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KORN'S INEQUALITIES FOR PIECEWISE H^1 VECTOR FIELDS

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ABSTRACT. Korn's inequalities for piecewise H^1 vector fields are established. They can be applied to classical nonconforming finite element methods, mortar methods and discontinuous Galerkin methods.

1. INTRODUCTION

In this paper we use a boldface italic lower-case Roman letter such as $\boldsymbol{v}$ to denote a vector (or vector function) with components v_j $(1 \leq j \leq d)$ and a boldface lower-case Greek letter such as $\boldsymbol{\eta}$ to denote a $d \times d$ matrix (or matrix function) with components η_{ij} $(1 \leq i, j \leq d)$. The Euclidean norm of the vector $\boldsymbol{v}$ (resp. the Frobenius norm of the matrix $\boldsymbol{\eta}$) will be denoted by $|\boldsymbol{v}|$ (resp. $|\boldsymbol{\eta}|$).

Let Ω be a bounded connected open polyhedral domain in $\mathbb{R}^d$ ($d = 2$ or 3). The classical Korn inequality (cf. [8], [14], [5] and the references therein) states that there exists a (generic) positive constant C_Ω such that

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega\big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\big) \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d, \tag{1.1}$$

where the strain tensor $\boldsymbol{\epsilon}(\boldsymbol{u})$ is the $d \times d$ matrix with components

$$\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for } 1 \leq i, j \leq d, \tag{1.2}$$

and the (semi)norms are defined by

$$|\boldsymbol{u}|^2_{H^1(\Omega)} = \sum_{j=1}^{d} |u_j|^2_{H^1(\Omega)} = \sum_{j=1}^{d} \int_\Omega |\nabla u_j|^2\, dx, \qquad \|\boldsymbol{\epsilon}(\boldsymbol{u})\|^2_{L_2(\Omega)} = \int_\Omega |\boldsymbol{\epsilon}(\boldsymbol{u})|^2\, dx,$$

$$\|\boldsymbol{u}\|^2_{L_2(\Omega)} = \int_\Omega |\boldsymbol{u}|^2\, dx \quad \text{and} \quad \|\boldsymbol{u}\|^2_{H^1(\Omega)} = |\boldsymbol{u}|^2_{H^1(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)}.$$

Let $\mathbf{RM}(\Omega)$ be the space of (infinitesimal) rigid motions on Ω defined by

$$\mathbf{RM}(\Omega) = \{\boldsymbol{a} + \boldsymbol{\eta}\boldsymbol{x} :\ \boldsymbol{a} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\eta} \in \mathfrak{so}(d)\}, \tag{1.3}$$

where $\boldsymbol{x} = (x_1, \ldots, x_d)^t$ is the position vector function on Ω and $\mathfrak{so}(d)$ is the Lie algebra of anti-symmetric $d \times d$ matrices. The space $\mathbf{RM}(\Omega)$ is precisely the kernel of the strain tensor; i.e., for $\boldsymbol{v} \in [H^1(\Omega)]^d$,

$$\boldsymbol{\epsilon}(\boldsymbol{v}) = \mathbf{0} \quad \Longleftrightarrow \quad \boldsymbol{v} \in \mathbf{RM}(\Omega). \tag{1.4}$$

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Let Φ be a seminorm on $[H^1(\Omega)]^d$ with the following properties:

$$\Phi(\boldsymbol{v}) \leq C_\Phi \|\boldsymbol{v}\|_{H^1(\Omega)} \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega)]^d,$$

where C_Φ is a generic positive constant depending on Φ, and

$$\Phi(\boldsymbol{m}) = 0 \quad \text{and} \quad \boldsymbol{m} \in \mathbf{RM}(\Omega) \quad \Longleftrightarrow \quad \boldsymbol{m} = \text{a constant vector.}$$

