} \mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} : \mathcal{V}_{1/D}^{\underline{\boldsymbol{\lambda}}} \to \mathcal{V}_{1/D}^{\boldsymbol{\lambda}^+}.$$

Similarly, the quantum trace functor is the right adjoint to this given by

$$\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} = \mathrm{RHom}_{T^{\underline{\boldsymbol{\lambda}}}}(\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}, -)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)] \colon \mathcal{V}_{1/D}^{\boldsymbol{\lambda}^+} \to \mathcal{V}_{1/D}^{\underline{\boldsymbol{\lambda}}}.$$

The evaluation and quantum cotrace are defined similarly.

As with the functors $\mathbb{B}$, these functors can be worked with using their relationship with standardization. Let $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}$ be the usual standardization functor and $\mathbb{S}_+^{\boldsymbol{\lambda}^+}$ denote the standardization functor where (μ,μ^*) one of the subsequences and all others are singletons.

Lemma 7.19

$$\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}} \cong \mathbb{S}_+^{\boldsymbol{\lambda}^+} \circ \mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \qquad \mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}_+^{\boldsymbol{\lambda}^+} \cong \mathbb{S}^{\underline{\boldsymbol{\lambda}}} \circ \mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$$

PROOF. The 0th cohomology of both $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}}$ and $\mathbb{S}_+^{\boldsymbol{\lambda}^+} \circ \mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ are given by tensor product with the bimodule given by the quotient of $\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ by standardly violating strands. Thus we need to show that $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}^{\underline{\boldsymbol{\lambda}}}$ applied to a projective gives a module. The proof of this using exact sequences is sufficiently similar to Lemma 6.9 that we leave the details to the reader.

The argument for $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ is a variation on this. Consider the functors $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}_+^{\boldsymbol{\lambda}^+}$ and $\mathbb{S}^{\underline{\boldsymbol{\lambda}}} \circ \mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ applied to a module of the form $P = P_1 \boxtimes \cdots \boxtimes P_{j-1} \boxtimes I \boxtimes P_j \boxtimes \cdots \boxtimes P_\ell$ with P_i projective and I injective. This may seem like a strange module, but it appears naturally as $\mathbb{B}_{j-1}^2$ applied to a usual standard module.

The functor $\mathbb{S}^{\underline{\boldsymbol{\lambda}}} \circ \mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ sends P to $\mathbb{S}^{\underline{\boldsymbol{\lambda}}}(P_1 \boxtimes \cdots \boxtimes P_\ell) \otimes \mathrm{Hom}(L_\lambda, I)$. An element of $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}_+^{\boldsymbol{\lambda}^+}$ gives an element of $\mathbb{S}^{\underline{\boldsymbol{\lambda}}} \circ \mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ by considering the image of diagrams with no crossings or dots. We apply the same induction argument to show that

$\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ \mathbb{S}_+^{\boldsymbol{\lambda}^+}$ has no higher cohomology as for $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$, but now viewing $\mathbb{S}_+^{\boldsymbol{\lambda}^+}(P)$ as a quotient of $\mathbb{B}_{j-1}^2$ applied to a projective, with the kernel filtered by $\mathbb{B}_{j-1}$ applied to standards. $\square$

Since $\mathfrak{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ is projective as a left module, tensor with it gives an exact functor. The quantum trace functor, however, is very far from being exact.

Proposition 7.20 $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ *categorifies the coevaluation and* $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ *the quantum trace.*

Proof. We need only prove the former, since the latter follows by adjunction. Furthermore, we may reduce to the case where μ is added at the end of the sequence, since all other cases are obtained from this by the action of $\mathcal{U}$.

In this case, consider $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}(S_{\mathbf{i}}^{\kappa})$. The resulting module is isomorphic to the standardization

$$\mathbb{S}^{\underline{\boldsymbol{\lambda}};\mu,\mu^*}(S_{\mathbf{i}}^{\kappa} \boxtimes L_\mu)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]$$

by Lemma 7.19.

This reduces to the case where $\underline{\boldsymbol{\lambda}} = \emptyset$, which we have covered in Propositions 7.5 and 7.13. $\square$

Proposition 7.21 *The functors* $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ *and* $\mathbb{T}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ *are strongly equivariant.*

Proof. By taking adjoint, one can reduce to just the case $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$. The proof is essentially the same as Lemma 6.7; the composition of functors $u \circ \mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+}$ and $\mathbb{K}_{\underline{\boldsymbol{\lambda}}}^{\boldsymbol{\lambda}^+} \circ u$ are both given by tensor product with honest modules by the exactness of u and the bimodules are easily identified. The difference is that in the first bimodule grabs strands below the maximum, whereas the second grabs them above it. These are equivalent by the relations (7.5) and (7.6). $\square$

The most important property of these functors is that they satisfy the obvious isotopy. To see this, consider the two functors

$$S_1 = \mathbb{T}_{\underline{\boldsymbol{\lambda}}_1\lambda\underline{\boldsymbol{\lambda}}_2}^{\underline{\boldsymbol{\lambda}}_1\lambda;\lambda^*,\lambda;\underline{\boldsymbol{\lambda}}_2}\mathbb{K}_{\underline{\boldsymbol{\lambda}}_1\lambda\underline{\boldsymbol{\lambda}}_2}^{\underline{\boldsymbol{\lambda}}_1;\lambda,\lambda^*;\lambda\underline{\boldsymbol{\lambda}}_2} \qquad S_2 = \mathbb{T}_{\underline{\boldsymbol{\lambda}}_1\lambda\underline{\boldsymbol{\lambda}}_2}^{\underline{\boldsymbol{\lambda}}_1;\lambda,\lambda^*;\lambda\underline{\boldsymbol{\lambda}}_2}\mathbb{K}_{\underline{\boldsymbol{\lambda}}_1\lambda\underline{\boldsymbol{\lambda}}_2}^{\underline{\boldsymbol{\lambda}}_1\lambda;\lambda^*,\lambda;\underline{\boldsymbol{\lambda}}_2}$$

which come from adding a pair of the representations are added on the left of an entry λ, and removing them on the right of λ or *vice versa*. These functors are depicted in more usual topological form in Figure 5.

Proposition 7.22 *The functors* S_1 *and* S_2 *are isomorphic to the identity functor.*

Proof. One can use Lemma 7.19 to reduce to the case where $\underline{\boldsymbol{\lambda}}_1 = \underline{\boldsymbol{\lambda}}_2 = \emptyset$. Furthermore, by Lemma 7.21, it suffices to check that $S_1 P_\emptyset \cong S_2 P_\emptyset \cong P_\emptyset \cong \Bbbk$, since any choice of isomorphism between these objects will induce isomorphisms between the functors. To prove the result for S_2, we must check that

$$\mathbb{S}^{\lambda;\lambda^*,\lambda}(P_\emptyset \boxtimes L_\lambda) \overset{L}{\otimes}_{T^{\underline{\boldsymbol{\lambda}}}} \mathbb{S}^{\lambda,\lambda^*;\lambda}(\dot{L}_\lambda \boxtimes \dot{P}_\emptyset)(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)] \cong \Bbbk$$

Applying the dot involution to switch left/right, the symmetry of tensor product shows that S_1 reduces to the same calculation.

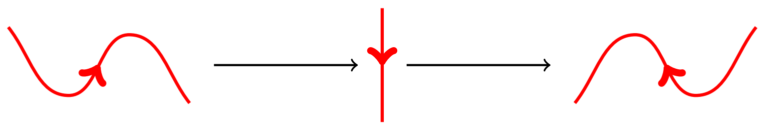


FIGURE 5. The "S-move"

We can use Lemma 7.8 to expand L_λ into a complex, and then use the spectral sequence attached to tensoring these complexes. The E^2-page of this spectral sequence has entries

$$E^2_{k,m} = \bigoplus_{i+j=m} \mathrm{Tor}^k \Big(\mathbb{S}^{\lambda;\lambda^*,\lambda}(P_\emptyset \boxtimes M_i), \mathbb{S}^{\lambda,\lambda^*;\lambda}(\dot{M}_j \boxtimes \dot{P}_\emptyset)(2\langle\lambda,\rho\rangle)\Big).$$

By the Tor-vanishing discussed in the proof of 6.12, this will be 0 unless the two factors lie in the same piece of the semi-orthogonal decomposition, that is, if $i = 0, j = 2\rho^\vee(\lambda)$ and $k = 0$. This term is exactly

$$\mathbb{S}^{\lambda;\lambda^*;\lambda}(P_\emptyset \boxtimes P_{\mathbf{i}_\lambda} \boxtimes P_\emptyset)) \otimes_{T^{\underline{\lambda}}} \mathbb{S}^{\lambda,\lambda^*;\lambda}(\dot{P}_\emptyset \boxtimes \dot{P}_{\mathbf{i}_\lambda} \boxtimes \dot{P}_\emptyset)[-2\rho^\vee(\lambda)] \cong \Bbbk[-2\rho^\vee(\lambda)].$$

The homological shift above is cancelled by the fact that $m = j = 2\rho^\vee(\lambda)$. Thus, the result follows. $\square$

It is extremely tempting to conclude that this proposition shows that the functors $\mathbb{K}$ and $\mathbb{T}$ are biadjoint; in fact, they are not always, though the adjunction on one side is clear from the definition. Rather, this is reflecting some sort of biadjunction between the 2-functors of "tensor with $\mathfrak{V}^\lambda$" and "tensor with $\mathfrak{V}^{\lambda^*}$" on the 2-category of representations of $\mathcal{U}$. While there is not a unified construction of a tensor product of two $\mathcal{U}$ categories, one can easily generalize the definition of $\mathfrak{V}^{\underline{\lambda}}$ to describe auto-2-functors of $\mathcal{U}$ representations given by adding one red line; we will discuss this construction in more detail in forthcoming work [**Web15**].

