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KNOTS ARE DETERMINED BY THEIR COMPLEMENTS

C. McA. GORDON AND J. LUECKE

Two (smooth or PL) knots K, K' in S^3 are *equivalent* if there exists a homeomorphism $h\colon S^3 \to S^3$ such that $h(K) = K'$. This implies that their complements $S^3 - K$ and $S^3 - K'$ are homeomorphic. Here we announce the converse implication.

THEOREM 1. *If two knots have homeomorphic complements then they are equivalent.*

This answers a question apparently first raised by Tietze [T, p. 83].

It was previously known that there were at most two knots with a given complement [CGLS, Corollary 3].

Whitten [W] has shown that prime knots with isomorphic groups have homeomorphic complements. Hence we have

COROLLARY 1.1. *If two prime knots have isomorphic groups then they are equivalent.*

The notion of equivalence of knots can be strengthened by saying that K and K' are *isotopic* if the above homeomorphism h is isotopic to the identity, or equivalently, orientation-preserving. The analog of Theorem 1 holds in this setting too: *if two knots have complements which are homeomorphic by an orientation-preserving homeomorphism, then they are isotopic.*

Theorem 1 and its orientation-preserving version are easy consequences of the following theorem concerning Dehn surgery.

THEOREM 2. *Nontrivial Dehn surgery on a nontrivial knot never yields* S^3.

The arguments used to prove Theorem 2 also lead to restrictions on when Dehn surgery on a knot yields a reducible manifold. (It is conjectured that this happens only with torus knots and cable knots.)

THEOREM 3. *If a 3-manifold obtained by Dehn surgery on a nontrivial knot is reducible then it has a lens space as a connected summand.*

COROLLARY 3.1. *Any homology 3-sphere obtained by Dehn surgery on a knot is irreducible.*

Theorem 3 also gives a new proof of the following result of Gabai [Ga], which includes the Property R Conjecture.

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COROLLARY 3.2 (GABAI). *Any 3-manifold obtained by a 0-framed surgery on a nontrivial knot is irreducible.*

SKETCH PROOF OF THEOREM 2. Let K be a knot in S^3, with tubular neighborhood $N(K)$, and let $X = S^3 - \mathring{N}(K)$ be the exterior of K. Let ρ be a *slope* on ∂X, that is, the unoriented isotopy class of an essential simple loop on ∂X. Let $K(\rho)$ denote the closed 3-manifold obtained by ρ-*Dehn surgery* on K, in other words, the result of attaching a solid torus V to X so that ρ bounds a disk in V. Let μ be the slope of a meridian of K. Then the *trivial* Dehn surgery yields $K(\mu) \cong S^3$. Let μ' be another slope on ∂X, having minimal geometric intersection number $n \geq 1$ with μ.

The starting point of the proof of Theorem 2 is the following proposition. David Gabai has independently proved this proposition.

PROPOSITION 1. *If $K(\mu')$ is homeomorphic to S^3, then there exist planar surfaces P, P' properly embedded in X such that*

(i) *$\partial P(\partial P')$ consists of parallel copies of μ (resp. μ');*

(ii) *P and P' are in mutual general position, and each component of ∂P intersects each component of $\partial P'$ in n points;*

(iii) *no arc of $P \cap P'$ is boundary-parallel in either P or P'.*

The construction of P and P' is based on [**Ga**, §4(A)]. P is the intersection with X of a suitable level 2-sphere in S^3 for a height function h with respect to which K is in *thin presentation* [**Ga**, p. 491]. Since $K(\mu')$ is homeomorphic to S^3, the core of the attached solid torus V is a knot K', say, and P' likewise comes from a level 2-sphere of a height function h' with respect to which K' is in thin presentation. Given P', the argument in [**Ga**, §4(A)] produces P such that (i) and (ii) hold and no arc of $P \cap P'$ is boundary-parallel in P'. Similarly, given P, one can find P' satisfying (i) and (ii) such that no arc on $P \cap P'$ is boundary-parallel in P. The additional content of Proposition 1 is that we can find P and P' so that these conditions hold simultaneously.

To do this, we pick a 1-parameter family of level 2-spheres of h in $K(\mu)$ between an adjacent local maximum and local minimum of K. This family becomes a 1-parameter family $\{P(\lambda)\}$ of punctured 2-spheres in X. We put the family $\{P(\lambda)\}$ in general position with respect to the height function $h'|X$. This means that for all but finitely many λ, $h'|P(\lambda)$ is a Morse function, and each $P(\lambda)$ such that $h'|P(\lambda)$ is not Morse has a single singularity corresponding to a birth, death or an exchange of tangencies.

Assume for a contradiction that Proposition 1 is false. The argument in [**Ga**, §4(A)], using the thin presentation of K', allows one to associate to each $P(\lambda)$ such that $h'|P(\lambda)$ is Morse, a punctured level 2-sphere of h', say P'_λ, which intersects $P(\lambda)$ transversely and is such that $P(\lambda) \cap P'_\lambda$ contains an arc which is boundary-parallel in P'_λ. If the corresponding arc in $\partial P'_\lambda$ lies above (below) $P(\lambda)$ in $K(\mu)$ then $P(\lambda)$ is called low (high, respectively), as in [**Ga**, §4(A)]. One observes that as λ increases, $P(\lambda)$ starts off high and ends up low. By the thinness of K under h, a change from high to low in

$\{P(\lambda)\}$ can only occur at a λ_0 such that $h'|P(\lambda_0)$ is not Morse. One analyses what happens at $P(\lambda_0)$ using the special way in which P'_λ is constructed for $P(\lambda)$, and eventually arrives at a contradiction to the thinness of K under h.

