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KNOT MODULES. I([1])

BY

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ABSTRACT. For a differentiable knot, i.e. an imbedding $S^n \subset S^{n+2}$, one can associate a sequence of modules $\{A_q\}$ over the ring $Z[t, t^{-1}]$, which are the source of many classical knot invariants. If X is the complement of the knot, and $\tilde{X} \to X$ the canonical infinite cyclic covering, then $A_q = H_q(\tilde{X})$. In this work a complete algebraic characterization of these modules is given, except for the Z-torsion submodule of A_1.

In classical knot theory there are many "abelian" invariants which have proved useful in distinguishing knots, e.g. knot-polynomials, "elementary" ideals, homology and linking pairings in the finite cyclic branched coverings, ideal classes (see [F], [FS]). It is known (see [T]) that these invariants can all be extracted from a certain module A over the ring $\Lambda = Z[t, t^{-1}]$ and a "Hermitian" pairing on A taking values in $Q(\Lambda)/\Lambda$ ($Q(\Lambda)$ is the quotient field of Λ). The construction of A and $\langle\ ,\ \rangle$ carries over to higher-dimensional knots and, in certain cases, are enough to classify the knot up to isotopy (see [L], [T1], [K]).

In general, there is a finite collection $A_1, \ldots, A_n$ of such modules associated to an n-dimensional knot in $(n + 2)$-space, and $\langle\ ,\ \rangle$ exists on A_k when $n = 2k - 1$. Our first purpose in this work will be to give an algebraic characterization of these objects. There is already a great deal known in this direction (see [K], [Ke], [L3], [G]). Our results, which will be complete except for some problems with the Z-torsion part of A_1, will extend and reformulate these known results. For this purpose we will find it necessary to define a new pairing $[\ ,\ ]$ in the Z-torsion part of A_k, when $n = 2k$.

In the second part of this work, we will make an algebraic study of the modules and forms which have arisen from Part I. Our approach is to consider new modules and forms, derived from the original ones, over rings with a good structure theory: polynomial rings over fields, and rings of algebraic integers. The structure theory then classifies the derived object via invariants in these rings. These invariants include most of the "classical" knot

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invariants. Our main results will characterize in many cases the invariants which can arise from knots of a given dimension. For example, the *rational* knot modules (i.e. $A_k \otimes Q$) with their product structure can often be completely characterized–this has consequences for knot cobordism realizability **[L4]**. In addition many integral invariants appear–including the ideal class invariants of **[FS]**, but also many new ones–and are characterized.

Some of the work in this paper in Part I is a redoing of known results, referred to above, in an effort to give a more unified and simple presentation of the entire subject. For example, we have been able to give a "coordinate free" formulation of some of the results of **[Ke]** and **[G]**. The middle-dimensional realization results of **[K]** are also given a different proof which has obvious applications to construction of more general manifolds with $\pi_1 = Z$.

Some of the results of this work have been previously announced in **[L1]**. However, let me take this opportunity to point out certain errors in **[L1]**. In §6, part (i) of the first proposition and part (iii) of the second proposition are wrong; in fact, there are easy counterexamples.

We will work in the category of *smooth* knots, but the same results hold for piecewise-linear knots (see **[W5]**). If the dimension of the knot is different from 4, this is also true of topological (locally flat) knots (see **[CS]**). Only the argument of §15 does not apply to topological knots.

1. A (*smooth*) *n-knot* (of codimension two) will be a smooth closed oriented submanifold $K^n \subset S^{n+2}$, where K^n is homeomorphic to S^n. Most of our arguments apply to locally-flat piecewise-linear or topological knots. The *complement* of the knot is the space $X = S^{n+2} - K^n$. It follows directly from Alexander duality that X is a homology circle, i.e. $H_*(X) \approx H_*(S^1)$. Abelianization defines an epimorphism $\varepsilon\colon \pi_1(X) \to Z$, where a preferred isomorphism $H_1(X) \approx Z$ is defined by the orientation of K and the duality isomorphism $H_1(X) \approx H^n(K)$. The covering space $\tilde{X} \to X$ associated to $\operatorname{Ker} \varepsilon \subset \pi_1(X)$ is the universal abelian covering of X. The group $\mathcal{C}$ of covering translations of $\tilde{X}$ admits a preferred isomorphism with Z via ε. In other words $\mathcal{C}$ has a preferred generating diffeomorphism T. The induced automorphism T_* of $H_q(\tilde{X})$ defines a unique module structure on $H_q(\tilde{X})$ over the ring $\Lambda = Z[t, t^{-1}]$ by setting $t = T_*$. We will use the notation $A_q = H_q(\tilde{X})$ and the sequence $\{A_q\}$ will be referred to as the *Alexander modules* of the knot K. Later on we will impose additional product structure on these modules.

PROPOSITION (1.1). *A_q is a finitely-generated Λ-module.*

Let X_0 be the complement of an open tubular neighborhood of K. Thus X_0 is a smooth compact manifold and, hence, admits a finite triangulation. Since $X_0 \subset X$ is a deformation retract, there is an infinite cyclic covering space $\tilde{X}_0$ of X_0, which is an equivariant deformation retract of $\tilde{X}$. The chain complex $C_*(\tilde{X}_0)$ is made up of free Λ-modules on the cells of X_0, and, therefore, is of finite type. Since Λ is Noetherian, this implies that $H_q(\tilde{X}_0) \approx H_q(\tilde{X})$ is a finitely-generated Λ-module. For most purposes, we may replace X by X_0–and we do so for the remainder of this work.

Let $\varepsilon\colon \Lambda \to Z$ be the canonical augmentation defined by $\varepsilon(t) = 1$.

PROPOSITION (1.2). *If ε is used to define a Λ-module structure on Z (i.e. $\lambda \cdot m = \varepsilon(\lambda)m$) then $A_q \otimes_\Lambda Z = 0$ for all $q > 0$. Equivalently, multiplication by $t - 1$ defines an automorphism of A_q, $q > 0$.*

Consider the Cartan-Leray spectral sequence of the infinite cyclic covering $\tilde{X} \to X$. In our case, $E^2_{k,l} = H_k(Z; H_l(\tilde{X}))$, where the (local) coefficients are equipped with the Z-action defined by the covering transformations. Using the free resolution

$$\Lambda \xrightarrow{t-1} \Lambda \xrightarrow{\varepsilon} Z \to 0$$

we see that $E^2_{k,l} = 0$ for $k \neq 0, 1$, $E^2_{0,l} = H_l(\tilde{X}) \otimes_\Lambda Z$ and

$$E^2_{1,l} = \text{Kernel}\{t - 1\colon H_l(\tilde{x}) \to H_l(\tilde{X})\}.$$

Since X is a homology circle, $E^2_{0,l} = 0$ for $l > 1$.

To obtain the case $q = 1$, note that $E^\infty_{0,1} = E^2_{0,1} = H_1(\dot{X}) \otimes_\Lambda Z$ is a subgroup of $H_1(X) \approx Z$, with quotient $E^\infty_{1,0} = E^2_{1,0} = \text{Kernel}\{t - 1\colon H_0(\tilde{X}) \to H_0(\tilde{X})\}$. Since $\tilde{X}$ is connected and $t = 1$ on $H_0(\tilde{X})$, the desired result follows.

COROLLARY (1.3). *A_q is a Λ-torsion module.*

Let $\alpha_1, \ldots, \alpha_m$ generate A_q, by Proposition (1.1). By Proposition (1.2), we may write $\alpha_i = (t - 1)\sum_{j=1}^m \lambda_{ij}\alpha_j$ for some $\lambda_{ij} \in \Lambda$. Rearranging this we may write:

$$\sum_{j=1}^{m} \mu_{ij}\alpha_j = 0, \qquad i = 1, \ldots, m,$$

where

$$\mu_{ij} = \begin{cases} (t-1)\lambda_{ij}, & i \neq j, \\ (t-1)\lambda_{ii} - 1, & i = j. \end{cases}$$

Let $\Delta = \det(\mu_{ij})$; then $\Delta\alpha_i = 0$ for $i = 1, \ldots, m$. Thus $\Delta A_q = 0$. Since $\varepsilon(\mu_{ij}) = -\delta_{ij}$, we have $\varepsilon(\Delta) = \pm 1$, and so $\Delta \neq 0$.

2. The deepest properties of the $\{A_q\}$ are derived from duality. To pursue this we summarize here the necessary facts which can be found in some detail in [M]. Let X be a compact piecewise-linear n-dimensional manifold with boundary and $\tilde{X} \to X$ a regular covering space with π = group of covering transformations. Using a triangulation of X and left action of π, the chain

groups $C_*(\tilde{X}, \partial\tilde{X})$ are free left $Z[\pi]$-modules with basis corresponding to the cells of X not in ∂X. If X^1 is defined by the dual cell-complex of X, then $C_*(\tilde{X}^1)$ are free left $Z[\pi]$-modules corresponding to the cells of X^1. An intersection pairing: $C_q(\tilde{X}, \partial\tilde{X}) \times C_{n-q}(\tilde{X}^1) \to Z[\pi]$, $(\alpha, \beta) \mapsto \alpha \cdot \beta$ is defined with the following properties:

(i) bilinear over Z,

(ii) $(g\alpha) \cdot \beta = g(\alpha \cdot \beta)$ for $g \in \pi$, $\alpha \in C_q(\tilde{X}, \partial\tilde{X})$, $\beta \in C_{n-q}(\tilde{X}^1)$,

(iii) $\alpha \cdot \beta = (-1)^{q(n-q)}\overline{\beta \cdot \alpha}$, where $\lambda \mapsto \bar{\lambda}$ is the antiautomorphism of $Z[\pi]$ defined by $\bar{g} = g^{-1}$ for any $g \in \pi$,

(iv) $(\partial\alpha) \cdot \beta = (-1)^q \alpha \cdot (\partial\beta)$, where ∂ is the boundary operator of the appropriate chain complex.

Let R, S be rings with unit, C a left R-module and G an $(R\text{-}S)$-bimodule. Then $\mathrm{Hom}_R(C, \ )$ is a right S-module. If ϕ: $C_1 \to C_2$ is a homomorphism of left R-modules, then the induced homomorphism ϕ^*: $\mathrm{Hom}_R(C_2, G) \to \mathrm{Hom}_R(C_1, G)$ is a right S-homomorphism. So for example if $C = \{C_q\}$ is a chain-complex of left R-modules, then $\mathrm{Hom}_R(C, G) = \{\mathrm{Hom}_R(C_q, G)\}$ is a cochain-complex of right S-modules. If $R = S = G$, we denote by $C^* = \mathrm{Hom}_R(C, R)$ the resulting right R-module.

By (i), (ii) above, the function $\alpha \mapsto \alpha \cdot \beta$ defines an element $[\beta] \in C_q(\tilde{X}, \partial\tilde{X})^*$, for any $\beta \in C_{n-q}(X^1)$. Furthermore, the function ϕ: $C_{n-q}(X^1) \to C_q(\tilde{X}, \partial\tilde{X})^*$ defined by $\phi(\beta) = [\beta]$ satisfies the property $\phi(\lambda\beta) = \phi(\beta)\bar{\lambda}$ according to (i), (ii), (iii).

THEOREM (2.1). *ϕ is bijective.*

See [M] for a proof.

Property (iv) implies that ϕ is a chain homomorphism, up to sign. Therefore ϕ induces an *additive* isomorphism:

$$(*) \qquad H_{n-q}(X^1) \approx H_e^q(\tilde{X}, \partial\tilde{X})$$

where the right-hand side denotes the cohomology of the cochain-complex $\mathrm{Hom}_{Z[\pi]}(C(\tilde{X}, \partial\tilde{X}), Z[\pi])$.

Formula $(*)$ relates a left $Z[\pi]$-module with a right $Z[\pi]$-module. To properly record the correspondence between these module structures, we use the usual technique of changing a right $Z[\pi]$-module A to a left $Z[\pi]$-module $\bar{A}$ by: $\lambda\alpha = \alpha\bar{\lambda}$, for $\lambda \in Z[\pi]$, $\alpha \in A$.

The final result is now:

COROLLARY (2.2). *ϕ induces an isomorphism of left $Z[\pi]$-modules*

$$H_{n-q}(\tilde{X}^1) \approx \overline{H_e^q(\tilde{X}, \partial\tilde{X})}.$$

In order to apply duality to obtain intrinsic information about the knot modules, we have to deal with a "universal coefficient" problem. Specifically, let R, S be rings with unit, C a free left chain complex over R and G an $(R\text{-}S)$-bimodule. Then $H^*(C; G) = H_*(\operatorname{Hom}_R(C, G))$ is a graded right S-module, as are $\operatorname{Hom}_R\{H_*(C), G\}$ and, in general, $\operatorname{Ext}_R^q(H_*(C), G)$.

These are related, in general, by a spectral sequence.

THEOREM (2.3). *Given R, S, C, G as above, there exists a spectral sequence converging to $H^*(C; G)$ with $E_2^{p,q} \approx \operatorname{Ext}_R^q(H_p(C), G)$ and differential d^r of degree $(1 - r, r)$. More specifically, there is a filtration*:

$$H^m(C; G) = J_{m,0} \supset J_{m-1,1} \supset \cdots \supset J_{0,m} \supset J_{-1,m+1} = 0$$

with $J_{p,q}/J_{p-1,q+1} \approx E_\infty^{p,q}$. All objects and isomorphisms are as right S-modules.

This spectral sequence is constructed in much greater generality in e.g. [E, Chapter XVII]. For the convenience of the reader we will outline its construction in our special case.

Choose an $(R\text{-}S)$-resolution of G, injective over R:

$$0 \to G \to Q_0 \to Q_1 \to \cdots$$

and define $X_{p,q} = \operatorname{Hom}_R(C_p, Q_q)$. Then $X = \{X_{p,q}\}$ is a double complex of right S-modules with differentials d_1, d_2 of degree $(1, 0)$ and $(0, 1)$, respectively, which give rise to two bigraded right S-modules $H_1(X)$ and $H_2(X)$. Each of these admits an induced differential from d_2, d_1, respectively, giving rise to homology S-modules $H_2H_1(X)$ and $H_1H_2(X)$ which inherit a bigrading from X.

In general, given a double complex X as above, there exists a cohomology spectral sequence with $E_2 = H_2H_1(X)$ converging to $H^*(\bar{X})$, where $\bar{X}$ is the cochain complex, with differential operator $\bar{d}$, defined by $\bar{X}_m = \Sigma_{p+q=m} X_{p,q}$, $\bar{d} = d_1 + d_2$. This arises from the filtration of $\bar{X}$ defined by $F_m = \Sigma_{q \geq m} X_{p,q}$. Notice that $E_1^{p,q} = H^{p+q}(F_q/F_{q+1}) = H_1^{p,q}(X)$ and the differential d^1 on $E_1^{p,q}$ coincides with that induced by d_2 on $H_1^{p,q}(X)$. One may check also that d^r has degree $(1 - r, r)$.

In our special case, it is immediate that $H_1^{p,q}(X) = H^p(C; Q_q) \approx \operatorname{Hom}_R(H_p(C), Q_q)$, since Q_q is R-injective. Therefore $H_2H_1(X)^{p,q} = \operatorname{Ext}_R^q(H_p(C), G)$ and this is the desired spectral sequence if it can be proved that $H^*(\bar{X}) = H^*(C; G)$.

But, by symmetry, there also exists a cohomology spectral sequence with $E_2 = H_1H_2(X)$ converging to $H^*(\bar{X})$. Since C_p is free, it is immediate that

$$H_2^{p,q} = \begin{cases} 0, & q > 0, \\ \operatorname{Hom}(C_p, G), & q = 0, \end{cases}$$

and then

$$H_1H_2(X)^{p,q} = \begin{cases} 0, & q > 0, \\ H^p(C; G), & q = 0. \end{cases}$$

It follows that the spectral sequence collapes and so $E_2^{p,q} = E_\infty^{p,q}$ and $H^p(\overline{X}) = E_\infty^{p,0} = H^p(C; G)$.

The edge homomorphism $H^m(C; G) \to E_\infty^{m,0} \subset E_2^{m,0} \approx \mathrm{Hom}_R(H_m(C), G)$ is the usual evaluation.

If R has homological dimension $\leqslant 1$, e.g. if R is a principal ideal domain, then $E_2^{p,q} = 0$ if $q \neq 0, 1$. Therefore the spectral sequence collapses: $E_2 = E_\infty$ and $J_{p,q} = 0$ if $q \neq 0, 1$. The result is a family of short exact sequences:

$$0 \to E_\infty^{m-1,1} = J_{m-1,1} \to H^m(C; G) \to E_\infty^{m,0} \to 0.$$

This is the usual universal coefficient theorem.

A similar reduction occurs under the following

Assumption. Homological dimension $R = 2$ and $\mathrm{Hom}_R\{H_p(C), G\} = 0$ for all R. In this case $E_2^{p,q} = 0$ for $q \neq 1, 2$ and the result is a family of short exact sequences:

$$0 \to J_{m-2,2} \to J_{m-1,1} \to E_\infty^{m-1,1} \to 0.$$

But

$$J_{m-2,2} = E_\infty^{m-2,2} = E_2^{m-2,2} = \mathrm{Ext}_R^2(H_{m-2}(C), G),$$
$$J_{m-1,1} = J_{m,0} = H^m(C; G),$$
$$E_\infty^{m-1,1} = E_2^{m-1,1} = \mathrm{Ext}_R^1(H_{m-1}(C), G)$$

and we have the following short exact sequence of right S-modules

$$0 \to \mathrm{Ext}_R^2(H_{m-2}(C), G) \to H^m(C; G) \to \mathrm{Ext}_R^1(H_{m-1}(C), G) \to 0.$$

We now specialize to the case of interest. Let $C = C_*(\tilde{X}_0, \partial\tilde{X}_0)$, where X_0 is the complement of an open tubular neighborhood of an n-knot and $\tilde{X}_0 \to X_0$ the universal abelian covering. Let $G = R = S = \Lambda$ and we regard Λ as an $(\Lambda\text{-}\Lambda)$-bimodule via the ring structure. Λ has homological dimension 2 by the Hilbert Szyzygy Theorem (see [Mc]). Since $H_p(\tilde{X}_0)$ is Λ-torsion by Corollary (1.3) and $\partial\tilde{X}_0 = R \times K$, it follows easily that $H_p(\tilde{X}_0, \partial\tilde{X}_0)$ is Λ-torsion and, therefore, $\mathrm{Hom}_\Lambda(H_p(\tilde{X}_0, \partial\tilde{X}_0), \Lambda) = 0$. We have now proved

Proposition (2.4). *If X_0 is the complement of an open tubular neighborhood of an n-knot, there exist short exact sequences of right Λ-modules*:

$$0 \to \mathrm{Ext}_\Lambda^2\big(H_{m-2}(\tilde{X}_0, \partial\tilde{X}_0), \Lambda\big) \to H_e^m(\tilde{X}_0, \partial\tilde{X}_0)$$
$$\to \mathrm{Ext}_\Lambda^1\big(H_{m-1}(\tilde{X}_0, \partial\tilde{X}_0), \Lambda\big) \to 0.$$

We can now combine this exact sequence with the duality isomorphism $H_e^m(\tilde{X}_0, \partial\tilde{X}_0) \approx \overline{H_{n+2-m}}(\tilde{X}_0^1)$, since $\dim X = n + 2$, to obtain:

$$0 \to \mathrm{Ext}_\Lambda^2\big(H_{m-2}(\tilde{X}_0, \partial\tilde{X}_0), \Lambda\big) \to \overline{H_{n+2-m}}(\tilde{X})$$
$$\to \mathrm{Ext}_\Lambda^1\big(H_{m-1}(\tilde{X}_0, \partial\tilde{X}_0), \Lambda\big) \to 0.$$

LEMMA (2.5). *$H_i(\tilde{X}_0) \approx H_i(\tilde{X}_0, \partial\tilde{X}_0)$ for $0 < i \leqslant n$.*

Since $\partial\tilde{X}_0 = K \times R$, this is immediate if $i < n$. When $i = n$, we have an exact sequence

$$H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0) \to H_n(\partial\tilde{X}_0) \to H_n(\tilde{X}_0) \to H_n(\tilde{X}_0, \partial\tilde{X}_0) \to 0.$$

It will clearly suffice to show that $H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0) \to H_n(\partial\tilde{X}_0)$ is onto. Consider the diagram:

$$\begin{array}{ccccc} H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0) & \longrightarrow & H_n(\partial\tilde{X}_0) & \longrightarrow & H_n(\tilde{X}_0) \\ \downarrow & & \downarrow & & \downarrow \\ H_{n+1}(X_0, \partial X_0) & \longrightarrow & H_n(\partial X_0) & \longrightarrow & H_n(X_0) \end{array}$$

where the vertical maps are induced by the covering map. The composition $H_n(\partial\tilde{X}_0) \to H_n(\partial X_0) \to H_n(X_0)$ is zero, since the image of a generator of $H_n(\partial\tilde{X}_0)$ is represented by a translate of K whose linking number with K–in the case $n = 1$–is zero. But $H_n(X_0) = 0$ if $n \neq 1$, while the isomorphism $H_1(X_0) \approx Z$ can be defined by linking number with K. Since $H_n(\partial\tilde{X}_0) \to H_n(\partial X_0)$ is injective, it suffices to show that $H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0) \to H_{n+1}(X_0, \partial X_0)$ is surjective.

