Analysis of Censored Data
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KOLMOGOROV-SMIRNOV TYPE TESTS FOR NB(W)UE ALTERNATIVES UNDER CENSORING SCHEMES*†

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Abstract

For testing the null hypothesis that a life distribution is exponential against the NBUE (or NWUE) class, the structure of progressively censored TTT statistics is critically examined and incorporated in the study of some linear and Kolmogorov-Smirnov type tests essentially proposed by Koul (1978b) and Kumazawa (1989). Their distribution-freeness property under the null hypothesis (conditionally, under some censoring schemes) is established, and related asymptotics are presented.

1. Introduction. Nonparametric notions of aging have been popular and useful for modeling degradation in performance in a wide variety of contexts ranging from reliability engineering to biomedical applications. Correspondingly, the development and investigation of statistical tests, based on complete or censored life-test observations, for testing the null hypothesis of exponentiality against various aging alternatives has been an active area of research. In particular, the need for statistical inference with *censored data* occur when observing all units in the sample is not feasible.

Among the standard aging notions (viz., Barlow and Proschan 1991), the NBUE (*New Better than Used in Expectation*) is a relatively weak form of aging assumption, often made when one is unwilling to invoke a stronger assumption such as IFR, IFRA or NBU. A continuous life d.f. F on $\mathbf{R}^+ = [0, \infty)$, with a finite mean μ and survival function $\bar{F} = 1 - F$, is said to be (strictly) NBUE if

$$\int_t^\infty \bar{F}(x)dx(<) \leq \mu\bar{F}(t), \quad \forall t \geq 0. \tag{1.1}$$

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The dual property NWUE (W ≡ Worse) is defined by reversing the inequality. The nonparametric class of d.f.s so defined includes the exponential distributions, characterized by pointwise equality in (1.1).

We intend to test for the null hypothesis H_0 : F is exponential, against the class of alternatives H^* : $F \in$ NBUE class. Tests for this and other nonparametric alternatives available in the literature (viz., Hollander and Proschan 1972, 1975, Koul 1978a, b, Koul and Susarla 1980, and Kumazawa 1986a, b, c, 1989) are all based on the *total-time-on-test* (TTT) statistics at successive failures, and they exploit the asymptotic normality of the TTT-statistics under H_0. Our main objective here is to focus on various types of censoring, commonly encountered in practice, and critically examine the role of *progressively censored* (PC) TTT statistics in this context.

The standard results and large sample properties of various estimators and tests based on either the KS-type statistics or the Kaplan-Meier empirical process arising in random censoring have been extensively studied in the literature; we may refer to Shorack and Wellner (1986), Fleming and Harrington (1991) and Andersen et al. (1993), among others. However, these tests are considered either in a goodness of fit or in a two sample setup; whereas our proposed tests relate specifically to NB(W)UE alternatives, and hence a different approach is needed.

In the uncensored case, treated in Section 2, our motivation leads to an alternative formulation of a KS-type test which was derived earlier by Koul (1978b). With Type-I and Type-II censoring, considered in Section 3, an appropriately modified version of the test statistic in the uncensored case is shown to remain distribution-free (conditionally in Type-I censoring) under H_0. Issues concerning the computation of 'Bahadur efficiency' under such censorings are explored. In the random censoring case, Koul and Susarla (1980) proposed a different type of test. Kumazawa (1986a,b,c; 1989) extended their test and formulated a KS-type statistic for the NB(W)UE alternatives. However, his treatment is based on some stringent regularity assumptions. Moreover, in order to apply his statistic for actual testing, the related distribution theory needs further exploration for the random censoring case. We intend to fill in this gap (in Section 4) by incorporating suitable resampling methods and more appropriate asymptotics to match less stringent regularity assumptions. We propose suitably jackknifed and bootstrapped versions of KS-type and linear test statistics and explore their properties using functional jackknifing and bootstrap methodology.

2. Preliminary notions. We summarize some standard results to facilitate the presentation in subsequent sections. Note that by (1.1) F is (strictly)

NBUE iff

$$(2.1) \qquad \xi_F(t) \;=\; \mu^{-1}\{\mu\bar{F}(t) - \int_t^\infty \bar{F}(u)du\} \;(>)\; \geq 0, \;\; \forall t \in \mathbf{R}^+$$

The opposite inequality holds for a NWUE d.f. F. Further, $\xi_F(t) \equiv 0$ iff F is exponential. The functional $\xi_F = \{\xi_F(t), t \in \mathbf{R}^+\}$ under complete or censored life-tests naturally leads us to our various test statistics.

