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KORN'S CONSTANT FOR A SPHERICAL SHELL*

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Abstract. Upon invoking the variational characterization of Korn's constant and Dafermos' technique to reduce it to a boundary value problem, the Korn constant of a spherical shell of arbitrary thickness has been evaluated. The classical result of Payne and Weinberger for the sphere is recovered as the special case of vanishing interior radius, while as the thickness of the shell tends to zero, Korn's constant tends to infinity in a nonuniform sense.

1. Introduction. Korn's inequalities have a long history which extends for a period of almost eighty years. They play an important role in the theory of static elasticity mainly in connection with existence theory, stability, and the qualitative study of solutions. An extensive list of references can be found in [6, 7, 9]. The optimum constant in Korn's inequality is known as Korn's constant [4]. Its value is known to be equal to 4 for a circle and to 56/13 for a sphere [10]. The Korn constant for an ellipse, as well as for a general two-dimensional domain which can be conformally mapped onto the unit disc by a rational mapping, is given by Horgan [8]. Dafermos [2] used the variational characterization of Korn's constant and reduced the problem to the equation of the eigenvalues of a certain boundary value problem in potential theory. In fact, utilizing the coincidence of the Euler equation with the Navier equation of linear elastostatics, as well as the completeness of the Papkovitch–Neuber representation [3], he showed that the method of eigenfunction expansion can be used to solve the appropriate boundary value problem.

Dafermos applied his method in the case of a circular ring, and so he evaluated Korn's constant for a two-dimensional not-simply connected region. His constant tends to 4 whenever the inner diameter tends to zero, while when the thickness of the ring tends to zero Korn's constant tends to infinity.

The purpose of this paper is to apply Dafermos' technique to the case of a spherical shell which is the three-dimensional analogue of the ring problem. Using general expansions in spherical harmonics and the Papkovitch–Neuber representation [4] for the elastostatic displacement field, we evaluate the spectrum of the corresponding boundary value problem, and the Korn constant is obtained from the maximum value of this spectrum. Long and tedious calculations had to be performed in order

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to bring the terms of the expansions for the Papkovitch potentials in a comparable form, so that the orthogonality of the surface harmonic can be applied. Nevertheless, the key point of our work is the appropriate use we made of the dependence of the four Papkovitch potentials [3] to weaken the coupling between the surface harmonics. This particular choice of the scalar potential in terms of the vector potential decouples the infinite system of spherical harmonics to an infinite set of algebraic systems of finite order, whose solvability is reduced to the vanishing of certain determinants.

Besides the eigenvalues 1 and 2 that have infinite multiplicity which are independent of the geometry of the elastic body, we also obtain the eigenvalue 4 which is an eigenvalue for the case of the sphere as well. Furthermore, just as in the sphere case, the asymptotic behaviour of the eigenvalues for large n has the form $4 + O(n^{-1})$ for any shell of fixed thickness. If we denote by $\delta \in (0, 1)$ the ratio of the inner to the outer radius of the shell, then, apart from the eigenvalues 1, 2, and 4, the rest of the spectrum depends on δ. In other words, we obtain a one-parameter family of spectra, parametrized by the thickness parameter δ. As a consequence, the maximum eigenvalue depends on δ and Korn's constant increases quadratically as $\delta \to 1-$. Therefore, the approach of the eigenvalues to 4, as $n \to +\infty$, is not uniform. As $\delta \to 0+$ the spectrum of the shell problem coincides with the corresponding spectrum of the sphere problem as it was given by Payne and Weinberger [10]. Some numerical values of the Korn constant for shells of different thickness are given at the end of Sec. 3. It is observed there, that for a shell of inner radius up to 1/3 of the outer radius, the Korn constant varies approximately between 4.3 and 5 which is very close to the 56/13 value for the sphere. On the other hand, if the inner radius becomes 0.995 of the outer, then Korn's constant assumes the value 692.227. Furthermore, for $\delta \in (0, 0.85)$, Korn's constant occurs either on the second, the third, or the fourth eigenvalue (for the sphere it occurs on the third), while for $\delta = 0.995$ Korn's constant is attained at the twenty-sixth eigenvalue.