Note that such a seminorm is invariant under the addition of a constant vector $\boldsymbol{c}$; i.e.,

$$\Phi(\boldsymbol{v}+\boldsymbol{c}) = \Phi(\boldsymbol{v}) \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega)]^d. \tag{1.5}$$

Then (1.1), (1.4), (1.5) and the compactness of the embedding of $H^1(\Omega)$ into $L_2(\Omega)$ imply that

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Phi \big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \Phi(\boldsymbol{u})\big) \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d. \tag{1.6}$$

In particular, the inequality (1.6) implies

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|Q\boldsymbol{u}\|_{L_2(\Omega)}\Big) \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d, \tag{1.7}$$

where

$$Q\boldsymbol{u} = \boldsymbol{u} - \frac{1}{|\Omega|}\int_\Omega \boldsymbol{u}\,dx$$

is the orthogonal projection operator from $[L_2(\Omega)]^d$ onto the orthogonal complement of the constant vector functions;

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_{\Omega,\Gamma} \Bigg(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \sup_{\substack{\boldsymbol{m}\in\mathbf{RM}(\Omega) \\ \|\boldsymbol{m}\|_{L_2(\Gamma)}=1,\, \int_\Gamma \boldsymbol{m}\,ds=\mathbf{0}}} \int_\Gamma \boldsymbol{u}\cdot\boldsymbol{m}\,ds\Bigg) \tag{1.8}$$

for all $\boldsymbol{u} \in [H^1(\Omega)]^d$, where ds is the infinitesimal $(d-1)$-dimensional volume and Γ is a measurable subset of $\partial\Omega$ with a positive $(d-1)$-dimensional volume; and

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\int_\Omega \nabla\times\boldsymbol{u}\,dx\Big|\Big) \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d, \tag{1.9}$$

where $\nabla\times\boldsymbol{u}$ is the vector function (the curl of $\boldsymbol{u}$) defined by

$$\nabla\times\boldsymbol{u} = \Big(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)^t$$

when $d = 3$, and the scalar function (the rotation of $\boldsymbol{u}$) defined by

$$\nabla\times\boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

when $d = 2$.

Remark 1.1. The inequality (1.7) is of course equivalent to Korn's inequality (1.1). Inequalities (1.8) and (1.9) imply Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_{\Omega,\Gamma}\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d,\ \boldsymbol{u}\big|_\Gamma = \mathbf{0},$$

and Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega)]^d,\ \Big|\int_\Omega \nabla\times\boldsymbol{u}\,dx\Big| = 0.$$

Henceforth we will also refer to (1.9) as Korn's second inequality.

In this paper we establish analogs of (1.7), (1.8) and (1.9) for piecewise H^1 vector fields (functions) with respect to a partition $\mathcal{P}$ of Ω consisting of nonoverlapping polyhedral subdomains, which is not necessarily a triangulation of Ω. In other words, we only assume that

$$D \cap D' = \emptyset \text{ if } D \text{ and } D' \text{ are distinct members of } \mathcal{P}, \text{ and } \overline{\Omega} = \bigcup_{D \in \mathcal{P}} \overline{D}.$$

Typical two- and three-dimensional examples of partitions are depicted in Figure 1, where the square is partitioned into 7 subdomains and the cube is partitioned into 5 subdomains.

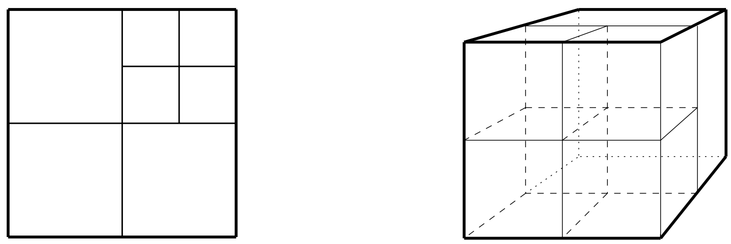

FIGURE 1. Examples of general partitions

The space $[H^1(\Omega,\mathcal{P})]^d$ of piecewise H^1 vector fields (functions) is defined by