In the uncensored case, with observed lifetimes $X_1, \ldots, X_n$, the (sample) *empirical d.f.* $F_n(x) = n^{-1}\sum_{i=1}^n I(X_i \leq x), x \in \mathbf{R}^+$, is an unbiased optimal estimator of F. Here $I(A)$ stands for the indicator function of the set A. Let $X_{n:0} = 0 < X_{n:1} < \ldots < X_{n:n} < X_{n:n+1} = \infty$ be the *order statistics* corresponding to $X_1, \ldots, X_n$ (ties neglected with probability one). Then $F_n(x) = (k-1)/n$, for $X_{n:k-1} \leq x < X_{n:k}, k = 1, \ldots, n+1$. To estimate ξ_F, it is quite appealing to construct the empirical functional $\xi_{F_n} = \{\xi_{F_n}(t), t \in \mathbf{R}^+\}$, by replacing F with F_n in (2.1). Here, μ is replaced by $\bar{X}_n = \int_0^\infty \bar{F}_n(x)dx$, where $\bar{F}_n = 1 - F_n$. Thus the plug-in estimator of $\xi_F(t)$ is

$$(2.2) \quad \hat{\xi}_n(t) \;=\; \xi_{F_n}(t) \;=\; \bar{X}_n^{-1}\{\bar{X}_n\bar{F}_n(t) - \int_t^\infty \bar{F}_n(u)du\}, \;\; \forall t \in \mathbf{R}^+.$$

The *normalized spacings* $d_{n0} = 0$, $d_{nk} = (n-k+1)\{X_{n:k} - X_{n:k-1}\}, k = 1, \ldots, n$ lead to the cumulative normalized spacings $D_{nk} = \sum_{j \leq k} d_{nj} = \sum_{j \leq k} X_{n:j} + (n-k)X_{n:k}$, $0 \leq k \leq n$. The *total time on test* (TTT) at time point $t \in [X_{n:k}, X_{n:k+1})$ is defined as

$$(2.3) \qquad D_n(t) = D_{nk} + (n-k)(t - X_{n:k}), \text{ for } k = 0, 1, ..., n.$$

Thus

$$(2.4) \quad \hat{\xi}_n(t) = \{D_n(t)/D_{nn} - F_n(t)\}, \; X_{n:k} \leq t < X_{n:k+1}, \; 0 \leq k \leq n.$$

Let us examine the nature of the stochastic process $\{\hat{\xi}_n(t), t \geq 0\}$. Note that F_n is a step function with jumps of magnitude n^{-1} at the points $X_{n:1}, \ldots, X_{n:n}$, while $D_n(t)$ is continuous and piecewise linear though its first derivative is discontinuous at the failure points $X_{n:1}, \ldots, X_{n:k}$. Therefore, motivated by (2.1) and (2.4), we may consider a KS-type test for NBUE alternatives based on the statistic

$$(2.5) \;\; K_n^+ \;= \sup_{t \geq 0}\hat{\xi}_n(t) = \max_{0 \leq k \leq n}\{\hat{\xi}_n(X_{n:k})\} = \max_{0 \leq k \leq n}\{D_{nk}/D_{nn} - k/n\}.$$

For NWUE alternatives, the appropriate test statistic is

$$(2.6) \qquad K_n^- \;=\; \max\{(k/n - D_{nk}/D_{nn} : 0 \leq k \leq n\}.$$

The statistics K_n^+, K_n^- were considered earlier by Koul (1978b) and revisited by Kumazawa (1989). In the following we present the relevant distribution theory from a different perspective wherein we have a slightly different normalizing factor. Note that under H_0(F is exponential with a finite mean), the d_{nk} are i.i.d. exponential, so that the joint density of $\mathbf{U}_n = D_{nn}^{-1}(d_{n1}, \ldots, d_{nn})$ is given by

$$\Gamma(n)du_1...du_n, \quad u_j \geq 0, \quad u_1 + \cdots + u_n = 1, \tag{2.7}$$

which does not depend on the mean μ. Thus, under H_0, K_n^+ (or, K_n^-) is *distribution-free* and its exact distribution can be obtained by direct enumeration. We introduce a stochastic process $W_n = \{W_n(t), t \in [0,1]\}$ by letting $W_n(t) = (n-1)^{1/2}\{D_{n[nt]} - tD_{nn}\}/D_{nn}, \quad t \in [0,1]$, where $[s]$ = largest integer $\leq s$. Then $K_n^+ = \sqrt{n-1}[\sup\{W_n(t) : t \in [0,1]\}]$, and a similar representation holds for K_n^-. By an appeal to (2.7), it can be easily seen that under H_0,

$$W_n \to^D W^0, \text{ in the } J_1\text{-topology on } D[0,1], \tag{2.8}$$

where W^0 is a Brownian bridge on $[0,1]$. Thus, we have

$$\lim_{n\to\infty} P\{(n-1)^{1/2}K_n^+ > \lambda \mid H_0\} = e^{-2\lambda^2}, \forall\lambda \geq 0, \tag{2.9}$$

and the same holds for K_n^-. It is worth pointing out here that based on the multivariate beta distribution in (2.7), our normalizing factor is $\sqrt{n-1}$ instead of the conventional $\sqrt{n}$, and this adjustment may make the asymptotic distribution closer to the exact one for moderate sample sizes as well. The asymptotic non-null distribution theory, including the *Bahadur efficiency* results, studied in detail in Koul (1978b) remain pertinent to our scheme as well, and hence, we avoid this repetition.