CHAPTER 8

Knot invariants

As in Chapter 7, we assume that $\mathfrak{g}$ is finite dimensional in this chapter.

1. Constructing knot and tangle invariants

Now, we will use the functors from the previous chapter to construct tangle invariants. Using these as building blocks, we can associate a functor $\Phi(T)\colon \mathcal{V}^{\boldsymbol{\lambda}}_{1/D} \to \mathcal{V}^{\boldsymbol{\mu}}_{1/D}$ to any diagram of an oriented labeled ribbon tangle T with the bottom ends given by $\underline{\boldsymbol{\lambda}} = \{\lambda_1, \dots, \lambda_\ell\}$ and the top ends labeled with $\underline{\boldsymbol{\mu}} = \{\mu_1, \dots, \mu_m\}$.

As usual, we choose a projection of our tangle such that at any height (fixed value of the x-coordinate) there is at most a single crossing, single cup or single cap. This allows us to write our tangle as a composition of these elementary tangles.

For a crossing, we ignore the orientation of the knot, and separate crossings into positive (right-handed) and negative (left-handed) according to the upward orientation we have chosen on $\mathbb{R}^2$.

- To a positive crossing of the i and $i+1$st strands, we associate the braiding functor $\mathbb{B}_{\sigma_i}$.
- To a negative crossing, we associate its right adjoint $\mathbb{B}_{\sigma_i^{-1}}$ (the left and right adjoints are isomorphic, since $\mathbb{B}$ is an equivalence).

For the cups and caps, it is necessary to consider the orientation, following the pictures of Figures 1 and 4.

- To a clockwise oriented cup, we associate the coevaluation.
- To a clockwise oriented cap, we associate the quantum trace.
- To a counter-clockwise cup, we associate the quantum cotrace.
- To a counter-clockwise cap, we associate the evaluation.

Proposition 8.1 *The map induced by $\Phi(T) : \mathcal{V}^{\boldsymbol{\lambda}}_{1/D} \to \mathcal{V}^{\boldsymbol{\mu}}_{1/D}$ on the Grothendieck groups $V_{\underline{\boldsymbol{\lambda}}}^{1/D} \to V_{\underline{\boldsymbol{\mu}}}^{1/D}$ is that assigned to a ribbon tangle by the structure maps of the category of $U_q(\mathfrak{g})$ with the ST ribbon structure.*

In particular, the graded Euler characteristic of the complex $\Phi(T)(\mathbb{k})$ for a closed link is the quantum knot invariant for the ST ribbon element.

Proof. We need only check this for each elementary tangle, which was done in Corollary 6.11, Chapter 2 and Proposition 7.20. $\square$

Theorem 8.2 *Consider a link L. The cohomology of $\Phi(L)(\Bbbk)$ is finite-dimensional in each homological degree, and each graded degree is a complex with finite dimensional total cohomology. In particular the bigraded Poincaré series*

$$\varphi(L)(q,t) = \sum_i (-t)^{-i} \dim_q H^i(\Phi(L)(\Bbbk))$$

is a well-defined element of $\mathbb{Z}[q^{1/D}, q^{-1/D}]((t))$.

PROOF. We note that the category $\mathcal{V}^{\emptyset}_{1/D}$ is the category of complexes of graded finite dimensional vector spaces

$$\cdots \longleftarrow M^{i+1} \longleftarrow M^i \longleftarrow M^{i-1} \longleftarrow \cdots$$

such that $M^i = 0$ for $i \gg 0$ and for some k, the vector space M^i is concentrated in degrees above $k - i$. Thus, $\Phi(L)(\Bbbk)$ lies in this category. In particular, each homological degree and each graded degree of $\Phi(L)(\Bbbk)$ is finite-dimensional. □

The only case where the invariant is known to be finite dimensional is when the representations $\underline{\boldsymbol{\lambda}}$ are minuscule; recall that a weight μ is called **minuscule** if every weight with a non-zero weight space in V_μ is in the Weyl group orbit of μ.

Proposition 8.3 *If all λ_i are minuscule, then the cohomology of $\Phi(T)(\Bbbk)$ is finite-dimensional.*

PROOF. If all λ_i are minuscule, then the preorder on standard modules is a true partial order, since there are never two standard modules with the same weight in each component. Furthermore, since every weight space of the categorification of a minuscule is equivalent to the category of vector spaces, $\mathrm{End}(S) \cong \Bbbk$ for any indecomposable standard.

These properties show that $T^{\underline{\boldsymbol{\lambda}}}$-mod is a highest weight category. Any highest weight category with finitely many simples has finite homological dimension (in fact, the homological dimension is no more than twice the number of simple objects).

Thus, in this case, the functor given by RHom or $\overset{L}{\otimes}$ with a finite dimensional module preserves being quasi-isomorphic to a finite length complex. □

2. The unknot for $\mathfrak{g} = \mathfrak{sl}_2$

Unfortunately, the cohomology of the complex $\Phi(T)(\Bbbk)$ is not always finite-dimensional. This can be seen in examples as simple as the unknot U for $\mathfrak{g} = \mathfrak{sl}_2$ and label 2.

In this case, the module L_2 has a standard resolution of the form

$$0 \longrightarrow S^{(0,0)}_{1^{(2)}}(-2) \longrightarrow S^{(0,1)}_{1,1}/(y_1+y_2)(-1) \longrightarrow S^{(0,2)}_{1^{(2)}} \longrightarrow L_2 \longrightarrow 0.$$

We let $A = \mathrm{End}_{\mathcal{V}^{2,2}}(S^{(0,1)}_{1,1}, S^{(0,1)}_{1,1}) \cong \Bbbk[y_1,y_2]/(y_1^2,y_2^2)$; the middle piece of the semi-orthogonal decomposition is equivalent to representations of this algebra.

Taking $\otimes$ of this resolution with its dual, we observe that all Tor's vanish between terms that do not lie in the same piece of the semi-orthogonal decomposition, so

$$\begin{aligned}\operatorname{Tor}^\bullet(L_\lambda, L_\lambda) &= \operatorname{Tor}^\bullet(S^{(0,2)}_{1^{(2)}}, (S^{(0,2)}_{1^{(2)}})^\star) \\ &\oplus \operatorname{Tor}^\bullet(S^{(0,1)}_{1,1}/(y_1+y_2), (S^{(0,1)}_{1,1}/(y_1+y_2))^\star)[2](-2) \oplus \operatorname{Tor}^\bullet(S^{(0,2)}_{1^{(2)}}, (S^{(0,2)}_{1^{(2)}})^\star)[4](-4) \\ &\cong \Bbbk \oplus \operatorname{Tor}^\bullet_A(A/(y_1+y_2), A/(y_1+y_2))[2](-2) \oplus \Bbbk[4](-4)\end{aligned}$$

The module $A/(y_1+y_2)A$ has a minimal projective resolution given by

$$\cdots \xrightarrow{y_1+y_2} A(-4) \xrightarrow{y_1-y_2} A(-2) \xrightarrow{y_1+y_2} A \longrightarrow A/(y_1+y_2)A \longrightarrow 0.$$

After taking $\otimes$, this becomes

$$\cdots \xrightarrow{0} A/(y_1{+}y_2)(-4) \xrightarrow{y_1-y_2} A/(y_1{+}y_2)(-2) \xrightarrow{0} A/(y_1{+}y_2) \xrightarrow{\sim} A/(y_1{+}y_2) \longrightarrow 0.$$

Thus, we have that

$$\operatorname{Tor}^i_A(A/(y_1+y_2), A/(y_1+y_2)) \cong \begin{cases} A/(y_1+y_2) & i=0 \\ \Bbbk(-2i) & i>0,\ \text{odd} \\ \Bbbk(-2i-2) & i>0,\ \text{even}\end{cases}$$

Thus, we have that

Proposition 8.4 $\varphi(U) = q^{-2}t^2 + 1 + q^2t^{-2} + \dfrac{q^{-2}-q^{-2}t}{1-t^2q^{-4}}.$

It is easy to see that the Euler characteristic is $q^{-2}+1+q^2 = [3]_q$, the quantum dimension of V_2. As this example shows, infinite-dimensionality of invariants is extremely typical behavior, and quite subtle. This same phenomenon of infinite dimensional vector spaces categorifying integers has also appeared in the work of Frenkel, Sussan and Stroppel [**FSS12**], and in fact, their work could be translated into the language of this paper using the equivalences of Chapter 9; it would be quite interesting to work out this correspondence in detail.

Conjecture 8.5 *The invariant $\Phi(L)$ for a link L is only finite-dimensional if all components of L are labeled with minuscule representations.*

3. Independence of projection

While Theorem 8.1 shows the action on the Grothendieck group is independent of the presentation of the tangle, it doesn't establish this for the functor $\Phi(T)$ itself.

Theorem 8.6 *The functor $\Phi(T)$ does not depend (up to isomorphism) on the projection of T.*

Proof. We have already proved the ribbon Reidemeister moves in at least one position: RI in Proposition 7.11 and RII and RIII as part of Theorem 6.15, and also the "S-move" shown in Figure 5 in Proposition 7.22. There is only one move of importance left for us to establish: the pitchfork move, shown in Figure 2.

