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LETTER TO THE EDITOR

Knotted solutions of the Maxwell equations in vacuum

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Abstract. It is shown that there exist electromagnetic knots, which are standard electromagnetic fields in which any pair of magnetic lines whatsoever (or of electric lines) are linked and the magnetic helicity has the value $\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r} = n\mathscr{a}$, n being a topological constant of the motion given by a Hopf index and $\mathscr{a}$ a certain action constant. A family of $n=1$ knots is obtained, the projection of spin on three-momentum being equal to $\mathscr{a}$.

In a recent letter [1], a model of the electromagnetic field in a vacuum was proposed which is derived from an underlying topological structure. Its basic field is a scalar ϕ which represents a map $S^3 \times R \to S^2$ and (after identifying the spheres S^3 and S^2 with R^3 and C, via stereographic projection) obeys the highly nonlinear equations

$$\delta^{\mu}\left(\frac{\delta_{\mu}\phi^{*}\delta_{\nu}\phi - \delta_{\nu}\phi^{*}\delta_{\mu}\phi}{(1+\phi^{*}\phi)^2}\right) = 0. \tag{1}$$

By defining the electromagnetic tensor $F_{\mu\nu}$ as

$$F_{\mu\nu} = f_{\mu\nu}(\phi) = \frac{\mathscr{a}^{1/2}}{2\pi\mathrm{i}} \frac{\delta_{\mu}\phi^{*}\delta_{\nu}\phi - \delta_{\nu}\phi^{*}\delta_{\mu}\phi}{(1+\phi^{*}\phi)^2} \tag{2}$$

where $\mathscr{a}$ is an action constant, which necessarily must be introduced in order that $f_{\mu\nu}$ will have suitable dimensions for electromagnetic fields, equation (1) transforms into the second pair of Maxwell equations, the first pair being automatically verified. The magnetic lines are the level curves of the scalar ϕ. As ϕ represents at any time a map from S^3 to S^2, the solutions can be classified in homotopy classes labelled by the Hopf index n, an integer which is equal to the linking number of any pair of magnetic lines (or of electric lines) and takes the value [2–6]

$$n = \frac{1}{\mathscr{a}} \int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r} \tag{3}$$

where $\boldsymbol{A}$ is the vector potential. This has the important consequence that the magnetic helicity $\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3\boldsymbol{r}$ (or the electric one $\int \boldsymbol{C}\cdot\boldsymbol{E}\,\mathrm{d}^3\boldsymbol{r}$, where curl $C = E$) is quantized, being equal to $n\mathscr{a}$, *even if the fields are c-numbers.* (The magnetic helicity is used in plasma physics [7] and is formally akin to the helicity used in fluid dynamics where the velocity $\boldsymbol{v}$ and the vorticity ω play the roles of $\boldsymbol{A}$ and $\boldsymbol{B}$ [8].) Although the electromagnetic tensor always satisfies the Maxwell equations in vacuum, not all the solutions of Maxwell equations are admissible in this model, since not all of them can be obtained from a scalar ϕ through (2). However, the fields $F_{\mu\nu}$ of this model are dense in the set of Maxwell solutions verifying $\boldsymbol{E}\cdot\boldsymbol{B} = 0$ for weak enough fields (i.e.,

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they are in the same relation as the rationals and the reals). In this sense, the model is equivalent to standard theory in the weak field case, although it does not admit all the strong Maxwell fields (see [1] for details).

Quite independently of its validity as a realistic representation of the electromagnetic field, this model provides a procedure to obtain electromagnetic knots, defined as topological solutions of the Maxwell equations in vacuum in which any two magnetic lines (or electric lines) are linked, the degree of the linking being characterized by an integer n which is both a Hopf index and a constant of the motion. Families of such knots can be obtained as follows.

(i) Identify the three-space R^3 and the complex plane C with the spheres S^3 and S^2, respectively, via stereographic projection. In this way a complex function $\phi(r)$ represents a map $S^3 \to S^2$.

(ii) Take the Hopf map [2, 3]

$$\phi_{\rm H}(x, y, z) = \frac{2(x+{\rm i}y)}{2z+{\rm i}(r^2-1)} \tag{4}$$

where $r^2 = x^2 + y^2 + z^2$, which has unit Hopf index. Any two level curves of $\phi_{\rm H}$ are linked once. For instance, this is the case with the lines $\phi_{\rm H} = 0$ and $\phi_{\rm H} = \infty$ which are the z axis and the circle $r = 1$, $z = 0$.