Consider the Cartan-Leray spectral sequence for the covering $(\tilde{X}_0, \partial\tilde{X}_0) \to (X_0, \partial X_0)$. $E^2_{p,q} = H_p(Z; H_q(\tilde{X}_0, \partial\tilde{X}_0)) = 0$, if $p \neq 0, 1$. Therefore the spectral sequence collapses. Since $H_m(X_0, \partial X_0) \approx H_m(S^{n+2}, K) = 0$ for $m \leqslant n$, $E^2_{p,q} = 0$ for $p + q \leqslant n$. In particular

$$\begin{aligned} 0 = E^2_{0,n} &\approx H_0(Z; H_n(\tilde{X}_0, \partial\tilde{X}_0)) \\ &= \operatorname{Coker}\{t - 1: H_n(\tilde{X}_0, \partial\tilde{X}_0) \to H_n(\tilde{X}_0, \partial\tilde{X}_0)\}. \end{aligned}$$

Since Λ is Noetherian, it follows by a standard argument that any endomorphism of a finitely generated Λ-module which is onto must also be one-one. Therefore $E^2_{1,n} \approx \operatorname{Kernel}\{t - 1: H_n(\tilde{X}_0, \partial\tilde{X}_0) \to H_n(\tilde{X}_0, \partial\tilde{X}_0)\} = 0$. This implies that the edge-homomorphism $H_0(Z; H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0)) = E^2_{0,n+1} \to H_{n+1}(X_0, \partial X_0)$ is an isomorphism; in particular, $H_{n+1}(\tilde{X}_0, \partial\tilde{X}_0) \to H_{n+1}(X_0, \partial X_0)$ is onto.

THEOREM (2.6). *If $\{A_q\}$ are the Alexander modules of an n-knot, we have exact sequences of right Λ-modules:*

$$0 \to \operatorname{Ext}^2_\Lambda(A_{n-q}, \Lambda) \to \bar{A}_q \to \operatorname{Ext}^1_\Lambda(A_{n+1-q}, \Lambda) \to 0$$

for $0 < q \leqslant n$, and $A_q = 0$ for $q > n$.

Only the cases $0 < q < n$ follow formally from the lemma, but we can include $q = n$ by observing that $\operatorname{Ext}^2_\Lambda(A_0, \Lambda) = 0 = \operatorname{Ext}^2(0,\Lambda)$, since $A_0 \approx Z$

with the trivial Λ-module structure. The last statement follows directly from the preceding short exact sequence.

3. The short exact sequences of Theorem (2.6) may seem, at first thought, to be a rather obscure and unmanageable property of the $\{A_q\}$, but we will now reinterpret it to give a pleasant final statement of duality for Alexander modules. The vehicle for this reinterpretation is consideration of the Z-torsion submodules.

We first do some algebra.

DEFINITION. We will say a Λ-module A is of *type K* if it is finitely-generated and multiplication by $t - 1$ induces an automorphism of A.

DEFINITION. For any Λ-module A, define $t(A)$ to be the Z-torsion submodule and $f(A) = A/t(A)$.

So any Alexander module is of type K–note that the argument of Corollary (1.3) shows that any module of type K is a Λ-torsion module. Also any submodule or quotient module of a module of type K is again of type K.

LEMMA (3.1). *If A is of type K, then $t(A)$ is finite.*

We will show that any Z-torsion Λ-module T of type K is finite. Since T is finitely generated over Λ, there exists a positive integer m such that $mT = 0$. Suppose that m is prime. Then we may regard T as a (finitely-generated) module over $\Lambda_m = Z/(m)[t, t^{-1}]$. Since Λ_m is a principal-ideal domain, T is a direct sum of cyclic Λ_m-modules. But any cyclic Λ_m-module is finite unless it is free–this possibility is excluded, however, by the condition that $t - 1$ is an automorphism.

The general case now follows by induction: Let p be any prime dividing m. By induction pT is finite, and T/pT is finite by the preceding argument.

We introduce the abbreviated notation:

$$e^1(A) = \mathrm{Ext}^1_\Lambda(A, \Lambda), \qquad e^2(A) = \mathrm{Ext}^2_\Lambda(A, \Lambda)$$

for any Λ-module A. Note that these are Λ-modules, since Λ is commutative, and this agrees with the right Λ-module structure considered in §2. If A is of type K, then so are $e^i(A)$. Finitely generated follows from Λ Noetherian.

PROPOSITION (3.2). *Let A be a Λ-module of type K.*
(i) *$e^1(A) \approx e^1(f(A))$ is Z-torsion free.*
(ii) *$e^2(A) \approx e^2(t(A))$ is Z-torsion.*

This will follow from:

LEMMA (3.3). *Let A be a Λ-module of type K. Then*
(i) *$e^1(A) = 0$ if $f(A) = 0$,*
(ii) *$e^2(A) = 0$ if $t(A) = 0$.*

PROOF OF LEMMA. If $f(A) = 0$, then $mA = 0$, for some positive integer m. Since multiplication by m in $e^1(A)$ is induced by multiplication by m in A, $me^1(A) = 0$. On the other hand multiplication by m in $e^1(A) = \text{Ext}^1_\Lambda(A, \Lambda)$, is also induced by multiplication by m in Λ. From the short exact sequence $0 \to \Lambda \to^m \Lambda \to \Lambda_m \to 0$, we may extract:

$$\to \text{Hom}_\Lambda(A, \Lambda_m) \to e^1(A) \xrightarrow{m} e^1(A) \to \ldots.$$

But $\text{Hom}_\Lambda(A, \Lambda_m) = 0$. In fact the argument of Corollary (1.3) shows there is $\Delta \in \Lambda$ such that $\varepsilon(\Delta) = \pm 1$ and $\Delta A = 0$, for any Λ-module A of type K. If ϕ: $A \to \Lambda_m$ is a Λ-homomorphism, then $\Delta\phi(\alpha) = \phi(\Delta\alpha) = 0$ for any $\alpha \in A$. But multiplication by any primitive Δ is injective in Λ_m.

Now we have shown that multiplication by m in $e^1(A)$ is injective and zero–thus $e^1(A) = 0$.

To prove (ii) consider the exact sequence $0 \to A \to^m A \to A/mA \to 0$ for *any* nonzero integer m. This implies the exact sequence:

$$e^2(A) \xrightarrow{m} e^2(A) \to e^3(A/mA) = 0.$$

In other words, $e^2(A) = me^2(A)$, for any integer m. It now suffices to prove $me^2(A) = 0$, for some integer $m \neq 0$.

Let I be the annihilator ideal of $e^2(A)$–i.e. $\lambda \in I$ if and only if $\lambda e^2(A) = 0$. By the argument of Corollary (1.3), we may construct, for any integer $m \neq 0$, $\Delta_m \in I$ such that $\Delta_m \equiv 1 \mod m$. In fact, if $\{\alpha_i\}$ generate $e^2(A)$, write $\alpha_i = m\Sigma\lambda_{ij}\alpha_j$. Then $\det(\delta_{ij} - m\lambda_{ij}) = \Delta_m$. We want to show that I contains some nonzero scalar. If not then I is contained in some *proper* principal ideal (ϕ), $\deg \phi > 0$. In particular $\phi|\Delta_m$, for all m. Therefore ϕ is a unit in Λ_m, for every m. If m is prime, this means all nonscalar terms of ϕ are divisible by m–the only possibility is that ϕ is a scalar.

This proves Lemma (3.3).

To prove Proposition (3.2), consider the exact sequence:

$$0 \to t(A) \to A \to f(A) \to 0$$

and the associated long exact sequence:

$$\begin{aligned} \ldots \to \text{Hom}_\Lambda(t(A), \Lambda) \to e^1(f(A)) \to e^1(A) \to e^1(t(A)) \\ \to e^2(f(A)) \to e^2(A) \to e^2(t(A)) \to e^3(f(A)) = 0. \end{aligned}$$

We have proved $\text{Hom}_\Lambda(t(A), \Lambda) = e^1(t(A)) = e^2(f(A)) = 0$, which implies much of the proposition.

If m is a nonzero integer, then we have an exact sequence $0 \to f(A) \to^m f(A) \to f(A)/mf(A) \to 0$. This gives the exact sequence:

$$e^1(f(A)/mf(A)) \to e^1(f(A)) \xrightarrow{m} e^1(f(A)).$$

Since $f(A)/mf(A)$ is Z-torsion, $e^1(f(A)/mf(A)) = 0$ and $e^1(f(A))$ has no m-torsion. Therefore $e^1(f(A))$ is Z-torsion free.

If $mt(A) = 0$, then m: $e^2(t(A)) \to e^2(t(A))$ is induced by the zero homomorphism and, hence, is zero. This proves the proposition.

We now reconsider the exact sequence of Theorem (2.5).

$$0 \to e^2(A_{n-q}) \to \bar{A}_q \to e^1(A_{n+1-q}) \to 0.$$

The first term is Z-torsion while the last term is Z-torsion free. It follows immediately that we may reformulate this exact sequence as follows:

THEOREM (3.4). *Let $\{A_q\}$ be the Alexander modules of an n-knot. If $T_q = t(A_q)$ and $F_q = f(A_q)$, then:*

(i) $\bar{T}_q \approx e^2(A_{n-q}) = e^2(T_{n-q})$,

(ii) $\bar{F}_q \approx e^1(A_{n+1-q}) = e^1(F_{n+1-q})$,

and $A_q = 0$ for $q > n$.

Thus we get a familiar type of duality relationship between complementary Z-torsion submodules and complementary Z-torsion free quotients.

A useful by-product of the above considerations is the following

PROPOSITION (3.5). *Let A be a Λ-module of type K. The following conditions on A are equivalent:*

(a) $t(A) = 0$,

(b) $e^2(A) = 0$,

(c) homological dimension $A = 1$.

PROPOSITION (3.6). *Let A be a Λ-module of type K. Then $e^1e^1(A) \approx f(A)$, $e^2e^2(A) \approx t(A)$.*

Let $0 \to F_2 \to^e F_1 \to^d F_0 \to A \to 0$ be a projective resolution of A. By [S], the F_i are free. Consider the dual complex:

$$F_0^* \xrightarrow{d^*} F_1^* \xrightarrow{e^*} F_2^*;$$

Ker $d^* = 0$; Cok $e^* \approx e^2(A)$; Ker $e^*/\text{Im } d^* \approx e^1(A)$. If $e^2(A) = 0$, then e^* is a split epimorphism and so e is a split monomorphism. Thus A has homological dimension 1 and (b) $\Rightarrow$ (c).

The implication (a) $\Rightarrow$ (b) follows from Proposition (3.2) while (c) $\Rightarrow$ (a) will follow from Proposition (3.6).

Suppose A is Z-torsion free; we now know A has homological dimension 1. If $0 \to F_1 \to^d F_0 \to A \to 0$ is a free resolution of A, then $e^1(A) \approx \text{Cok } d^*$, i.e. $0 \to F_0^* \to^{d^*} F_1^* \to e^1(A) \to 0$ is a free resolution of $e^1(A)$. The first part of Proposition (3.6) is now evident, since $e^1(A) = e^1(f(A))$ by Proposition (3.2).

Suppose A is Z-torsion. If $0 \to F_1 \to^e F_1 \to^d F_0 \to A \to 0$ is a free resolution of A, then, since $e^1(A) = A^* = 0$, we have an exact sequence

$$0 \to F_0^* \xrightarrow{d^*} F_1^* \xrightarrow{e^*} F_2^* \to e^2(A) \to 0.$$

The second part of Proposition (3.6) is now evident, since $e^2(A) \approx e^2(t(A))$.

EXAMPLES. For classical knot theory, i.e. $n = 1$, we have a single Alexander module A_1. It follows from Theorem (3.4) that $T_1 = 0$ (see [C]) while $\bar{F}_1 \approx e^1(F_1)$. For 2-knots there are Alexander modules A_1, A_2 and we conclude that $T_2 = 0$, $\bar{F}_2 \approx e^1(F_1)$ and $\bar{T}_1 \approx e^2(T_1)$.

As an illustration, let us consider the case of a cyclic module $A = \Lambda/I$, for some ideal $I \subset \Lambda$. It is not hard to check that $t(A) \approx (\lambda)/I$ and $f(A) \approx \Lambda/(\lambda)$, where (λ) is the minimal principal ideal containing I. Furthermore A is of type K if and only if $\varepsilon(I) = Z$. One can also check easily that $e^1(A) \approx f(A)$. We can now conclude that a necessary condition for A to be the module of a 1-knot is that I be principal and $\bar{I} = I$.

It is harder to compute $e^2(A)$, but if I is restricted to be generated by no more than two elements, then one can check directly that $e^2(A) \approx t(A)$. Therefore, in order that A be the first Alexander module of a 2-knot it is necessary that $\bar{\lambda} I = \lambda \bar{I}$–the second Alexander module must then be isomorphic to $\Lambda/(\bar{\lambda})$. We have no restriction on $f(A)$.

In general we may observe that $\{T_q: 2q < n\}$ determine $\{T_q: 2q > n\}$, and $\{F_q: 2q \leqslant n\}$ determine $\{F_q: 2q \geqslant n + 2\}$. In addition we have the self-duality properties:

$$\text{(3.7)} \qquad \begin{aligned} \bar{T}_q &\approx e^2(T_q), \qquad n = 2q, \\ \bar{F}_q &\approx e^1(F_q), \qquad n = 2q - 1. \end{aligned}$$

4. These self-duality properties can be strengthened by the existence of a certain product structure. First we observe the following algebraic facts.

PROPOSITION (4.1). *If A is a Λ-torsion module, there is a natural isomorphism of Λ-modules $e^1(A) \approx \mathrm{Hom}^1_\Lambda(A, Q(\Lambda)/\Lambda)$.*

PROPOSITION (4.2). *If A is a finite Λ-module of type K, then there is a natural isomorphism of Λ-modules: $e^2(A) \approx \mathrm{Hom}_Z(A, Q/Z)$, where $\mathrm{Hom}_Z(A, Q/Z)$ inherits its Λ-module structure from that on A.*

By "natural" we mean that the isomorphism is defined as a natural transformation between functors of A.

Proposition (4.1) is a standard fact implied by the long exact sequence:

$$\begin{aligned} 0 = \mathrm{Hom}_\Lambda(A, Q(\Lambda)) &\to \mathrm{Hom}_\Lambda(A, Q(\Lambda)/\Lambda) \\ &\to \mathrm{Ext}^1_\Lambda(A, \Lambda) \to \mathrm{Ext}^1_\Lambda(A, Q(\Lambda)) = 0 \end{aligned}$$

derived from the short exact sequence $0 \to \Lambda \to Q(\Lambda) \to Q(\Lambda)/\Lambda \to 0$.

The proof of Proposition (4.2) will be more complicated.

If A is a finite Λ-module, then there exists a positive integer k such that $t^k(\alpha) = \alpha$ for all $\alpha \in A$. Let θ denote the quotient ring $\Lambda/(t^k - 1)$, which we consider as a Λ-module. We will establish a sequence of isomorphisms:

$$e^2(A) \approx \operatorname{Ext}^1_\Lambda(A, \theta) \approx \operatorname{Hom}_\Lambda(A, Q/Z \otimes_Z \theta) \approx \operatorname{Hom}_Z(A, Q/Z).$$

Consider the short exact sequence $0 \to \Lambda \xrightarrow{t^k - 1} \Lambda \to \theta \to 0$ and the derived long exact sequence

$$\operatorname{Ext}^1_\Lambda(A, \Lambda) \to \operatorname{Ext}^1_\Lambda(A, \theta) \to \operatorname{Ext}^2_\Lambda(A, \Lambda) \xrightarrow{t^k - 1} \operatorname{Ext}^2_\Lambda(A, \Lambda).$$

Since A is finite, $\operatorname{Ext}^1_\Lambda(A, \Lambda) = 0$ by Lemma (3.3) The endomorphism $t^k - 1$ of $\operatorname{Ext}^2_\Lambda(A, \Lambda)$ is induced by $t^k - 1$ on A, which is zero. This establishes the first isomorphism.

Consider the short exact sequence $0 \to Z \to Q \to Q/Z \to 0$ and the derived exact sequence: $0 \to \theta \to Q \otimes_Z \theta \to Q/Z \otimes_Z \theta \to 0$. Note that θ is free as a Z-module. From this short exact sequence we obtain another derived exact sequence:

$$\operatorname{Hom}_\Lambda(A, Q \otimes_Z \theta) \to \operatorname{Hom}_\Lambda(A, Q/Z \otimes_Z \theta) \to \operatorname{Ext}^1_\Lambda(A, \theta) \to \operatorname{Ext}^1_\Lambda(A, Q \otimes_Z \theta).$$

Since A is finite and $Q \otimes_Z \theta$ is Z-divisible, ther exists an integer m such that $mA = 0$. But m: $Q \otimes_Z \theta \approx Q \otimes_Z \theta$. Therefore, any additive functor of two variables is zero on $(A, Q \otimes_Z \theta)$ since multiplication by m is both zero and an isomorphism. In particular $\operatorname{Hom}_\Lambda(A, Q \otimes_Z \theta) = 0 = \operatorname{Ext}^1_\Lambda(A, Q \otimes_Z \theta)$ and we have the second isomorphism.

The final isomorphism we construct explicitly. Let ϕ: $A \to Q/Z \otimes_Z \theta$ be a Λ-homomorphism. Write $\phi(\alpha) = \sum_{i=0}^{k-1} \phi_i(\alpha) \otimes t^i$. It is clear that each ϕ_i is a Z-homomorphism and $\phi_i(\alpha) = \phi_{i+1}(t\alpha)$, where i is taken mod k. Conversely, given ϕ_0: $A \to Q/Z$ a Z-homomorphism, we may define ϕ: $A \to Q/Z \otimes_Z \theta$ by

$$\phi(\alpha) = \sum_{i=0}^{k-1} \phi_0(t^{k-i}\alpha) \otimes t^i.$$

It is straightforward to check that $\phi \to \phi_0$ defines the desired isomorphism.

It remains to verify that the composite isomorphism is independent of k. In fact, if $k|l$ and $\theta^1 = \Lambda/(t^l - 1)$, then multiplication by $(t^l - 1)/(t^k - 1)$ defines a homomorphism $\theta \to \theta^1$ which induces a commutative diagram:

$$\begin{array}{ccccc} & \operatorname{Ext}^1_\Lambda(A, \theta) & \approx & \operatorname{Hom}_\Lambda(A, Q/Z \otimes_Z \theta) & \\ \nearrow\!\!\approx & \big\downarrow & & \big\downarrow & \approx\!\!\searrow \\ e^2(A) & & & & \operatorname{Hom}_Z(A, Q/Z) \\ \approx\!\!\searrow & \big\downarrow & & \big\downarrow & \nearrow\!\!\approx \\ & \operatorname{Ext}^1_\Lambda(A, \theta^1) & \approx & \operatorname{Hom}_\Lambda(A, Q/Z \otimes \theta^1) & \end{array}$$

From this, independence of k follows immediately.

This completes the proof of Proposition (4.2).

COROLLARY (4.3). *If A is a finite Λ-module of type K in which every element has order p, where p is prime, then $e^2(A) \approx A$.*

We may then regard A as a Λ_p-module and the corollary follows readily from Proposition (4.2), using the structure theorem for modules over the principal ideal domain Λ_p.

It follows from Propositions (4.1) and (4.2) that we may rewrite the self-duality properties (3.7) in the following way:

COROLLARY (4.4). *Let $\{A_q\}$ be the Alexander modules of an n-knot and T_q, F_q the Z-torsion submodule and Z-torsion free quotient of A_q, respectively. Then there exist pairings:*

$$\langle\, ,\rangle: F_q \times F_{n+1-q} \to Q(\Lambda)/\Lambda, \qquad [\, ,]: T_q \times T_{n-q} \to Q/Z$$

satisfying the following properties:

conjugate linear: $\langle \lambda\alpha, \beta\rangle = \lambda\langle\alpha, \beta\rangle = \langle\alpha, \bar{\lambda}\beta\rangle$ *for* $\lambda \in \Lambda$; $\alpha \in F_q$; $\beta \in F_{n+1-q}$;

nonsingular: *the adjoint to* $\langle\, ,\, \rangle$, *which is a homomorphism* (*by* (i)) $\bar{F}_q \to \mathrm{Hom}_\Lambda(F_{n+1-q}, Q(\Lambda)/\Lambda)$ *is bijective*;

Z-linear: $[m\alpha, \beta] = [\alpha, m\beta] = m[\alpha, \beta]$ *for* $m \in Z$; $\alpha \in T_q$; $\beta \in T_{n-q}$;

conjugate selfadjoint: $[\lambda\alpha, \beta] = [\alpha, \bar{\lambda}\beta]$ *for* $\lambda \in \Lambda$; $\alpha \in T_q$; $\beta \in T_{n-q}$;

nonsingular: *the adjoint to* $[\, ,\,]$ *which is a homomorphism* (*by* (c), (d)) $\bar{T}_q \to \mathrm{Hom}_Z\{T_{n-q}, Q/Z\}$ *is bijective.*

The pairings $\langle\, ,\rangle$ and $[\, ,]$ are just the adjoints of the given isomorphisms of (3.7) in light of Propositions (4.1), (4.2). The other properties are just the translations of the algebraic properties of the isomorphisms.

5. It turns out that the existence of these pairings is almost, but not quite, sufficient to characterize the Alexander modules. To accomplish this, we will redefine the pairings $\langle\, ,\, \rangle$ and $[\, ,\,]$ in a more geometric manner in order to prove the following symmetry properties in the middle dimensions:

$(-1)^{q+1}$-*Hermitian*: $\langle\alpha, \beta\rangle = (-1)^{q+1}\overline{\langle\beta, \alpha\rangle}$, when $n = 2q - 1$.

$(-1)^{q+1}$-*symmetric*: $[\alpha, \beta] = (-1)^{q+1}[\beta, \alpha]$, when $n = 2q$.