Once we have established this move, we can easily show the others which are necessary. The illustrative example of the "χ-move" follows from the pitchfork and

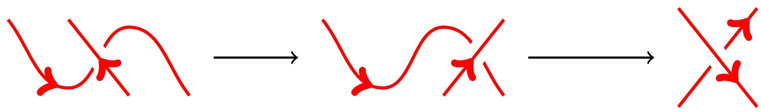


FIGURE 1. The "χ-move"

S-move, shown in Figure 1. The other moves in the list of Ohtsuki [**Oht02**, Theorem 3.3] follow in the same way.

So, let us turn to the pitchfork. We may assume that the pictured red strands are the only ones using Lemma 7.19 as in earlier proofs. We must prove that this move holds for all reflections and orientations. The vertical reflection of the version shown in Figure 2 follows from that illustrated by adjunction. We may assume that the cup is clockwise oriented, since the counter clockwise move can be derived from that one using Reidemeister moves II and III. The orientation of the "middle tine" is irrelevant, so we will ignore it. Thus, we have reduced to the case of Figure 2 and its reflection "through the page."

For the orientation shown in Figure 2, we need only show this move holds for $P_\emptyset$ again, since we again have equivariance for the $\mathcal{U}$ action by Lemma 7.21.

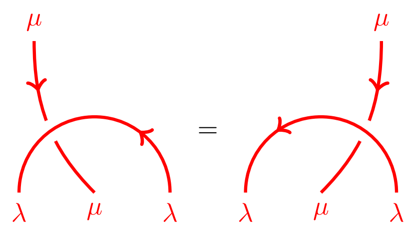


FIGURE 2. The "pitchfork" move

We have two functors $\mathcal{V}^{\mu}_{1/D} \to \mathcal{V}^{\lambda,\mu,\lambda^*}_{1/D}$ given by

$$\Pi_1 = \mathbb{B}_{\sigma_1^{-1}} \circ \mathbb{S}^{\mu,\lambda+\lambda^*}(P_\emptyset \boxtimes -) \qquad \Pi_2 = \mathbb{B}_{\sigma_2} \circ \mathbb{S}^{\lambda+\lambda^*,\mu}(- \boxtimes P_\emptyset).$$

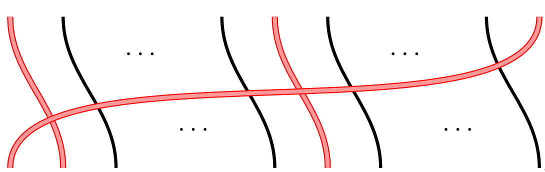
Lemma 8.7 *The functors* Π_1 *and* Π_2 *coincide.*

PROOF. First, we multiply both sides by $\mathbb{B}_{\sigma_1}$, so we must show that we have isomorphisms of functors

$$\mathbb{S}^{\mu,\lambda+\lambda^*}(P_\emptyset \boxtimes -) \cong \mathbb{B}_{\sigma_1} \circ \mathbb{B}_{\sigma_2} \circ \mathbb{S}^{\lambda+\lambda^*,\mu}(- \boxtimes P_\emptyset).$$

We need only exhibit a natural transformation and show it is an isomorphism when applied to projectives.

The isomorphism is given by the diagram



,

and is essentially the same as that of Proposition 6.10. We note that this element has degree zero because we are assuming that the roots on the black strands add to $\lambda + \lambda^*$. Any diagram in the module $\mathbb{B}_{\sigma_1}\mathbb{B}_{\sigma_2}\mathbb{S}^{\lambda+\lambda^*,\mu}(P_{\mathbf{i}}^{\kappa} \boxtimes P_{\emptyset})$ can be prefixed by this element, so the map is surjective. Any element which is sent to 0 by adjoining this diagram is easily seen to be 0, since the standardly violating strand can be slid downward to become a violating strand, so the map is also injective. □

The pitchfork move shown in Figure 2 follows from this lemma, since two sides of the depicted move are

$$- \otimes_T \Pi_1 L_\lambda(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)] \quad \text{and} \quad - \otimes_T \Pi_2 L_\lambda(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)].$$

The only variation remaining to check is the case where the move is reflected through the page (i.e. with the signs of the crossings given reversed), but this follows from the lemma as well since the two sides are

$$- \otimes_T (\Pi_1 L_\lambda)^\star(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)] \quad \text{and} \quad - \otimes_T (\Pi_2 L_\lambda)^\star(2\langle\lambda,\rho\rangle)[-2\rho^\vee(\lambda)]. \quad \square$$

Some care must be exercised with the normalization of these invariants, since as we noted in Chapter 2, they are the Reshetikhin-Turaev invariants for a slightly different ribbon element from the usual choice. However, the difference is easily understood. Let L be a link drawn in the blackboard framing, and let L_i be its components, with L_i labeled with λ_i. Recall that the **writhe** $\mathrm{wr}(K)$ of a oriented ribbon knot is the linking number of the two edges of the ribbon; this can be calculated by drawing the link the blackboard framing and taking the difference between the number of positive and negative crossings. Here we give a slight extension of the proposition of Snyder and Tingley relating the invariants for different framings [**ST09**, Theorem 5.21]:

Proposition 8.8 *The invariants attached to L by the standard and Snyder-Tingley ribbon elements differ by the scalar* $\prod_i (-1)^{2\rho^\vee(\lambda_i)\cdot(\mathrm{wr}(L_i)-1)}$.

PROOF. The proof is essentially the same as that of [**ST09**, Theorem 5.21] with a bit more attention paid to the case where the components have different labels. The proof is an induction on the crossing number of the link. The formula is correct for any framing of an unlink, which gives the base case of our induction.

Now note that the ratio between the knot invariants only depends on the number of rightward oriented cups and caps, so both the ratio between the invariants for the usual and ST ribbon structures and the formula given are insensitive to Reidemeister II and III as well as crossing change (which changes the writhe, but by an even number). These operations can be used to reduce any link to an unlink with some framing. Since we have already considered this case, we are done. □

One of the main reasons for interest in these quantum invariants of knots is their connection to Chern-Simons theory and invariants of 3-manifolds, so it is natural to ask:

Question 8.9 *Can these invariants glue into a categorification of the Witten-Reshetikhin-Turaev invariants of 3-manifolds?*

Remark 8.10 *The most naive ansatz for categorifying Chern-Simons theory, following the development of Reshetikhin and Turaev* [**RT91**] *would associate*

- *a category $\mathcal{C}(\Sigma)$ to each surface Σ, and*
- *an object in $\mathcal{C}(\Sigma)$ to each isomorphism of Σ with the boundary of a 3-manifold*

such that

- *the invariants $\mathcal{K}$ we have given are the Ext-spaces of this object for a knot complement with fixed generating set of $\mathcal{C}(T^2)$ labeled by the representations of $\mathfrak{g}$, and*
- *the categorification of the WRT invariant of a Dehn filling is the Ext space of this object with another associated to the torus filling.*

4. Functoriality

One of the most remarkable properties of Khovanov homology is its functoriality with respect to cobordisms between knots [**Jac04**]. This property is not only theoretically satisfying but also played an important role in Rasmussen's proof of the unknotting number of torus knots [**Ras10**]. Thus, we certainly hope to find a similar property for our knot homologies. While we cannot present a complete picture at the moment, there are promising signs, which we explain in this chapter. We must restrict ourselves to the case where the weights λ_i are minuscule, since even the basic results we prove here do not hold in general. We will assume this hypothesis throughout this section.

The weakest form of functoriality is putting a Frobenius structure on the vector space associated to a circle. This vector space, as we recall, is

$$A_\lambda = \mathrm{Ext}^\bullet(L_\lambda, L_\lambda)[2\rho^\vee(\lambda)](2\langle\lambda,\rho\rangle).$$

This algebra is naturally bigraded by the homological and internal gradings. The algebra structure on it is that induced by the Yoneda product. Recall that $\mathfrak{S}$ denotes the right Serre functor of $\mathcal{V}^{(\lambda,\lambda^*)}_{1/D}$, discussed in Chapter 2.

Theorem 8.11 *For a minuscule weights λ, we have a canonical isomorphism*

$$\mathfrak{S}L_\lambda \cong L_\lambda(-4\langle\lambda,\rho\rangle)[-4\rho^\vee(\lambda)].$$

Thus, the functors $\mathbb{K}$ and $\mathbb{T}$ are biadjoint up to shift.

In particular, $\mathrm{Ext}^{4\langle\lambda,\rho\rangle}(L_\lambda, L_\lambda) \cong \mathrm{Hom}(L_\lambda, L_\lambda)^*$, and the dual of the unit

$$\iota^*\colon\ \mathrm{Ext}^{4\langle\lambda,\rho\rangle}(L_\lambda, L_\lambda) \to \mathbb{k}$$

is a symmetric Frobenius trace on A_λ of degree $-4\langle\lambda,\rho\rangle$

One should consider this as an analogue of Poincaré duality, and thus is a piece of evidence for A_λ's relationship to cohomology rings.

Proof. As we noted in the proof of 8.3, $T^{\underline{\boldsymbol{\lambda}}}$ has finite global dimension if the weights $\underline{\boldsymbol{\lambda}}$ are minuscule. The result then follows immediately from Proposition 6.21. □

It would be enough to show that this algebra is commutative to establish the functoriality for flat tangles; we simply use the usual translation between 1+1 dimensional TQFTs and commutative Frobenius algebras (for more details, see the book by Kock [**Koc04**]). At the moment, not even this very weak form of functoriality is known.

Question 8.12 *Is there another interpretation of the algebra A_λ? Is it the cohomology of a space?*

One natural guess, based on the work of Mirković-Vilonen [**MV07**] and the symplectic duality conjecture of the author and collaborators [**BLPW**], is that A_λ is the cohomology of the corresponding Schubert variety $\overline{\mathrm{Gr}_\lambda}$ in the Langlands dual affine Grassmannian.