2. Statement of the problem. Let Ω be a three-dimensional open and bounded region having a smooth boundary $\partial\Omega$. If $\mathbf{u}$ is a continuously differentiable vector field, satisfying the normalization condition

$$\int_\Omega (\nabla\mathbf{u} - \nabla\mathbf{u}^\tau)\, dv = \tilde{\mathbf{0}}, \tag{1}$$

where "τ" denotes transposition, then we will say that Korn's inequality holds for Ω if there exists a number $K > 0$ such that

$$4\int_\Omega \|\nabla\mathbf{u}\|^2\, dv \le K \int_\Omega \|\nabla\mathbf{u} + \nabla\mathbf{u}^\tau\|^2\, dv, \tag{2}$$

with the norm of the dyadic field $\mathbf{a} \otimes \mathbf{b}$ defined by

$$\|\mathbf{a} \otimes \mathbf{b}\|^2 = \sum_{i,j=1}^{3} a_i^2 b_j^2. \tag{3}$$

The smallest value of K for which (2) holds is defined to be the Korn's constant for Ω. The variational characterization of (2) leads to the Euler equation [2]

$$(2 - K)\Delta\mathbf{u}(\mathbf{r}) - K\nabla(\nabla \cdot \mathbf{u}(\mathbf{r})) = \mathbf{0}, \quad \mathbf{r} \in \Omega \tag{4}$$

and the boundary condition

$$(2-K)\hat{\mathbf{n}}\cdot\nabla\mathbf{u}(\mathbf{r})-K(\nabla\mathbf{u}(\mathbf{r}))\cdot\hat{\mathbf{n}}=\mathbf{0},\quad \mathbf{r}\in\partial\Omega, \tag{5}$$

where $\hat{\mathbf{n}}$ denotes the outward unit normal on Ω. Equation (4) is the Navier equation of elastostatics [2] whenever

$$K=\frac{2(\lambda+\mu)}{\lambda}=\nu^{-1} \tag{6}$$

where λ and μ are the Lamé constants of linearized and isotropic elasticity and ν denotes Poisson's ratio.

It has been proved [2, 10] that the values $K=1$ and $K=2$ are eigenvalues of (4), (5) of infinite multiplicity and that if K_n, $n\in N$, are all the eigenvalues of (4), (5) besides 1 and 2, then Korn's constant is given by

$$K=\sup_{n\in\mathbf{N}}\{1,2,K_n\}. \tag{7}$$

In the next section we evaluate the spectrum of eigenvalues for (4), (5) in the case of a spherical shell. Then (7) will provide the corresponding Korn constant.

3. The spherical shell. Let Ω be the spherical shell having inner radius $b>0$ and outer radius $a>b$. The boundary condition (5) can be written as

$$2(1-K)\hat{\mathbf{r}}\cdot\nabla\mathbf{u}-K\hat{\mathbf{r}}\times(\nabla\times\mathbf{u})=\mathbf{0}, \tag{8}$$

where $\hat{\mathbf{r}}$ points into Ω^c.

Any solution of Eq. (4) in Ω, for $K\neq 1,2$, has the Papkovitch–Neuber representation [3]

$$\mathbf{u}(\mathbf{r})=\mathbf{A}(\mathbf{r})+\frac{K}{4(1-K)}\nabla(\mathbf{r}\cdot\mathbf{A}(\mathbf{r})+B(\mathbf{r})), \tag{9}$$

where the three components of $\mathbf{A}$ and B are harmonic functions in Ω. Three out of these four harmonic functions are independent [3]. Nevertheless, we keep all four in the representation (9), in order to be able to effectively perform the necessary calculations.

A complete expansion of the harmonic fields $\mathbf{A}$ and B in terms of solid spherical harmonics [5] yields

$$\mathbf{A}(\mathbf{r})=\sum_{n=0}^{\infty}[\mathbf{A}_n(\mathbf{r})+\mathbf{A}_{-(n+1)}(\mathbf{r})] \tag{10}$$

$$B(\mathbf{r})=\sum_{n=0}^{\infty}[B_n(\mathbf{r})+B_{-(n+1)}(\mathbf{r})] \tag{11}$$

where $\mathbf{A}_n, B_n$ are interior harmonics of degree n and $\mathbf{A}_{-(n+1)}, B_{-(n+1)}$ are exterior harmonics of degree n. We decompose the nonharmonic terms in the representation (9) as follows:

$$\nabla(\mathbf{r}\cdot\mathbf{A}_n)=\frac{1}{2n+1}\mathbf{W}_n+\frac{1}{2n-1}r^2\nabla\nabla\cdot\mathbf{A}_n, \tag{12}$$