$$[H^1(\Omega,\mathcal{P})]^d = \{\boldsymbol{v} \in [L_2(\Omega)]^d : \boldsymbol{v}_{D} = \boldsymbol{v}\big|_D \in [H^1(D)]^d \quad \forall\, D \in \mathcal{P}\},$$

and the seminorm $|\cdot|_{H^1(\Omega,\mathcal{P})}$ is given by

$$|\boldsymbol{v}|_{H^1(\Omega,\mathcal{P})} = \Big(\sum_{D\in\mathcal{P}} |\boldsymbol{v}|^2_{H^1(D)}\Big)^{1/2}.$$

We also use the notation $\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{v})$ to denote the matrix function defined by

$$\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{v})\big|_D = \boldsymbol{\epsilon}(\boldsymbol{v}_{D}) \qquad \forall\, D \in \mathcal{P}. \tag{1.10}$$

Let $S(\mathcal{P},\Omega)$ be the set of all the (open) sides (i.e., edges ($d = 2$) or faces ($d = 3$)) common to two subdomains in $\mathcal{P}$. For example, there are 10 such edges in the two-dimensional example in Figure 1 and 8 such faces in the three-dimensional example. (Precise definitions of $S(\mathcal{P},\Omega)$ will be given in Section 4 and Section 5.) For $\sigma \in S(\mathcal{P},\Omega)$, we denote by π_σ the orthogonal projection operator from $[L_2(\sigma)]^d$ onto $[P_1(\sigma)]^d$, the space of vector polynomial functions on σ of degree ≤ 1.

The following are analogs of the classical Korn inequalities for $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^d$:

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|Q\boldsymbol{u}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\mathrm{diam}\ \sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.11}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \sup_{\substack{\boldsymbol{m}\in\mathbf{RM}(\Omega)\\ \|\boldsymbol{m}\|_{L_2(\Gamma)}=1,\, \int_\Gamma \boldsymbol{m}\,ds=\boldsymbol{0}}} \Big(\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{m}\,ds\Big)^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\mathrm{diam}\ \sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.12}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\,\sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.13}$$

where $[\boldsymbol{u}]_\sigma$ is the jump of $\boldsymbol{u}$ across the side σ and the positive constant C depends only on the shape regularity of the partition $\mathcal{P}$. In particular these inequalities are valid for partitions that are not quasi-uniform. (More details on the shape regularity assumptions are given in Section 4 and Section 5.)

Inequalities (1.11)–(1.13) imply

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|Q\boldsymbol{u}\|_{L_2(\Omega)}\Big), \tag{1.14}$$

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \sup_{\substack{\boldsymbol{m}\in\mathbf{RM}(\Omega)\\ \|\boldsymbol{m}\|_{L_2(\Gamma)}=1,\ \int_\Gamma \boldsymbol{m}\,ds=\mathbf{0}}} \int_\Gamma \boldsymbol{u}\cdot\boldsymbol{m}\,ds\Big), \tag{1.15}$$

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|\Big), \tag{1.16}$$

provided $\pi_\sigma[\boldsymbol{u}]_\sigma = \mathbf{0}$ for all $\sigma\in S(\mathcal{P},\Omega)$, or equivalently

$$\int_\sigma [\boldsymbol{u}]_\sigma\cdot\boldsymbol{l}\,ds = 0 \qquad \forall\,\sigma\in S(\mathcal{P},\Omega) \quad \text{and} \quad \boldsymbol{l}\in[P_1(\sigma)]^d. \tag{1.17}$$

Thus we immediately obtain (1.14)–(1.16) for certain classical nonconforming finite elements (cf. [11], [10], [9], [13]). With some modifications, these estimates can also be applied to certain mortar elements (cf. [18]). Details will be carried out elsewhere.