Another candidate algebra is the multiplication induced on V_λ by the quantized "shift of function algebra" $\mathcal{A}_f$ for a regular nilpotent element f studied by Feigin, Frenkel, and Rybnikov [**FFR10**].

We can use the biadjunction of $\mathbb{K}$ and $\mathbb{T}$ to give a rather simple prescription for functoriality: for each embedded cobordism in $I \times S^3$ between knots in S^3, we can isotope so that the height function is a Morse function, and thus decompose the cobordism into handles. Furthermore, we can choose this so that the projection goes through these handle attachments at times separate from the times it goes through Reidemeister moves. We construct the functoriality map by assigning

- to each Reidemeister move, we associate a fixed isomorphism of the associated functors.
- to the birth of a circle (the attachment of a 2-handle), we associate the unit of the adjunction $(\mathbb{K}, \mathbb{T})$ or $(\mathbb{C}, \mathbb{E})$, depending on the orientation.
- to the death of a circle (the attachment of a 0-handle), we associate the counits of the opposite adjunctions $(\mathbb{T}, \mathbb{K})$ or $(\mathbb{E}, \mathbb{C})$ (i.e., the Frobenius trace).
- to a saddle cobordism (the attachment of a 1-handle), we associate (depending on orientation) the unit of the second adjunction above, or the counit of the first.

Conjecture 8.13 *This assignment of a map to a cobordism is independent of the choice of Morse function, i.e. this makes the knot homology theory $\mathcal{K}(-)$ functorial.*

In the case of $\mathfrak{sl}_2$, there is a homology theory which we believe to coincide with ours, defined by Cooper, Hogancamp and Krushkal [**CK12**, **CHK11**]. A version of functoriality for this theory has been given by Hogancamp [**Hog**], overcoming some of the difficulties posed by the failure of finite global dimension this case, but still not giving an answer for every cobordism between knots.

CHAPTER 9

Comparison to category $\mathcal{O}$ and other knot homologies

Now, we specialize to the case where $\mathfrak{g} \cong \mathfrak{sl}_n$ and $\Bbbk = \mathbb{C}$. In this case, we can reinterpret our results in terms of the work of Brundan and Kleshchev [**BK08**, **BK09**] who have shown that the cyclotomic Khovanov-Lauda algebras for $\mathfrak{sl}_n$ are isomorphic to cyclotomic degenerate affine Hecke algebras (cdAHA). Proposition 5.33 allows us to embed the category of projectives over $T^{\underline{\boldsymbol{\lambda}}}$ in the category of all T^{λ}-modules. Transporting structure via Brundan and Kleshchev's isomorphism, we obtain a subcategory of modules over the degenerate affine Hecke algebra. We will show that this subcategory is also the image of the embedding of a block of parabolic category $\mathcal{O}$ via a well-known functor. In particular, this will allow us to match our categories $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ with blocks of category $\mathcal{O}$ in type A and compare the knot homologies constructed in Chapter 8 to those constructed using category $\mathcal{O}$ by Mazorchuk, Stroppel and Sussan [**MS09**, **Sus07**].

1. Cyclotomic degenerate Hecke algebras

Definition 9.1 *The degenerate affine Hecke algebra (dAHA) H_d is the algebra generated by the polynomial ring $\Bbbk[x_1, \dots, x_d]$ and the group ring $\Bbbk[S_d]$ subject to the relations*

$$s_i x_j = x_{s_i \cdot j} s_i - \delta_{j,i} + \delta_{j,i+1} \qquad\qquad x_i x_j = x_j x_i$$

for the simple reflections in $s_i \in S_d$.

We have a natural action of H_d on the $\mathfrak{gl}_N$ module $P \otimes V^{\otimes d}$ for any $\mathfrak{gl}_N$ representation P, where $V = \mathbb{C}^N$ is the defining representation of $\mathfrak{gl}_N$:

- S_d acts on the d copies of V, and
- x_1 acts by $C \otimes 1^{\otimes d-1}$ where C is the Casimir element of $\mathfrak{gl}_N$.

We'll be interested in applying this result in one particular context. Fix a parabolic $\mathfrak{p} \subset \mathfrak{gl}_N$. Without loss of generality, we can assume that $\mathfrak{p}$ is the precisely this subalgebra of block upper-triangular matrices attached to a composition $\pi = (\pi_1, \dots, \pi_\ell)$. These can be used to define a weight $\lambda = \sum_i \omega_{\pi_i} \in Y(\mathfrak{g})$; that is, $\lambda^j = \#\{i | \pi_i = j\}$.

Definition 9.2 *Parabolic category $\mathcal{O}$, which we denote $\mathcal{O}^{\mathfrak{p}}$, is the full subcategory of $\mathfrak{gl}_N$-modules with a weight decomposition where $\mathfrak{p}$ acts locally finitely.*

Since induction sends finite-dimensional modules to $\mathfrak{p}$-locally finite modules, $P \otimes V^{\otimes d} \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (W \otimes V^{\otimes d})$ lies in this category for W any finite dimensional $\mathfrak{p}$-representation.

We'll index the parabolic Verma module in $\mathcal{O}^{\mathfrak{p}}$ by their ρ-shifted highest weight. That is, we'll let $M^{\mathfrak{p}}(a_1,\dots,a_N)$ be the parabolic Verma module where the diagonal elementary matrix e_{ii} acts by a_i+i-1, and $L(a_1,\dots,a_N)$ be the simple $\mathfrak{gl}_N$ module with this highest weight. We'll only consider the case where a_i is an integer in this paper. For example, the trivial module is $L(0,-1,\dots,-N+1)$. Of course, for certain highest weights, $L(a_1,\dots,a_N)$ will not lie in $\mathcal{O}^{\mathfrak{p}}$. In this case, by convention, $M^{\mathfrak{p}}(a_1,\dots,a_N)=0$. For example, the module $L(a_1,\dots,a_N)$ will be in $\mathcal{O}^{\mathfrak{gl}_N}$ if and only if the entries a_i are *strictly* increasing.

More generally, $L(a_1,\dots,a_N)$ will be in $\mathcal{O}^{\mathfrak{p}}$ if and only if the associated highest weight is dominant when restricted to the Levi $\mathfrak{l}$ of block triangular matrices. That is, if we have that $a_1>\cdots>a_{\pi_1}, a_{\pi_1+1}>\cdots>a_{\pi_1+\pi_2}$, etc.

Following Brundan and Kleshchev [**BK08**, §4.2], we can conveniently package this condition by thinking of the numbers a_i as the column reading of the entries of a tableau on the Young pyramid for the composition $\boldsymbol{\pi}$. To fix conventions, we read the columns from top to bottom and in order from left to right. The inequalities above are the statement that the tableau is *column-strict*, i.e. its entries increase in each column decrease when read from top to bottom. Thus, we have that:

Lemma 9.3 *The simple module $L(a_1,\dots,a_N)$ is in $\mathcal{O}^{\mathfrak{p}}$ if* **a** *is the column reading of a column-strict tableau.*

This labeling is particularly convenient, since two simples $L(a_1,\dots,a_N)$ and $L(a_1',\dots,a_N')$ are in the same block of $\mathcal{O}^{\mathfrak{p}}$ if and only if $a_i=a'_{w(i)}$ for some permutation w. From now on, we let $P=M^{\mathfrak{p}}(\pi_1,\dots,1,\pi_2,\dots,1,\dots,\pi_\ell,\dots,1)$. The corresponding "ground state" tableau fills each box with its height. Note that this is the only column-strict tableau with these entries, so there are no other simples in the same block as P. Thus P is simple, projective and injective in $\mathcal{O}^{\mathfrak{p}}$.

Now, consider the action of dAHA on $\oplus_d P\otimes V^{\otimes d}$. This action is not faithful, but there is a very convenient description of its kernel:

Definition 9.4 *The* **cyclotomic degenerate affine Hecke algebra** *is the quotient of the dAHA given by*

$$H_d^\lambda = H_d\Big/\Big\langle \prod_{i=1}^{n}(x_1-i)^{\lambda^i}\Big\rangle \qquad H^\lambda \cong \bigoplus_{d\geq 0} H_d^\lambda.$$

We let e_d be the idempotent which projects to H_d^λ in H^λ.

Theorem 9.5 (Brundan-Kleshchev [**BK08**, Th. B]) *When*

$$P=M(\pi_1,\dots,1,\pi_2,\dots,1,\dots)$$

as above, the action of dAHA on $P\otimes V^{\otimes d}$ factors through a faithful action of H_d^λ.

Thus, we have a functor $\mathrm{Hom}_{\mathfrak{gl}_N}(P\otimes V^{\otimes d},-):\mathcal{O}^{\mathfrak{p}}\to H^\lambda$-mod. This functor is very far from being an equivalence, but on each block of $\mathcal{O}^{\mathfrak{p}}$ it is either 0, or fully faithful on projectives by [**BK08**, 6.10]. Thus, certain blocks of $\mathcal{O}^{\mathfrak{p}}$ can be described in terms of endomorphism rings of modules over H^λ, as in [**BK08**, Th. C].

The center of H_d^λ is generated by the symmetric polynomials in the alphabet x_i. Particular, this algebra decomposes into summands according to the joint spectrum of these symmetric polynomials. For any list $(a_1, \dots, a_d)$ of integers, we have a summand

$$H_d^\lambda(a_1, \dots, a_d) = \{m \in H_d^\lambda | (f(\mathbf{x}) - f(\mathbf{a}))^j m = 0$$

for $j \gg 0$ and any symmetric polynomial f. The projection to this summand is given by multiplication by a central idempotent $e(\mathbf{a})$ of H_d^λ, since it is an idempotent bimodule endomorphism of H_d^λ.