To prove Theorem 2 we now carry out a combinatorial analysis of the intersection of the planar surfaces P and P', ultimately deriving a contradiction. More precisely, we cyclically number the boundary components of $P(P')$ in the order they occur on ∂X, and label the endpoints of the arcs of $P \cap P'$ in $P(P')$ with the corresponding boundary component of $P'(P)$. Assigning (arbitrary) orientations to P and P' allows us to refer to $+$ and $-$ boundary components of P and P', according to the direction of the induced orientation of a boundary component as it lies on ∂X. Then for any arc α of $P \cap P'$, the boundary components of P joined by α on P have the same sign if and only if the boundary components of P' joined by α on P' have opposite sign. Capping off the boundary components of $P(P')$ with disks, we regard these disks as forming the 'fat' vertices of a graph $\Gamma(\Gamma')$ in S^2, the edges of $\Gamma(\Gamma')$ corresponding to the arcs of $P \cap P'$ in $P(P')$. We thus obtain two labeled graphs in S^2, whose edges are in one-one correspondence, such that the labeling satisfies the sign condition noted above, and such that neither graph contains a trivial loop (by condition (iii) of Proposition 1).

A (disk) face of Γ' corresponds to a subdisk of P' which we may regard as lying in $K(\mu)$ with its boundary contained in $P \cup \partial N(K)$. Similarly, the faces of Γ may be regarded as lying in $K(\mu')$. This allows us to infer topological properties of $K(\mu)$ ($K(\mu')$) from graph-theoretic properties of $\Gamma'(\Gamma)$.

For $n \geq 2$, this program is already carried out in **[CGLS**, §2.5 and 2.6; see Proposition 2.5.6]. There it is shown that given a pair of graphs Γ, Γ' as above, one of them, say Γ, contains a special kind of face (a *Scharlemann cycle*), which implies that $K(\mu')$ contains a punctured lens space.

We therefore focus on the case $n = 1$, where the above assertion is false and more delicate graph-theoretic arguments are needed. To describe the result, note that a disk face of Γ' (say) corresponds to a disk E in P' whose boundary ∂E can be expressed as a sequence of arcs $a_1, b_1, \ldots, a_k, b_k$, where a_i lies in $P \cap P'$ and b_i lies in $\partial N(K)$, $1 \leq i \leq k$. If the boundary components of P are labeled $1, \ldots, p$ (in order on $\partial N(K)$), then each b_i joins some consecutive pair of boundary components $j_i, j_i + 1$. Orienting ∂E, we regard b_i as representing $(j_i, j_i + 1)$ or $-(j_i, j_i + 1)$ according as b_i runs from j_i to $j_i + 1$ or vice versa. In this way, ∂E represents an element $\alpha(E) = (\alpha_1(E), \ldots, \alpha_p(E))$ of the free abelian group on the set of consecutive label-pairs $(1, 2), (2, 3), \ldots, (p-1, p), (p, 1)$. We say that Γ' *represents all types* if there exists a collection $\mathscr{E}$ of disk faces of Γ' such that

(1) for each $E \in \mathscr{E}$ and for each label-pair $(j, j + 1)$, all occurrences of $(j, j + 1)$ in ∂E have the same sign;

(2) for each sequence $(\varepsilon_1, \dots, \varepsilon_p)$, where $\varepsilon_i = \pm$, $1 \leq i \leq p$, there exists $E \in \mathscr{E}$ and $\eta = \pm$ such that

$$\operatorname{sign} \alpha_i(E) = \eta\varepsilon_i \text{ for all } i \text{ such that } \alpha_i(E) \neq 0.$$

(Clearly (1) and (2) are independent of the orientation of ∂E.)

We remark that if Γ' contains a Scharlemann cycle E, then taking $\mathscr{E} = \{E\}$ shows that Γ' represents all types.

We prove the following result.

PROPOSITION 2. *Let Γ, Γ' be a pair of graphs as described above. Either Γ contains a Scharlemann cycle or Γ' represents all types.*

The final step in the proof of Theorem 2 is supplied by the following proposition (in which we do not assume that $K(\mu')$ is S^3).

PROPOSITION 3. *Suppose that X contains properly embedded planar surfaces P, P' satisfying conditions* (i), (ii) *and* (iii) *of Proposition* 1, *where P is the intersection with X of a level* 2*-sphere in a thin presentation of K. Let Γ, Γ' be the associated graphs. Then Γ' does not represent all types.*

Proposition 3 is proved by showing that under the given hypotheses, a collection $\mathscr{E}$ of faces of Γ' representing all types would ultimately lead to the existence of a punctured lens space in $K(\mu) \cong S^3$, which is absurd.

Propositions 2 and 3 imply that $K(\mu')$ contains a punctured lens space. Since, in the context of Theorem 2, $K(\mu')$ is also homeomorphic to S^3, this contradiction completes the proof.

SKETCH PROOF OF THEOREM 3. Let μ' be a slope on ∂X such that $K(\mu')$ is reducible. Then there exists a properly embedded, incompressible (and non-boundary-parallel) planar surface P' in X whose boundary components have slope μ'. By **[Ga**, §4(A)**]** there is a planar surface P in X, coming from a level sphere in a thin presentation of K, such that P and P' satisfy conditions (i), (ii), and (iii) of Proposition 1. Exactly as in the proof of Theorem 2, Propositions 2 and 3 now show that $K(\mu')$ contains a punctured lens space.

We understand that David Gabai and Will Kazez, using a similar approach, have independently obtained some partial results on Theorem 2.

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Department of Mathematics, University of Texas at Austin, Austin, Texas 78712

Department of Mathematics, Courant Institute, 251 Mercer Street, New York, New York 10012

Current address (J. Luecke): Department of Mathematics, University of Texas at Austin, Austin, Texas 78712