3. Type I and II censoring. In a Type-I censored or right truncation model life-testing is confined only to a finite time interval $(0, t^*]$. Thus, the failures occurring in $(0, t^*]$ are observed, while the units surviving at $t^*(< \infty)$ and the corresponding unobserved survival times are regarded as censored at that point. In Type-II censoring, for a prefixed positive integer $r^*(\leq n)$, life-testing experiment is curtailed at (a stochastic) time point $X_{n:r^*}$ corresponding to the r^*th failure. In Type-I censoring $r_n^*(= nF_n(t^*))$, the number of failures occurring in $(0, t^*)$, is stochastic though t^* is pre-assigned (nonstochastic). In Type-II censoring, generally, we choose r^* such that $n^{-1}r^* = p^* \in (0,1)$.

In Type-I censoring, $F_n(t^*) = n^{-1}r_n^* = p_n^*$, is stochastic, and $p_n^* \to p^* = F(t^*)$ a.s., as $n \to \infty$. For testing $H_0 : F$ is exponential against NBUE alternatives, by analogy with (2.5), we consider the test-statistic

$$K_n^+(t^*) = \max\{D_{nk}/D_{nr_n^*} - k/r_n^* : 0 \le k \le r_n^*\}. \tag{3.1}$$

The conditional density of $\mathbf{U}_n^* = D_{nr_n^*}^{-1}(d_{n1}, \dots, d_{nr_n^*})$, given r_n^*, is

$$\Gamma(r_n^*)du_1 \cdots du_{r_n^*}, \quad u_j \ge 0, \quad u_1 + \cdots + u_{r_n^*} = 1. \tag{3.2}$$

Therefore, under H_0, $K_n^+(t^*)$ is conditionally (given $r_n^* > 0$) distribution-free. Conventionally, we let $K_n^*(t^*) = 0$ when $r_n^* = 0$, which occurs with probability $\{\bar{F}(t)\}^n \to 0$ as $n \to \infty$. Parallel to (2.7)-(2.8), we set here $W_n^* = \{W_n^*(t) = \sqrt{r_n^* - 1}\{D_{n[tr_n^*]}/D_{nr_n^*} - t\}, t \in [0,1]\}$. Then, as in (2.8), under H_0, given r_n^*, W_n^* weakly converges to a Brownian bridge W_n^0, and as $r_n^*/n \to p^*(> 0)$ a.s., as $n \to \infty$, we have under H_0,

$$W_n^* \to^D W^0, \text{ in the Skorokhod } J_1\text{-topology on } D[0,1]. \tag{3.3}$$

Thus, we obtain that for every $\lambda \ge 0$,

$$\lim_{n\to\infty} P\{(r_n^* - 1)^{1/2} K_n^+(t^*) \ge \lambda \mid H_0\} = e^{-2\lambda^2}. \tag{3.4}$$

Likewise, we modify K_n^- in (2.7), define $K_n^-(t^*)$, proceed as in (3.2)-(3.4), and show that (3.4) applies to the asymptotic null distribution of $K_n^-(t^*)$.

Let us place the Type-II censoring case side by side. Here, r^* is non-stochastic, so that we may define the test-statistic $K_{nr^*}^+$ as

$$K_{nr^*}^+ = \max\{(D_{nk}/D_{nr^*}) - k/r^* : 0 \le k \le r^*\}, \tag{3.5}$$

and an analogous expression holds for $K_{nr^*}^-$. In this setup, $K_{nr^*}^+$ (or $K_{nr^*}^-$) is distribution-free under H_0 and the same limiting distribution in (3.4) holds.

The Bahadur efficiency results obtained by Koul (1978b) in the uncensored case need considerable modifications for both Type-I and II censoring schemes. Fortunately, the null distributions are conformable in both the uncensored and censored cases in the sense that they differ only via their normalizing factor, and the same limiting distribution prevails. Therefore, the essential task is to study the a.s. convergence properties of K_n^+, $K_{nr^*}^+$ etc. when the null hypothesis may not hold. We deal here only with Type-II censoring case as the other one can be handled similarly.

We define the *p-value* (or *level attained*) of $K_{nr^*}^+$ (at t) by $L_n(t) = P\{K_{nr^*}^+ > t \mid H_0\}$, $t \ge 0$, so that by (3.4) we have

$$L_n(K_{nr^*}^+) = \exp\{-2(r^* - 1)(K_{nr^*}^+)^2[1 + o(1)]\} \text{ a.s.} \tag{3.6}$$

Note that $n^{-1}r^* \to p^* > 0$ as $n \to \infty$. Thus, under the alternative hypothesis, whenever $K_{nr^*}^+ \to w_{p^*}$ a.s., as $n \to \infty$, we may conclude that the

Bahadur slope for $K^+_{nr^*}$ is $4p^*w^2_{p^*} = c(p^*, w)$, say. To formulate w_{p^*} suitably we recall that $n^{-1}D_{nk} = \int_0^{X_{n:k}} x dF_n(x) + \bar{F}(X_{n:k})X_{n:k}$, $k = 0, 1, ..., n$. Also for every $u \in (0,1)$, let

$$(3.7)\quad \delta(u) = \int_0^{F^{-1}(u)} \{\bar{F}^{-1}(s)ds + (1-u)F^{-1}(u) = \int_0^{F^{-1}(u)} \bar{F}(x)dx.$$