The pairing $\langle\, ,\, \rangle$ will be a straightforward generalization of the usual linking pairing defined on elements of finite order in the homology of a compact manifold. We place ourselves in a somewhat more general situation.

Let X be a compact n-dimensional manifold and $\tilde{X} \to X$ a regular covering whose group π of covering transformations is free abelian. We have an intersection pairing:

$$C_q(\tilde{X}, \partial\tilde{X}) \times C_{n-q}(\tilde{X}^1) \to \Lambda = Z[\pi]$$

as outlined in §2.

Suppose $\alpha \in H_q(\tilde{X}, \partial\tilde{X})$, $\beta \in H_{n-q-1}(\tilde{X})$ are Λ-torsion elements. Let $z \in C_q(\tilde{X}, \partial\tilde{X})$, $w \in C_{n-q-1}(\tilde{X}^1)$ be representative cycles of α, β respectively. Then we may write $\lambda z = \partial c$ for some $\lambda \in \Lambda$ and $c \in C_{q+1}(\tilde{X}, \partial\tilde{X})$. Set $\langle \alpha, \beta \rangle = (c \cdot w)/\lambda \bmod \Lambda$. Exactly as in the classical case $\langle\ ,\ \rangle$ is a well-defined pairing. Conjugate linearity and $(-1)^{q+1}$-Hermitian follow from the corresponding properties of the intersection pairing (note this is the special case $q = n - q - 1$ of a more general Hermitian property) by the usual arguments (see [M]). Alternatively, $\langle\ ,\ \rangle$ may be defined by the composition:

$$\overline{H}_q(\tilde{X}, \partial\tilde{X}) \to H_e^{n-q}(\tilde{X}) \overset{\delta^*}{\leftarrow} H_e^{n-q-1}(\tilde{X}; Q(\Lambda)/\Lambda)$$
$$\overset{\psi_2}{\to} \operatorname{Hom}_\Lambda\big(H_{n-q-1}(\tilde{X}), Q(\Lambda)/\Lambda\big).$$

The first map is the duality isomorphism (Corollary 2.2).

The second map δ^* is part of the exact cohomology sequence derived from the short exact sequence of coefficients $0 \to \Lambda \to Q(\Lambda) \to Q(\Lambda)/\Lambda \to 0$ and ψ_2 is the evaluation map.

Consider the diagram of exact rows:

$$\begin{array}{ccccccc}
 & & & & 0 \longrightarrow H_{n-q}(\tilde{X})^* & \longrightarrow & \operatorname{Hom}_\Lambda\{H_{n-q}(\tilde{X}), Q(\Lambda)\} \\
 & & & & \uparrow \psi_3 & & \uparrow \psi_4 \ \wr \\
H_e^{n-q-1}(\tilde{X}; Q(\Lambda)) & \longrightarrow & H_e^{n-q-1}(\tilde{X}; Q(\Lambda)/\Lambda) & \xrightarrow{\delta^*} & H_e^{n-q}(\tilde{X}) & \longrightarrow & H_e^{n-q}(\tilde{X}; Q(\Lambda)) \\
\psi_1 \downarrow \wr & & \psi_2 \downarrow & & & & \\
\operatorname{Hom}_\Lambda\{H_{n-q-1}(\tilde{X}), Q(\Lambda)\} & \longrightarrow & \operatorname{Hom}_\Lambda\{H_{n-q-1}(\tilde{X}), Q(\Lambda)/\Lambda\} & & & &
\end{array}$$

The vertical maps $\{\psi_i\}$ are evaluation maps; ψ_1 and ψ_4 are isomorphisms because $Q(\Lambda)$ is injective (e.g. the spectral sequence of Theorem (2.3) collapses). Now let π be infinite cyclic.

We may analyze ψ_2 with the help of the universal coefficient spectral sequence (Theorem (2.3)). Since $\operatorname{Ext}^2_\Lambda(A, Q(\Lambda)/\Lambda) = \operatorname{Ext}^3_\Lambda(A, \Lambda) = 0$, for any Λ-module A, we see that the E^2 diagram has two nonzero rows and by an argument similar to that following Theorem (2.3), we obtain a short exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda\big(H_i(\tilde{X}), Q(\Lambda)/\Lambda\big) \to H_e^i\big(\tilde{X}; Q(\Lambda)/\Lambda\big)$$
$$\overset{\psi_4}{\to} \operatorname{Hom}_\Lambda\big(H_i(\tilde{X}), Q(\Lambda)/\Lambda\big) \to 0.$$

It follows by a simple diagram-chase that $\langle\ ,\ \rangle$ is well defined on Λ-torsion elements, since $\psi_3 = 0$ on Λ-torsion, and any homomorphism $H_{n-q}(\tilde{X}) \to Q(\Lambda)$ must be zero on Λ-torsion.

We now specialize to the case where X is a knot complement and $\dim X = n + 2$; in particular, $H_i(\tilde{X}) = A_i$ is of type K for $i > 0$. Since A_i are Λ-torsion modules, $A_i^* = 0 = \mathrm{Hom}_\Lambda(A_i, Q(\Lambda))$. It now follows easily that we have an exact sequence:

$$0 \to \mathrm{Ext}^1_\Lambda(A_{n-q-1}, Q(\Lambda)/\Lambda) \to \overline{H}_q(\tilde{X}, \partial\tilde{X}) \xrightarrow{l} \mathrm{Hom}_\Lambda(A_{n-q-1}, Q(\Lambda)/\Lambda) \to 0$$

where l is the adjoint of the pairing $\langle\ ,\ \rangle$. Since $\mathrm{Ext}^1_\Lambda(A, Q(\Lambda)/\Lambda) \approx \mathrm{Ext}^2_\Lambda(A, \Lambda)$ is finite when A is of type K, we have an isomorphism induced by l:

$$f\left(\overline{H}_q(\tilde{X}, \partial\tilde{X})\right) \approx \mathrm{Hom}_\Lambda(A_{n-q-1}, Q(\Lambda)/\Lambda).$$

LEMMA (5.1). *If A is a module of type K, then any Λ-homomorphism ϕ: $A \to Q(\Lambda)/\Lambda$ is zero on Z-torsion.*

If $\alpha \in A$ has finite order, then $\phi(\alpha)$ lifts to an element $\lambda \in Q[t, t^{-1}] \subset Q(\Lambda)$. But since $\Delta A = 0$, where $\Delta \in \Lambda$ satisfies $\varepsilon(\Delta) = 1$, it must be true that $\Delta \cdot \lambda \in \Lambda$. Since Δ is primitive, this implies $\lambda \in \Lambda$.

COROLLARY (5.2). *If A is of type K, then* $\mathrm{Hom}_\Lambda(f(A), Q(\Lambda)/\Lambda) \approx \mathrm{Hom}_\Lambda(A, Q(\Lambda)/\Lambda)$.

We have now proved that the pairing $\langle\ ,\ \rangle$ induces a nonsingular pairing $\langle\ ,\ \rangle$: $F_q \times F_{n+1-q} \to Q(\Lambda)/\Lambda$ where the $\{F_q\}$ are the Z-torsion free part of the modules associated to an n-knot (note that the complement X has dimension $n + 2$). When $n = 2q - 1$, we refer to this pairing as the *Blanchfield pairing*–it reduces to the pairing of the same name in **[K]** and was first considered for knots of dimension $n = 1$ in **[B]**. We have now proved

THEOREM (5.3). *The Blanchfield pairing on the Z-torsion free part of qth Alexander module of a $(2q - 1)$-knot is conjugate-linear, $(-1)^{q+1}$-Hermitian and nonsingular.*

6. We now turn to the pairing $[\ ,\ ]$.

We define a preliminary pairing $\{\ ,\ \}$. Choose an integer k large enough to satisfy a finite number of conditions to be itemized in the remainder of the discussion. Set $\theta = \Lambda/(t^k - 1)$ and let $I(\theta)$ be the Λ-injective hull of θ. We will define

$$\{\ ,\ \}: T_q \times T_{n-q} \to I(\theta)/\theta$$

satisfying:

conjugate-linear: $\{\lambda\alpha, \beta\} = \{\alpha, \bar{\lambda}\beta\} = \lambda\{\alpha, \beta\}$,

nonsingular: the adjoint homomorphism $\bar{T}_q \to \mathrm{Hom}_\Lambda(T_{n-q}, I(\theta)/\theta)$ is bijective,

$(-1)^{q(n-q)+1}$-*Hermitian*: $\{\alpha, \beta\} = (-1)^{q(n-q)+1}\overline{\{\beta, \alpha\}}$.

Assuming, for the moment, such a $\{\ ,\ \}$ exists, we show how to construct the desired $[\ ,\ ]$.

Define $Q\theta = Q \otimes_Z \theta$. The inclusion $\theta \subset I(\theta)$ extends to an injection $Q\theta \subset I(\theta)$ which is unique because $I(\theta)$ is Z-torsion free. To see this merely notice that $t(I(\theta))$, if nontrivial, must meet θ nontrivially, since $I(\theta)$ is an essential extension (see [Mc]) of θ–but θ is Z-torsion free. It is clear that $Q\theta/\theta = t(I(\theta)/\theta)$. This implies that

$$\mathrm{Hom}_\Lambda(A, Q\theta/\theta) = \mathrm{Hom}_\Lambda(A, I(\theta)/\theta)$$

if A is a Z-torsion module. In particular, $\{\ ,\ \}$ takes values in $Q\theta/\theta$. Now we have shown in (4.2) that

$$e\colon \mathrm{Hom}_\Lambda(A, Q\theta/\theta) \approx \mathrm{Hom}_Z(A, Q/Z)$$

as Λ-modules, when A is finite, by $e(\phi) = \phi_0$ if $\phi(\alpha) = \sum_{i=0}^{k-1}\phi_i(\alpha) \otimes t^i$. If we define $[\ ,\ ] = e \circ \{\ ,\ \}$, the desired properties of $[\ ,\ ]$ follow from those of $\{\ ,\ \}$.

We now construct $\{\ ,\ \}$. Note first that

$$\bar{T}_{n-q} = t\big(\overline{H_{n-q}}(\tilde{X})\big) = t\big(\overline{H_{n-q}(\tilde{X}, \partial\tilde{X})}\big) = t\big(H_e^{q+2}(\tilde{X})\big)$$

by duality. So we want to construct an isomorphism:

$$t\big(H_e^{q+2}(\tilde{X})\big) \approx \mathrm{Hom}_\Lambda(T_q, I(\theta)/\theta).$$

Consider the following chain of homomorphisms:

$$\begin{aligned}(6.1)\qquad & H_e^{q+2}(\tilde{X}) \xleftarrow{\delta'} H_e^{q+1}(\tilde{X}; \theta) \xleftarrow{\delta''} H_e^{q}\big(\tilde{X}; I(\theta)/\theta\big) \\ & \xrightarrow{e'} \mathrm{Hom}_\Lambda\{A_q, I(\theta)/\theta\} \xrightarrow{r} \mathrm{Hom}_\Lambda\{T_q, I(\theta)/\theta\}.\end{aligned}$$

δ', δ'' are the coboundary maps of the exact cohomology sequences derived from the short exact sequences of coefficients:

$$0 \to \Lambda \xrightarrow{t^k-1} \Lambda \to \theta \to 0,$$
$$0 \to \theta \to I(\theta) \to I(\theta)/\theta \to 0.$$

e' is an evaluation map and r is defined by restriction. Define $D \subset H_e^{q+2}(\tilde{X})$ to be Image $\delta' \cdot \delta''$. We define a homomorphism $\phi\colon D \to \mathrm{Hom}_\Lambda(T_q, I(\theta)/\theta)$ by setting $\phi(\alpha) = re'(\alpha')$, if $\delta'\delta''(\alpha') = \alpha$.

We first show that ϕ is well defined, i.e. different choices of α' do not affect the value of $re'(\alpha')$. Our choice of α' can be altered by any element in the image of the coefficient homomorphism

$$H_e^q\big(\tilde{X}; I(\theta)\big) \to H_e^q\big(\tilde{X}; I(\theta)/\theta\big)$$

and so we must show that a certain composition of maps

$$H_e^q\big(\tilde{X}; I(\theta)\big) \to \mathrm{Hom}_\Lambda(T_q, I(\theta)/\theta)$$

is zero. But it is easily seen that this can be factored through $\operatorname{Hom}_\Lambda(T_q, I(\theta))$ which is zero since $I(\theta)$ is Z-torsion free.

Our choice of $\delta''(\alpha')$ may be altered by an element in the image of the coefficient homomorphism

$$H_e^{q+1}(\tilde{X}) \to H_e^{q+1}(\tilde{X}; \theta).$$

Consider the chain of homomorphisms:

$$(6.2)\qquad \begin{aligned} H_e^{q+1}(\tilde{X}) &\overset{\bar{\delta}''}{\longleftarrow} H_e^q(\tilde{X}; Q(\Lambda)/\Lambda) \overset{\bar{e}'}{\longrightarrow} \operatorname{Hom}_\Lambda(A_q, Q(\Lambda)/\Lambda) \\ &\overset{\bar{r}}{\longrightarrow} \operatorname{Hom}_\Lambda(T_q, Q(\Lambda)/\Lambda) \end{aligned}$$

analogous to (6.1). In fact this sequence is mapped into (6.1) by obvious coefficient homomorphisms

$$\begin{array}{ccc} Q(\Lambda) & \longrightarrow & I(\theta) \\ \cup & & \cup \\ \Lambda & \longrightarrow & \theta \end{array}$$

Now $\bar{\delta}''$ is an isomorphism, since $H_e^i(\tilde{X}; Q(\Lambda)) \approx \operatorname{Hom}_\Lambda(A_i, Q(\Lambda)) = 0$, so any change in $\delta''(\alpha')$ produces a change in $re'(\alpha')$ which lies in the image $\operatorname{Hom}_\Lambda(T_q, Q(\Lambda)/\Lambda) \to \operatorname{Hom}_\Lambda(T_q, I(\theta)/\theta)$. But by Lemma (5.1), $\operatorname{Hom}_\Lambda(T_q, Q(\Lambda)/\Lambda) = 0$. We next show that ϕ is an isomorphism.

Suppose $\alpha \in t(H_e^{q+2}(\tilde{X}))$ and we have α' such that $\delta'\delta''(\alpha') = \alpha$ and $re'(\alpha') = 0$. We would like to show that $\delta''(\alpha')$ is the image of an element of $H_e^{q+1}(\tilde{X})$ or, equivalently, that $\alpha' \in H_e^q(\tilde{X}; I(\theta)/\theta)$ lies in the submodule generated by the images of $H_e^q(\tilde{X}; I(\theta))$ and $H_e^q(\tilde{X}; Q(\Lambda)/\Lambda)$ under the appropriate coefficient homomorphisms.

Note from the universal coefficient spectral sequence (2.3) that, for any Λ-module G with injective dimension one we have the usual exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda\big(H_{i-1}(\tilde{X}), G\big) \to H_e^i(\tilde{X}; G) \to \operatorname{Hom}_\Lambda(A_i, G) \to 0.$$

In particular this holds for $G = I(\theta)$, $I(\theta)/\theta$ or $Q(\Lambda)/\Lambda$. Therefore we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Ext}^1_\Lambda(A_{q-1}, Q(\Lambda)/\Lambda) & \longrightarrow & H_e^q(\tilde{X}; Q(\Lambda)/\Lambda) & \overset{\bar{e}'}{\longrightarrow} & \operatorname{Hom}_\Lambda(A_q, Q(\Lambda)/\Lambda) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}^1_\Lambda(A_{q-1}, I(\theta)/\theta) & \longrightarrow & H_e^q(\tilde{X}; I(\theta)/\theta) & \overset{\bar{e}'}{\longrightarrow} & \operatorname{Hom}_\Lambda(A_q, I(\theta)/\theta) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & H_e^q(\tilde{X}; I(\theta)) & \overset{\approx}{\longrightarrow} & \operatorname{Hom}_\Lambda(A_q, I(\theta)) & \longrightarrow & 0
\end{array}$$

Furthermore we have the commutative diagram:

$$\begin{array}{ccccc} \operatorname{Ext}^1_\Lambda(A, Q(\Lambda)/\Lambda) & \longrightarrow & \operatorname{Ext}^1_\Lambda(A, I(\theta)/\theta) & & \\ \wr & & \wr & & \\ \operatorname{Ext}^2_\Lambda(A, \Lambda) & \longrightarrow & \operatorname{Ext}^2_\Lambda(A, \theta) & \longrightarrow & 0 \end{array}$$

for any Λ-module A, where the bottom row is a portion of the long exact sequence derived from the short exact sequence $0 \to \Lambda \xrightarrow{t^k-1} \Lambda \to \theta \to 0$, since $\operatorname{Ext}^3_\Lambda(A, \Lambda) = 0$. From this it follows that it suffices to show that $e'(\alpha')$ lies in the submodule generated by the images of $\operatorname{Hom}_\Lambda(A_q, Q(\Lambda)/\Lambda)$ and $\operatorname{Hom}_\Lambda(A_q, I(\theta))$ in $\operatorname{Hom}_\Lambda(A_q, I(\theta)/\theta)$. The assumption $re'(\alpha') = 0$ means that $e'(\alpha')$ is induced by an element of $\operatorname{Hom}_\Lambda(F_q, I(\theta)/\theta)$. We will complete the proof that (6.1) defines a monomorphism by observing that $\operatorname{Hom}_\Lambda(F, I(\theta)/\theta)$ is generated by the images of $\operatorname{Hom}_\Lambda(F, Q(\Lambda)/\Lambda)$ and $\operatorname{Hom}_\Lambda(F, I(\theta))$, for any Z-torsion free Λ-module F of type K. To see this just examine the commutative diagram:

$$\begin{array}{ccccccc} & & \operatorname{Hom}_\Lambda(F, Q(\Lambda)/\Lambda) & \longrightarrow & \operatorname{Ext}^1_\Lambda(F, \Lambda) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ \operatorname{Hom}_\Lambda(F, I(\theta)) & \longrightarrow & \operatorname{Hom}_\Lambda(F, I(\theta)/\theta) & \longrightarrow & \operatorname{Ext}^1_\Lambda(F, \theta) & & \\ & & & & \downarrow & & \\ & & & & \operatorname{Ext}^2_\Lambda(F, \Lambda) = 0 & & \end{array}$$

with exact rows and columns.

To prove surjectivity of the homomorphism defined by (6.1), we need only note that e' and r are both surjective. For e' we have already made this observation above from the universal coefficient spectral sequence. For r we need only observe that $\operatorname{Ext}^1_\Lambda(F_q, I(\theta)/\theta) = 0$. But $\operatorname{Ext}^1_\Lambda(A, I(\theta)/\theta) \approx \operatorname{Ext}^2_\Lambda(A, \theta)$, for any A, while the exact sequence $0 \to \Lambda \xrightarrow{t^k-1} \Lambda \to \theta \to 0$ implies a surjection $\operatorname{Ext}^2_\Lambda(A, \Lambda) \to \operatorname{Ext}^2_\Lambda(A, \theta) \to 0$. If $A = F_q$, then $\operatorname{Ext}^2_\Lambda(A, \Lambda) = 0$.

We will now show that k may be chosen so that $D = t(H_e^{q+2}(\tilde{X}))$. Certainly $D \subset t(H_e^{q+2}(\tilde{X}))$ for any k, because T_q is finite and ϕ is injective. To obtain equality we choose k as follows.

First choose an integer m such that $mt(H_e^{q+2}(\tilde{X})) = 0$. Now consider $H_e^{q+2}(\tilde{X}; \underline{\Lambda_m})$, a Z-torsion module. It is also of type K if $q < n$ because $H_e^i(\tilde{X}) \approx \bar{A}_{n+2-i}$ is of type K if $i \neq n + 2$, and we may apply the five-lemma to the exact sequence

$$H_e^{q+1}(\tilde{X}) \xrightarrow{m} H_e^{q+1}(\tilde{X}) \to H_e^{q+1}(\tilde{X}; \Lambda_m) \to H_e^{q+2}(\tilde{X}) \xrightarrow{m} H_e^{q+2}(\tilde{X}).$$

Therefore $H_e^{q+1}(\tilde{X}; \Lambda_m)$ is finite and we may choose k so that $t^k - 1$ annihilates it.

Let $\alpha \in t(H_e^{q+2}(\tilde{X}))$. We will construct α' such that $\alpha = \delta'\delta''(\alpha')$.

LEMMA (6.3). *Let R be a unique factorization domain; λ, μ relatively prime elements in R and C a free chain complex over R. Then there are short exact sequences:*

(1) $$0 \to R/(\lambda) \xrightarrow{\mu} R/(\lambda) \to R/(\lambda, \mu) \to 0,$$

(2) $$0 \to R/(\mu) \xrightarrow{\lambda} R/(\mu) \to R/(\lambda, \mu) \to 0,$$

(3) $$0 \to R \xrightarrow{\mu} R \to R/(\mu) \to 0,$$

(4) $$0 \to R \xrightarrow{\lambda} R \to R/(\lambda) \to 0.$$

If δ_i is the connecting homomorphism of the exact cohomology sequence of C associated with (i), then $\delta_4 \circ \delta_1 = -\delta_3 \circ \delta_2$.