$$\nabla(\mathbf{r}\cdot\mathbf{A}_{-(n+1)})=-\frac{1}{2n+1}\mathbf{W}_{-(n+1)}-\frac{1}{2n+3}r^2\nabla\nabla\cdot\mathbf{A}_{-(n+1)}, \tag{13}$$

where

$$\mathbf{W}_n = -\nabla\left[r^{2n+3}\nabla\cdot\left(\frac{\mathbf{A}_n}{r^{2n+1}}\right)\right] - \frac{2}{2n-1}r^{2n+1}\nabla\left[\frac{1}{r^{2n-1}}\nabla\cdot\mathbf{A}_n\right] \tag{14}$$

is a homogeneous interior harmonic of degree n, and where

$$\mathbf{W}_{-(n+1)} = -\nabla\left[\frac{1}{r^{2n-1}}\nabla\cdot(r^{2n+1}\mathbf{A}_{-(n+1)})\right] + \frac{2}{2n+3}\frac{1}{r^{2n+1}}\nabla[r^{2n+3}\nabla\cdot\mathbf{A}_{-(n+1)}] \tag{15}$$

is a homogeneous exterior harmonic of degree n. Formulae (13) and (15) can be formally recovered from (12) and (14) respectively via the substitution $n \to -(n+1)$. Hence, we can combine the expansion for $\mathbf{u}$ into the form

$$\begin{aligned}\mathbf{u}(r,\theta,\varphi) = \sum_{n=-\infty}^{+\infty}\Bigg\{&\left[\mathbf{A}_n(r,\theta,\varphi) + \frac{K}{4(1-K)(2n+1)}\mathbf{W}_n(r,\theta,\varphi)\right]\\ &+\frac{K}{4(1-K)}\left[\frac{r^2}{2n+3}\nabla\nabla\cdot\mathbf{A}_{n+2}(r,\theta,\varphi) + \nabla B_{n+1}(r,\theta,\varphi)\right]\Bigg\},\end{aligned} \tag{16}$$

where $\mathbf{W}_n$ is given by (14). With the exception of the term $r^2\nabla\nabla\cdot\mathbf{A}_{n+2}$, all the other terms in the n-th term of the series are solid harmonics of degree n. The term $r^2\nabla\nabla\cdot\mathbf{A}_{n+2}$ is a surface harmonic of degree n, but it fails to be a solid harmonic because of the factor r^2. On the other hand, in the interest of satisfying the boundary condition (8) the r^2 factor is immaterial since it becomes a constant for $r = a$, or $r = b$. Therefore, every term in the expansion (16) is a surface harmonic function of degree n. Consequently, the orthogonality of the surface spherical harmonics can be directly applied. Next, we use (16) to satisfy the boundary conditions. In view of (9), condition (8) yields

$$2(1-K)\frac{\partial}{\partial r}\mathbf{u} - K\hat{\mathbf{r}}\times(\nabla\times\mathbf{A}) = \mathbf{0}, \tag{17}$$

and since

$$\hat{\mathbf{r}}\times(\nabla\times\mathbf{A}) = (\nabla\mathbf{A})\cdot\hat{\mathbf{r}} - \frac{\partial}{\partial r}\mathbf{A} = \frac{1}{r}\nabla(\mathbf{A}\cdot\mathbf{r}) - \left(\frac{\partial}{\partial r} + \frac{1}{r}\right)\mathbf{A}, \tag{18}$$

we arrive at

$$2(1-K)r\frac{\partial}{\partial r}\mathbf{u} + Kr\frac{\partial}{\partial r}\mathbf{A} + K\mathbf{A} - K\nabla(\mathbf{A}\cdot\mathbf{r}) = \mathbf{0}, \tag{19}$$

for $r = a$ and $r = b$.

Substituting (10), (12), (13), and (16) into (19) and using Euler's theorem for the derivative of a homogeneous function, we derive the following form of the boundary condition (8).

$$\begin{aligned}\sum_{n=-\infty}^{+\infty}\Bigg\{&[2n - K(n-1)]\mathbf{A}_n(r,\theta,\varphi) + \frac{K(n-2)}{2(2n+1)}\mathbf{W}_n(r,\theta,\varphi)\\ &+\frac{Kn}{2}\nabla B_{n+1}(r,\theta,\varphi) + \frac{Kn}{2(2n+3)}r^2\nabla\nabla\cdot\mathbf{A}_{n+2}\Bigg\} = \mathbf{0}\end{aligned} \tag{20}$$

for $r = a$ and $r = b$.