We let $e_{\mathfrak{g}}$ be the idempotent projecting to the subalgebra

$$\bigoplus_{(a_1,\dots,a_d)\in[1,n]^d} H_d^\lambda(a_1, \dots, a_d).$$

We can alternately describe this as projection to the kernel of $\prod_{i=1}^d \prod_{j=1}^n (x_i - j)^N$ for $N \gg 0$.

In this chapter, we use the polynomials Q_{ij} as defined in the previous chapter for a fixed orientation of the type A (or later, affine type A) quiver. The most obvious choice is

$$Q_{ij}(u, v) = \begin{cases} 1 & i \neq j \pm 1 \\ u - v & i = j + 1 \\ v - u & i = j - 1 \end{cases}$$

Proposition 9.6 ([**BK09**]) *There is an isomorphism* $\Upsilon \colon T^\lambda \to e_{\mathfrak{g}} H^\lambda e_{\mathfrak{g}} \overset{\text{def}}{=} H^{\lambda,n}$.

Under this map, we have that $\Upsilon(y_j e(\mathbf{i})) = e(\mathbf{i})(x_j - i_j)$, and $\Upsilon^{-1}(s_i)$ is in a linear combination of $y_i^a y_{i+1}^b \psi_i e(\mathbf{i})$ and $y_i^a y_{i+1}^b e(\mathbf{i})$ by [**BK09**, (3.41-42)].

2. Comparison of categories

First, let us endeavor to understand how we can translate the $T^{\boldsymbol{\lambda}}$-modules $y_{\mathbf{i},\kappa} T^\lambda$ defined in Chapter 5 into the language of the cdAHA using Υ. It's immediate from Proposition 9.6 that

$$\Upsilon(y_{\mathbf{i},\kappa}) = e(\mathbf{i}) \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} (x_k - i_k)^{\lambda_j^{i_k}}.$$

However, the strong dependence of this element on $e(\mathbf{i})$ makes it problematic for use in the Hecke algebra. We first specialize to the case where all the weights λ_j are fundamental. That is, we have $\lambda_j = \omega_{\pi_j}$ for some π_j. As suggested by the notation, we will later want to think of π_j as a composition. This bit of notation allows us to associate to each κ an element of $H^{\lambda,n}$ (note that there is no dependence on $\mathbf{i}$):

$$z_\kappa = \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} (x_k - \pi_j) \tag{9.1}$$

We let $M_{\mathbf{i}}^\kappa = e(\mathbf{i}) z_\kappa H^{\lambda,n}$ and $M^\kappa = z_\kappa H^{\lambda,n}$.

Proposition 9.7 *If $\lambda_j = \omega_{\pi_j}$, then for all $\mathbf{i}$, we have $\Upsilon(y_{\mathbf{i},\kappa}) H^{\lambda,n} = M_{\mathbf{i}}^\kappa$. In particular, we have an isomorphism $T^{\boldsymbol{\lambda}} \cong \operatorname{End}(\oplus_\kappa M^\kappa)$.*

PROOF. If $a \neq i_k$, then we can rewrite $e(\mathbf{i})$ as

$$e(\mathbf{i}) = (x_k - a)e(\mathbf{i})\left(\frac{-1}{a-i_k} - \frac{x_k - i_k}{(a-i_k)^2} - \frac{(x_k-i_k)^2}{(a-i_k)^3} - \cdots\right)$$

since $(x_k - i_k)e(\mathbf{i})$ is nilpotent. It follows that

$$e(\mathbf{i})(x_k - \pi_j)H^{\lambda,n} = e(\mathbf{i})(x_k - i_k)^{\lambda_j^{i_k}} H^{\lambda,n} \tag{9.2}$$

since $\lambda_j^{i_k} = \delta_{\pi_j, i_k}$. Thus, applying (9.2) to each term in z_κ, the result follows. $\square$

We note that the modules M^κ are closely related to the permutation modules discussed by Brundan and Kleshchev in [**BK08**, §6]. Each way of filling π as a tableau such that the column sums are $\kappa(i) - \kappa(i-1)$ results in a permutation module which is a summand of M^κ.

Now we wish to understand how the modules M^κ are related to parabolic category $\mathcal{O}$. Let $N = \sum_j \pi_j$ be the number of boxes in π. As before, the π_i give a composition of N, and thus a parabolic subgroup $\mathfrak{p} \subset \mathfrak{gl}_N$, which is precisely the operators preserving a flag $V_1 \subset V_2 \subset \cdots \subset V$. If, as usual, κ is a weakly increasing function on $[1,\ell]$ with non-negative integer values and further $\kappa(\ell) \leq d$, then we let

$$V_\kappa^d = V_1^{\otimes \kappa(1)} \otimes V_2^{\otimes \kappa(2)-\kappa(1)} \otimes \cdots \otimes V^{d-\kappa(\ell)}$$

as a $\mathfrak{p}$-representation. We can induce this representation to an object in $\mathcal{O}^\mathfrak{p}$ which we denote

$$\mathsf{P}_d^\kappa \cong U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V_\kappa^d),$$

where $\mathbb{C}_{-\rho}$ is the 1-dimensional $\mathfrak{p}$-module defined in [**BK08**, pg. 4]. These modules contain as summands the divided power modules

$$U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes \mathrm{Sym}^{\kappa(1)}(V_1) \otimes \mathrm{Sym}^{\kappa(2)-\kappa(1)}(V_2) \otimes \cdots \otimes \mathrm{Sym}^{d-\kappa(\ell)}(V))$$

defined by Brundan and Kleshchev in [**BK08**, §4.5].

All the objects P_d^κ live in the subcategory we denote $\mathcal{O}_{>0}^\mathfrak{p}$ which is generated by all parabolic Verma modules whose corresponding tableau has positive integer entries. We also consider a much smaller subcategory which has only finitely many simple objects: let $\mathcal{O}_n^\mathfrak{p}$ be the subcategory of $\mathcal{O}^\mathfrak{p}$ generated by all parabolic Vermas whose corresponding tableau only uses the integers $[1,n]$. Let $\mathrm{pr}_n : \mathcal{O}^\mathfrak{p} \to \mathcal{O}_n^\mathfrak{p}$ be the projection to this subcategory ($\mathcal{O}_n^\mathfrak{p}$ is a sum of blocks, so there is a unique projection).

Proposition 9.8 *If one ranges over all κ and all integers d, then $\oplus_{\kappa,d}\mathsf{P}_\kappa^d$ is a projective generator for $\mathcal{O}_{>0}^\mathfrak{p}$.*

PROOF. This follows from a simple modification of the proof of [**BK08**, Theorem 4.14]. In the notation of that proof, we have that $\mathsf{P}_d^\kappa \cong R(\mathsf{P}_{\kappa(\ell)}^{\kappa^-} \otimes \mathbb{C}_{-\rho}) \otimes V^{\otimes d-\kappa(\ell)}$, where κ^- is the restriction of κ to $[1,\ell-1]$. As noted in that proof, by induction, this is two functors which preserve projective modules applied to a projective module; thus P_d^κ is projective.

Each of Brundan and Kleshchev's divided power modules is a summand in one of the P_d^κ, as we noted earlier. Since any indecomposable projective of $\mathcal{O}^\mathfrak{p}$ is a summand of a divided power module, the same is true of the P_d^κ's. $\square$

Proposition 9.9 *For all d, κ, we have*

$$z_\kappa H^\lambda e_d \cong \operatorname{Hom}(P \otimes V^{\otimes d}, \mathsf{P}^d_\kappa)$$
$$M^\kappa e_d \cong \operatorname{Hom}(P \otimes V^{\otimes d}, \operatorname{pr}_n(\mathsf{P}^d_\kappa)).$$

Proof. This rests on a single computation, which is that the image in $P \otimes V$ of the action of $\prod_{i=j+1}^{\ell}(x_1 - \pi_i)$ is

$$U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V_j) \subset U(\mathfrak{gl}_n) \otimes_{U(\mathfrak{p})} (\mathbb{C}_{-\rho} \otimes V) \cong P \otimes V;$$

this follows from [**BK08**, Lemma 3.3]. This shows that the image of z_κ acting on $P \otimes V^{\otimes d}$ is P^d_κ, so by the projectivity of $P \otimes V^{\otimes d}$, every homomorphism to P^d_κ factors through this one.

We can identify those homomorphisms whose image is in $\operatorname{pr}_n(\mathsf{P}^d_\kappa) \subset \mathsf{P}^d_\kappa$ as those killed by some power of $\chi^n_j = \prod_{i=1}^n (x_j - i)$ for each j (if a number m appears in a tableau, then $x_j - m$ is nilpotent for some j, and so if $m \notin [1, n]$, then χ^n_j is invertible for that j). Thus, this homomorphism space is the subspace of $z_\kappa H^\lambda e_d$ on which all χ^n_j act nilpotently, which is precisely $M^\kappa e_d$. □

Corollary 9.10 *For the sequence of weights $\underline{\boldsymbol{\lambda}} = (\omega_{\pi_1}, \dots, \omega_{\pi_\ell})$, we have an equivalence $\Xi : \mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \xrightarrow{\cong} \mathcal{O}^{\mathfrak{p}}_n$.*

We can generalize this statement a bit further: let us now consider the case where the weights λ_i are not fundamental. In this case, to each weight λ_i we have a unique Young diagram given by writing it as a sum of fundamental weights, and we obtain a pyramid π by concatenating these horizontally (this is the pyramid associated earlier to the refinement of $\underline{\boldsymbol{\lambda}}$ into fundamental weights). We associate a parabolic $\mathfrak{p}$ with the pyramid as on the previous page.