Note that the D_{nk} are all appropriate L-statistics, and hence, the general asymptotic results including *Bahadur representations* for such L-statistics [studied in details in Chapter 4 of Jurečková and Sen (1995)] can be incorporated to conclude that

$$(3.8)\quad \max\{|n^{-1}D_{nk} - \delta(k/n)| : 0 \le k \le n\} \to 0 \text{ a.s., as } n \to \infty.$$

Moreover $\delta(u)$ is nonnegative, $\nearrow$ in $u \in (0,1)$ with $\delta(1) = \mu$. Therefore, from the above discussion, it follows that for every $p^* > 0$,

$$(3.9)\quad w^*_p = \sup\{\delta(u)/\delta(p^*) - (u/p^*) : 0 \le u \le p\} \equiv w^0_{p^*}/\delta(p^*), \text{ say,}$$

When $p^* = 1$, one may proceed as in Koul (1978b) and compute the Bahadur slope in specific cases. However, for $p^* < 1$, the situation is more complex, mainly due to the fact that for NBUE distributions, $\delta(u) \ge \mu u$ for all $u \in (0,1)$, but the excess $\delta(u) - \mu u$ may not be monotone or even a simple (e.g., linear) function of u. For this reason, when $p^* < 1$, we may compute the *limiting Bahadur efficiency* by considering a sequence $\{F_j, j \ge 1\}$ of NBUE distributions, with the same mean $\mu < \infty$, converging to an exponential. Corresponding to the sequence of d.f.s F_j, define the functions $\delta_j(u)$, $0 \le u \le 1$, $j = 1, 2, ...$ as in (3.7), and assume that $\delta_j(u) = \mu u + g_j(u), 0 \le u \le 1$, $j \ge 1$, where $g_j(u) \to 0$ as $j \to \infty$, for each $u \in (0,1)$. Then writing $w_{p^*,j}$ for w_{p^*} when $F = F_j$, we have

$$(3.10)\quad w_{p^*,j} = (p^{*2}\mu)^{-1}\sup\{p^*g_j(u) - ug_j(p^*) : 0 \le u \le p^*\}[1 + o(1)],$$

and this may be incorporated in the study of the limiting Bahadur efficiency of $\{K^+_{nr^*}\}$ with respect to $\{K^+_n\}$. A similar picture holds for $\{K^-_{nr^*}\}$ (with respect to $\{K^-_n\}$) for NWUE alternatives with a finite mean. In passing, we may remark that NBUE (NWUE) alternatives are not of parametric nature (e.g., location / scale purturbations), and hence the conventional definition of Pitman relative efficiency may not apply here. Also, for a sequence F_j of NBUE (NWUE) d.f.s converging to an exponential F, the asymptotic distribution theory of KS-type statistics may not conform to a simple pattern for which some other conventional measure of asymptotic relative efficiency is readily adoptable. Large deviation results for NBUE (NWUE) classes of distributions are therefore of considerable interest in the study of such asymptotic properties of KS-type tests.

4. Random censoring. In random censoring schemes, the set of censoring variables $T_1, ..., T_n$ are i.i.d. according to a d.f. G defined on $\mathbf{R}^+$, such that T_i and X_i are independent for each $i = 1, \ldots, n$. Both G and F are assumed to be continuous, so that ties can be neglected with probability 1. It is customary to let the censoring d.f. G belong to a general family $\mathcal{G}$. The observable random elements are

$$(4.1) \qquad Z_i = \min(T_i, X_i), \quad I_i = I(Z_i = X_i), \quad i = 1, \ldots, n.$$

In this random censoring scheme, for testing H_0 of exponentiality of F against NBUE (NWUE) alternatives, G is a nuisance parameter (functional). Note that the $Z_i, i = 1, \ldots, n$ are i.i.d. with d.f. $H(z) = 1 - \bar{H}(z)$, where $\bar{H}(z) = \bar{F}(z)\bar{G}(z)$. Hence the exponential vs. NBUE (NWUE) property of F may not be preserved by H when the censoring distribution G is arbitrary. Therefore, in order to apply the methodology developed in sections 2 and 3, to develop a test based on the observed responses (Z_i, I_i), an alternative approach based on weak convergence results is used and presented below.