(iii) Define the functions $\phi(r)$ and $\theta(r)$ as

$$\phi(r) = \phi_{\rm H}(\lambda x, \lambda y, \lambda z) \qquad \theta(r) = [\phi_{\rm H}(\lambda y, \lambda z, \lambda x)]^* \tag{5}$$

where λ is any inverse length. They are two Hopf maps.

(iv) Define the initial values of the electric and magnetic fields as

$$B_i = -\tfrac{1}{2}\varepsilon_{ijk} f_{jk}(\phi) \qquad E_i = -\tfrac{1}{2}\varepsilon_{ijk} f_{jk}(\theta) \tag{6}$$

where the tensor $f_{ij}(\phi)$ is defined by (2), $\hbar$ being now any quantity with dimensions of action. The vectors $\boldsymbol{B}$ and $\boldsymbol{E}$ are mutually orthogonal and tangent to the lines $\phi(r) = \text{constant}$ and $\theta(r) = \text{constant}$, respectively. Moreover, because of the way they are deduced from the Hopf map (4), two arbitrary $\boldsymbol{B}$-lines are knotted once, the same thing happening to any pair of $\boldsymbol{E}$-lines. Moreover, as the magnetic lines turn around the z axis and the electric ones turn around the x axis, (6) represents a magnetic and an electric vortex.

(v) The solution of the Maxwell equations with initial data given by (6) will have its field strength lines linked once for all time, since this property is invariant under continuous variations of ϕ and θ. It is therefore a knotted solution. A family of knots with $n = 1$ can be obtained from it by applying Lorentz transformations.

(vi) To obtain knots with Hopf index equal to n, just take the nth power of (4) and repeat the same procedure [2]. In the general case, maps $\phi(\boldsymbol{r})$ and $\theta(\boldsymbol{r})$ can be chosen such that, instead of (5),

$$(\nabla\phi^* \times \nabla\phi) \cdot (\nabla\theta^* \times \nabla\theta) = 0 \tag{7}$$

which means that the curves $\phi(\boldsymbol{r}) = \text{constant}$ and $\theta(\boldsymbol{r}) = \text{constant}$ are orthogonal to each other [1].

Changing the value of $\hbar$ produces a rescaling of the fields but does not modify their topological properties.

A comment on the terminology is perhaps in order. The word knot refers usually to a line which is tied to itself, while two intertwined curves are said to be linked. The solutions which we are calling knots could also be termed self-linked, since the magnetic lines are linked one to each other. However, the use of the term knot is justified as the electromagnetic field as a whole is tied to itself.

The energy E, momentum $\boldsymbol{p}$ and spin $\boldsymbol{J}$ of the knot can be calculated from the Cauchy data (6) by integration over all space of the corresponding tensors, and their value turns out to be

$$E = 2a\lambda \qquad \boldsymbol{p} = (0, a\lambda, 0) \qquad \boldsymbol{J} = (0, a, 0). \tag{8}$$

This means that the knot helicity in the sense used in particle physics (that is, the projection of the spin in the direction of the three-momentum $\boldsymbol{p}$) is equal to the Hopf index times the action constant a. We can thus write, in this case,

$$h = \int_{R^3} \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3 r = \frac{\boldsymbol{J} \cdot \boldsymbol{p}}{p} = n a. \tag{9}$$

It is to be stressed that the two meanings of helicity coincide in the knot and that the topological constant is closely related to an observable quantity. Note that the knot is a finite-energy solution.

Making use of well known Fourier transform techniques, it is easy to show that the solution with initial data given by (6) has the expression

$$\boldsymbol{B}(\boldsymbol{r}, t) = \lambda^2 a^{1/2}(Q\boldsymbol{H}_1 + P\boldsymbol{H}_2) \qquad \boldsymbol{E}(\boldsymbol{r}, t) = \lambda^2 a^{1/2}(P\boldsymbol{H}_1 - Q\boldsymbol{H}_2) \tag{10}$$

where the vectors $\boldsymbol{H}_k$ are

$$\begin{aligned}\boldsymbol{H}_1 &= \frac{4}{\pi D^3}[2(Y+T-XZ), -2(X+(Y+T)Z), -(1+Z^2-X^2-(Y+T)^2)] \\ \boldsymbol{H}_2 &= \frac{4}{\pi D^3}[-(1-Z^2+X^2-(Y+T)^2), 2(Z-X(Y+T)), -2(Y+T+XZ)]\end{aligned} \tag{11}$$