For each collection of semi-standard[1] tableaux T_i on each of these diagrams which only use the integers $[1, n]$, this gives a tableau on π (now just column-strict). Such a tableau can be converted into a module in $\mathcal{O}^{\mathfrak{p}}$ for $\mathfrak{gl}_N$ (where $N = \sum \pi_i$) by taking the projective cover of the $\mathfrak{p}$-parabolic Verma module corresponding to this tableau. Let $\mathcal{O}^{\mathfrak{p}}_{\underline{\boldsymbol{\lambda}}}$ be the subcategory of modules presented by these projectives.

Proposition 9.11 *The functor Ξ induces an equivalence of $\mathcal{O}^{\mathfrak{p}}_{\underline{\boldsymbol{\lambda}}}$ and $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$.*

Proof. Let π_i be a composition chosen so that $\underline{\boldsymbol{\lambda}}' = (\omega_{\pi_1}, \dots, \omega_{\pi_q})$ is one way of splitting $\underline{\boldsymbol{\lambda}}$ into fundamental weights. By Lemma 4.21, we have an embedding $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \hookrightarrow \mathfrak{V}^{\underline{\boldsymbol{\lambda}}'}$ as the objects represented by $P^\kappa_{\mathbf{i}}$ where κ is constant on the blocks of fundamental weights obtained by breaking up λ_i.

Corollary 9.10 thus shows that $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is equivalent to the subcategory of $\mathcal{O}^{\mathfrak{p}}_{\underline{\boldsymbol{\lambda}}'}$ consisting of objects presented by projectives $\operatorname{pr}_n(\mathsf{P}^d_\kappa)$ where κ is constant on the blocks of fundamental weights obtained by breaking up λ_i. In terms of category $\mathcal{O}$, these are the result of inducing finite-dimensional $\mathfrak{p}$-modules obtained by tensoring the vector spaces which appear in a particular flag preserved by $\mathfrak{p}$, the gaps of which encode the sequence $\underline{\boldsymbol{\lambda}}$.

That is, the indecomposable projectives of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ are sent to the indecomposable projectives which appear as summands of these $\operatorname{pr}_n(\mathsf{P}^\kappa_d)$. Thus these are in bijection,

[1] In [**BK08**], these are called "standard."

and there can only be $\dim V_{\underline{\boldsymbol{\lambda}}}$ of the latter. Since there is exactly that number of tableaux which are semi-standard in blocks as described above, we need only show that these occur as summands.

This follows from the relationship between the crystal structure on tableaux and projectives in category $\mathcal{O}$. Specifically, since any tableau which is semi-standard in blocks can be obtained from the empty tableau by the operations of attaching a fresh Young diagram filled with the ground state tableau and of applying crystal operators, the argument from [**BK08**, Corollary 4.6] shows that the projective corresponding to such a tableau is a summand of an appropriate P^{κ}_{d}. □

We note that this shows that our categorification agrees with that for twice fundamental weights of $\mathfrak{sl}_n$ recently given by Hill and Sussan [**HS10**].

The category $\mathcal{O}^{\mathfrak{p}}$ has a natural endofunctor given by tensoring with V. Restricting to $\mathcal{O}^{\mathfrak{p}}_n$, we can take the functor $f_\bullet = \mathrm{pr}_n(-\otimes V)$. This functor has a natural decomposition $f_\bullet = \oplus_{i=1}^n f_i$ in terms of the generalized eigenspaces of x_1 acting on $-\otimes V$; we need only take $i \in [0,n]$ since these are the only eigenvalues of x_1 on the projection to $\mathcal{O}^{\mathfrak{p}}_n$.

Proposition 9.12 *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{O}^{\mathfrak{p}}_n \arrow[r, "f_i"] & \mathcal{O}^{\mathfrak{p}}_n \\
\mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"] \arrow[r, "\mathfrak{F}_i"'] & \mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"']
\end{tikzcd}$$

Proof. The functor $f_\bullet$ corresponds to tensoring a $H^{\lambda,n}_d$-module with $H^{\lambda,n}_{d+1}$. By Proposition 9.6, this corresponds to tensoring over $T^{\underline{\boldsymbol{\lambda}}}_{\mu}$ with $\oplus_i T^{\underline{\boldsymbol{\lambda}}}_{\mu-\alpha_i}$ via the map $\oplus \nu_i$. This is, of course, the functor $\oplus_{i=1}^n \mathfrak{F}_i$. Via Brundan and Kleshchev's isomorphism, x_n acts on $\mathfrak{F}_i M$ for any M by $y_n + i$; that is, $x_n - i$ acts invertibly on $\mathfrak{F}_j M$ for $j \neq i$ and nilpotently on $\mathfrak{F}_i M$. This shows the desired isomorphism. □

For any parabolic subalgebra $\mathfrak{q} \supset \mathfrak{p}$ with Levi $\mathfrak{l} = \mathfrak{q}/\mathrm{rad}\,\mathfrak{q}$, we have an induction functor

$$\mathrm{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N} \overset{def}{=} U(\mathfrak{gl}_N) \otimes_{U(\mathfrak{q})} - : \mathcal{O}^{\mathfrak{p}}(\mathfrak{l}) \to \mathcal{O}^{\mathfrak{p}}$$

where $\mathcal{O}^{\mathfrak{p}}(\mathfrak{l})$ denotes the parabolic category $\mathcal{O}$ for $\mathfrak{l}$ and the parabolic $\mathfrak{p}/\mathrm{rad}\,\mathfrak{q}$ (here $\mathfrak{l}$-representations are considered as $\mathfrak{q}$ representations by pullback).

Choices of $\mathfrak{q}$ are in bijection with partitions of $\underline{\boldsymbol{\lambda}}$ into consecutive blocks $\underline{\boldsymbol{\lambda}}_1, \dots, \underline{\boldsymbol{\lambda}}_k$. Let $\Xi_{\mathfrak{l}} : \mathfrak{V}^{\underline{\boldsymbol{\lambda}}_1;\dots;\underline{\boldsymbol{\lambda}}_k} \to \mathcal{O}^{\mathfrak{p}}(\mathfrak{l})$ be the comparison functor analogous to Ξ for $\mathfrak{l}$.

Proposition 9.13 *We have a commutative diagram*

$$\begin{tikzcd}
\mathcal{O}_n^{\mathfrak{p}}(\mathfrak{l}) \arrow[rr, "\operatorname{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N}"] & & \mathcal{O}_n^{\mathfrak{p}} \\
\mathfrak{V}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_k} \arrow[u, "\Xi_{\mathfrak{l}}"] \arrow[rr, "\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1,\dots,\underline{\boldsymbol{\lambda}}_k}"'] & & \mathfrak{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"']
\end{tikzcd}$$

Proof. We know that both functors are exact, by Proposition 5.7; thus need only check this on projectives. Consider a representation of $\mathfrak{l}$ given by an exterior product of projectives in category $\mathcal{O}$ for each of its $\mathfrak{gl}_j$-factors

$$P = \mathsf{P}_1 \boxtimes \cdots \boxtimes \mathsf{P}_k.$$

Then the induction $\operatorname{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N} P$ is a quotient of the projective P' corresponding to the concatenation T of the tableaux T_i for the P_i. The kernel is the image of all maps from projectives higher than T in Bruhat order through a series of transpositions which change the content of at least one of the T_i.

Similarly, the standardization $\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1;\cdots;\underline{\boldsymbol{\lambda}}_k}(\Xi_{\mathfrak{l}}^{-1}(P))$ is a quotient of $\Xi^{-1}(P')$; the kernel is the image of all maps from projectives that correspond to idempotents for sequences where at least one black strand has been moved left from one block to the other. Thus, these functors agree on the level of projective objects.

Now, we must show that they agree on morphisms; that is, we must show that the action of $T^{\underline{\boldsymbol{\lambda}}_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_k}$ induced on $\operatorname{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N}(\Xi(T^{\underline{\boldsymbol{\lambda}}_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_k}))$ agrees with that on $\Xi(\mathbb{S}^{\underline{\boldsymbol{\lambda}}_1,\dots,\underline{\boldsymbol{\lambda}}_k}(T^{\underline{\boldsymbol{\lambda}}_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_k}))$ under an isomorphism between these objects. Since $T^{\underline{\boldsymbol{\lambda}}_1}_{\alpha_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_k}_{\alpha_k}$ is the full-endomorphism algebra of $S_{\boldsymbol{\alpha}}$, it is also the full endomorphism algebra of $\Xi(S_{\boldsymbol{\alpha}})$. Thus, in fact, any isomorphism $\Xi(S_{\boldsymbol{\alpha}}) \cong \operatorname{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N}(\Xi(T^{\underline{\boldsymbol{\lambda}}_1}_{\alpha_1} \otimes \cdots \otimes T^{\underline{\boldsymbol{\lambda}}_k}_{\alpha_k}))$ induces an isomorphism of functors. □

Some care is required here on the subject of gradings. Brundan and Kleshchev's results relating category $\mathcal{O}$ to Khovanov-Lauda algebras are ungraded; they imply no connection between the usual graded lift of $\tilde{\mathcal{O}}^{\mathfrak{p}}$ of category $\mathcal{O}$ and the graded category of modules over $T^{\underline{\boldsymbol{\lambda}}}$. Luckily, the uniqueness of Koszul gradings proven in [**BGS96**, 2.5.2] implies that any Morita equivalence between two Koszul graded algebras can be lifted to a graded equivalence.