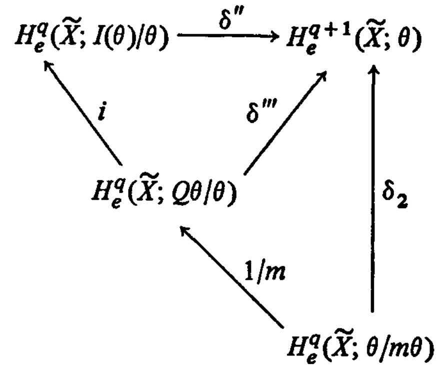
PROOF. The exactness of (1), (2) follow easily from λ, μ being relatively prime. Given $\alpha \in H^*(C; R/(\lambda, \mu))$, lift a representative cycle to a cochain z with coefficients in R (since C is free). Then $\delta z = \lambda z_1 + \mu z_2$, for some cochains z_1, z_2. It follows that $\delta_1(\alpha)$ has a representative cocycle z_2 and $\delta_2(\alpha)$ has representative cocycle z_1. Then $\delta_4 \circ \delta_1(\alpha)$ and $\delta_3 \circ \delta_2(\alpha)$ have representative cocycles $\delta z_2/\lambda$ and $\delta z_1/\mu$, respectively. But $0 = \delta^2 z = \lambda\delta z_1 + \mu\delta z_2$, and the lemma follows. We apply this lemma to the case $\lambda = m$, $\mu = t^k - 1$. Consider the diagram:

$$\begin{array}{ccccc}
 & & H_e^{q+2}(\tilde{X}) & & \\
 & & \uparrow m & & \\
H_e^{q+1}(\tilde{X}; \theta) & \xrightarrow{\delta'} & H_e^{q+2}(\tilde{X}) & \xrightarrow{t^k - 1} & H_e^{q+2}(\tilde{X}) \\
\uparrow \delta_2 & & \uparrow \delta_4 & & \uparrow \delta_4 \\
H_e^{q}(\tilde{X}; \theta/m\theta) & \xrightarrow{\delta'} & H_e^{q+1}(\tilde{X}; \Lambda_m) & \xrightarrow{t^k - 1} & H_e^{q+1}(\tilde{X}; \Lambda_m)
\end{array}$$

The rows and columns are exact. The left square is anticommutative by the lemma while commutativity of the right square is well known. Since $m\alpha = 0$, choose $\alpha_1 \in H_e^{q+1}(\tilde{X}; \Lambda_m)$ such that $\delta_4(\alpha_1) = \alpha$. Since $t^k - 1$ annihilates $H_e^{q+1}(\tilde{X}; \Lambda_m)$, we may choose $\alpha_2 \in H_e^q(\tilde{X}; \theta/m\theta)$ such that $\alpha_1 = \delta'(\alpha_2)$.

We now derive α' from α_2 as follows.

Consider the commutative diagram:



where δ_2, δ'', δ''' are connecting homomorphisms in the appropriate exact cohomology sequence and the remaining maps are induced by coefficient homomorphisms:

$$\theta/m\theta \overset{1/m}{\to} Q\theta/\theta \subset I(\theta)/\theta.$$

Choosing $\alpha' = -\, i(1/m(\alpha_2))$, it follows that $\delta'\delta''(\alpha') = -\ \delta'\delta_2(\alpha_2) = \delta_4\delta_1(\alpha_2) = \alpha$, as desired.

Note that conjugate linearity of $\{\ ,\ \}$ corresponds to the fact that ϕ is a homomorphism.

Finally, we will verify the Hermitian property of $\{\ ,\ \}$. This will depend on the Hermitian property of the intersection pairing and so we make the usual chain level formulation. It will be convenient to use the second definition of $\{\ ,\ \}$, namely using the two sequences of homomorphisms:

(6.4) $$H_e^{q+2}(\tilde{X}) \overset{\delta_4}{\leftarrow} H_e^{q+1}(\tilde{X}; \Lambda_m) \overset{\delta_1}{\leftarrow} H_e^q(\tilde{X}; \theta/m\theta)$$
$$\overset{e'}{\to} \operatorname{Hom}_\Lambda\{A_q, \theta/m\theta\} \overset{r}{\to} \operatorname{Hom}_\Lambda\{T_q, \theta/m\theta\} \overset{i}{\to} \operatorname{Hom}\{T_q, I(\theta)/\theta\}.$$

If $\alpha \in T_{n-q}$, $\beta \in T_q$, then $\{\alpha, \beta\}$ is defined to be $ire'(\alpha') \cdot \beta$, where $\alpha' \in H_e^q(\tilde{X}; \theta/m\theta)$ satisfies $-\delta_4\delta_1(\alpha') = \bar{\alpha}$, the dual of α. The above diagram has shown that these two definitions of $\{\ ,\ \}$ agree.

This translates into the following. Let z, w be representative cycles of α, β. Then mz is null-homologous and we may write $mz = \partial z'$, for some $(n - q + 1)$-chain z'. Now $(t^k - 1)z'$ is null-homologous mod m and so we may write $(t^k - 1)z' = \partial z'' + mz_0$, where z'' is an $(n - q + 2)$-chain and z_0 an $(n - q + 1)$-chain. Finally we may write $\{\alpha, \beta\} = (-z'' \cdot w)/m$, or rather its image under $Q \otimes_z \Lambda \to Q\theta \subset I(\theta) \to I(\theta)/\theta$, where $z'' \cdot w$ is the intersection number in Λ.

Similarly we may choose chains w', w'' and w_0 of dimensions $q + 1$, $q + 2$ and $q + 1$, respectively, so that $mw = \partial w'$ and $(t^k - 1)w' = \partial w'' + mw_0$, and

$\{\beta, \alpha\} = (-w'' \cdot z)/m$. First note that

$$m\partial z_0 = \partial(mz_0) = \partial(\partial z'' + mz_0) = \partial(t^k - 1)z' = (t^k - 1)\partial z' = (t^k - 1)mz.$$

Since Λ is Z-torsion free, we have $\partial z_0 = (t^k - 1)z$. Similarly $\partial w_0 = (t^k - 1)w$. Now $(t^k - 1)z'' \cdot w = -t^k(z'' \cdot (t^k - 1)w) = -t^k(z'' \cdot \partial w_0)' = (-1)^{n-q+1}t^k(\partial z'' \cdot w_0)$ using the well-known equality $x \cdot \partial y = (-1)^d \partial x \cdot y$, where $d = \dim x$. Now $\partial z'' \cdot w_0 = ((t^k - 1)z' - mz_0) \cdot w_0$, and so $(t^k - 1)z'' \cdot w \equiv (-1)^{n-q+1}t^k(t^k - 1)z' \cdot w_0 \bmod m$. Since m and $t^k - 1$ are relatively prime, $z'' \cdot w \equiv (-1)^{n-q+1}t^k z' \cdot w_0 \bmod m$. Now $mz' \cdot w_0 = z' \cdot ((t^k - 1)w' - \partial w'') \equiv -z' \cdot \partial w'' \pmod{t^k - 1} = (-1)^{n-q}\partial z' \cdot w'' = (-1)^{n-q} mz \cdot w''$. Since m and $t^k - 1$ are relatively prime, we have $z' \cdot w_0 \equiv (-1)^{n-q} z \cdot w''$ mod $t^k - 1$. Combining these results we have

$$z'' \cdot w \equiv -z \cdot w'' \quad \operatorname{mod}(m, t^k - 1).$$

Therefore

$$\{\alpha, \beta\} = -\frac{z'' \cdot w}{m} = \frac{z \cdot w''}{m} \quad (\text{in } Q\theta/\theta)$$
$$= (-1)^{q(n-q)} \frac{\overline{w'' \cdot z}}{m} = (-1)^{q(n-q)+1} \overline{\{\beta, \alpha\}}.$$

The pairing $[\ ,\ ]$: $T_q \times T_{n-q} \to Q/Z$ has now been shown to have the desired properties. When $n = 2q$, we will refer to this pairing as the *torsion-pairing* and our results can be summarized in

THEOREM (6.5). *The torsion pairing on the Z-torsion subgroup of the qth Alexander module of a* $(2q)$*-knot is Z-linear, conjugate selfadjoint,* $(-1)^{q+1}$*-symmetric and nonsingular.*

7. It is enlightening to examine the torsion pairing in the special case of a fibered knot. We shall see that it is equivalent to the usual linking pairing on the middle homology of the fiber.

As definition of a (smooth) *fibered knot* we will use the formulation of [**La**]. A knot $K \subset S^{n+2}$ is fibered if there exists a smooth map ϕ: $S^{n+2} \to R^2$ satisfying

(i) 0 is a regular value and $K = \phi^{-1}(0)$,

(ii) $\phi/|\phi|$: $S^{n+2} - K \to S^1$ is a smooth fibration.

Then any fiber F_0 of $\phi/|\phi|$ is an unbounded manifold whose closure E_0 in S^{n+2} is a smooth submanifold with boundary K. Furthermore, if X is the complement of a suitable open tubular neighborhood of K, X fibers over S^1 and the fiber is a smooth compact manifold F with ∂F diffeomorphic to K. The fibration $X \to S^1$ determines a diffeomorphism T: $F \to F$, such that $T|\partial F$ = identity, and we can describe X as $F \times I$ with identifications $(x, 0) \leftrightarrow (T(x), 1)$. This allows us to identify $\tilde{X}$ with $F \times R$ and the generating covering transformation t of $\tilde{X}$ with the map $(x, u) \mapsto (T(x), u + 1)$.

There is an obvious isomorphism $H_*(\tilde{X}) \approx H_*(F)$ under which the action of $t \in \Lambda$ on $H_*(\tilde{X})$ corresponds to the automorphism T_* on $H_*(F)$.

PROPOSITION (7.1). *The pairing* $[\ ,\ ]$: $T_q \times T_{n-q} \to Q/Z$ *corresponds to the standard linking pairing* $(\ ,\ )$ *defined on elements of* $H_*(F)$ *of finite order.*

Recall that the latter pairing is defined as follows [Se]. Assume F has a PL-structure and F' is the dual structure, let z, w be, respectively, a q-cycle in F and an $(n - q)$-cycle in F' ($\dim F = n + 1$) representing homology classes α, β of finite order. Then $mz = \partial c$, for some $(q + 1)$-chain in F and $(\alpha, \beta) = \operatorname{int}(c, w)/m$, where $\operatorname{int}(\ ,\ )$ is the usual intersection number.

We may define a PL-structure on $\tilde{X} = F \times R$ using cells of the form $\{\sigma \times i, \sigma \times [i, i + 1]\}$ where σ is a cell of F and $i \in z$. A dual structure will be made up of cells of the form $\{\tau \times (i + \frac{1}{2}), \tau \times [i - \frac{1}{2}, i + \frac{1}{2}]\}$, where τ is a cell of F' and $i \in Z$. The boundary operator is given by:

$$\partial(\sigma \times i) = \partial\sigma \times i,$$
$$\partial(\sigma \times [i, i + 1]) = \sigma \times (i + 1) - \sigma \times i - \partial\sigma \times [i, i + 1]$$

where σ represents an oriented cell, or any chain, of F.

We now choose a PL-structure on F so that T is a PL-automorphism (see [Mu]). Then the covering transformation t of $\tilde{X}$ is PL and the cells map as follows:

$$t(\sigma \times i) = T_\#(\sigma) \times (i + 1),$$
$$t(\sigma \times [i, i + 1]) = T_\#(\sigma) \times [i + 1, i + 2].$$

Let z, w be cycles in F as above–then $z \times 0$, $w \times \frac{1}{2}$ are corresponding cycles in $\tilde{X}$ representing α', β'. We want to show $[\alpha', \beta'] = (\alpha, \beta) = \operatorname{int}(c, w)/m$. We use the definition of $[\ ,\ ]$ in (6.4). Note that

$$m(z \times 0) = \partial(c \times 0).$$

We must now find z'' such that $\partial z'' = (t^k - 1)(c \times 0) \bmod m$ and then $[\alpha', \beta']$ is the constant term of $(z'' \cdot (w \times \frac{1}{2}))/m \in Q\theta/\theta = Q/Z \otimes_Z \theta$. In other words $[\alpha', \beta']$ is the sum of the coefficients of all t^{ik}, $i \in Z$. Now k has been chosen so that $t^k = 1$ on $H_*(\tilde{X}; Z/m)$; therefore $(T_*)^k = 1$ on $H_*(F; Z/m)$. We may write $T_\#^k(c) - c = \partial\Phi \bmod m$, where Φ is a $(q + 2)$-chain in F. Now set

$$z'' = \Phi \times 0 + \sum_{i=0}^{k-1} T_\#^k(c) \times [i, i + 1].$$

So

$$\partial z'' = \partial\Phi \times 0 + \sum_{i=0}^{k-1} \big(T_{\#}^{k}(c) \times (i+1) - T_{\#}^{k}(c) \times i - T_{\#}^{k}(\partial c) \times [i, i+1]\big)$$

$$= T_{\#}^{k}(c) \times 0 - c \times 0 + T_{\#}^{k}(c) \times k - T_{\#}^{k}(c) \times 0 - m \sum_{i=0}^{i-1} \big(T_{\#}^{k}(z) \times [i, i+1]\big)$$

$$= -c \times 0 + t^{k}(c \times 0) \pmod{m}$$

as desired. Now

$$[\alpha', \beta'] = \frac{z'' \cdot \left(w \times \frac{1}{2}\right)}{m} = \frac{1}{m} \sum_{i=0}^{i-1} t^{i}\big(\mathrm{int}(T_{\#}^{k} c, w)\big),$$

whose constant term is

$$\frac{\mathrm{int}(T_{\#}^{k} c, w)}{m} = \frac{\mathrm{int}(c, T_{\#}^{-k} w)}{m} = \frac{\mathrm{int}(c, w)}{m} = [\alpha, \beta]$$

since $T_{\#}^{k}(w)$ is homologous to w.

8. We now examine the "group of the knot" $\pi_1(X)$. In [Ke] Kervaire obtains an algebraic description of those groups which can arise as groups of a knot of dimension > 3. Our main reason for reopening this subject is its relevance to the characterization of knot modules, but we will also, in passing, obtain some further information on $\pi_1(X)$.

We first recall

THEOREM (8.1) (KERVAIRE). *If π is the group of an n-knot, then π is finitely-presented and satisfies*:

(a) $H_1(\pi) \approx Z$,

(b) $H_2(\pi) = 0$,

(c) *π is of "weight one", i.e. there is $\alpha \in \pi$ such that π is the normal closure (in π) of α.*

OUTLINE OF PROOF. π is finitely-presented, since X is a finite cell-complex. $H_1(\pi) = \pi/\pi' \approx H_1(X) \approx Z$. $H_2(\pi)$ is the cokernel of the Hurewicz homomorphism $\pi_2(X) \to H_2(X)$, according to [H], but $H_2(X) = 0$. Finally, if $\langle\alpha\rangle$ is the normal closure of α in π, then $\pi/\langle\alpha\rangle$ is the fundamental group of $X \cup_\alpha e^2 = X$ with a 2-cell attached by a map $f\colon \partial e^2 \to X$ representing α. If we choose α to be the homotopy class of a "meridian" of the knot K, i.e. the boundary circle of a fiber in a tubular neighborhood of K, then $X \cup_\alpha e^2$ is a deformation retract of $S^{n+2} - \{\text{point}\}$. We also notice the following connection:

PROPOSITION (8.2). *If π is finitely-generated and satisfies* (8.1)(a), (c), *X is a space such that $\pi_1(X) \approx \pi$, $\tilde{X}$ the infinite cyclic covering of X associated to $\pi' \subset \pi$, then $H_1(\tilde{X})$ is a Λ-module of type K.*

An alternative, purely algebraic description of $H_1(\tilde{X})$ is of interest. As an abelian group $H_1(\tilde{X}) \approx \pi'/\pi'' = H_1(\pi')$ and the Λ-module structure is given as follows. Consider the short exact sequence:

$$1 \to H_1(\pi') \to \pi/\pi'' \to H_1(\pi) \to 1.$$

If $\alpha \in H_1(\pi)$, then conjugation in $H_1(\pi')$ by a pull-back of α in π/π'' is easily checked to be independent of the particular pull-back, since $H_1(\pi')$ is abelian. This defines a group action of $H_1(\pi)$ on $H_1(\pi')$. Choosing a particular isomorphism $H_1(\pi) \approx Z$ gives an action of Z on $H_1(\pi')$ which then extends to a Λ-module structure in the usual way.

We leave it to the reader to check that this coincides with the Λ-module structure on $H_1(\tilde{X})$ defined by covering transformations.

Let $\{\alpha_i\} \subset \pi$ be a finite set of generators. If $\alpha \in \pi$ is as in (8.1)(c), it is easy to check that α defines a generator of π/π'. Set $\beta_i = \alpha_i\alpha^{m_i}$, where m_i is chosen so that $\beta_i \in \pi'$. Then $\{\alpha, \beta_i\}$ generate π. From this we can conclude that $\{\alpha^m\beta_i\alpha^{-m}: m \in Z\}$ is a set of generators of π'. If t is the generator of Λ corresponding to α, this implies that $\{t^m\bar{\beta}_i: m \in Z\}$ generate $H_1(\pi')$, as an abelian group, where $\bar{\beta}_i$ is the reduction of β_i to $H_1(\pi')$. But then $\{\bar{\beta}_i\}$ generate $H_1(\pi')$ as a Λ-module.

(8.1)(c) means π is generated by $\{\beta\alpha\beta^{-1}: \beta \in \pi\}$. Therefore π' is generated by $\{\alpha^m\beta\alpha\beta^{-1}\alpha^{-1-m}: m \in Z, \beta \in \pi\}$. Since we can write each $\beta = \alpha^k\gamma$, where $\gamma \in \pi'$, $k \in Z$, π' is generated by $\{\alpha^m\gamma\alpha\gamma^{-1}\alpha^{-1-m}: m \in Z, \gamma \in \pi'\}$. If $\bar{\gamma}$ is the reduction of γ to $H_1(\pi')$, then the reduction of $\alpha^m\gamma\alpha\gamma^{-1}\alpha^{-1-m}$ is $t^m\bar{\gamma} - t^{m+1}\bar{\gamma}$. In particular we see that $(t-1)H_1(\pi') = H_1(\pi')$, which implies $H_1(\pi')$ is of type K.

We now observe one more condition on π, which arises when we bring in the second Alexander module.

PROPOSITION (8.3). *If π is the group of a knot and A_2 is the 2nd Alexander module, then there is an epimorphism ρ: $A_2 \to H_2(\pi')$, where π' is the commutator subgroup of π.*

PROOF. This is a direct consequence of the Hopf theorem [H] applied to the space $\tilde{X}$ whose fundamental group is π'.

9. Our first realization theorem will omit the delicate middle-dimensional duality–this requires entirely different techniques. On the other hand we will

include the group of the knot in our results.

THEOREM (9.1). *Let $n > 2$, π a finitely-presented group and $A_1, \ldots, A_n$ Λ-modules of type K satisfying*:

(i) $H_1(\pi) \approx Z$,

(ii) $H_2(\pi) = 0$,

(iii) *π has weight one*,

(a) *$A_1 \approx H_1(\pi')$ with the action of t defined by conjugation by an element of π generating $H_1(\pi)$*,

(b) $\overline{f(A_q)} \approx e^1(f(A_{n+1-q}))$,

(c) $t(A_q) \approx e^2(t(A_{n-q}))$,

(d) *there is an epimorphism ρ: $A_2 \twoheadrightarrow H_2(\pi')$.*

According to our previous results, these are necessary conditions for π, $\{A_i\}$ to be the group and Alexander modules of an n-knot.

Suppose the following extra conditions are satisfied:

(1) *if $n = 2q - 1$, A_q is Z-torsion,*

(2) *if $n = 2q$, A_q is Z-torsion free,*

(3) *if $n = 2$, π has a presentation with one more generator than relation,*

(4) *A_2 has a Z-torsion free summand F such that $\rho(F) = H_2(\pi')$.*

Then there exists a (*smooth*) *n-knot with group π and Alexander modules $\{A_i\}$. Moreover the knot may be required to be diffeomorphic to S^n if $n > 3$. If $n = 2$, the ambient sphere may only be a homotopy 4-sphere.*

REMARKS. Conditions (1), (2) serve to trivialize the middle-dimensional self-duality (§§5, 6). Conditions (3), (4) are certainly not necessary (see [**Ke**])–we will shed a bit more light on (3) below in (9.2). Note that, if $n = 3$, we have, by (1), (4), forced $H_2(\pi') = 0$. I do not know whether $\rho | T_2$ can be nonzero, in general.

To clear up the relation between (2), (3), (4) when $n = 2$, we first prove

PROPOSITION (9.2). *Let π satisfy* (9.1)(i), (iii), (3). *Then π also satisfies* (ii), *$H_1(\pi')$ is Z-torsion free and $H_2(\pi') = 0$.*

PROOF. Construct the complex P_2 as below in the proof of (9.3). Thus $\pi_1(P_2) \approx \pi$ and the infinite cyclic covering $\tilde{P}_2$ associated with $\pi' \subset \pi$ has a chain complex:

$$C_2(\tilde{P}_2) \xrightarrow{d_2} C_1(\tilde{P}_2) \xrightarrow{d_1} C_0(\tilde{P}_2)$$

made up of free Λ-modules of ranks $n - 1$, n, and 1, respectively. (π has a presentation with n generators and $n - 1$ relations.) Since $H_1(\pi') \approx H_1(\tilde{P}_2)$ is a module of type K, by Proposition (8.2), this chain complex will become exact after $\otimes_\Lambda Q(\Lambda)$–by (1.3). Therefore, by considerations of rank, d_2 must be a monomorphism and $H_2(\tilde{P}_2) = 0$. By the Hopf theorem [H], $H_2(\pi') = 0$. Furthermore, since

$$\text{Image } d_1 = \text{Kernel}\big(C_0\big(\tilde{P}_2\big) \approx \Lambda \to Z\big)$$

is free, we have the free resolution $0 \to C_2(\tilde{P}_2) \to \text{Ker } d_1 \to H_1(\tilde{P}_2) \to 0$ and, therefore, by Proposition (3.5), $H_1(\tilde{P}_2) = H_1(\pi')$ is Z-torsion free. Finally, we conclude $H_2(P_2) = 0$ from the argument of Proposition (1.2) and, therefore, by **[H]**, $H_2(\pi) = 0$. See also **[Ke]**.