By means of the orthogonality of the surface spherical harmonics the boundary conditions (20) are reduced to the following weakly coupled infinite system:

$$\left(\frac{a}{r}\right)^n (\mathbf{L}_n + \mathbf{M}_n) + \left(\frac{r}{a}\right)^{n+1} (\mathbf{L}_{-(n+1)} + \mathbf{M}_{-(n+1)}) = -\frac{Kna^2}{2(2n+3)} \left(\frac{a}{r}\right)^n \nabla\nabla \cdot \mathbf{A}_{n+2} - \frac{K(n+1)a^2}{2(2n-1)} \left(\frac{r}{a}\right)^{n+1} \nabla\nabla \cdot \mathbf{A}_{-n+1}, \quad (21)$$

$$\left(\frac{b}{r}\right)^n (\mathbf{L}_n + \mathbf{M}_n) + \left(\frac{r}{b}\right)^{n+1} (\mathbf{L}_{-(n+1)} + \mathbf{M}_{-(n+1)}) = -\frac{Knb^2}{2(2n+3)} \left(\frac{b}{r}\right)^n \nabla\nabla \cdot \mathbf{A}_{n+2} - \frac{K(n+1)b^2}{2(2n-1)} \left(\frac{r}{b}\right)^{n+1} \nabla\nabla \cdot \mathbf{A}_{-n+1} \quad (22)$$

for $n = 0, 1, 2, \ldots$, where

$$\mathbf{L}_n = [2n - K(n-1)]\mathbf{A}_n(r, \theta, \varphi) + \frac{K(n-2)}{2(2n+1)}\mathbf{W}_n(r, \theta, \varphi), \quad (23)$$

$$\mathbf{L}_{-(n+1)} = [K(n+2) - 2(n+1)]\mathbf{A}_{-(n+1)}(r, \theta, \varphi) + \frac{K(n+3)}{2(2n+1)}\mathbf{W}_{-(n+1)}(r, \theta, \varphi), \quad (24)$$

$$\mathbf{M}_n = \frac{Kn}{2}\nabla B_{n+1}(r, \theta, \varphi), \quad (25)$$

$$\mathbf{M}_{-(n+1)} = -\frac{K(n+1)}{2}\nabla B_{-n}(r, \theta, \varphi). \quad (26)$$

Solving Equations (21), (22) with respect to $\mathbf{L}_n + \mathbf{M}_n$ and $\mathbf{L}_{-(n+1)} + \mathbf{M}_{-(n+1)}$ and combining the resulting expression into a single one, we derive the formulae

$$\mathbf{L}_n = -\frac{K(n+1)}{2(2n-1)} r^{2n+1} \frac{a^2 - b^2}{a^{2n+1} - b^{2n+1}} \nabla\nabla \cdot \mathbf{A}_{-(n-1)}(r, \theta, \varphi) - \frac{Kn}{2}\nabla \left[B_{n+1}(r, \theta, \varphi) + \frac{1}{2n+3} \frac{a^{2n+3} - b^{2n+3}}{a^{2n+1} - b^{2n+1}} \nabla \cdot \mathbf{A}_{n+2}(r, \theta, \varphi) \right] \quad (27)$$

for $n = 0, \pm 1, \pm 2, \ldots$.

At this stage, we utilize the freedom we have to choose one of the potentials A_1, A_2, A_3, B at will and set

$$B_{n+1}(r, \theta, \varphi) = -\frac{1}{2n+3} \frac{a^{2n+3} - b^{2n+3}}{a^{2n+1} - b^{2n+1}} \nabla \cdot \mathbf{A}_{n+2}(r, \theta, \varphi) \quad (28)$$

for $n = 0, \pm 1, \pm 2, \ldots$. Therefore, if we denote by

$$\delta = \frac{b}{a} \in (0, 1), \quad (29)$$

(27) yields $\mathbf{L}_0 = \mathbf{0}$ and

$$\mathbf{L}_n = -\frac{K(n+1)}{2(2n-1)} a^2 \frac{1 - \delta^2}{1 - \delta^{2n+1}} \left(\frac{r}{a}\right)^{2n+1} \nabla\nabla \cdot \mathbf{A}_{-(n-1)}(r, \theta, \varphi) \quad (30)$$