There are now two proofs in the literature that in the type A case, when all weights are fundamental, these algebras are Koszul. Hu and Mathas have shown that their quiver Schur algebra is Koszul [**HM**, Th. C]; thus, we may use the Morita equivalence of Theorem 5.34 to transport this result to $T^{\underline{\boldsymbol{\lambda}}}$. The author has also given a direct geometric proof in [**Webc**, Th. B], by directly constructing a graded isomorphism of $T^{\underline{\boldsymbol{\lambda}}}$ with an Ext-algebra in the Koszul dual of $\mathcal{O}_n^{\mathfrak{p}}$.

Proposition 9.14 *When $\mathfrak{g} = \mathfrak{sl}_n$ and $\underline{\boldsymbol{\lambda}}$ is a list of fundamental weights, the algebra $T^{\underline{\boldsymbol{\lambda}}}_{\mu}$ is Koszul.*

If $\underline{\boldsymbol{\lambda}}$ is not a list of fundamental weights, then we expect that $T^{\underline{\boldsymbol{\lambda}}}$ will never be Koszul.

Corollary 9.15 *The equivalence Ξ has a graded lift.*

We note that both the action of projective functors and of induction functors on $\mathcal{O}^{\mathfrak{p}}$ have graded lifts which are unique up to grading shift, and thus are determined by their action on the Grothendieck group. Thus the graded lifts given by the action of $\mathcal{U}$ and $\mathbb{S}$ agree, up to an easily understood shift, with those used in other papers on graded category $\mathcal{O}$ (most importantly for us, this is used in the work of Mazorchuk-Stroppel [**MS08**] and Sussan [**Sus07**] on link homologies, which we build upon later).

3. The affine case

We note that the constructions of the previous section generalize in an absolutely straightforward way to the affine case by simply replacing the results of Section 3 of [**BK09**] with Section 4.

We let $\hat{H}_d$ denote the affine Hecke algebra (not the degenerate one we considered earlier). Fix an element $\zeta \in \overline{\Bbbk}$, the separable algebraic closure of $\Bbbk$ such that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0,$$

and n is smallest integer for which this holds (for example, if $\Bbbk$ is characteristic 0, these means that ζ is a primitive nth root of unity). The **cyclotomic affine Hecke algebra** or **Ariki-Koike algebra** (introduced in [**AK94**]) for λ is the quotient

$$\hat{H}^\lambda = \bigoplus_d \hat{H}_d / \langle (X_1 - \zeta^i)^{\alpha_i^\vee(\lambda)} \rangle.$$

where we adopt the slightly strange convention that if $\zeta \in \mathbb{Z}$, then $\zeta^i = \zeta + i$, and otherwise it is the usual power operation.

Theorem 9.16 ([**BK09**, Main Theorem]) *When $\mathfrak{g} \cong \widehat{\mathfrak{sl}}_n$, there is an isomorphism $T^\lambda \cong \hat{H}^\lambda$.*

This symmetric Frobenius algebra has a natural quasi-hereditary cover, called the **cyclotomic q-Schur algebra**, defined by Dipper, James and Mathas [**DJM98**]. Indecomposable projectives over this algebra are indexed by ordered $k = \sum_{i=0}^n \alpha_i^\vee(\lambda)$-tuples of partitions.

Proposition 9.17 *When $\mathfrak{g} = \widehat{\mathfrak{sl}}_n$, then $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ is equivalent to the subcategory of representations of the cyclotomic q-Schur algebra consisting of objects presented by certain projective modules.*

If all λ_i are fundamental, then these are exactly the projectives for the multipartitions where each constituent partition is n-regular.

The results [**Webd**, 5.5& 5.8] actually allow one to write an explicit isomorphism between $T^{\underline{\boldsymbol{\lambda}}}$ and an endomorphism ring over projectives for the cyclotomic q-Schur algebra, giving a more explicit version of this theorem.

Proof. By Corollary 5.32, $T^{\underline{\boldsymbol{\lambda}}}$ is the endomorphism algebra of certain modules over T^λ, which one can see by the same arguments as Proposition 9.9 are of the form $\hat{z}_\lambda T^\lambda$ where

$$\hat{z}_\kappa = \prod_{j=1}^{\ell} \prod_{k=\kappa(j)+1}^{n} (x_k - \zeta^{\pi_j}).$$

These are permutation modules for the Ariki-Koike algebra, exactly those corresponding to the multipartitions where the kth component is of the form (1^{m_k}).

Corresponding to the summands of these modules, we have a subset S of the indecomposable projectives over the cyclotomic q-Schur and the corresponding simple quotients. The modules over the cyclotomic q-Schur algebra carry a categorical action of $\widehat{\mathfrak{sl}}_n$ as argued in [**Wad**, 5.8]. This is coincides with the action defined by Shan [**Sha11**] under an equivalence of categories by [**Wad**, 6.3]. Thus, we can transport Shan's crystal structure to simple modules over the cyclotomic q-Schur algebra; by [**Sha11**, 6.3], this crystal is the tensor product of ℓ copies of a level 1 Fock crystal.

The simples S are a subcrystal of this structure. Furthermore, if we consider all ranks together, this set is closed under the operation sending an $(\ell-1)$-multipartition $\nu^{(i)}$ to an ℓ-multipartition with $\nu^{(\ell)} = \emptyset$. There is only one such subset: the ℓ-multipartitions where all components are n-regular. $\square$

If λ_i is a general weight, as before, we can define $\underline{\boldsymbol{\lambda}}'$ by breaking every λ_i into fundamental weights. In this case, $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ will be equivalent to the subcategory presented by projectives where the first $k_1 = \sum_{i=1}^{n}(\lambda_1^i)$ partitions, the next k_2, etc, for an n-Kleshchev multipartition.

Thus, our categorification can be seen a generalization of the Ariki categorification theorem [**Ari96**]. As mentioned in the introduction, the author and Stroppel address the question of how to describe the entirety of the cyclotomic q-Schur algebra diagrammatically and obtain categorifications of other interesting objects in affine representation theory in [**SW**, **Webe**, **Webd**].

4. Comparison to other knot homologies

A great number of other knot homologies have appeared on the scene in the last decade, and obviously, we would like to compare them to ours. In this chapter we check the isomorphism which seems most straightforward based on the similarity of constructions: we describe an isomorphism to the invariants constructed by Mazorchuk-Stroppel and Sussan for the fundamental representations of $\mathfrak{sl}_n$.

In order to compare knot homologies, we must compare the functors we have described on our categories $\mathcal{V}^{\underline{\boldsymbol{\lambda}}}$ and those on $\tilde{\mathcal{O}}_n^{\mathfrak{p}}$. In order to keep combinatorics simpler, we consider our fundamental weights as weights of $\mathfrak{gl}_n$; this only affects the inner products between elements of the weight lattice, neither of which are in the root lattice. This has the advantage of assuring that all inner products between weights are integral, so we have no need of fractional gradings.

For simplicity, in this chapter we will assume that $\underline{\boldsymbol{\lambda}}$ is a sequence of fundamental weights. In this paper, we are only concerned about commuting up to isomorphism of functors; thus when we say a diagram of functors "commutes" we mean that the functors for any two paths between the same points are isomorphic.

First, let us consider the braiding functors. Associated to each permutation of N letters, we have a derived twisting functor $T_w \colon D^b(\mathcal{O}_n) \to D^b(\mathcal{O}_n)$ (see [**AS03**] for more details and the definition). We let T_w also denote the graded lift of this functor, which is normalized so that the Verma module for a dominant weight μ

generated in degree 0 is sent to that of highest weight $w(\lambda+\rho)-\rho$ also generated in degree 0.

Proposition 9.18 *When $\underline{\boldsymbol{\lambda}} = (\omega_1, \cdots, \omega_1)$, then $\mathfrak{p} = \mathfrak{b}$ and we have a commutative diagram*

$$\begin{tikzcd} D^b(\tilde{\mathcal{O}}_n) \arrow[r, "T_v"] & D^b(\tilde{\mathcal{O}}_n) \\ \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"] \arrow[r, "\mathbb{B}_v"'] & \mathcal{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"'] \end{tikzcd}$$

Proof. We note that the functors T_v commute with translation functors by [**AS03**, Lemma 2.1(5)]. The same holds for $\Xi \circ \mathbb{B}_v \circ \Xi^{-1}$ by Propositions 6.7 and 9.12.

Every projective object in $\tilde{\mathcal{O}}_n$ is a summand of a composition of translation functors applied to a dominant Verma module, and every morphism is the image of a natural transformation between these. The we need only compute the behavior of the functors T_v and on Verma modules $\Xi \circ \mathbb{B}_v \circ \Xi^{-1}$ on the level of objects in order to check isomorphisms of functors.