The first step in the proof of (9.1) is the construction of CW-complexes with certain properties. These will be used to construct the desired knots as in **[W]**.

LEMMA (9.3). *Let π be a finitely-presented group satisfying* (9.1)(i)–(iii), *F a Z-torsion free Λ-module of type K and ρ: $F \to H_2(\pi')$ an epimorphism. Then there exists a connected* 3-*dimensional CW-complex P with $\pi_1(P) \approx \pi$, $H_2(\tilde{P}) \approx F$ (where $\tilde{P}$ is the ∞-cyclic covering of P associated to $\pi' \subset \pi$), ρ corresponding to the "Hopf epimorphism" of $\tilde{P}$ i.e. the epimorphism $H_2(\tilde{P}) \twoheadrightarrow H_2(\pi_1(\tilde{P}))$ which exists by the Hopf theorem* [H] *and $H_3(\tilde{P}) = 0$.*

If π also satisfies (9.1)(3), *and $F = 0$, we may choose P to be* 2-*dimensional.*

PROOF. Choose a presentation of π: $\{x_1, \dots, x_n; r_1 = 0, \dots, r_m = 0\}$ where the r_i are elements of the free group on $\{x_i\}$. Define P_1 to be the one-point union of n circles. If we identify x_i with the element of $\pi_1(P_1)$ represented by the ith circle, then we may represent each r_j by a map f_j: $S^1 \to P_1$. Construct P_2 from P_1 by attaching m 2-cells using the $\{f_j\}$. Clearly $\pi_1(P_2) \approx \pi$. Let $\tilde{P}_2$ be the infinite cyclic covering of P_2 associated with $\pi' \subset \pi$.

$H_2(\tilde{P}_2)$ is a free Λ-module, since it is the kernel of a homomorphism between free modules, the boundary operator $C_2(\tilde{P}_2) \to C_1(\tilde{P}_2)$, Λ has homological dimension 2 and every projective Λ-module is free. We have the exact sequence $\pi_2(\tilde{P}_2) \to H_2(\tilde{P}_2) \to H_2(\pi') \to 0$ from Hopf's theorem. Since $H_2(\tilde{P}_2)$ is free and ρ: $F \to H_2(\pi')$ is onto, we may construct a homomorphism ρ' to give a commutative diagram

$$\begin{array}{ccccccc}
 & & & \overset{\rho'}{\nearrow} & F & & \\
 & & & & \downarrow \rho & & \\
\pi_2(\tilde{P}_2) & \longrightarrow & H_2(\tilde{P}_2) & \longrightarrow & H_2(\pi') & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array}$$

We may, furthermore, make ρ' onto as follows. Notice that ρ' is onto F if and only if $\text{Ker } \rho \subset \text{Im } \rho'$. Choose a (finite) set of generators of Ker ρ–recall

Λ is Noetherian. For each of these attach a trivial 2-cell to P_2, i.e. form the one-point union with a 2-sphere. This has the effect of adding a copy of Λ to $H_2(\tilde{P}_2)$ which maps to 0 in $H_2(\pi')$. Now we just extend ρ' over these new summands by mapping them to the corresponding generators of Ker ρ.

Since F is Z-torsion free, it has homological dimension 1, by Proposition (3.5). Therefore, Ker ρ' is a free Λ-module. Moreover, by Hopf's theorem, the elements of Ker ρ are spherical. Choose a basis of Ker ρ' and represent these by maps $S^2 \to P_2$. Construct P from P_2 by attaching 3-cells using these maps. It follows easily that $H_2(\tilde{P}) \approx F$. $H_3(\tilde{P}) = 0$ since the boundary operator:

$$C_3(\tilde{P}) \approx \operatorname{Ker} \rho' \subset H_2(\tilde{P}_2) \subset C_2(\tilde{P}_2) = C_2(\tilde{P})$$

is injective.

This proves the first part of Lemma (9.3).

In case $n = m + 1$, we can conclude that $H_2(\tilde{P}_2) = 0$, using the argument of (9.2). We may, therefore, take $P = P_2$.

LEMMA (9.4). *Let A be a Λ-module of type K and $q \geqslant 2$ an integer. Then there exists a connected $(q + 2)$-dimensional CW-complex P such that $\pi_1(P) \approx Z$, $H_q(\tilde{P}) \approx A$, where $\tilde{P}$ is the universal covering of P, and $H_i(\tilde{P}) = 0$ for $i = \neq 0, q$.*

If A is Z-torsion free, we may make P $(q + 1)$-dimensional.

PROOF. Choose a free resolution of A:

$$0 \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to A \to 0.$$

The F_i are finitely-generated. Let P_q be the one-point union of a circle and a copy of S^q for each member of a basis $\{x_i\}$ of F_0. Then $\pi_q(P_q) \approx H_q(\tilde{P}_q) = C_q(\tilde{P}_q) \approx F_0$. Let $\{y_i\}$ be a basis of F_1 and choose f_i: $S^q \to P_q$ to represent the element of $\pi_q(P_q)$ corresponding to $d_1(y_i)$. Construct P_{q+1} by attaching $(q + 1)$-cells to P_q using $\{f_i\}$. Then $C_{q+1}(\tilde{P}_{q+1}) \approx F_1$, $C_q(\tilde{P}_{q+1}) = C_q(\tilde{P}_q) \approx F_0$ and the boundary operator $C_{q+1}(\tilde{P}_{q+1}) \to C_q(\tilde{P}_{q+1})$ corresponds to d_1. Now $H_{q+1}(\tilde{P}_{q+1}) \subset C_{q+1}(\tilde{P}_{q+1})$ corresponds to Ker d_1 and, therefore, d_2 defines an isomorphism $F_2 \approx H_{q+1}(\tilde{P}_{q+1})$. By the theorem of G. Whitehead [**Hu**, p. 167], the Hurewicz homomorphism $\pi_{q+1}(\tilde{P}_{q+1}) \to H_{q+1}(\tilde{P}_{q+1})$ is onto. Therefore we can represent a basis of $H_{q+1}(\tilde{P}_{q+1})$ by maps $S^{q+1} \to \tilde{P}_{q+1}$ which project to maps $S^{q+1} \to P_{q+1}$. P is now constructed from P_{q+1} by attaching $(q + 2)$-cells using these maps.

The chain complex of $\tilde{P}$ is now up to isomorphism:

$$\dots \to 0 \to F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \to 0 \to \dots \to \Lambda \xrightarrow{t-1} \Lambda$$

which implies the desired homological behavior.

If A is Z-torsion free then it has homological dimension 1 (Proposition (3.5)) and we may choose $F_2 = 0$. In this case $P = P_{q+1}$.

10. The next step in the construction of knots is

LEMMA (10.1). *Let P be a finite d-dimensional complex such that $\pi = \pi_1(P)$ satisfies* (9.1)(i)–(iii), *and $H_i(\tilde{P})$ are Λ-modules of type K, for $i > 0$, where $\tilde{P}$ is the infinite cyclic covering of P. Then P imbeds in R^{n+3} where*

$$n \geqslant \sup\{2, 2d - 3\}.$$

Let Y be the boundary of a regular neighborhood of P in R^{n+3}. Then

(i) $\pi_1(Y) \approx \pi_1(P)$,

(ii) $H_*(Y) \approx H_*(S^1 \times S^{n+1})$,

(iii) $H_i(\tilde{Y}) \approx H_i(\tilde{P})$ *if $i \leqslant n + 1 - d$, where $\tilde{Y}$ is the infinite cyclic covering of Y,*

(iv) $H_{d-1}(\tilde{Y})$ *is finite if $n = 2d - 3$ and $H_{d-1}(\tilde{P})$ is finite.*

PROOF. An argument similar to the proof of Proposition (1.2) shows that P is a homology circle. Therefore $H_d(P) = 0$ and the imbeddability follows from a theorem of van Kampen [V]. If N is a regular neighborhood of P in R^{n+3}, then Y is a deformation retract of $N - P$. Therefore $\pi_i(N, Y) \approx \pi_i(N, N - P) = 0$ for $i < n + 3 - d$, by general position. Since $n + 3 - d \geqslant 3$ and $N \simeq P$, we conclude (i). The Hurewicz theorem implies (iii).

Since (N, Y) is $(n + 2 - d)$-connected, it follows easily that $H_i(Y) \approx H_i(P)$ for $i \leqslant \frac{1}{2}(n + 2)$–which implies (ii)–except for the case $n = 2d - 3$, $i = d - 1$. For this we examine the homology sequence of (N, Y):

$$H_d(N, Y) \to H_{d-1}(Y) \to H_{d-1}(N) = 0.$$

Now $H_d(N, Y) \approx H^d(N) = H^d(P) = 0$.

To prove (iv), we consider the homology sequence of $(\tilde{N}, \tilde{Y})$:

$$H_d(\tilde{N}, \tilde{Y}) \to H_{d-1}(\tilde{Y}) \to H_{d-1}(\tilde{N}).$$

It suffices to prove $H_d(\tilde{N}, \tilde{Y})$ is finite. But $H_d(\tilde{N}, \tilde{Y}) \approx H_e^d(\tilde{N}) \approx H_e^d(\tilde{P})$. Since $H_*(\tilde{P})$ is of type K, we have, from the universal coefficient spectral sequence, just as in Proposition (2.4), the short exact sequence

$$0 \to \operatorname{Ext}^2_\Lambda\big(H_{d-2}(\tilde{P}), \Lambda\big) \to H_e^d(\tilde{P}) \to \operatorname{Ext}^1_\Lambda\big(H_{d-1}(\tilde{P}), \Lambda\big) \to 0.$$

Since $H_{d-1}(\tilde{P})$ is finite, $\operatorname{Ext}^1_\Lambda(H_{d-1}(\tilde{P}), \Lambda) = 0$ (see Lemma (3.3)). The finiteness of $\operatorname{Ext}^2_\Lambda(H_{d-2}(\tilde{P}), \Lambda)$ (see Proposition (3.2)) gives the desired result.

If $\alpha \in \pi_1(Y)$ is an element satisfying property (iii), represent α by an imbedded circle $S \subset Y$. S has a trivial normal bundle, since Y is orientable, and so has a tubular neighborhood T diffeomorphic to $S^1 \times D^{n+1}$.

Define $X = \overline{Y - T}$. By general position, $\pi_i(Y, X) = 0$ for $i \leqslant n$. Therefore $\pi_1(X) \approx \pi_1(Y)$, if $n \geqslant 2$, $H_i(X) \approx H_i(Y)$ and $H_i(\tilde{X}) \approx H_i(\tilde{Y})$ for $i < n$. Since ∂X is diffeomorphic to $S^1 \times S^n$ we may attach to X a copy of $D^2 \times S^n$ via a diffeomorphism between their boundaries. Let Σ denote the resulting manifold.

CLAIM. Σ is a homotopy $(n + 2)$-sphere if $n \geqslant 2$.

It follows from van Kampen's theorem and property (9.1)(iii) of π, that Σ is simply-connected. Now $H_i(\Sigma, X) \approx H_i((D^2, S^1) \times S^n) \approx H_{i-2}(S^n)$. Therefore $H_i(X) \approx H_i(\Sigma)$ for $i < n$, $i \neq 1, 2$. To check $i = 1, 2$ we examine part of the homology sequence:

$$H_3(\Sigma, X) \to H_2(X) \to H_2(\Sigma) \to H_2(\Sigma, X) \xrightarrow{\partial} H_1(X) \to H_1(\Sigma) \to 0.$$

$H_2(\Sigma, X)$ is infinite cyclic and a generator maps by ∂ onto the elements of $H_1(X) \approx H_1(Y)$ represented by α. Since $H_1(Y)$ is inifnite cyclic generated by α, ∂ is an isomorphism. This shows $H_1(\Sigma) = 0$ and $H_2(X) \approx H_2(\Sigma)$. Now $H_i(\Sigma) \approx H_i(X) \approx H_i(Y) = 0$ for $2 \leqslant i < n$. Since $n > 2$ this proves, by duality, that $H_*(\Sigma) \approx H_*(S^{n+2})$. The claim now follows by a theorem of J.H.C. Whitehead.

If $n > 3$, then Σ is, in fact, a piecewise-linear sphere by **[Sm]** and, therefore, by changing the attaching diffeomorphism between X and $D^2 \times S^n$, on an $(n + 1)$-disk, we may arrange that Σ be diffeomorphic to S^{n+2}.

By setting $K = S^n \times 0 \subset S^n \times D^2 \subset \Sigma$, we have constructed an n-knot whose complement is X.

We now construct, by these techniques, some particular knots.

(10.2) Given π satisfying (9.1)(i)–(iii), (3), there exists a 2-knot in a homotopy 4-sphere with $\pi_1(X) \approx \pi$.

Let P be the 2-dimensional complex constructed in Lemma (9.3) such that $\pi_1(P) \approx \pi$, $H_2(\tilde{P}) = 0$. By Lemma (10.1) we may imbed $P \subset R^5$; the knot derived from this embedding will do.

(10.3) Let π satisfy (9.1)(i)–(iii) and F be a Z-torsion free Λ-module of type K with an epimorphism ρ: $F \to H_2(\pi')$. If $n > 4$, or $n = 3$ and $F = 0$, there exists a n-knot (diffeomorphic to S^n) in S^{n+2} such that $\pi_1(X) \approx \pi$, $H_q(\tilde{X}) = 0$ for $3 \leqslant q \leqslant [(n + 1)/2]$ and $H_2(\tilde{X}) \approx F$ if $n > 4$, is finite if $n = 3$.

Let P be the 3-dimensional complex constructed in Lemma (9.3) with $\pi_1(P) \approx \pi$, $H_2(\tilde{P}) \approx F$, $H_3(\tilde{P}) = 0$. By Lemma (10.1) we may imbed $P \subset R^{n+3}$ and the resulting knot is readily checked to satisfy the above properties.

(10.4) Given a Λ-module A of type K and integers $q > 2$ and $n > 2q$, with A Z-torsion free if $n = 2q$, there exists an n-knot (diffeomorphic to S^n) in S^{n+2} such that $\pi_1(X) \approx Z$, $H_q(\tilde{X}) \approx A$, $H_i(\tilde{X}) = 0$ if $i \neq q$, $0 < i \leqslant [n/2]$ and $H_i(\tilde{X})$ finite if $i = (n + 1)/2$.

Let P be the complex constructed in Lemma (9.4), of dimension $q + 1$ if $n = 2q$. Then we may imbed $P \subset R^{n+3}$ and the resulting knot will satisfy the desired conditions as a consequence of Lemma (10.1).

It is clear that the knots produced in (10.3) and (10.4) must be, somehow, combined to get the general knot which will satisfy Theorem (9.1) for $n > 3$. This combination is achieved by the operation of "connected sum" (see **[Ha]**).

Let $K_1 \subset S_1$, $K_2 \subset S_2$ be smooth n-knots. Choose orientation-preserving imbeddings $D^{n+2} \subset S_1$, $D^{n+2} \subset S_2$ such that $D^{n+2} \cap K_i = D^n$ and the resulting imbeddings $D^n \subset K_i$ are orientation-preserving. Now choose an orientation-*reversing* diffeomorphism ϕ: $D^{n+2} \to D^{n+2}$ such that $\phi(D^n) = D^n$ and reverses the orientation of D^n. Define the *connected sum*

$$S = \overline{S_1 - D^{n+2}} \cup_\phi \overline{S_2 - D^{n+2}}, \quad K = \overline{K_1 - D^n} \cup_\phi \overline{K_2 - D^n}.$$

S has an orientation consistent with the orientations of $S_i - D^{n+2} \subset S$; similarly for K. It is easy to see that $K \subset S$ is an n-knot.

LEMMA (10.5). *Suppose $K_i \subset S_i$, $i = 1, 2$, are n-knots with complements X_i and $K \subset S$ is the connected sum with complement X. Then $H_q(\tilde{X}) \approx H_q(\tilde{X}_1) \oplus H_q(\tilde{X}_2)$, for $q > 0$. If $\pi_1(X_2) \approx Z$, then $\pi_1(X) \approx \pi_1(X_1)$.*

PROOF. An examination of the connected sum construction shows that X can be described, up to homotopy, as the union of X_1 and X_2 identifying a meridian of K_1 with a meridian of K_2, preserving orientations. In $\tilde{X}_1$ and $\tilde{X}_2$ the meridians lift to copies of the real line–therefore $\tilde{X} \simeq \tilde{X}_1 \vee \tilde{X}_2$. The statements about homology follow easily. The statement about fundamental group follows from the van Kampen theorem and the fact that a meridian of K_2 represents a generator of $\pi_1(X_2) \approx Z$.

We can now complete the proof of Theorem (9.1) The desired knot will be the connected sum of knots $K_i \subset S_i$, $i = 1, \ldots, [n/2]$. We may assume $n \geqslant 3$, since, if $n = 2$, a single knot of type (10.2) will suffice.

Let $K_1 \subset S_1$ be a knot of type (10.3) with π, F as given–note that, if $n = 3$, conditions (9.1)(1), (4) force $F = 0$. By (9.1)(4), we may write $A_2 = F \oplus B_2$. Let $K_2 \subset S_2$ be a knot of type (10.4) with $q = 2$, $A = B_2$. For $i \geqslant 3$, let $K_i \subset S_i$ be a knot of type (10.4) with $q = i$ and $A = A_q$. Properties (9.1)(b), (c) follow from Theorem (3.4), while the remaining properties follow in a straightforward manner.

11. As a consequence of Theorem (9.1), we will prove the following realization theorem concerning only Alexander modules.

THEOREM (11.1). *Let $n \geqslant 2$, and $A_1, \ldots, A_n$ a sequence of Λ-modules of type K satisfying*:

(a) $\overline{f(A_q)} \approx e^1(f(A_{n+1-q}))$,

(b) $t(A_q) \approx e^2(t(A_{n-q}))$,

(1) *if* $n = 2q - 1, f(A_q) = 0$,

(2) *if* $n = 2q$, $t(A_q) = 0$,

(3) $t(A_1) = 0$.

Then there exists a smooth n-knot (*diffeomorphic to* S^n) *in* S^{n+2} *with* $A_q = H_q(\tilde{X})$.

REMARK. Restrictions (1) and (2) will be removed when we consider realization of the product structure below, but restriction (3) will persist as a result of the difficulties with the group of the knot that I have not been able to overcome. It is not difficult to realize various special $t(A_1)$, by constructing suitable π or by twist-spinning (see [Z])–but I will not go into this question any further in the present work.

PROOF OF THEOREM (11.1). Let A_1 be a Z-torsion free Λ-module of type K. We will construct a group π satisfying (9.1)(i)–(iii) such that $H_1(\pi') = A_1$, $H_2(\pi') = 0$. By choosing $F = 0$, we can then conclude Theorem (11.1) directly from (9.1) for $n \geqslant 3$. To cover the case $n = 2$, we will also require π to satisfy (9.1)(3). In light of Proposition (9.2), Theorem (11.1) will now follow from (9.1) for $n = 2$ also, except that the 2-knot will lie in a *homotopy* 4-sphere. To repair this fault, we will want to impose the further condition on π that the construction of (10.2) produces S^4 as the ambient sphere.

Suppose $0 \to F_1 \xrightarrow{d} F_0 \to A_1 \to 0$ is a free resolution of A_1. Since A_1 is Λ-torsion, F_1 and F_0 have the same rank and d has a square matrix representation (λ_{ij}). Since A_1 is of type K, $(\varepsilon(\lambda_{ij}))$ is a unimodular integral matrix. We may change $(\varepsilon(\lambda_{ij}))$ to the identity matrix (δ_{ij}) by elementary row and column operations. Since these operations can be lifted to (λ_{ij}), it follows that, by changing bases of F_0 and F_1, we may assume d has a matrix representative (λ_{ij}) with $\varepsilon(\lambda_{ij}) = \delta_{ij}$.

Write $\lambda_{ij} = \sum_k a_{ijk} t^k$. We now define π to be the group with the following presentation:

$$\pi = \left(\{x_i\}, \tau\colon \prod_{j,k} \left(\tau^k x_j \tau^{-k}\right)^{a_{ijk}} = 1, \text{ for all } i \right).$$

The order of multiplication is defined by *lexicographically* ordering the index pairs (j, k), i.e. $(j, k) \leqslant (j', k')$ iff $j < j'$ or $j = j'$ and $k < k'$.

$H_1(\pi)$ has a presentation: $(\{X_i\},\ t\colon \sum_{j,k} a_{ijk} X_j = 0)$ written in additive notation. Since $\sum_k a_{ijk} = \varepsilon(\lambda_{ij}) = \delta_{ij}$, this becomes $(\{X_i\},\ t\colon X_i = 0)$, which is the infinite cyclic group generated by t, the image of τ. Furthermore we see that π' is the normal closure of $\{x_i\}$ in π. In fact $\{\tau^k x_j \tau^{-k}\}$ form a set of generators of π'. It follows that a presentation of π' is

$$\left\{ \{x_{jk}\}\colon \prod_{j,k} x_{j,k+l}^{a_{ijk}} = 1, \text{ for all } i, l \right\}$$

by setting $x_{jk} = \tau^k x_j \tau^{-k}$ and taking all conjugates by powers of τ of the relations in π. Therefore, a presentation of $H_1(\pi')$ is given by

$$\left(\{X_{jk}\}\colon \sum_{j,k} a_{ijk} X_{j,k+l} = 0, \text{ all } i, l \right).$$

If we write $X_{jk} = t^k X_j$, using the Λ-module structure of $H_1(\pi')$, we get the presentation, as a Λ-module:

$$\left(X_j \colon t^l \left(\sum_j \lambda_{ij} X_j \right) = 0, \text{ all } i, l \right)$$

which is also a presentation of A_1.