and $(X, Y, Z, T, R) = \lambda(x, y, z, t, r)$, $C = 1 + R^2 - T^2$, $Q = C^3 - 12T^2C$, $P = T(6C^2 - 8T^2)$ and $D = C^2 + 4T^2$. The energy density and the Poynting vector are then

$$T^{00} = \frac{16\lambda^4 a}{\pi^2 D^3}(1 + X^2 + (Y+T)^2 + Z^2)^2 \tag{12}$$

$$\begin{aligned}\boldsymbol{S} = {} & \frac{16\lambda^4 a}{\pi^2 D^3}(1 + X^2 + (Y+T)^2 + Z^2) \\ & \times [2(Z + X(Y+T)), (1 - X^2 + (Y+T)^2 - Z^2), -2(X - (Y+T)Z)].\end{aligned} \tag{13}$$

The electromagnetic field (10) verifies $\int \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}^3\boldsymbol{r} = a$ and any pair of magnetic lines (or electric ones) is knotted once. The expression of T^{00} is interesting. It has two maxima, a principal one which asymptotically approaches the point $x = z = 0$, $y = t$, and a secondary one which approaches the point $x = z = 0$, $y = -t$. For instance for $T = 10, 100, 1000$, they are placed at 9.9833, 99.998, 999.99 (principal maximum) and −9.8456, −99.985, −999.99 (secondary one), respectively. The values of the energy density are very close to $T^{00} = (16\lambda^4 a/\pi^2)/(1+4T^2)$ and $T^{00} = (16\lambda^4 a/\pi^2)/(1+4T^2)^3$, respectively. Most of the energy is around the principal one, whose velocity approaches asymptotically to the one along the y axis. As the knot proceeds it spreads, the secondary maximum decreasing much faster than the principal one (their ratio goes as $1/T^4$).

The Fourier expansion of the knot fields $\boldsymbol{E}$ and $\boldsymbol{B}$ is given by

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{r}, t) &= \frac{a^{1/2}}{(2\pi)^{3/2}} \lambda^2 \int [\boldsymbol{R}(\boldsymbol{k}) \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) + \boldsymbol{R}'(\boldsymbol{k}) \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \, \mathrm{d}^3\boldsymbol{k} \\ \boldsymbol{E}(\boldsymbol{r}, t) &= \frac{a^{1/2}}{(2\pi)^{3/2}} \lambda^2 \int [\boldsymbol{R}'(\boldsymbol{k}) \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) - \boldsymbol{R}(\boldsymbol{k}) \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \, \mathrm{d}^3\boldsymbol{k}\end{aligned} \tag{14}$$

where the vectors $\boldsymbol{R}(\boldsymbol{k})$ and $\boldsymbol{R}'(\boldsymbol{k})$ are given by

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{k}) &= \frac{\mathrm{e}^{-\omega}}{(2\pi)^{1/2}} \left(\frac{k_1 k_3}{\omega}, \frac{k_2 k_3}{\omega} + k_3, -\frac{k_1^2 + k_2^2}{\omega} - k_2 \right) \\ \boldsymbol{R}'(\boldsymbol{k}) &= \frac{\mathrm{e}^{-\omega}}{(2\pi)^{1/2}} \left(\frac{k_2^2 + k_3^2}{\omega} + k_2, -\frac{k_1 k_2}{\omega} - k_1, -\frac{k_1 k_3}{\omega} \right).\end{aligned}$$

It is easy to see that $|\boldsymbol{R}| = |\boldsymbol{R}'|$ and that $\boldsymbol{R}, \boldsymbol{k}, \boldsymbol{R}'$ form a positively oriented orthogonal trihedron. The vector potential can be chosen as

$$\begin{aligned}A^0 &= 0 \\ \boldsymbol{A}(\boldsymbol{r}, t) &= \frac{a^{1/2}}{(2\pi)^{3/2}} \lambda^2 \int \frac{1}{\omega} [\boldsymbol{R}(\boldsymbol{k}) \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) + \boldsymbol{R}'(\boldsymbol{k}) \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)] \, \mathrm{d}^3 k.\end{aligned}$$

The quantity $R^2(\boldsymbol{k}) + R'^2(\boldsymbol{k})$ turns out to be equal to

$$R^2(\boldsymbol{k}) + R'^2(\boldsymbol{k}) = \frac{4}{\pi} \mathrm{e}^{-2\omega/\lambda} \frac{(\omega + k_2)^2}{\lambda^2} \tag{15}$$

from which can be seen that the knot is not monochromatic, but a wavepacket. The frequency which gives the greatest contribution is the maximum of (15), which turns out to take place at $\omega = \omega_0 = \lambda$, $\boldsymbol{k} = \boldsymbol{k}_0 = (0, \lambda, 0)$. Consequently, the wavepacket is peaked at ω_0, $\boldsymbol{k}_0$, the energy, the momentum and the spin being given by (8). The helicity, in the sense of projection of the spin on the direction of motion, is equal to the action constant a, as mentioned above. Note that, as this field is a non-monochromatic packet, E is not equal to p. If we define the mass of the packet as $m^2 = E^2 - p^2$, its value is $m = \sqrt{3}\, a\lambda$ (remember that we are dealing with a c-number field).