For this purpose, we invoke the usual *product-limit* (PL) estimator of $\bar{F}$ to estimate the criterion functional $\xi_F()$ in (2.1) and modify the estimate $\hat{\xi}_n(.)$ in (2.4). Letting $N_n(t) = \sum_{i \leq n} I(Z_i > t), \quad t \in \mathbf{R}^+$, and setting $\alpha_i(t) = I(Z_i \leq t, I_i = 1), \quad t \geq 0, \quad i = 1, \ldots, n$, and $\tau_n = \max\{Z_i : 1 \leq i \leq n\}$, the Kaplan-Meier (1958) PL-estimator of $\bar{F}$, based on the observations $(Z_i, I_i), i = 1, \ldots, n$ is defined as

$$(4.2) \ \bar{P}_n(t) = \prod_{i=1}^{n} \Big\{\frac{N_n(Z_i)}{N_n(Z_i)+1}\Big\}^{\alpha_i(t)} I(t \leq \tau_n) = \prod_{\{i: Z_i \leq t\}} \Big\{\frac{n\bar{H}_n(Z_i)}{n\bar{H}_n(Z_i)+1}\Big\}^{I_i},$$

where $\bar{H}_n(t) = n^{-1}\sum_{i \leq n} I(Z_i > t) = n^{-1}N_n(t), t \geq 0$. The estimator of the mean μ, based on $\bar{P}_n(.)$, is $\hat{\mu}_n^* = \int_0^\infty \bar{P}_n(u)du = \int_0^{\tau_n} \bar{P}_n(u)du$, and hence, we consider here the empirical measure $\hat{\xi}_n^* = \{\hat{\xi}_n^*(t), t \geq 0\}$, where

$$(4.3) \qquad \hat{\xi}_n^*(t) = (\hat{\mu}_n^*)^{-1}\{\hat{\mu}_n^* \bar{P}_n(t) - \int_t^{\tau_n} \bar{P}_n(u)du\} I(t \leq \tau_n).$$

Let $(Z_{n:0} = 0) < Z_{n:1} < \cdots < Z_{n:n} (< Z_{n:n+1} = +\infty)$ be the order statistics corresponding to the observations $Z_i, \ i \leq n$. These order statistics correspond to both failure points (for which $I_i = 1$) and censored points (where $I_i = 0$). Let $m_n = \sum_{i \leq n} \alpha_i(\tau_n)$ be the total number of failure points, and let these ordered failure points be denoted by $Z_{n:1}^* \ldots, Z_{n:m_n}^*$. We write equivalently $Z_{n:j}^* = Z_{n:S_j}, \ j = 1, \ldots, m_n$, where $\{S_1, \ldots, S_{m_n}\} \subset \{1, \ldots, n\}$. Then clearly, $N_n(Z_{n:j}^*) = (n - S_j)$, for $j = 1, \ldots, m_n$. Hence, by (4.2),

$$(4.4) \quad \bar{P}_n(t) = \prod_{i \leq j} \Big(\frac{n - S_j}{n - S_j + 1}\Big), \quad \text{for } Z_{n:j}^* \leq t < Z_{n:j+1}^*, \ 0 \leq j \leq m_n,$$

where $Z^*_{n:0} = 0$ and $Z^*_{n:m_n+1} = +\infty$. Consequently, $\bar{P}_n(.)$ is also a step-function with downward steps only at the failure points $Z^*_{n:j}, 1 \le j \le m_n$. Therefore, in (4.3), $\bar{P}_n$ is a step-function, while $\int_0^{\tau_n} \bar{P}_n(u)du$ is continuous and piecewise linear (between successive $Z^*_{n:j}$). Hence, as in the uncensored case, we may consider the following KS-type test statistic

$$K^+_{nR} = \max\{\hat{\xi}^*_n(Z^*_{n:j}) : 0 \le j \le m_n\}. \tag{4.5}$$

A variant form of this statistic has been considered by Kumazawa (1989), although not as explicitly by incorporating (4.3) to structurize the picture. Since G is arbitrary, K^+_{nR} is not genuinely (or, even conditionally) distribution-free. Nevertheless, it is possible to consider some asymptotic tests for H_0 vs. NBUE alternatives, based on the following weak convergence and resampling methodology results. In passing we may remark that without this additional consideration of the asymptotics, the statistic in (4.5) is *not* by itself usable as a test statistic. Further, in view of the exponentiality of F, not all the regularity assumptions in Theorem 2.1 of Kumazawa (1989) are needed in the current context.

We rewrite $\xi_F(t) = \xi(\bar{F};t)$, $t \in [0,\infty), \bar{F} \in \mathcal{F}$ where $\xi(.) : \mathcal{F} \times \mathbf{R}^+ \to \mathbf{R}$, is a functional of $\bar{F} \in \mathcal{F}$ and a real valued function of $t \in \mathbf{R}^+$. Following Ren and Sen (1991), we may call $\xi(.)$ an *extended statistical functional.* In the same vein, we let $\hat{\xi}^*_n(t) = \xi(\bar{P}_n;t), \quad t \in \mathbf{R}^+$. Recall that $\xi_F(t) = \bar{F}(t) - (\int_t^\infty \bar{F}(y)dy)/(\int_0^\infty \bar{F}(y)dy), t \in \mathbf{R}^+$, so that for F defined on $\mathbf{R}^+$ with $\mu_F = \int_0^\infty \bar{F}(y)dy < \infty$, $\xi_F(t)$ is a bounded and differentiable functional. As such, we follow the usual *differentiable statistical functional* approach as in Ren and Sen (1991), and arrive at the following:

$$\hat{\xi}^*_n(t) = \xi(\bar{F},t) + \int_{\mathbf{R}^+} \xi_1(y;t,\bar{F})d[\bar{P}_n(y) - \bar{F}(y)] + \mathrm{Rem}(\bar{P}_n,\bar{F};t), \tag{4.6}$$

for $t \in \mathbf{R}^+$, where $\xi_1(y;t,\bar{F})$ stands for the *Hadamard* or *compact* derivative of $\xi(\bar{F};t)$ at $(\bar{F},t)$, and the remainder term

$$\mathrm{Rem}(\bar{P}_n,\bar{F};t) \;=\; o(||\bar{P}_n - \bar{F}||), \text{ uniformly in } t \in \mathbf{R}^+. \tag{4.7}$$

At this stage, we can appeal to the usual weak convergence results on the PL-estimator [viz., Shorack and Wellner (1986) and Pollard (1991)], and claim that under the assumed regularity conditions, as $n \to \infty$,

$$n^{1/2}(\bar{P}_n - \bar{F}) \;\to^D\; W^*_0 = \{W^*_0(t), t \in \mathbf{R}^+\}, \tag{4.8}$$

where the Gaussian W^*_0 can be expressed in terms of a standard Brownian motion $B = \{B(t), t \ge 0\}$ by the relation $W^*_0(t) = \bar{F}(t)B(C(t)), t \in \mathbf{R}^+$, where $C(t)$ is nonnegative, nondecreasing and is defined by $C(t) = \int_0^t \{\bar{F}^2(y)\bar{G}(y)\}^{-1}dF(y)$, $t \in \mathbf{R}^+$. Note that in the uncensored case, $\bar{G}(y) =$

$1 \forall y < \infty$, so that $C(t) = F(t)/\bar{F}(t)$, $t \geq 0$. Hence, in this case, we have, $W_o^*(t) = \bar{F}(t) B(F(t)/\bar{F}(t)) = B_0(F(t))$, $t \geq 0$, where $\{B_0(t), t \in [0,1]\}$ is a standard Brownian bridge.

In statistical modeling and analysis of censored data, the Cox (1972) *proportional hazard model* (PHM) is often invoked to yield manageable (albeit generally nonrobust) solutions. Here we set

$$\bar{G}(x) = \{\bar{F}(x)\}^{\lambda}, \quad x \in \mathbf{R}^+, \quad \lambda > 0. \tag{4.9}$$

We can verify that there exists a monotone time transformation $s = h(t)$ with $h(0) = 0$ and $h(\infty) = 1$, such that $W_0^*(h^{-1}(s)) \equiv W_0(s) \equiv B_0(s)$, $s \in [0,1]$. This simplification, also noted by Kumazawa (1989), may not generally hold in random censoring sans the PHM assumption. In any case, to define $C(t)$ properly, we need that $\bar{G}(y) > 0 \forall\ y$ for which $\bar{F}(y) > 0$. Since F is exponential under H_0, we therefore assume that G does not have a finite upper endpoint.

The process $\hat{\xi}_n^*(t) - \xi_F(t)$, underlying our proposed tests, differs from the conventional empirical process in (4.8) or its two sample versions. However, we note that under H_0,

$$\hat{\xi}_n^*(t) - \xi_F(t) = \{\bar{P}_n(t) - \bar{F}(t)\} - [(\hat{\mu}_n^*)^{-1} \int_t^{\infty} \bar{P}_n(y) dy - e^{-t/\mu}], \tag{4.10}$$

where $\hat{\mu}_n^* = \int_0^{\infty} \bar{P}_n(u) du$. Since F is exponential (μ) under H_0, the PHM condition on G asserts that G is exponential with mean $\nu = \mu/\lambda$, and this characterization of G under the PHM makes it redundant to impose some other conditions such as in Theorem 2.1 of Kumazawa (1989). For random censoring with an arbitrary G with $G(x) < 1$ for all $x < \infty$, such a reduction (in law) to a Brownian bridge by a time-transformation may not be taken for granted. Consequently, there is a genuine need to find out the asymptotic distribution of K_{nR}^+ in (4.5), under the null hypothesis without the PHM assumption. It seems quite possible to address this issue by incorporating some *resampling methods*, and in the rest of this section, we probe into such possibilities.

It is worth mentioning in this context that for this specific NBUE testing problem, Koul and Susarla (1980) proposed a linear test statistic with the property of asymptotic normality. Their findings were extended by Kumazawa (1986a,b; 1989), who also studied asymptotic efficiency results. These test statistics can be expressed, in a unified form, as a linear functional $L_n = \int_0^{\infty} \hat{\xi}_n^*(t) d\Pi_n(t)$, where $\Pi_n(t)$ is $\uparrow$ in $t \in \mathbf{R}^+, \Pi_n(0) = 0, \Pi_n(\infty) = 1$, and is a step-function with jumps $a_{nk} (\geq 0)$ at the points $Z_{n:k}^*, k = 1, \ldots, m_n$. Thus, L_n can be expressed as