To check (9.1)(iii) we add the relation $\tau = 1$, to get

$$\left(\{x_i\} \colon \prod_{j,n} x_j^{a_{ijk}} = 1, \text{ all } i \right) \quad \text{or}$$

$$\left(\{x_i\} \colon \prod_j x_j^{\Sigma_k a_{ijk}} = 1, \text{ all } i \right),$$

using the fact that the order of multiplication is defined lexicographically on (j, k). Since $\Sigma_k a_{ijk} = \varepsilon(\lambda_{ij}) = \delta_{ij}$, we get the trivial group.

Obviously π satisfies (9.1)(3). It now follows from [Ke] or (10.2) that there is a 2-knot in some homotopy 4-sphere with group π. We would like to demonstrate, however, that such a knot exists in S^4. For this we use the construction of Kervaire in [Ke]. Given a group π satisfying (9.1)(i), (iii), (3) $\pi = \{x_1, \ldots, x_{n+1} \colon r_1 = 1, \ldots, r_n = 1\}$, where r_i are words in $\{x_i\}$, we consider

$$V_0 = \underbrace{S^1 \times S^3 \# S^1 \times S^3 \# \cdots \# S^1 \times S^3}_{(n+1) \text{ times}}.$$

$\pi_1(V_0)$ is the free group on $\{x_1, \ldots, x_{n+1}\}$, where x_i is represented by a loop around the ith $S^1 \times S^3$. If V is constructed from V_0 by surgery on imbeddings $f_i \colon S^1 \times D^3 \subset V_0$ representing $\{r_i\}$, then $\pi_1(V) \approx \pi$. Let $\tau \in \pi$ be an element such that the relation $\tau = 1$ kills π. Surgery on V using any imbedding $f \colon S^1 \times D^3 \subset V$ representing τ will produce a homotopy 4-sphere Σ (see the argument in [Ke]) and the desired knot in Σ is just $f(* \times S^2)$.

Suppose $x_{n+1} = \tau$. Then we may take f to be a standard imbedding $S^1 \times D^3 \subset S^1 \times S^3$ into the $(n+1)$st $S^1 \times S^3$ in V_0 (missing the disk removed by connected sum). If we do this surgery on τ *before* the surgeries on $\{r_i\}$, we obtain

$$V_1 = \underbrace{S^1 \times S^3 \# \cdots \# S^1 \times S^3 \# S^4}_{n\text{-times}} \approx \underbrace{S^1 \times S^3 \# \cdots \# S^1 \times S^3}_{n\text{-times}}.$$

Suppose $r_i(x_1, \ldots, x_n, 1) = x_i$. Then $f_i \colon S^1 \times D^3 \subset V_1$ is homotopic to a standard imbedding $S^1 \times D^3 \subset S^1 \times S^3$ into the ith $S^1 \times S^3$.

Therefore, by general position f_i is *isotopic* to this standard imbedding or one obtained from it by twisting the normal framing. But, if the latter is the case, we may change f_i by untwisting the normal framing. It follows now that surgery on f_I essentially changes the ith $S^1 \times S^3$ to S^4 and the resulting $\Sigma = S^4 \# \cdots \# S^4 = S^4$.

Since our group π satisfies these two conditions, the proof is complete.

12. We now turn to the middle-dimensional Alexander modules. The problem here is to realize a given Z-torsion free or Z-torsion Λ-module with the appropriate product structure as the middle Alexander module of an odd- or even-dimensional knot, respectively. We first discuss the Z-torsion free case.

THEOREM (12.1). *Let F be a Z-torsion free Λ-module of type K equipped with a pairing $\langle\ ,\ \rangle$: $F \times F \to Q(\Lambda)/\Lambda$ which is conjugate linear, nonsingular and $(-1)^{q+1}$-Hermitian (see* §4, 5), *for some positive integer q.*

If $n = 2q - 1 > 3$, then there is an n-knot with qth Alexander module isomorphic to F, Blanchfield pairing $\langle\ ,\ \rangle$ and all other Alexander modules 0.

We will discuss the cases $n = 1, 3$ below.

To prove Theorem (12.1), we will, as usual, construct the complement of the desired knot. This construction is the content of

LEMMA (12.2). *Let $\lambda = (\lambda_{ij})$ and $\mu = (\mu_{ij})$ be $(m \times m)$-matrices with entries in Λ satisfying:*

(a) $\det \lambda \neq 0$,

(b) $\lambda \bar{\mu}^T = (-1)^{q+1} \mu \bar{\lambda}^T$,

(c) *the diagonal entries of $\lambda\bar{\mu}^T$ are zero.*

Then, if $q > 2$, there exists a compact $(2q + 1)$-dimensional smooth manifold M with $\pi_1(M) \approx Z$, $H_i(\tilde{M}) = 0$ for $i \neq 0, q$, where $\tilde{M}$ is the universal covering of M, and $H_q(\tilde{M})$ has λ as a relation matrix with respect to a set of generators $\{e_i\}$ while the linking pairing:

$$\langle\ ,\ \rangle: H_q(\tilde{M}) \times H_q(\tilde{M}) \to Q(\Lambda)/\Lambda$$

defined above in §5, *has $\lambda^{-1}\mu$ as matrix representative with respect to $\{e_i\}$.*

More precisely, the relations of $\{\Sigma_j \lambda_{ij} e_j = 0,\ i = 1, \ldots, m\}$ generate all the relations in $H_q(\tilde{M})$–it follows that $H_q(\tilde{M})$ is a Λ-torsion module, since $\Delta =$ $\det \lambda$ *annihilates every element–and, if $\lambda^{-1}\mu = \gamma = (\gamma_{ij})$–so $\gamma_{ij} \in Q(\Lambda)$–then $\langle e_i, e_j \rangle = \gamma_{ij}$* mod Λ.

PROOF. Define

$$M_0 = \underbrace{S^1 \times D^n \# S^q \times D^{q+1} \# \cdots \# S^q}_{m \text{ copies}}.$$

Then $\pi_1(\partial M_0) \approx \pi_1(M_0) \approx Z$, $\pi_i(\partial M_0) = 0 = \pi_i(M_0)$ for $1 < i < q$, $\pi_q(M_0)$ is a free Λ-module of rank m and $\pi_q(\partial M_0)$ is a free Λ-module of rank $2m$. A basis $\{x_1, \ldots, x_m; y_1, \ldots, y_m\}$ for $\pi_q(\partial M_0)$ is defined by letting x_i, y_i be the homotopy classes of $S^q \times *$, $* \times S^q$ in the ith $S^q \times D^{q+1}$, where $* \in S^q$. The inclusion $\pi_q(\partial M_0) \to \pi_q(M_0)$ maps y_i to zero and $\{x_1, \ldots, x_m\}$ to a basis $\{e_i\}$ of $\pi_q(M_0)$.

Define $\alpha_i \in \pi_q(\partial M_0)$ by $\alpha_i = \Sigma_j \lambda_{ij} x_j + \Sigma_j \mu_{ij} y_j$. Consider the intersection pairing $H_q(\partial M_0) \times H_q(\partial M_0) \to \Lambda$ (see §2). Clearly $x_i \cdot y_j = (-1)^q y_j \cdot x_i = \delta_{ij}$, and $x_i \cdot x_j = y_i \cdot y_j = 0$ if $S^q \times *$, $* \times S^q$ are oriented appropriately. Therefore

$$\alpha_i \cdot \alpha_j = \sum_k \left(\lambda_{ik}\bar{\mu}_{jk} + (-1)^q \mu_{ik}\bar{\lambda}_{jk}\right) = 0,$$

by (b). Set $\alpha_i' = \Sigma_j \lambda_{ij} x_j$ and $\alpha_i'' = \Sigma_j \mu_{ij} y_j$. Then

$$\alpha_i' \cdot \alpha_i'' = \sum_j \lambda_{ij}\bar{\mu}_{ij} = 0$$

by (c).

Now α_i' and α_i'' may be represented by imbedded spheres. For example α_i' is represented by taking connected sums of disjoint sections $\{S^q \times u_r \colon u_r \in S^q\}$ using tubes which wrap around $S^1 \times S^{n-1}$. Since $\alpha_j' \cdot \alpha_i'' = 0$, and $q > 2$, we may use the Whitney process to assume α_i' and α_i'' are disjoint (see **[Ke1]**). By taking an appropriate connected sum, we may, therefore, represent $\alpha_i = \alpha_i' + \alpha_i''$ by an imbedded sphere. Since $\alpha_i \cdot \alpha_j = 0$, we may use the Whitney process, again, to assume these spheres are disjoint.

Furthermore we may assume that the imbedded spheres representing $\{\alpha_i\}$ have trivial normal bundles, since this is true of the representatives of $\{x_i\}$ and a connected sum of two submanifolds with trivial normal bundle has a trivial normal bundle.

We now construct M from M_0 by adding handles along the imbedded framed spheres in ∂M_0 representing $\{\alpha_i\}$. Since the image of α_i in $\pi_q(M_0)$ is $\Sigma_j \lambda_{ij} e_j$, it is clear that $\pi_q(M) \approx H_q(\tilde{M})$ is as desired. It is also clear that $\pi_1(M)$ and $H_i(\tilde{M})$, $i \neq q$, are as desired. It remains to check the linking pairing $\langle\ ,\ \rangle$.

Let $\Delta = \det \lambda$; then $\Delta I = \lambda\hat{\lambda}$, where I is the identity matrix and $\hat{\lambda} = (\hat{\lambda}_{ij})$ is the matrix of cofactors of λ. Let z_i be the cycle in int M_0 carried by $S^q \times 0$ in the ith copy of $S^q \times D^{q+1}$, which represents $e_i \in \pi_q(M_0)$. Let z_i' be the cycle in ∂M_0 representing x_i carried by $S^q \times *$ in the ith copy of $S^q \times D^{q+1}$, $* \in \partial D^{q+1}$. We want to construct a chain F_i in M such that $\partial F_i = \Delta z_i'$ and the matrix of intersection numbers $(F_i \cdot z_j) = \hat{\lambda}\mu$. From this we can then conclude:

$$\langle e_i, e_j\rangle \equiv \frac{1}{\Delta}(F_i \cdot z_j) = \frac{1}{\Delta}\hat{\lambda}\mu = \lambda^{-1}\mu$$

which is the desired result.

Let w_i be the cycle in ∂M_0 representing y_i, carried by $* \times \partial D^{q+1}$ in the ith copy of $S^q \times D^{q+1}$, and D_i the chain in $\overline{M - M_0}$ carried by the core disk in the handle attached along α_i. Clearly $\sum_j(\lambda_{ij}z'_j + \mu_{ij}w_j)$ is homologous to ∂D_i in ∂M_0, by definition of α_i. Therefore

$$\sum_j (\lambda_{ij}z'_j + \mu_{ij}w_j) = \partial(D_i + E_i)$$

where E_i is a chain in ∂M_0.

If u_i is a chain in M_0 carried by $* \times D^{q+1}$, then $\partial u_i = w_i$. Therefore:

$$\Delta z'_i = \sum_{j,k} \hat{\lambda}_{ij}\lambda_{jk}z'_k = \sum_j \hat{\lambda}_{ij}\partial\left(D_j + E_j - \sum_k \mu_{jk}u_k\right)$$

and we may set

$$F_i = \sum_j \hat{\lambda}_{ij}\left(D_j + E_j - \sum_k \mu_{jk}u_k\right).$$

Now $D_i \cdot z_j = E_i \cdot z_j = 0$, since z_j is carried by int M_0 while D_i, E_i are carried by $\overline{M - M_0}$. Furthermore $u_i \cdot z_j$ is the same as $x_i \cdot y_j$, up to some fixed sign, which is δ_{ij}. Therefore $F_i \cdot z_j = \pm \sum_j \hat{\lambda}_{ij}\mu_{jk}$.

Finally the sign may be changed by reorienting M.

REMARK. (1) The same argument proves the analogous theorem for simply-connected manifolds, replacing Λ by Z and $\tilde{M}$ by M.

(2) A useful reformulation of the hypotheses would replace μ by a prescribed matrix γ with entries in $Q(\Lambda)$ satisfying: $\gamma = (-1)^{q+1}\bar{\gamma}^T$, $\lambda\gamma$ integral (i.e. has entries in Λ), and $\lambda\gamma\bar{\lambda}^T$ has *zero* diagonal. The conclusion would be that $\langle\ ,\ \rangle$ has γ as a matrix representative with respect to $\{e_i\}$.

Suppose we are given a Λ-torsion module A of homological dimension *one* together with a pairing: $\langle\ ,\ \rangle: A \times A \to Q(\Lambda)/\Lambda$ which is conjugate linear and ε-Hermitian ($\varepsilon = \pm 1$).

To apply Theorem (12.2), we need to find for some generator set $\{e_i\}$, a relation matrix λ for A and a representative matrix γ for $\langle\ ,\ \rangle$ satisfying the conditions in Remark (2) (replacing $(-1)^{q+1}$ by ε).

PROPOSITION (12.3). *Given $(A, \langle\ ,\ \rangle)$ as above, there exists a well-defined element of*

$$\begin{Bmatrix} \operatorname{Hom}_\Lambda(A, Z/2) & \text{if } \varepsilon = -1 \\ \operatorname{Ext}^1_\Lambda(A, Z/2) & \text{if } \varepsilon = +1 \end{Bmatrix},$$

where $Z/2$ has the trivial Λ-structure, which is zero if and only if the desired matrices λ and γ exist.

PROOF. Let $0 \to F_1 \xrightarrow{d} F_0 \to A \to 0$ be a free resolution with bases $\{e_i\}$ of F_0, $\{e'_i\}$ of F_1, such that λ is the matrix representative of d, i.e. $d(e'_i) = \sum_j \lambda_{ij}e_j$.

Let $(\, , \,): F_0 \times F_0 \to Q(\Lambda)$ be a conjugate linear lift of $\langle \, , \, \rangle$–this can be done by lifting $\langle e_i, e_j \rangle \in Q(\Lambda)/\Lambda$ to arbitrary $(e_i, e_j) \in Q(\Lambda)$ and then extending the definition of $(\, , \,)$ bilinearly. We may also choose $(e_i, e_j) = \varepsilon\overline{(e_j, e_i)}$ for $i \neq j$. If we could also do this for $i = j$, $(\, , \,)$ would be ε-Hermitian. Now $\langle e_i, e_i \rangle = \varepsilon\overline{\langle e_i, e_i \rangle}$, which means any lift λ of $\langle e_i, e_i \rangle$ to $Q(\Lambda)$ satisfies $\rho = \lambda - \varepsilon\bar{\lambda} \in \Lambda$. If $\varepsilon = +1$, we may write $\rho = \sigma - \bar{\sigma}$, for some $\sigma \in \Lambda$ and then set $(e_i, e_i) = \lambda - \sigma$. If $\varepsilon = -1$, it is necessary to be able to decompose $\rho = \sigma + \bar{\sigma}$, for some $\sigma \in \Lambda$. This is possible if and only if the scalar term of ρ is *even*, a condition which is *independent of* λ: if λ_0 is also a lift of $\langle e_i, e_i \rangle$ and $\rho_0 = \lambda_0 + \bar{\lambda}_0 \in \Lambda$, then $\lambda - \lambda_0 \in \Lambda$ and $\rho - \rho_0 = (\lambda - \lambda_0) + (\bar{\lambda} - \bar{\lambda}_0)$, which has an even scalar term.

Define a homomorphism $\phi: A \to Z/2$ by $\phi(\alpha) =$ mod 2 reduction of the scalar term of $\lambda + \bar{\lambda}$, where λ is any lift to $Q(\Lambda)$ of $\langle \alpha, \alpha \rangle$. By the arguments above, ϕ is well defined and $\phi = 0$ is a sufficient condition to lift $\langle \, , \, \rangle$ to the desired $(\, , \,)$. Conversely it is easy to see directly that the existence of such $(\, , \,)$ implies $\phi = 0$.

If γ is defined to be the matrix representative of $(\, , \,)$ with respect to $\{e_i\}$, the only condition possibly not satisfied is that $\lambda\gamma\bar{\lambda}^T$ have zero diagonal. As a first step, let us ask that the diagonal entries have *even* scalar term. If $\varepsilon = -1$, such a diagonal entry δ satisfies $\bar{\delta} = -\delta$; which certainly assumes this property. If $\varepsilon = +1$, we measure the obstruction by a homomorphism ψ: $F_1 \to Z/2$ defined by $\psi(\alpha) =$ mod 2 reduction of the scalar term of $(d(\alpha), d(\alpha))$. Now two different lifts $(\, , \,)$ of $\langle \, , \, \rangle$ differ by a pairing $F_0 \times F_0 \to \Lambda$. The associated ψ will then differ by the restriction of a homomorphism $F_0 \to Z/2$ to F_1 by d. Conversely, any such change in ψ can be "lifted" to a change in $(\, , \,)$. If we consider the exact sequence:

$$\mathrm{Hom}_\Lambda(F_0, Z/2) \xrightarrow{d^*} \mathrm{Hom}_\Lambda(F_1, Z/2) \to \mathrm{Ext}^1_\Lambda(A, Z/2) \to 0$$

we see that the image of ψ in $\mathrm{Ext}^1_\Lambda(A, Z/2)$ is a well-defined obstruction to choosing $(\, , \,)$ with the desired behavior.

We leave it to the reader to check that these obstructions are independent of the particular resolution of A.

The proposition will now follow from

LEMMA (12.4). *Let ν be a square matrix over Λ satisfying $\nu^T = \varepsilon\bar{\nu}$, for some $\varepsilon = \pm 1$, whose diagonal entries have even scalar terms. Then, for some m, the block sum*:

$$\nu_0 = \nu \oplus \underbrace{\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}}_{m \text{ copies}}$$

is congruent, over Λ, *to a matrix with zero diagonal–i.e. there exists a unimodular matrix* ρ, *over* Λ, *such that* $\rho\nu_0\bar{\rho}^T$ *has zero diagonal.*

Assuming this lemma, set $\nu = \lambda\gamma\bar{\lambda}^T$. Using the ρ which satisfies the lemma, we may set

$$\gamma_0 = \gamma \oplus \underbrace{\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}}_{m}$$

and $\lambda_0 = \lambda \oplus I_{2m}$, where I_{2m} is the $(2m \times 2m)$-identity matrix. Now $\rho\lambda_0$, γ_0 will satisfy the conclusions of Proposition (12.3).

PROOF OF LEMMA. Suppose ν is a $(k \times k)$-matrix. Let δ be a $(k \times k)$-matrix such that $\varepsilon\delta + \bar{\delta}$ has the same diagonal entries as ν–e.g. we may choose δ to be a diagonal matrix. Then the block sum

$$\gamma_0 = \nu \oplus \begin{pmatrix} 0 & I_k \\ \varepsilon I_k & 0 \end{pmatrix},$$

where I_k is the $(k \times k)$-identity matrix, is congruent to a matrix with zero diagonal. In fact, if we set:

$$\rho = \begin{bmatrix} I_k & I_k & -\delta \\ 0 & I_k & 0 \\ 0 & 0 & I_k \end{bmatrix}$$

then

$$\rho\nu_0\bar{\rho}^T = \begin{bmatrix} \nu - \varepsilon\delta - \bar{\delta} & -\varepsilon\delta & I_k \\ -\bar{\delta} & 0 & I_k \\ \varepsilon I_k & \varepsilon I_k & 0 \end{bmatrix}.$$

If A is a Λ-module of type K, then $\mathrm{Hom}_\Lambda(A, Z/2) = 0 = \mathrm{Ext}^1_\Lambda(A, Z/2)$. In fact $t - 1$ defines an automorphism of A, but zero on $Z/2$. Combining this observation with Lemma (12.4) and Proposition (12.3), we have

PROPOSITION (12.5). *Let* A *be a* Z*-torsion free* Λ*-module of type* K, *and* $\langle\, ,\rangle$: $A \times A \to Q(\Lambda)/\Lambda$ *conjugate linear and* ε*-Hermitian. Then for any* $q > 2$ *such that* $\varepsilon = (-1)^{q+1}$, *there exists a compact smooth* $(2q+1)$*-dimensional manifold* M *such that* $\pi_1(M) \approx Z$, $H_i(\tilde{M}) = 0$ *for* $i \neq 0$, q, *and* $A \approx H_q(\tilde{M})$ *so that* $\langle\, ,\rangle$ *corresponds to the linking pairing*: $H_q(\tilde{M}) \times H_q(\tilde{M}) \to Q(\Lambda)/\Lambda$.

LEMMA (12.6). *Suppose* M *is as in Proposition* (12.5) *and* $\langle\, ,\rangle$ *is nonsingular. Then*

(a) ∂M *is diffeomorphic to* $\Sigma \times S^1$, *for some topological* $(2q-1)$*-sphere* Σ, *and*

(b) $\hat{\Sigma} = M \cup \Sigma \times D^2$, *where* ∂M *is identified with* $\partial(\Sigma \times D^2)$ *by a suitable diffeomorphism, is diffeomorphic to* S^{2q+1}.

It is clear that the $(2q-1)$-knot $\Sigma \times 0 \subset \hat{\Sigma}$ has $(A, \langle\ ,\ \rangle)$ as its qth Alexander module and Blanchfield pairing, while the other Alexander modules are zero and the group of the knot is Z. This will prove Theorem (12.1).