The inequalities (1.11)–(1.13) also immediately imply

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|Q\boldsymbol{u}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\,\sigma)^{-1}\|[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\Big), \tag{1.18}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Gamma)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\,\sigma)^{-1}\|[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\Big), \tag{1.19}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\,\sigma)^{-1}\|[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\Big), \tag{1.20}$$

for all $\boldsymbol{u}\in[H^1(\Omega,\mathcal{P})]^d$. These estimates are useful for the analysis of discontinuous Galerkin methods for elasticity problems (cf. [15], [6], [12] and the references therein).

Remark 1.2. Note that classical Korn's inequalities can also be expressed in terms of the full H^1 norm (cf. [8], [14], [5]). In view of (1.11)–(1.13) and the following

Poincaré-Friedrichs inequalities (cf. [2]):

$$\|\boldsymbol{u}\|^2_{L_2(\Omega)} \le C\Big(|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} + \Big|\int_\Gamma \boldsymbol{u}\,ds\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\sigma)^{-1}\|\pi_{\sigma,0}[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\Big),$$

$$\|\boldsymbol{u}\|^2_{L_2(\Omega)} \le C\Big(|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} + \Big|\int_\Omega \boldsymbol{u}\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\sigma)^{-1}\|\pi_{\sigma,0}[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\Big),$$

for all $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^d$, where $\pi_{\sigma,0}$ is the orthogonal projection operator from $[L_2(\sigma)]^d$ onto $[P_0(\sigma)]^d$, the space of constant vector functions on σ, we also have the following "full norm" versions of Korn's inequalities for piecewise H^1 vector fields:

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.21}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \sup_{\substack{\boldsymbol{m}\in\mathbf{RM}(\Omega)\\ \|\boldsymbol{m}\|_{L_2(\Gamma)}=1}} \Big(\int_\Gamma \boldsymbol{u}\cdot\boldsymbol{m}\,ds\Big)^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.22}$$

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\int_\Omega \boldsymbol{u}\,dx\Big|^2 + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big), \tag{1.23}$$

for all $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^d$. The "full norm" versions of (1.14)–(1.16) and (1.18)–(1.20) can be readily derived from (1.21)–(1.23).

Remark 1.3. Let $P_1(\Gamma)$ be the restriction of $P_1(\mathbb{R}^d)$ to Γ and let π_Γ be the orthogonal projection operator from $[L_2(\Gamma)]^d$ onto $[P_1(\Gamma)]^d$. Then Korn's first inequalities (1.8) and (1.15) (resp. (1.12) and (1.22)) remain valid if the terms involving the integral over Γ in these inequalities are replaced by $\|\pi_\Gamma u\|_{L_2(\Gamma)}$ (resp. $\|\pi_\Gamma u\|^2_{L_2(\Gamma)}$).

The rest of the paper is organized as follows. First we derive Korn's inequalities for piecewise linear and piecewise H^1 vector fields with respect to simplicial triangulations of Ω. These are carried out in Section 2 and Section 3. Korn's inequalities for piecewise H^1 vector fields with respect to general partitions are then established in Section 4 for two-dimensional domains and in Section 5 for three-dimensional domains. A generalization of the result in Section 2 to piecewise polynomial vector fields is given in Section 6, which can be used to derive (1.14)–(1.16) for some nonconforming finite elements that violate (1.17). The appendix contains a discussion of the dependence of the constant in Korn's second inequality (1.9) on the underlying domain, which is used in Section 3 and Section 5.

Throughout this paper we use $|S|$ to denote the k-dimensional volume of a k-dimensional geometric object S in a Euclidean space.

2. A generalized Korn's inequality for piecewise linear vector fields with respect to simplicial triangulations

In this and the next two sections we restrict our attention to the case where the partition is actually a triangulation $\mathcal{T}$ of Ω by simplexes (i.e., triangles for $d = 2$ and tetrahedra for $d = 3$). The intersection of the closures of any two simplexes in $\mathcal{T}$ is therefore either empty, a vertex, a closed edge or a closed face. In this case $S(\mathcal{T}, \Omega)$ coincides with the set of interior open edges ($d = 2$) or open faces ($d = 3$). The minimum angle of the triangles or tetrahedra in $\mathcal{T}$ will be denoted by $\theta_{\mathcal{T}}$.