By Proposition 6.20, $\mathbb{B}_v$ sends the exceptional collection of standard objects to its mutation by using v to reorder the root functions $\boldsymbol{\alpha}$ given by the sum of the roots that appear between the red lines. On the other hand, the functor T_v sends the exceptional collection of Verma modules to its mutation by the change of order associated to the action of v on tableaux. By Proposition 9.13, these changes of partial order are intertwined by the correspondence between standard modules and Verma modules given by Ξ. Thus the mutations also match under Ξ, so the diagram commutes. $\square$

Finally, we turn to describing the functors associated to cups and caps. If π has a column of height n in the kth position, then any block of category $\tilde{\mathcal{O}}_n^{\mathfrak{p}}$ is equivalent to the block of category $\tilde{\mathcal{O}}_n^{\mathfrak{p}'}$ associated to π', the diagram π with that column of height n removed. The content of the tableaux in the new block is that of the original block with the multiplicity of each number in $[1, n]$ reduced by 1. The effect of this functor on the simples, projectives and Vermas is simply removing that column of height n (which by column strictness must be the numbers $[1, n]$ in order). The functor that realizes this equivalence $\zeta : \tilde{\mathcal{O}}_n^{\mathfrak{p}} \to \tilde{\mathcal{O}}_n^{\mathfrak{p}'}$ is the **Enright-Shelton equivalence**, which is developed in the form most useful for us in [**Sus07**, §3.2].

Having already developed the equivalence Ξ, this functor is actually quite easy to describe. Let P_d^{κ} denote the module attached to κ and d for $\mathfrak{p}'$ as above, and let $Q_d^{\kappa+}$ be the module attached in the same way to $\mathfrak{p}$, where

$$\kappa_+(j) = \begin{cases} \kappa(j) & j \leq k \\ \kappa(j-1) & j > k. \end{cases}$$

We already have equivalences of $\mathfrak{V}^{\underline{\boldsymbol{\lambda}}}$ with the category generated by $\mathrm{pr}_n(\mathsf{P}_d^{\kappa})$ and with that generated by $\mathrm{pr}_n(Q_d^{\kappa})$; under these two equivalences, $\mathrm{pr}_n(\mathsf{P}_d^{\kappa})$ and $\mathrm{pr}_n(Q_d^{\kappa})$

are sent to the same projective. The functor ζ is the composition of the second equivalence with the inverse of the first.

We will also use also have **Zuckerman functors**, which are the derived functors of sending a module in $\tilde{\mathcal{O}}$ to its largest quotient which is locally finite for $\mathfrak{p}$. These are left adjoint to the forgetful functor $D^b(\tilde{\mathcal{O}}^{\mathfrak{p}}) \to D^b(\tilde{\mathcal{O}})$.

Begin with a pyramid π, and assume π' is obtained from π by replacing a pair of consecutive columns whose lengths add up to n (a pair of consecutive dual representations in the sequence $\underline{\boldsymbol{\lambda}}$), with one of length n, and π'' is obtained by deleting them altogether.

Definition 9.19 *The* **ES-cup functor** $K\colon \tilde{\mathcal{O}}^{\pi''} \to \tilde{\mathcal{O}}^{\pi}$ *is the composition of the inverse of the Enright-Shelton equivalence for* π'' *and* π' *with the forgetful functor from* $\tilde{\mathcal{O}}^{\pi'}$ *to* $\tilde{\mathcal{O}}^{\pi}$ *(which corresponds to an inclusion of parabolic subgroups).*

The **ES-cap functor** $T\colon \tilde{\mathcal{O}}^{\pi} \to \tilde{\mathcal{O}}^{\pi''}$ *is the composition of the Zuckerman functor from* $\tilde{\mathcal{O}}^{\pi}$ *to* $\tilde{\mathcal{O}}^{\pi'}$ *with the Enright-Shelton functor* $\zeta\colon \tilde{\mathcal{O}}^{\pi'} \to \tilde{\mathcal{O}}^{\pi''}$.

Proposition 9.20 *Both squares in the diagram below commute.*

$$\begin{tikzcd}
D^{\uparrow}(\tilde{\mathcal{O}}_n^{\mathfrak{p}'}) \arrow[r, bend left, "K"] & D^{\uparrow}(\tilde{\mathcal{O}}_n^{\mathfrak{p}}) \arrow[l, bend left, "T"] \\
\mathcal{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"] \arrow[r, bend left, "\mathbb{K},\mathbb{C}"] & \mathcal{V}^{\underline{\boldsymbol{\lambda}}^+} \arrow[u, "\Xi"'] \arrow[l, bend left, "\mathbb{T},\mathbb{E}"]
\end{tikzcd}$$

Proof. We need only check this for K, since in both cases, the functors above are in adjoint pairs.

Using the compatibility results for functors proved in Propositions 9.12 and 9.13, we can reduce to the case where the cup is added at the far right. Let $\mathfrak{l}$ is be the standard Levi of type $(N-n,n)$. In this case, the ES-equivalence is just given by $\operatorname{ind}_{\mathfrak{l}}^{\mathfrak{gl}_N}(-\otimes\mathbb{C}^n)$, since this sends $\operatorname{pr}_n(\mathsf{P}_d^{\kappa})$ to $\operatorname{pr}_n(Q_d^{\kappa})$. On the other hand, we already know by Proposition 9.13 that this is intertwined with $\mathbb{S}^{\underline{\boldsymbol{\lambda}},(\omega_1,\omega_{n-1})}(-,L_{\omega_1})$, which matches with $\mathbb{K}$ as shown in Lemma 7.19. $\square$

These propositions show that our work matches with that of Sussan [**Sus07**] and Mazorchuk-Stroppel [**MS09**], though the latter paper is "Koszul dual" to our approach above. Recall that each block of $\tilde{\mathcal{O}}_n$ has a Koszul dual, which is also a block of parabolic category $\mathcal{O}$ for $\mathfrak{gl}_N$ (see [**Bac99**]). In particular, we have a Koszul duality equivalence

$$\mathfrak{K}: D^{\uparrow}(\tilde{\mathcal{O}}_n^{\mathfrak{p}}) \to D^{\downarrow}({}^{n}_{\mathfrak{p}}\tilde{\mathcal{O}})$$

where ${}^{n}_{\mathfrak{p}}\tilde{\mathcal{O}}$ is the direct sum over all n part compositions μ (where we allow parts of size 0) of a block of $\mathfrak{p}_\mu$-parabolic category $\tilde{\mathcal{O}}$ for $\mathfrak{gl}_N$ with a particular central character depending on $\mathfrak{p}$.

Now, let T be an oriented tangle labeled with $\underline{\boldsymbol{\lambda}}$ at the bottom and $\underline{\boldsymbol{\lambda}}'$ at top, with all appearing labels being fundamental. Then, as before, associated to $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}$ we have parabolics $\mathfrak{p}$ and $\mathfrak{p}'$.

Proposition 9.21 *Assume $\underline{\boldsymbol{\lambda}}$ and $\underline{\boldsymbol{\lambda}}'$ only contain the fundamental weights ω_1 and ω_{n-1}. Then we have a commutative diagram*

$$\begin{tikzcd}
D^{\downarrow}({}^{n}_{\mathfrak{p}}\tilde{\mathcal{O}}) \arrow[r, "\mathcal{F}(T)"] & D^{\downarrow}({}^{n}_{\mathfrak{p}'}\tilde{\mathcal{O}}) \\
D^{\uparrow}(\tilde{\mathcal{O}}^{\mathfrak{p}}_{n}) \arrow[u, "\mathfrak{S}", "\sim"'] \arrow[r, "\mathbb{F}(T)"] & D^{\uparrow}(\tilde{\mathcal{O}}^{\mathfrak{p}'}_{n}) \arrow[u, "\mathfrak{S}"', "\sim"] \\
\mathcal{V}^{\underline{\boldsymbol{\lambda}}} \arrow[u, "\Xi"] \arrow[r, "\Phi(T)"] & \mathcal{V}^{\underline{\boldsymbol{\lambda}}'} \arrow[u, "\Xi"']
\end{tikzcd}$$

where $\mathbb{F}(T)$ is the functor for a tangle defined by Sussan in [**Sus07**] *and $\mathcal{F}(T)$ is the functor defined by Mazorchuk and Stroppel in* [**MS09**].

Our invariant $\mathcal{K}$ thus coincides with the knot invariants of both the above papers when all components are labeled with the defining representation. In particular, it coincides with Khovanov homology when $\mathfrak{g} = \mathfrak{sl}_2$ and Khovanov-Rozansky homology when $\mathfrak{g} = \mathfrak{sl}_3$.

Proof. We need only check that we define the same functors as Sussan and Mazorchuk-Stroppel on a single crossing of strands labeled ω_1 and on cups and caps. In [**Sus07**, §6], the action of crossings is given by twisting functors and in [**MS09**, §6] by shuffling functors; thus, Proposition 9.18 identifies our crossing with Sussan's and the duality of twisting and shuffling functors proven in [**RH04**] shows that it matches that of Mazorchuk and Stroppel.

Since Sussan's cup and cap functors defined in [**Sus07**, §3.2] are defined by applying a Zuckerman functor after the ES-equivalence $\mathcal{O}^{\mathfrak{p}}_n \cong \mathcal{O}^{\mathfrak{p}'}_n$ on objects, Proposition 9.20 shows that our functors agree with his; similarly, Mazorchuk and Stroppel's functor is an ES-equivalence Koszul dual to ours, followed by a translation functor, which matches our Zuckerman functor by [**RH04**]. □

We believe strongly that this homology agrees with that of Khovanov-Rozansky when one uses the defining representation for all n (this is conjectured in [**MS09**]), but actually proving this requires an improvement in the state of understanding of the relationship between the foam model of Mackaay, Stošić and Vaz [**MSV09**] and the model we have presented. Progress in this direction was recently made by Lauda, Queffelec and Rose[**LRQ**, **QR**] using skew Howe duality to relate foam categories and $\mathcal{U}$; the author and Mackaay will explain one version of this connection in future.

It would also be desirable to compare our results to those of Cautis-Kamnitzer for minuscule representations, and Khovanov-Rozansky for the Kauffman polynomial, but this will require some new ideas, beyond the scope of this paper.

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