By Lorentz-transforming the field (14), we can obtain a family of knots characterized by the speed of the centre of energy v, such that $E = \sqrt{3}\, \lambda a \gamma(v)$, $\boldsymbol{p} = \sqrt{3}\, \lambda a v \gamma(v)$. More precisely, if we go to a system S' travelling along the y axis with velocity u, the new values of these quantities are (with $\gamma = 1/\sqrt{(1-u^2)}$)

$$\begin{aligned}&E' = \gamma(2-u)\lambda a \qquad p' = (0, \gamma(1-2u)\lambda a, 0) \qquad v' = \frac{1-2u}{2-u} \\ &\omega_0' = \sqrt{\frac{1-u}{1+u}}\, \lambda \qquad h = a\end{aligned} \tag{16}$$

where ω_0' is the frequency at which the wavepacket is peaked. If u is very close to -1, we have, up to a factor such as $[1 + \mathrm{O}(1+u)]$,

$$E' = 3\gamma\lambda a \qquad |\boldsymbol{p}'| = 3\gamma\lambda a \qquad v' = 1 \tag{17}$$

and

$$E' = \tfrac{3}{2} a \omega_0' \qquad p' = \tfrac{3}{2} a k_0' \qquad E'^2 - p'^2 = 3\lambda^2 a^2. \tag{18}$$

The parameter λ may be chosen at liberty. In particular, we can take $\lambda = \omega^*\sqrt{[(1+u)/(1-u)]}$ with ω^* equal to any desired value whatsoever. In that case, by making u close enough to -1, the energy, momentum, velocity of the centre of energy v' and mass can be made arbitrarily close to the values:

$$E' = \tfrac{3}{2}\not{a}\omega^* \qquad p' = \tfrac{3}{2}\not{a}\omega^* \qquad v' = 1 \qquad p^\mu p_\mu = 0 \tag{19}$$

ω^* being the frequency at which the wavepacket is peaked. This means that, although the knots cannot be photons in general, since their four momenta are not in the light cone, the family of knots contains some wavepackets with the following appealing properties:

(*a*) the mass $m = \sqrt{(p^\mu p_\mu)}$ is arbitrarily close to zero;

(*b*) the velocity of the centre of energy is arbitrarily close to one;

(*c*) the energy is arbitrarily close to the action constant $\not{a}$ multiplied by $\frac{3}{2}$ times the frequency ω^* at which the packet is peaked and the linear momentum is arbitrarily close to $3\not{a}/2$ times the corresponding wavevector. The frequency ω^* can take any value.

Furthermore, the helicity of all the knots of the family, in the sense of the projection of the spin on the three-momentum $\boldsymbol{p}$, is exactly equal to $+\not{a}$, as can be easily seen by Lorentz-transforming the angular momentum tensor. There are also knots with $h = -\not{a}$.

These states are similar to the photons. This is curious and intriguing, but note that there are knots with properties clearly different from those of light quanta.

It is to be stressed that the knots (10) are solutions of Maxwell equations in vacuum and hence standard electromagnetic fields for any value of $\not{a}$. In the case of the model presented in [1], $\not{a}$ has a precise fixed value and this allows us to classify its solutions in homotopy classes. This topological structure has a suggestive consequence. Although the knots spread and their L^∞ norm goes to zero, there is a clear difference with the analogous property of finite energy solutions of the Maxwell equations or of wavepackets of the Schrödinger equation. For it happens that the knots and the vacuum are in different homotopy classes, always separated by the condition $\int \boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}^3 r = \not{a}$.

The fact that the Hopf index is closely related to an observable quantity such as the helicity suggests that the existence of electromagnetic knots could lead to a better understanding of the quantization process, as is discussed in [9] and [10]. It certainly deserves further consideration, for instance, to clarify under which conditions the magnetic helicity coincides with the particle physics helicity, since this does not happen generally.

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