PROOF OF (12.6). Note that M has a $(q+1)$-dimensional subcomplex K, defined by the cores of the handles, such that ∂M is a deformation retract of $M - K$. Since K has codimension $q > 2$, it follows, by general position, that $\pi_1(\partial M) \approx \pi_1(M - K) \approx \pi_1(M)$–these isomorphisms all induced by inclusion maps.

Now we will show that $\pi_i(\partial M) = 0$ for $1 < i < 2q - 1$. This is equivalent, by the Hurewicz theorem, to $H_i(\partial \tilde{M}) = 0$, $0 < i < 2q - 1$. Consider the exact sequence

$$\ldots \to H_{i+1}(\tilde{M}) \to H_{i+1}(\tilde{M}, \partial \tilde{M}) \to H_i(\partial \tilde{M}) \\ \to H_i(\tilde{M}) \to H_i(\tilde{M}, \partial \tilde{M}) \to \ldots .$$

Now $H_i(\tilde{M}, \partial \tilde{M}) \approx \overline{H_e^{2q+1-i}}(\tilde{M})$ by duality. By the universal coefficient spectral sequence–see Theorem (2.3)–we have a short exact sequence:

$$0 \to \operatorname{Ext}^2_\Lambda\big(H_{j-2}(\tilde{M}), \Lambda\big) \to H_e^j(\tilde{M}) \to \operatorname{Ext}^1_\Lambda\big(H_{j-1}(\tilde{M}), \Lambda\big) \to 0$$

and so $H_i(\tilde{M}, \partial \tilde{M}) = 0$ if $i \neq q-1, q, 2q$. But when $i = q - 1$, we have $H_{q-1}(\tilde{M}, \partial \tilde{M}) = \operatorname{Ext}^2_\Lambda(H_q(\tilde{M}), \Lambda) = 0$, since $H_q(\tilde{M}) \approx A$ is Z-torsion free. From the exact homology sequence we now conclude that $H_i(\partial \tilde{M}) = 0$ for $i \neq 0, q-1, q, 2q-1$ while, for these dimensions, $H_0(\partial \tilde{M}) \cong H_{2q-1}(\partial \tilde{M}) \approx Z$ and there is an exact sequence

$$0 \to H_q(\partial \tilde{M}) \to H_q(\tilde{M}) \xrightarrow{j^*} H_q(\tilde{M}, \partial \tilde{M}) \to H_{q-1}(\partial \tilde{M}) \to 0.$$

Furthermore $H_q(\tilde{M}, \partial \tilde{M}) \approx \overline{H_e^{q+1}}(\tilde{M}) \approx \overline{H_e^q}(\tilde{M}; Q(\Lambda)/\Lambda) \approx \operatorname{Hom}_\Lambda(\overline{H_q(\tilde{M})}, Q(\Lambda)/\Lambda)$, by the arguments in §5, since $H_q(\tilde{M})$ is Z-torsion free of type K, and $H_{q+1}(\tilde{M}) = 0$, and so the homomorphism $H_q(\tilde{M}) \to \operatorname{Hom}_\Lambda(\overline{H_q(\tilde{M})}, Q(\Lambda)/\Lambda)$ defined by j^* and this sequence of isomorphisms coincides with the adjoint of the linking pairing. Thus j^* is an isomorphism, when the linking pairing is nonsingular, and $H_{q-1}(\partial \tilde{M}) = 0 = H_q(\partial \tilde{M})$.

We now know that $\pi_1(\partial M) \approx Z$ and $\pi_i(\partial M) = 0$ for $1 < i < 2q - 1$. Let $f\colon S^1 \times D^{2q-1} \to \partial M$ be an imbedding representing a generator of $\pi_1(\partial M)$ and set $V = M \cup_f D^2 \times D^{2q-1}$, i.e. add a handle of index 2 to M via f. Since $\pi_1(M) \approx \pi_1(\partial M)$ and M is a homology circle, V is contractible and ∂V is simply-connected. Therefore by [Sm], ∂V is diffeomorphic to S^{2q}. Now

$$X = \partial V - D^2 \times S^{2q-2} = \partial M - f(S^1 \times D^{2q-1})$$

is the complement of the framed $(2q-2)$-knot $D^2 \times S^{2q-2} \subset \partial V$. By general position $\pi_i(X) \approx \pi_i(\partial M) \approx \pi_i(S^1)$ for $i < 2q-2$, which implies, by the unknotting theorem of [L2], that $D^2 \times S^{2q-2} \subset \partial V \approx S^{2q}$ is isotopic to a composition:

$$D^2 \times S^{2q-2} \xrightarrow{1 \times h} D^2 \times S^{2q-2} \xrightarrow{i} S^{2q}$$

where h is some diffeomorphism of S^{2q-2} and i is the standard imbedding defining the trivial framed knot. It follows readily that ∂M, which is obtained from ∂V by surgery along $i \circ (1 \times h)$, is diffeomorphic to $S^1 \times \Sigma_h$, where Σ_h is the topological sphere defined by identifying two copies of D^{2q-1} along their boundaries by h. This proves (a).

To prove (b) we proceed by standard arguments, using the van Kampen theorem and the Mayer-Vietoris sequence. We only need observe that M is a homology circle and $\pi_1(\partial M) \approx \pi_1(M)$.

The proof of Theorem (12.1) is now complete.

13. We now turn to the realization of a Z-torsion module as the middle-dimensional Alexander module of an even-dimensional knot.

THEOREM (13.1). *Let A be a finite Λ-module of type K and $[\ ,\ ]: A \times A \to Q/Z$ a Z-linear $(-1)^{q+1}$-symmetric, nonsingular, conjugate selfadjoint pairing. If $q > 1$, there exists a $2q$-knot with qth Alexander module and torsion pairing isomorphic to $(A, [\ ,\ ])$ and all other Alexander modules* 0.

We will construct a *fibered* knot as follows. If M is a closed $(q-1)$-connected $(2q+1)$-manifold with $H_q(M)$ finite, then $H_i(M) = 0$ for $i \neq 0, q, 2q+1$. Suppose h is an orientation-preserving diffeomorphism of M such that $h_* - 1$ is an automorphism of $H_q(M)$–we may suppose that h leaves some $(2q+1)$-disk $D \subset M$ fixed. Consider the "mapping torus" V of h, i.e. $M \times I$ with the ends attached according to the rule $(x, 0) = (h(x), 1)$. It is easily checked, if $q > 1$, that V is homology equivalent to $S^1 \times S^{2q+1}$–if $q = 1$, we need a more delicate condition on h (see [Ma]). Furthermore, if $\tilde{V}$ is the infinite cyclic covering of V defined by the obvious map $V \to S^1$, then $H_*(\tilde{V}) \approx H_*(M)$, and the Λ-module structure on $H_*(\tilde{V})$ is defined by identifying the action of $t \in \Lambda$ with h_*. Now $D \times I \subset M \times I$ determines an imbedding $D \times S^1 \subset V$. If we remove $D \times S^1$ and replace it with $\partial D \times D^2$, the resulting manifold Σ is a homotopy sphere. Furthermore $\partial D \times 0 \subset \Sigma$ is a $2q$-knot in Σ whose complement is $V - D \times S^1$. It is easy to see that the Alexander modules coincide with $H_*(\tilde{V})$. In case Σ is an exotic sphere we may form the connected sum, away from $\partial D \times 0$, with $-\Sigma$ to get a $2q$-knot in S^{2q+2}.

Since V fibers over S^1 with fiber M, the complement of $\partial D \times 0 \subset \Sigma$ fibers over S^1 with fiber M-point. By Proposition (7.1), the pairing $[\ ,\ ]$ coincides with the linking pairing of M.

To prove Theorem (13.1) by such a construction, we will therefore need only to prove

THEOREM (13.2). *Let A be a finite abelian group, ϕ an automorphism of A and* $[\ ,\ ]$: $A \times A \to Q/Z$ *a bilinear,* $(-1)^{q+1}$*-symmetric nonsingular pairing for which ϕ is an isometry.*

If $q > 1$ and $\phi - 1$ is an automorphism, then there exists a closed $(q-1)$*-connected* $(2q+1)$*-manifold M with $H_q(M) \approx A$ and linking pairing corresponding to* $[\ ,\ ]$, *and a diffeomorphism (orientation-preserving) h of M such that* $h_* = \phi$.

We will prove this by constructing a suitable q-connected $(2q+2)$-manifold W and diffeomorphism g of W–then setting $M = \partial W$, $h = g|M$. The homological relation between M and W is as follows. Suppose $\langle\ ,\ \rangle$ is the (nondegenerate) intersection pairing of W defined on $H = H_{q+1}(W)$. Then $\langle\ ,\ \rangle$ extends to a pairing on $H \otimes Q$ with values in Q. Define H^1 to be the *dual* of H in $H \otimes Q$, i.e. $H^1 = \{\alpha\colon \langle \alpha, H\rangle \subset Z\}$. Then $H^1 \supset H$ and $\langle\ ,\ \rangle$ induces a nonsingular pairing $[\ ,\ ]$ of $H^1/H = A$ to Q/Z. One can prove that $H_q(M) \approx A$ and the linking pairing on $H_q(M)$ corresponds to $[\ ,\ ]$ (see e.g. [W1]).

There is a further structure on M and W which we must consider. If q is even $\neq 2, 6$ and W is a parallelizable manifold, for simplicity, there is a quadratic function $\mu\colon H \to Z_2$, i.e. it satisfies the condition

$$\mu(\alpha+\beta) - \mu(\alpha) - \mu(\beta) \equiv \langle \alpha, \beta\rangle \mod 2. \tag{13.3}$$

The following is a special case of the results of Wall [W1].

THEOREM (13.4). *Suppose $\langle\ ,\ \rangle$ is a nondegenerate* $(-1)^{q+1}$*-symmetric bilinear form on the free abelian group H and $\mu\colon H \to Z_2$ satisfies* (13.3) *if q is even* $\neq 2, 6$. *Suppose also that $\langle\ ,\ \rangle$ is even if q is odd, i.e. $\langle \alpha, \alpha\rangle$ is even, for all* $\alpha \in H$.

If $q > 1$, there is a parallelizable q-connected manifold W, with $H_i(W) = 0$ for $i > q+1$, such that $H \approx H_{q+1}(W)$ with $\langle\ ,\ \rangle$ corresponding to the intersection pairing of W and μ corresponding to the quadratic function of W.

Such W is unique and, in fact, given any automorphism ψ of H preserving $\langle\ ,\ \rangle$ and μ–i.e. $\langle \psi(\alpha), \psi(\beta)\rangle = \langle \alpha, \beta\rangle$, $\mu \circ \psi = \mu$–there is a diffeomorphism g of W such that $g_* = \psi$.

We also recall that, if M is a $(q-1)$-connected $(2q+1)$-manifold with $A = H_q(M)$, linking pairing $[\ ,\ ]\colon A \times A \to Q/Z$, and q is odd $\neq 3, 7$, there is an associated quadratic function $b\colon A \to Q/Z$, i.e. $b(\alpha+\beta) - b(\alpha) - b(\beta) = [\alpha, \beta]$ and $2b(\alpha) = [\alpha, \alpha]$ for any $\alpha, \beta \in A$ (see [W2]).

We will now proceed as follows. Let A, $[\ ,\ ]$, ϕ be as in Theorem (13.2). We will construct H, $\langle\ ,\ \rangle$, μ, ψ so that $\langle\ ,\ \rangle$ is a $(-1)^{q+1}$-symmetric nondegenerate bilinear form on the free abelian group H, even if q is odd, $\mu: H \to Z_2$ is defined and quadratic with respect to $\langle\ ,\ \rangle$ if q is even, and ψ is an automorphism of H preserving $\langle\ ,\ \rangle$ and μ. The relation to A, $[\ ,\ ]$, ϕ will be: $H^1/H \approx A$, where H^1 is the dual of H with respect to $\langle\ ,\ \rangle$, $[\ ,\ ]$ coincides with the pairing induced by $\langle\ ,\ \rangle$, and ϕ is the isometry of $[\ ,\ ]$ induced by ψ. Assuming this, we can then construct W, g as in Theorem (13.4); setting $M = \partial W$, $h = g|\partial W$ will prove Theorem (13.2).

The existence of the desired H and $\langle\ ,\ \rangle$ follows from Wall [W3] if A, $[\ ,\ ]$ satisfy the following extra conditions:

(13.5) (i) if q is odd, $[\ ,\ ]$ must admit an associated quadratic $b: A \to Q/Z$ (then $b(\alpha) = \frac{1}{2}\langle\alpha, \alpha\rangle$, for $\alpha \in H'$),

(ii) if q is even, then $[\alpha, \alpha] = 0$ for all $\alpha \in A$ (note that $[\alpha, \alpha] = 0$ or $\frac{1}{2}$, by skew-symmetry).

We will show that (13.5) is a consequence of $\phi - 1$ being an automorphism. In fact, for (i), we can define

$$b(\alpha) = \left[(1-\phi)^{-1}(\alpha), (\alpha)\right].$$

If $\alpha = (1-\phi)(\alpha')$, $\beta = (1-\phi)(\beta')$ are any elements of A, we have

$$\begin{aligned}
[\alpha, \alpha] &= [(1-\phi)(\alpha'), (1-\phi)(\alpha')] = [\alpha', \alpha'] + [\phi\alpha', \phi\alpha'] - 2[\alpha', \phi\alpha'] \\
&= 2([\alpha', \alpha'] - [\alpha', \phi\alpha']), \quad \text{since } \phi \text{ is an isometry} \\
&= 2[(1-\phi)\alpha', \alpha'] = 2b(\alpha),
\end{aligned}$$

$$\begin{aligned}
b(\alpha+\beta) &= [\alpha' + \beta', (1-\phi)(\alpha' + \beta')] \\
&= [\alpha', (1-\phi)\alpha'] + [\beta', (1-\phi)\beta'] \\
&\quad + [\beta', (1-\phi)\alpha'] + [(1-\phi)\beta', \alpha'] \\
&= b(\alpha) + b(\beta) + 2[\beta', \alpha'] - ([\beta', \phi\alpha'] + [\phi\beta', \alpha']).
\end{aligned}$$

But $[\alpha, \beta] = [(1-\phi)(\alpha'), (1-\phi)(\beta')] = 2[\alpha', \beta'] - ([\phi\alpha', \beta'] + [\alpha', \phi\beta'])$ and we see that b is quadratic with respect to $[\ ,\ ]$.

Furthermore, ϕ preserves b, since

$$b(\phi\alpha) = \left[(1-\phi)^{-1}\phi\alpha, \phi\alpha\right] = \left[\phi(1-\phi)^{-1}\alpha, \phi\alpha\right] = \left[(1-\phi)^{-1}\alpha, \alpha\right] = b(\alpha).$$

Incidentally, this particular b is the *only* quadratic function associated to $[\ ,\ ]$ which preserves ϕ, since any two such b will differ by a *homomorphism* σ: $A \to Q/Z$ such that $\sigma \circ \phi = \sigma$–or $\sigma \circ (\phi - 1) = 0$. But $\phi - 1$ is onto, and so $\sigma = 0$.

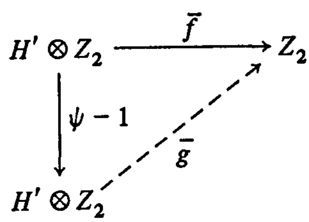
To check (ii), note that the function $\alpha \to \langle\alpha, \alpha\rangle$ defines a homomorphism $\sigma: A \to Z_2 = \{0, \frac{1}{2}\} \subset Q/Z$. $\sigma(\alpha + \beta) = [\alpha + \beta, \alpha + \beta] = [\alpha, \alpha] + [\beta, \beta] + [\alpha, \beta] + [\beta, \alpha] = \sigma(\alpha) + \sigma(\beta)$, since $[\ ,\ ]$ is skew-symmetric. Furthermore

$\sigma \circ \phi = \sigma$, since ϕ is an isometry. Therefore $\sigma \circ (\phi - 1) = 0$, which implies $\sigma = 0$ since $\phi - 1$ is onto.

We next want to lift ϕ to an isometry ψ of $\langle\, ,\rangle$ on H. In [W4], Wall shows this can be done *stably*, with the extra condition that, when q is odd, ϕ also preserves the quadratic function b: $A \to Q/Z$ defined by $b(\alpha) = \frac{1}{2}\langle\bar{\alpha}, \bar{\alpha}\rangle$, where $\bar{\alpha} \in H^1$ is any representative of $\alpha \in A \approx H^1/H$. But, in fact, the b we have defined has these properties.

Now we have an isometry ψ: $H \oplus H_1 \approx H \oplus H_2$ between the forms $\langle\, ,\rangle \oplus \langle\, ,\rangle_1$ and $\langle\, ,\rangle \oplus \langle\, ,\rangle_2$, where $\langle\, ,\rangle_i$ is a *unimodular* (even quadratic if q odd) form on H_i, $i = 1, 2$, which induces the given isometry ϕ when we identify $A \approx H^1/H$ with $(H \oplus H_i)'/H \oplus H_i = H' \oplus H_i/H \oplus H_i = H'/H$. It follows from the Witt theorem that $\langle\, ,\rangle_1$ and $\langle\, ,\rangle_2$ are rationally equivalent. Since they are unimodular and even (q odd), they are stably integrally equivalent (see e.g. [Hi]). By identifying $(H_1, \langle\, ,\rangle_1)$ with $(H_2, \langle\, ,\rangle_2)$ we may regard ψ as an automorphism of $H \oplus H_1$, preserving $\langle\, ,\rangle \oplus \langle\, ,\rangle_1$. Thus $H' = H \oplus H_1$ with $\langle\, ,\rangle' = \langle\, ,\rangle \oplus \langle\, ,\rangle_1$ and isometry ψ is the desired lift of A, $[\, ,]$, ϕ.

It remains only to construct μ: $H' \to Z_2$, quadratic with respect to $\langle\, ,\rangle'$ and preserving ψ, if q is even. First choose *any* μ': $H' \to Z_2$ quadratic with respect to $\langle\, ,\rangle'$. Set $f(\alpha) = \mu'(\psi\alpha) - \mu'(\alpha)$, for all $\alpha \in H'$, defining a homomorphism f: $H' \to Z_2$. If we can factor $f = g \circ (\psi - 1)$, for some homomorphism g: $H' \to Z_2$, then $\mu = \mu' - g$ is the desired quadratic function preserving ψ. Consider the diagram:



where $\bar{f}$ is induced by f. If $\bar{g}$ exists, it will induce the desired g. Since we are dealing here with vector spaces over Z_2, it suffices to check that

$$\bar{f}(\operatorname{Ker}(\psi - 1)) = 0.$$

In other words, if $\psi(\alpha) \equiv \alpha \bmod 2$, for some $\alpha \in H'$, then $f(\alpha) = \alpha$. But $f(\alpha) = \mu'(\psi\alpha) - \mu'(\alpha) = \mu'(\alpha + 2\beta) - \mu'(\alpha) \equiv \mu'(\alpha) + \mu'(2\beta) + \langle\alpha, 2\beta\rangle - \mu'(\alpha) \equiv \mu'(2\beta) \equiv \mu'(\beta) + \mu'(\beta) + \langle\beta, \beta\rangle \equiv 0 \bmod 2$.

This completes the proof of Theorem (13.2) and so Theorem (13.1).

14. If we combine Theorems (11.1), (12.1), and (13.1), using connected sums, we obtain

THEOREM (14.1). *Let* $n \geqslant 1$; $T_2, \dots, T_{n-2}$; $F_1, \dots, F_n$ *a collection of, respectively, finite and Z-torsion free* Λ*-modules of type K satisfying*

(1) $\bar{F}_i \cong e^1(F_{n+1-i})$,

(2) $\bar{T}_i \approx e^2(T_{n-i})$.

Furthermore, if $n = 2q - 1$, *let* $\langle\ ,\ \rangle: F_q \times F_q \to \Lambda$ *be a nonsingular, conjugate-linear,* $(-1)^{q+1}$*-Hermitian pairing, and if* $n = 2q > 2$, *let* $[\ ,\ ]: T_q \times T_q \to Q/Z$ *be a nonsingular, Z-linear, conjugate selfadjoint pairing.*

Then there exists an n-knot K with Alexander modules $\{A_i\}$ *such that* $f(A_i) \approx F_i$, $t(A_i) \approx T_i$, $t(A_1) = 0$, *and* $\langle\ ,\ \rangle$ *or* $[\ ,\ ]$ *corresponds to the Blanchfield pairing or torsion pairing of K, except for the following cases*:

(i) $n = 1$, *and*

(ii) $n = 3$ *with* $F_2 \neq 0$.

In order to remove these exceptions we will outline another approach to the study of Alexander modules, via "Seifert matrices." This idea for dealing with the low-dimensional cases was first used by Kearton in [K] for the case $n = 1$.

In our presentation, we will omit many details, which can be found in previous works (e.g. [L3]). Let $K \subset S^{n+2}$ be an n-knot–then there is an oriented submanifold $V^{n+1} \subset S^{n+2}$ such that $\partial V = K$ (see e.g. [L1]). This is called a *Seifert manifold* for K. If we set $Y = S^{n+2} - V$, then there are two maps $i^+, i^-: V \to Y$ defined by translating V off itself in the positive or negative normal direction. If $X = S^{n+2} - K$, as usual, and $\tilde{X}$ the infinite cyclic cover, then there is an exact sequence derived from a Mayer-Vietoris sequence:

$$(14.2)\qquad \begin{aligned} \dots \to H_q(V) \otimes_Z \Lambda &\xrightarrow{d} H_q(Y) \otimes_Z \Lambda \\ &\xrightarrow{e} H_q(\tilde{X}) \xrightarrow{\partial} H_{q-1}(V) \otimes_Z \Lambda \to \dots \end{aligned}$$

where $d(\alpha \otimes \lambda) = i_*^+(\alpha) \otimes t\lambda - i_*^-(\alpha) \otimes \lambda$.