for $n = \pm 1, \pm 2, \ldots$. By virtue of the identities

$$\nabla \cdot \mathbf{W}_n(r, \theta, \varphi) = \frac{2n(2n+1)}{2n-1} \nabla \cdot \mathbf{A}_n(r, \theta, \varphi) \quad (31)$$

and

$$\nabla \cdot [r^{2n+1} \nabla \nabla \cdot \mathbf{A}_{-(n-1)}(r, \theta, \varphi)] = -n(2n+1) r^{2n-1} \nabla \cdot \mathbf{A}_{-(n-1)}(r, \theta, \varphi) \tag{32}$$

the divergence of (30) assumes the form

$$2[2n(2n-1) - K(n^2 - n + 1)] \left(\frac{a}{r}\right)^{n-1} \nabla \cdot \mathbf{A}_n(r, \theta, \varphi)$$
$$= Kn(n+1)(2n+1) \frac{1-\delta^2}{1-\delta^{2n+1}} \left(\frac{r}{a}\right)^n \nabla \cdot \mathbf{A}_{-(n-1)}(r, \theta, \varphi), \tag{33}$$

for every $n = \pm 1, \pm 2, \ldots$.

The functions $(\frac{a}{r})^{n-1} \nabla \cdot \mathbf{A}_n$ and $(\frac{r}{a})^n \nabla \cdot \mathbf{A}_{-(n-1)}$ are surface spherical harmonics of degree $n-1$. The role of these functions is interchanged if we replace n in (33) by $-n+1$. Consequently, the n-th degree surface spherical harmonics satisfy the relation (33) as well as

$$Kn(n-1)(2n-1) \frac{1-\delta^2}{1-\delta^{2n-1}} \delta^{2n-1} \left(\frac{a}{r}\right)^n \nabla \cdot \mathbf{A}_{n+1}(r, \theta, \varphi)$$
$$= 2[2n(2n+1) - K(n^2 + n + 1)] \left(\frac{r}{a}\right)^{n+1} \nabla \cdot \mathbf{A}_{-n}(r, \theta, \varphi) \tag{34}$$

and this is true for every $n = 1, 2, 3, \ldots$.

Relations (33) and (34) have to hold simultaneously in order for the boundary conditions on $r = a$ and $r = b$ to be satisfied. Therefore, since the surface harmonics of degree $n-1$ for $n = 1, 2, \ldots$ are not zero, we obtain the characteristic equations

$$4[2(n+1)(2n+1) - K(n^2 + n + 1)][2n(2n+1) - K(n^2 + n + 1)]$$
$$- K^2 n(n+1)(2n+3)(n-1)(n+2)(2n-1) \frac{1-\delta^2}{1-\delta^{2n+3}} \frac{1-\delta^2}{1-\delta^{2n-1}} \delta^{2n-1} = 0 \tag{35}$$

for $n = 1, 2, \ldots$ as a consequence of the vanishing of the corresponding determinants. For $n = 1$ we obtain the eigenvalue $K_1 = 4$. For $n \geq 2$, we set

$$\Gamma_n(\delta) = \frac{(1-\delta^2)^2 \delta^{2n-1}}{(1-\delta^{2n+3})(1-\delta^{2n-1})}, \tag{36}$$

$$K_n(\delta) = \frac{4}{1 + \Lambda_n(\delta)}, \tag{37}$$

where $K_n(\delta)$ denotes a root of (35) corresponding to a particular value of n, and the n-th characteristic equation is written as

$$n(n+1)(2n+1)^2 \Lambda_n^2(\delta) - 2(2n+1)^2 \Lambda_n(\delta)$$
$$- (n-1)(n+2)[1 + n(n+1)(2n-1)(2n+3)\Gamma_n(\delta)] = 0. \tag{38}$$

Equation (38) has two real roots and since we are interested in the maximum value of $K_n(\delta)$, we consider only the smallest root of (38) which is given by

$$\Lambda_n(\delta) =$$
$$\frac{1}{n(n+1)} \left(1 - \frac{\sqrt{(n^2+n+1)^2 + n^2(n+1)^2(n-1)(n+2)(2n-1)(2n+3)\Gamma_n(\delta)}}{2n+1} \right) \tag{39}$$

for $n \geq 2$. For any given $\delta \in (0, 1)$,

$$\lim_{n \to \infty} \left(n^2(n+1)^2(n-1)(n+2)(2n-1)(2n+3)\Gamma_n(\delta) \right) = 0, \tag{40}$$

because of the exponential decay of Γ_n with respect to the thickness parameter δ. Consequently,