$$L_n = \sum_{k \leq m_n} a_{nk} \hat{\xi}_n^*(Z_{n:k}^*). \tag{4.11}$$

We assume that there exists a proper measure Π, such that $\Pi_n \to \Pi$ a.s.,as $n \to \infty$, where $\Pi(0) = 0$ and $\Pi(\infty) = 1$. Based on our weak convergence results and general asymptotic results on L-statistics (viz., Chapter 4 in Jurečková and Sen 1995), by letting $\lambda^* = \int_0^\infty \xi_F(t) d\Pi(t)$, it readily follows that

$$(4.12) \qquad n^{1/2}(L_n - \lambda^*) \to^D \quad N(0, \gamma^2(\Pi, F, G)),$$

where the asymptotic variance function $\gamma^2 \equiv \gamma^2(\Pi, F, G)$ depends on the known Π and unknown F and G. Even under H_0 (when F is exponential), as G is unspecified, γ^2 is not specified and may not be a scalar multiple of μ^2 or some other function of the exponential d.f. Therefore, even if one wants to use this asymptotically normal test statistic (with $\lambda^* = 0$ under exponentiality), there is a need to estimate γ^2 consistently from the experimental data set. For this purpose, both *jackknife* and *bootstrap* methods can be effectively incorporated.

First, consider the jackknife methodology. Let $L_{n-1}^{(i)}$ be the countrerpart of L_n for a sample of size $(n-1)$ obtained by deleting the i-th observation from the base sample, and let $L_{n,i} = nL_n - L_{n-1}^{(i)}$, $1 \leq i \leq n$, be the *pseudovalues* generated by the jackknifing method. Further let

$$(4.13) \qquad L_{nJ} = n^{-1} \sum_{i=1}^{n} L_{n,i}, \quad V_{nJ} = (n-1)^{-1} \sum_{i=1}^{n} (L_{n,i} - L_{nJ})^2.$$

Then, L_{nJ} is the jackknifed version of L_n and V_{nJ} is the (Tukey-) estimator of γ^2. From general results on functional jackknifing (viz., Sen 1988a, b), it follows that even if F is not exponential

$$(4.14) \qquad V_{nJ} \to \gamma^2(\Pi, F, G) \text{ a.s., as } n \to \infty.$$

Thus we may consider the test statistic $T_n = L_{nJ}/\sqrt{V_{nJ}}$, and select the (asymptotic) critical level as $\tau_\alpha = \Phi^{-1}(1-\alpha)$.

Next, consider the bootstrap version. From $\mathbf{O}_n = \{(Z_i, I_i),\ 1 \leq i \leq n\}$, the collection of the random elements in (4.1), draw with replacement (SRS) a sample of size n and denote this collection by $\mathbf{O}_n^* = \{(Z_i^*, I_i^*),\ 1 \leq i \leq n\}$. Then $P\{(Z_i^*, I_i^*) = (Z_\alpha, I_\alpha) \mid \mathbf{O}_n\} = 1/n$, for every $\alpha = 1, \ldots, n$ and $i = 1, \ldots, n$. The usual multinomial law can be brought in to describe the conditional probability structure of $\mathbf{O}_n^*$, given $\mathbf{O}_n$. Let $\bar{P}_n^*, \hat{\mu}_n^{**}$ denote the product limit estimator in (4.2) and its derived mean when $\mathbf{O}_n$ is replaced by $\mathbf{O}_n^*$ and incorporate them to define L_n^*, the bootstrap counterpart of L_n in (4.11). Then generate a large number M of such bootstrap samples (which are conditionally independent, given $\mathbf{O}_n$), and denote the corresponding bootstrap statistics by L_{nj}^*, $j = 1, \ldots, M$. Then a bootstrap d.f. of L_n can be formulated as

$$(4.15) \qquad H_{nM}^*(y) = M^{-1} \sum_{j \leq M} I\{\sqrt{n}(L_{nj}^* - L_n) \leq y\}, \quad y \in \mathbf{R}^+.$$

From the general results on bootstrapping (viz., Efron and Tibshirani 1993) and by virtue of the inherent asymptotic normality results, it follows that as n, M increases,

$$\text{(4.16)} \qquad \sup_y |H^*_{nM}(y) - \Phi(y/\gamma)| \to^P 0.$$

Therefore, the upper α-quantile of H^*_{nM} provides a consistent estimator of the critical level. Alternatively, one may define the bootstrap variance estimators

$$\text{(4.17)} \qquad V^*_{nM} = \frac{n}{M}\sum_{j=1}^{M}(L^*_{nj} - L_n)^2,$$

and consider the test statistic $T^*_n = \sqrt{n}L_n/\sqrt{V^*_{nM}}$, for which τ_α provides a consistent estimator of the critical level.