In fact this breaks up into short exact sequences since $\partial = 0$. To see this (also see [G]), suppose $\alpha = \sum_{i=k}^m \alpha_i \otimes t^i \in \partial H_q(\tilde{X})$, and $\alpha_k, \alpha_m \neq 0$–note that any element of $G \otimes_Z \Lambda$, where G is an abelian group, has a unique representation as a finite sum $\sum_i \alpha_i \otimes t^i$, $\alpha_i \in G$. Since $t - 1$ is an epimorphism of $H_q(\tilde{X})$, we may write $\alpha = (t-1)^N \beta$, for any positive integer N. If $\beta = \sum_{i=l}^r \beta_i \otimes t^i$ where $\beta_l, \beta_r \neq 0$, we may conclude that $\alpha_{r+N} = \beta_r \neq 0$, $\alpha_l = \pm \beta_l \neq 0$ and, therefore, $m - k = r + N - l \geqslant N$, which is impossible. Consequently, we have exact sequences

$$0 \to H_q(V) \otimes_Z \Lambda \xrightarrow{d} H_q(Y) \otimes_Z \Lambda \to H_q(\tilde{X}) \to 0.$$

Note that this argument works for any coefficient groups. Now consider the short exact sequence of coefficient groups $0 \to Z \xrightarrow{i} Q \xrightarrow{j} Q/Z \to 0$. Combining the associated exact homology sequence and the above short exact sequence, we get a commutative diagram with exact columns and rows:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H_{q+1}(V; Q/Z) \otimes_Z \Lambda & \xrightarrow{d} & H_{q+1}(Y; Q/Z) \otimes_Z \Lambda & \xrightarrow{e} & H_{q+1}(\tilde{X}; Q/Z) & \to & 0 \\
 & & \downarrow{\partial_* \otimes 1} & & \downarrow{\partial_* \otimes 1} & & \downarrow{\partial_*} & & \\
0 & \to & H_q(V) \otimes_Z \Lambda & \xrightarrow{d} & H_q^{\cdot}(Y) \otimes_Z \Lambda & \xrightarrow{e} & H_q(\tilde{X}) & \to & 0 \\
 & & \downarrow{i_* \otimes 1} & & \downarrow{i_* \otimes 1} & & \downarrow{i_*} & & \\
0 & \to & H_q(V; Q) \otimes_Z \Lambda & \xrightarrow{d} & H_q(Y; Q) \otimes_Z \Lambda & \xrightarrow{e} & H_q(\tilde{X}; Q) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

If we let T_q, F_q denote, respectively, the Z-torsion subgroup and Z-torsion free quotient of H_q, we may, in a straightforward manner, derive the following exact sequences, using the fact that $F_q \approx \text{Image } i_*$, $T_q = \text{Image } \partial_*$:

$$0 \to T_q(V) \otimes_Z \Lambda \xrightarrow{d} T_q(Y) \otimes_Z \Lambda \xrightarrow{e} T_q(\tilde{X}) \to 0,$$

$$0 \to F_q(V) \otimes_Z \Lambda \xrightarrow{d} F_q(Y) \otimes_Z \Lambda \xrightarrow{e} F_q(\tilde{X}) \to 0.$$

We will be particularly interested in the Z-torsion free case. It follows from Alexander duality that the linking pairing: l: $F_q(Y) \times F_{n+1-q}(V) \to Z$ is nonsingular. Let us consider the special case $n = 2q - 1$ and choose bases $\{\alpha_i\}$ of $B_q(V)$ and $\{\beta_i\}$ of $B_q(Y)$ which are dual with respect to l, i.e. $l(\beta_i, \alpha_j) = \delta_{ij}$. Define a bilinear pairing l': $F_q(V) \times F_q(V) \to Z$ by $l'(\alpha, \alpha') = l(i_*^+(\alpha), \alpha') = l(\alpha, i_*^-(\alpha'))$–this is called the *Seifert pairing*. It is now straightforward to check that, if A is the matrix representative of l' with respect to the basis $\{\alpha_i\}$, then the mapping d has a matrix representative $tA + (-1)^q A^T$ with respect to the bases $\{\alpha_i \otimes 1\}$, $\{\beta_i \otimes 1\}$ of the free Λ-modules $F_q(V) \otimes_Z \Lambda$, $F_q(Y) \otimes_Z \Lambda$ (see [G], [L3]).

The Blanchfield pairing on $F_q(\tilde{X}) = F_q$, can also be expressed in terms of the Seifert matrix A.

PROPOSITION (14.3). *With respect to the generators $\gamma_i = e(\beta_i \otimes 1)$ of F_q, the corresponding matrix representative of $\langle\ ,\ \rangle$ is $(1 - t)(tA + (-1)^q A^T)^{-1}$.*

PROOF. We need to recall a few more of the details of the argument which establishes (14.2). Under the projection p: $\tilde{X} \to X, p^{-1}(Y)$ is the disjoint union of $\{Y_i\}$, where $p|Y_i$ is a homeomorphism onto Y. Also $p^{-1}(V-K)$ is the disjoint union of $\{V_i\}$, where each $p\ |V_i$ is a homeomorphism onto $V-K$. If τ denotes the covering transformation inducing the action of t on $H_*(\tilde{X})$, we may assume $\tau(Y_i) = Y_{i+1}$, $\tau(V_i) = V_{i+1}$ and $\overline{Y}_i = Y_i \cup V_{i+1} \cup V_i$. The maps i_+, i_- lift to maps $V_i \to Y_i$ and $V_i \to Y_{i-1}$, respectively.

If $A = (a_{ij})$, i.e. $a_{ij} = l(i_*^+(\alpha_i), \alpha_j)$, one may check that $i_*^+(\alpha_i) = \sum_j a_{ij}\beta_j$, while $i_*^-(\alpha_i) = (-1)^{q+1}\sum_j a_{ji}\beta_j$. Let us identify Y, V with Y_0, $V_0 \subset \tilde{X}$. Then if $\bar{\alpha}_i$, $\bar{\beta}_i$ are cycles in $V = V_0$, $Y = Y_0$ representing α_i, β_i, we may conclude that $\bar{\alpha}_i$ is homologous to $\sum a_{ij}\bar{\beta}_j$ in $\overline{Y}_0$ and is homologous to $(-1)^{q+1}\sum_j a_{ji}\tau^{-1}(\bar{\beta}_j)$ in $\overline{Y}_{-1}$. Let c_i, c_i' be chains in $Y_0 = Y$ satisfying

$$\partial c_i = \bar{\alpha}_i - \sum_j a_{ij}\bar{\beta}_j,$$

$$\partial \tau^{-1} c_i' = \bar{\alpha}_i - (-1)^{q+1}\sum_j a_{ji}\tau^{-1}\left(\bar{\beta}_j\right).$$

Therefore

$$\partial\left(c_i' - \tau(c_i)\right) = (-1)^q \sum_j a_{ji}\bar{\beta}_j + \tau\left(\sum_j a_{ij}\bar{\beta}_j\right)$$
$$= \sum_j \left(\tau a_{ij} + (-1)^q a_{ji}\right)\bar{\beta}_j.$$

Define $B = (b_{ij}(t))$, the matrix of cofactors of $tA + (-1)^q A^T$. In other words $(tA + (-1)^q A^T)B = \det(tA + (-1)^q A^T)I$. Each $b_{ij}(t)$ is an element of Λ. If we set $\Delta(t) = \det(tA + (-1)^q A^T)$, then we have

$$\delta_{jk}\Delta(t) = \sum_i b_{ki}(t)\left(ta_{ij} + (-1)^q a_{ji}\right)$$

and so

$$\Delta(\tau)\bar{\beta}_k = \sum_j \delta_{jk}\Delta(\tau)\bar{\beta}_j = \sum_{j,i} b_{ki}(\tau)\left(\tau a_{ij} + (-1)^q a_{ji}\right)\bar{\beta}_j$$
$$= \sum_i b_{ki}(\tau)\partial\left(c_i' - \tau(c_i)\right) = \partial\left(\sum_i b_{ki}(\tau)(c_i' - \tau(c_i))\right).$$

We can now compute $\langle\gamma_k, \gamma_l\rangle$. By definition, we choose representative cycles, say $\bar{\beta}_k$ and $\bar{\beta}_l'$, in dual triangulations of X, and $\lambda \in \Lambda$ such that $\lambda\gamma_k = 0$. If $\lambda\bar{\beta}_k = \partial c$, then the intersection number $(c \cdot \bar{\beta}_l')/\lambda \equiv \langle\gamma_k, \gamma_l\rangle$ (mod Λ).

The above formula says we may choose $\lambda = \Delta(t)$ and then

$$c = \sum_i b_{ki}(\tau)(c_i' - \tau(c_i)).$$

Therefore

$$\langle \gamma_k, \gamma_l \rangle = \frac{\Sigma_i b_{ki}(t)\left(\langle c_i', \bar{\beta}_l' \rangle - t\langle c_i, \bar{\beta}_l' \rangle\right)}{\Delta(t)}.$$

Now $\langle c_i, \bar{\beta}_l' \rangle$ and $\langle c_i', \bar{\beta}_l' \rangle$ are the ordinary intersection numbers in $Y \subset S^{n+2}$ and therefore coincide with the linking numbers $l(\partial c_i, \bar{\beta}_l')$ and $l(\partial c_i', \bar{\beta}_l')$ in S^{n+2}. By definition of c_i and c_i', we have

$$\langle c_i, \bar{\beta}_l' \rangle = l(\bar{\alpha}_i, \bar{\beta}_l') - \sum_j a_{ij} l(\bar{\beta}_j, \bar{\beta}_l'),$$

$$\langle c_i', \bar{\beta}_l' \rangle = l(\bar{\alpha}_i, \bar{\beta}_l') - (-1)^{q+1} \sum_j a_{ji} l(\bar{\beta}_j, \bar{\beta}_l').$$

Since $\{\alpha_i\}$, $\{\beta_i\}$ are dual with respect to l, we have

$$\langle c_i, \bar{\beta}_l' \rangle = \delta_{il} - \sum_j a_{ij}\lambda_{jl}, \qquad \langle c_i', \bar{\beta}_l' \rangle = \delta_{il} - (-1)^{q+1} \sum_j a_{ji}\lambda_{jl}$$

where $\lambda_{ij} = l(\bar{\beta}_i, \bar{\beta}_j) \in Z$. So we have

$$\langle \gamma_k, \gamma_l \rangle = \frac{\Sigma_i b_{ki}(t)}{\Delta(t)} \left[(1-t)\delta_{il} - \sum_j \left(ta_{ij} + (-1)^q a_{ji}\right)\lambda_{jl} \right]$$

$$= \frac{b_{kl}(t)(1-t)}{\Delta(t)} - \sum_{i,j} \frac{b_{ki}(t)\left(ta_{ij} + (-1)^q a_{ji}\right)\lambda_{jl}}{\Delta(t)}.$$

But $\Sigma_i b_{ki}(t)(ta_{ij} + (-1)^q a_{ji}) = \delta_{kj}\Delta(t)$ and so we have

$$\langle \gamma_k, \gamma_l \rangle = (1-t)\frac{b_{kl}(t)}{\Delta(t)} - \sum_j \delta_{kj}\lambda_{jl} = (1-t)\frac{b_{kl}(t)}{\Delta(t)} \quad (\text{mod } \Lambda).$$

Since $B = \Delta(t)(tA + (-1)^q A^T)^{-1}$, this proves Proposition (14.3).

It is not hard to show that $A + (-1)^q A^T$ is a matrix representative of the intersection pairing of the $2q$-manifold V. Since $\partial V = K$, a sphere, the intersection pairing is unimodular. If $q = 2$, we have the existing condition that the signature of the intersection pairing $A + A^T$ (the *index* of V) is a multiple of 16 (see **[L3]** for all of this). Conversely, we have

THEOREM (14.4). *Let A be a square integral matrix such that $A + (-1)^q A^T$ is unimodular and, if $q = 1$,* signature$(A + A^T)$ *is a multiple of* 16. *If $n = 2q - 1 > 1$, there is an n-knot with Alexander modules $A_i = 0$ for $i \neq q$ (and, therefore, A_q Z-torsion free) and Seifert matrix A. If $n = 3$, we may have to add several blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to A.*

See **[Ke]**, **[L4]** for a proof.

15. We are now ready to deal with the exceptions in Theorem (14.1).

(i) $n = 1$. Suppose F is a Z-torsion free Λ-module of type K and $\langle\ ,\ \rangle$ a nonsingular, conjugate-linear, Hermitian pairing $F \times F \to Q(\Lambda)/\Lambda$; then there exists a $(2q-1)$-knot for any odd $q > 1$, realizing $(F, \langle\ ,\ \rangle)$ as its qth Alexander module and pairing. Therefore any Seifert matrix A, of this knot determines $(F, \langle\ ,\ \rangle)$ completely by Proposition (14.3).

But, according to Theorem (14.4), there exists a 1-knot with this Seifert matrix. It follows that this 1-knot must have 1st Alexander module $\approx F$ with Blanchfield pairing $\langle\ ,\ \rangle$.

Thus Theorem (14.1) is true for $n = 1$.

(ii) $n = 3$, $F_2 \neq 0$. This case is more complicated since we have a new obstruction given by the signature condition.

Let x: $Q(\Lambda)/\Lambda \to Q$ be the function defined by Trotter in [T1]. We recall some of its properties:

(15.1) (a) x is Q-linear,

(b) $x(\bar{\lambda}) = -x(\lambda)$,

(c) $x((t-1)\lambda) = \lambda(1)$.

It follows from (a) and (b) that, if $\langle\ ,\ \rangle$: $F \times F \to Q(\Lambda)/\Lambda$, is a conjugate-linear, ε-Hermitian pairing, then $x \circ \langle\ ,\ \rangle = \langle\ ,\ \rangle_x$ is a Q-linear, $(-\varepsilon)$-symmetric pairing $(F \otimes_Z Q) \times (F \otimes_Z Q) \to Q$.

Proposition (15.2). *If $(F_q, \langle\ ,\ \rangle)$ is the Z-torsion free part of the qth Alexander module of a $(2q-1)$-knot $K \subset S^{n+2}$, V is a Seifert manifold for K, and q is even, then* signature$(\langle\ ,\ \rangle_x) = -$index V.

Proof. Let A be the Seifert matrix derived from V. By Proposition (14.3) $(1-t)(tA + A^T)^{-1}$ is the matrix representative of $\langle\ ,\ \rangle$ with respect to a certain set of generators $\{\gamma_i\}$ of F_q, as a Λ-module. By property (15.1)(c), $-(A + A^T)^{-1}$ is the matrix representative of $\langle\ ,\ \rangle_x$ with respect to $\{\gamma_i\}$.

Claim. $\{\gamma_i\}$ generate $F_q \otimes_Z Q$ as a vector space over Q. Assuming this for the moment, we may then make a rational change of coordinates to replace the $\{\gamma_i\}$ by $\{\gamma_i'\}$ so that $\gamma_1', \ldots, \gamma_k'$ is a *basis* for $F_q \otimes_Z Q$ (as a vector space over Q) and $\gamma_i' = 0$ for $i > k$. Under this change of coordinates $\langle\ ,\ \rangle_x$ now has a representative matrix $C = (c_{ij})$ with respect to $\{\gamma_i'\}$. Clearly $c_{ij} = 0$ if i or $j > k$ and the submatrix $c' = (c_{ij} | i, j \leqslant k)$ is a matrix representative of $\langle\ ,\ \rangle_x$ with respect to a basis. Therefore

$$\text{signature}(\langle\ ,\ \rangle_x) = \text{signature } C' = \text{signature } C = -\text{signature}(A + A^T)^{-1}.$$

But signature$(A + A^T)^{-1}$ = signature$(A + A^T)$ and $A + A^T$ is a representative matrix of the intersection pairing of V.

Proof of Claim. The generators $\{\gamma_i\}$ have a relation matrix $tA + (-1)^q A^T$. In other words, there are *rational* linear combinations $\{\rho_i, \sigma_i\}$ of the $\{\gamma_i\}$ such that F_q is generated (over $\Lambda \otimes_Z Q$) by $\{\gamma_i\}$ subject to the relations: $t\rho_i = \sigma_i$. Let $\{\gamma_{i_1}, \ldots, \gamma_{i_k}\}$ be a *minimal* subset of $\{\gamma_i\}$ such that F_q admits such a description–i.e. there exists rational linear combinations $\{\rho_1, \ldots, \rho_k; \alpha_1, \ldots, \alpha_k\}$ of the $\{\gamma_{i_j}\}$ and F_q is generated (over $\Lambda \otimes_Z Q$) by the $\{\gamma_{i_j}\}$ subject to the relations $t\rho_i = \sigma_i$. Then the $\{\rho_i\}$ are linearly independent over Q. For if not, then some nontrivial rational linear combinations of the relations $t\rho_i = \sigma_i$ is a relations of the form $0 = \sigma$, for some rational linear combination σ of the σ_i, and, therefore, of the $\{\gamma_{i_j}\}$. There are two possibilities: (i) the $\{\gamma_{i_j}\}$ are linearly dependent over Q or (ii) this is the zero relation. In case (ii) we have a presentation (over $\Lambda \otimes_Z Q$) for $F_q \otimes_Z Q$ with more generators than relations, which is impossible since F_q is a Λ-torsion module. In case (i), we can eliminate one of the $\{\gamma_{i_j}\}$ and contradict the minimality of $\{\gamma_{i_j}\}$.

Thus the $\{\rho_i\}$ are linearly independent over Q and so we can express each γ_{i_j} as a rational linear combination of the $\{\rho_i\}$. This means we can replace the relations $t\rho_i = \sigma_i$ by relations $t\gamma_{i_j} = \mu_j$. Now it is clear that $F_q \otimes_Z Q$ has $\{\gamma_{i_j}\}$ as a vector space basis over Q and the relations define the action of t on the basis elements. This proves the claim.

COROLLARY (15.3). *If* $(F_1, \langle\ ,\ \rangle)$ *is the Z-torsion free part of the 2nd Alexander module of a 3-knot, then* signature$(\langle\ ,\ \rangle_x)$ *is a multiple of* 16.

Note that, for any even q, signature$(\langle\ ,\ \rangle_x)$ is a multiple of 8; this follows from §14, since the intersection pairing of V is an even, unimodular, integral quadratic form. Furthermore, for any even $q > 2$, there is a $(2q - 1)$-knot such that signature$(\langle\ ,\ \rangle_x) = 8$. This follows immediately from Proposition (15.2) and Theorem (14.4). Thus Corollary (15.3) represents a true restriction.

We can now show that Theorem (14.1) is true for $n = 3$ with the extra restriction that signature$(\langle\ ,\ \rangle_x)$ is a multiple of 16. We use the exact same arguement we used above to deal with the case $n = 1$. The desired $(F, \langle\ ,\ \rangle)$ produces a matrix A which, according to Theorem (14.1), corresponds to some 3-knot, after adding blocks of the form $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. This is the desired knot, since the extra blocks do not change $(F, \langle\ ,\ \rangle)$.

It is interesting to note that in [CS] Cappell and Shaneson have produced *topological* (locally flat) 3-knots such that signature$(\langle\ ,\ \rangle_x) = 8$ and have, in fact, shown that signature$\langle\ ,\ \rangle_x/8$, reduced mod 2, coincides with the Kirby-Siebemann obstruction [KS] to smoothing the knot. One might conjecture that Theorem (14.1) is true as stated for *topological* 3-knots. In fact, this conjecture seems to follow from the recent paper [SC].

16. For convenience we now state the final version of our realizability results.

THEOREM (16.1). *Let K be a (smooth) n-knot with Alexander modules $A_1, \ldots, A_n$, Blanchfield pairing $\langle\ ,\ \rangle$ on $f(A_q)$ if $n = 2q - 1$, torsion pairing*

[,] on $t(A_q)$ if $n = 2q$. Then

(a) *A_i are Λ-modules of type K,*

(b) *$\overline{f(A_i)} \approx e^1(f(A_{n+1-i}))$ for $i = 1, \ldots, n$,*

(c) *$\overline{t(A_i)} \approx e^2(t(A_{n-i}))$ for $i = 1, \ldots, n-1$; $t(A_n) = 0$,*

(d) *$\langle\, ,\, \rangle$ is conjugate-linear, $(-1)^{q+1}$-Hermitian and nonsingular,*

(e) *[,] is Z-linear, $(-1)^{q+1}$-symmetric, conjugate selfadjoint and nonsingular,*

(f) signature *$\langle\, ,\, \rangle_x$ is a multiple of* 16, *if $n = 3$.*

Conversely, given Z-torsion free Λ-modules $\{F_i\}$, Z-torsion Λ-modules $\{T_i\}$, all of type K, satisfying:

(b′) *$\overline{F_i} \approx e^1(F_{n+1-i})$,*

(c′) *$\overline{T_i} \approx e^2(T_{n-i})$,*

for some integer n, and pairing $\langle\, ,\, \rangle$ on F_q, if $n = 2q - 1$, or [,] on T_q, if $n = 2q$, satisfying (d), (e), (f), *then there exists an n-knot with Alexander modules $\{A_i\}$ such that $F_i \approx f(A_i)$, $T_i \approx t(A_i)$ and Blanchfield pairing corresponding to $\langle\, ,\, \rangle$, or torsion pairing corresponding to [,] except for the restriction $T_1 = 0$.*

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