$$K_n(\delta) = 4 + O(1/n), \quad n \to \infty \tag{41}$$

which implies, by virtue of $\Lambda_n(\delta) < 0$ for every $n \geq 2$, that the maximum value of $K_n(\delta)$ is obtained for some integer $n_0(\delta) \geq 2$. Obviously, the Korn constant depends on the thickness δ of the spherical shell and it occurs at an eigenvalue which also depends on δ. In the limit as $\delta \to 0+$

$$\Lambda_n(\delta) = \frac{1-n}{(n+1)(2n+1)} + O(\delta), \tag{42}$$

$$K_n(\delta) = \frac{2(n+1)(2n+1)}{n^2+n+1} + O(\delta) \tag{43}$$

and the maximum occurs for

$$K_3(\delta) = \frac{56}{13} + O(\delta), \tag{44}$$

which recovers the well-known result of Payne and Weinberger [10] for the Korn constant of the sphere.

On the other hand, as the thickness of the shell approaches zero,

$$\lim_{\delta \to 1-} \Gamma_n(\delta) = \frac{4}{(2n-1)(2n+3)}; \tag{45}$$

hence equation (35) can be written as

$$K^2(1) - 2(n^2+n+1)K(1) + 4n(n+1) = 0, \tag{46}$$

which has the roots

$$K_1(1) = 2, \qquad K_2(1) = 2n(n+1). \tag{47}$$

Therefore, the Korn constant tends to infinity quadratically with vanishing thickness of the shell.

In conclusion, for every $\delta \in (0, 1)$ the Korn constant is given by

$$K(\delta) = \max \left[\{1, 2, 4\} \cup \left\{ \frac{4}{1 + \Lambda_n(\delta)},\ n \geq 2 \right\} \right], \tag{48}$$

where $\Lambda_n(\delta)$ has the form (39). The eigenvalues 1, 2, and 4 are independent of δ, while the rest of the spectrum is a function of it. For a very thick shell Korn's constant tends to $56/13$ while for a thin shell it tends to infinity quadratically.

In the next table Korn's constant for different values of shell thickness is provided.

δ	$n_0(\delta)$	$K(\delta)$
.01	3	4.308
.05	3	4.308
.09	2	4.309
.10	2	4.318
.20	2	4.521
.30	2	4.976
.40	2	5.706
.50	2	6.720
.60	3	8.056
.70	3	11.188
.80	4	16.722
.85	4	22.465
.90	5	33.867
.95	8	68.715
.96	9	86.017
.97	10	114.835
.98	13	172.458
.99	18	345.859
.991	19	384.346
.995	26	692.227

We observe that for an interior hole with a radius up to one-third of the radius of the sphere, the spherical shell behaves approximately as a solid sphere, while an appreciable discrepancy appears for thickness of the shell less than one half of the exterior radius.

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References

[1] B. Bernstein and R. A. Toupin, *Korn inequalities for the sphere and circle*, Arch. Rat. Mech. Anal. **6**, 51–64 (1960)

[2] C. Dafermos, *Some remarks on Korn's inequality*, Z. Angew. Math. Phys. **19**, 913–920 (1968)

[3] R. A. Eubanks and E. Sternberg, *On the completeness of the Boussinesq–Papkovich stress functions*, J. Rat. Mech. Anal. **5**, 735–746 (1956)

[4] M. Gurtin, *The linear theory of elasticity*, in Handbuch der Physik, Band VIa/2, Berlin, 1972

[5] E. W. Hobson, *The theory of spherical and ellipsoidal harmonics*, Chelsea, 1955

[6] C. O. Horgan and J. K. Knowles, *Eigenvalue problems associated with Korn's inequalities*, Arch. Rat. Mech. Anal. **40**, 384–402 (1971)

[7] C. O. Horgan, *On Korn's inequality for incompressible media*, SIAM J. Appl. Math. **28**, 419–430 (1975)

[8] C. O. Horgan, *Inequalities of Korn and Friedrichs in elasticity and potential theory*, Z. Angew. Math. Phys. **26**, 155–164 (1975)

[9] C. O. Horgan and L. E. Payne, *On inequalities of Korn, Friedrichs and Babuška-Aziz*, Arch. Rat. Mech. Anal. **82**, 165–179 (1983)

[10] L. E. Payne and H. F. Weinberger, *On Korn's inequality*, Arch. Rat. Mech. Anal. **8**, 89–98 (1961)