In the current context, K^+_{nR} is a functional of $\hat{\xi}^*_n(.)$, which is attracted in law by a Gaussian process. As such, such resampling methods need some modifications. For bootstrapping, we proceed as in before. Based on $\mathbf{O}^*_\mathbf{n}$, we incorporate $\bar{P}^*_n, \hat{\mu}^{**}_n$ etc. in the construction of the KS-type statistic, which we denote by K^{*+}_{nR}. Then parallel to (4.15), define

$$\text{(4.18)} \qquad H^*_{nM}(y) = M^{-1}\sum_{j\le M} I\{\sqrt{n}K^{*+}_{nR} \le y\}, \quad y \in \mathbf{R}^+.$$

We take the upper α-quantile of this empirical bootstrap d.f. as the critical level of K^+_{nR}; its consistency may then be proved as in Hušková and Janssen (1993), who have considered similar functionals of degenerate U-statistics. Our basic weak convergence results again provide the desired theoretical justifications.

Finally, we consider a version of jackknifing which may be applied here under suitable regularity conditions. Recall that

$$\text{(4.19)} \qquad \hat{\xi}^*_n(t) = \bar{P}_n(t) - (\int_t^\infty \bar{P}_n(y)dy)/\hat{\mu}^*_n, \quad t \ge 0.$$

Let $\bar{P}^{(i)}_{n-1}$ and $\hat{\mu}^{(i)*}_{n-1}$ be the PL-estimator and its derived mean based on a sample of size $(n-1)$ obtained by dropping the ith observation from the base sample, and let $\hat{\xi}^{*(i)(t)}_{n-1} = \bar{P}^{(i)}_{n-1}(t) - (\hat{\mu}^{(i)*}_{n-1})^{-1}\int_t^\infty \bar{P}^{(i)}_{n-1}(y)dy, t \ge 0$, for $i = 1, \ldots, n$. Then the pseudovalues (processes) are denoted by

$$\text{(4.20)} \qquad \hat{\xi}^*_{n,i}(t) = n\hat{\xi}^*_n(t) - (n-1)\hat{\xi}^{*(i)}_{n-1}(t), \quad t \ge 0, \quad 1 \le i \le n.$$

By an appeal to functional jackknifing methodology (viz., Sen 1988a, b), we define

$$\hat{\xi}_{nJ}(t) = n^{-1}\sum_{i=1}^{n}\hat{\xi}^*_{n,i}(t), \quad t \ge 0,$$

$$\text{(4.21)} \qquad v_{nJ}(s,t) = \frac{1}{n-1}\sum_{i=1}^{n}\{\hat{\xi}^*_{n,i}(t) - \hat{\xi}_{n,J}(t)\}\{\hat{\xi}^*_{n,i}(s) - \hat{\xi}_{n,J}(s)\},$$

for $s, t \in \mathbf{R}^+$. We need to compute the above only at the observed failure points $Z^*_{n:J}, 1 \leq j \leq m_n$. Using the transformation $s \to \bar{P}_n(s), t \to \bar{P}_n(t)$, we write $v^*_{nJ}(a,b) = v_{nJ}(s,t)$, where $a = \bar{P}_n(s)$, and $b = \bar{P}_n(t)$. If $v^*_{nJ}(a,b), 0 \leq a < b \leq 1$ conforms to any bell shaped form (zero at the two extremities and positive otherwise), then a reduction of the process to a Brownian bridge is possible by time-transformation, and hence, the limit law in (2.9) still remains intact (as K^+_{nR} is invariant under time-transformation). Looking at (4.8), we see that such is the case when $\bar{G}$ dominates $\bar{F}$ in an appropriate sense. For example, under PHM, if in (4.9) we have $\lambda \leq 1$, so that G is stochastically larger than F, then we get a tied-down Gaussian process. On the other hand, if under PHM, $\bar{G} \leq \bar{F}$ pointwise, i.e., $\lambda > 1$, then we have $\bar{F}^2(t)C(t) = (1+\lambda)^{-1}\{(\bar{F}(t))^{1-\lambda} - \bar{F}^2(t)\} \uparrow \infty$ as $t \uparrow \infty$, so that in (4.8), we may not have a tied-down Gaussian function at the upper extremity. This is the main reason we make the assumption that

$$\bar{G}(x) \geq \bar{F}(x), \quad \text{for} \quad x \geq x_0, \quad \text{for some} \quad x_0 \geq 0. \tag{4.22}$$

In view of the null hypothesis of exponentiality, we are essentially assuming that the survival function of the censoring variable eventually lies above an exponential survival function. This then justifies (4.8) and also validates the jackknife methodology. In this case, there is even no need to compute $v^*_{nJ}(a,b)$, and the time-transformation takes care of the limit laws. In fact, bootstrapping is also therefore not needed under (4.22). The Bahadur ARE can be worked out as in the uncensored case under (4.22), but the slopes will depend on the unknown G. We conclude this section with a remark that as under the null hypothesis F is exponential (while G is arbitrary), we may even use a semi-parametric bootstrap method wherein we estimate the mean of F by using the PL-estimator of F and use parametric bootstrap for F, while a nonparametric one for G. This may have some advantage in the estimation of the critical levels, but essentially for asymptotic power studies such estimators may not be that robust. For this reason, we advocate the use of the proposed nonparametric bootstrapping methodology.

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