To avoid the proliferation of constants, we henceforth use the notation $A \lesssim B$ to represent the statement $A \leq \kappa(\theta_{\mathcal{T}})B$, where the (generic) function $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and independent of $\mathcal{T}$. The notation $A \approx B$ is equivalent to $A \lesssim B$ and $B \lesssim A$.

Let $V_{\mathcal{T}} = \{\boldsymbol{v} \in [L_2(\Omega)]^d : \boldsymbol{v}_T = \boldsymbol{v}\big|_T \in [P_1(T)]^d \quad \forall T \in \mathcal{T}\}$ be the space of piecewise linear vector fields and $W_{\mathcal{T}} = \{\boldsymbol{w} \in [H^1(\Omega)]^d : \boldsymbol{w}_T = \boldsymbol{w}\big|_T \in [P_1(T)]^d$ $\forall T \in \mathcal{T}\}$ be the space of continuous piecewise linear vector fields. We define a linear map $E : V_{\mathcal{T}} \longrightarrow W_{\mathcal{T}}$ as follows. Let $\mathcal{V}(\mathcal{T})$ be the set of all the vertices of $\mathcal{T}$. Then $E\boldsymbol{v}$ is defined by

$$\big(E\boldsymbol{v}\big)(p) = \frac{1}{|\chi_p|} \sum_{T \in \chi_p} \boldsymbol{v}_T(p) \qquad \forall\, p \in \mathcal{V}(\mathcal{T}), \tag{2.1}$$

where

$$\chi_p = \{T \in \mathcal{T} : p \in \partial T\}$$

is the set of simplexes sharing p as a common vertex and $|\chi_p|$ is the number of simplexes in χ_p. Note that

$$|\chi_p| \lesssim 1 \qquad \forall\, p \in \mathcal{V}(\mathcal{T}). \tag{2.2}$$

The following lemma contains the basic estimate for the operator E.

Lemma 2.1. *It holds that*

$$|(\boldsymbol{v}_T - E\boldsymbol{v})(p)|^2 \lesssim \sum_{\sigma \in \Xi_p} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \qquad \forall\, \boldsymbol{v} \in V_{\mathcal{T}},\ T \in \mathcal{T} \quad \text{and} \quad p \in \mathcal{V}(T), \tag{2.3}$$

where $\mathcal{V}(T)$ is the set of the vertices of the simplex T,

$$\Xi_p = \{\sigma \in S(\mathcal{T}, \Omega) : p \in \partial\sigma\}$$

is the set of interior sides sharing p as a common vertex, and $[\boldsymbol{v}]_\sigma$ is the jump of $\boldsymbol{v}$ across σ.

Proof. Let $\boldsymbol{v} \in V_{\mathcal{T}}$, $T \in \mathcal{T}$ and $p \in \mathcal{V}(T)$. We have, by (2.1),

$$(\boldsymbol{v}_T - E\boldsymbol{v})(p) = \frac{1}{|\chi_p|} \sum_{T' \in \chi_p} \big(\boldsymbol{v}_T(p) - \boldsymbol{v}_{T'}(p)\big). \tag{2.4}$$

Let T' be a simplex in χ_p. There exists a chain of simplexes $T_1, \ldots, T_m \in \chi_p$ such that (i) $T_1 = T$ and $T_m = T'$, and (ii) T_j and T_{j+1} share a common side $\sigma_j \in \Xi_p$. (A two-dimensional example is depicted in Figure 2.)

Note that (2.2) implies $m \lesssim 1$ and hence

$$|\boldsymbol{v}_T(p) - \boldsymbol{v}_{T'}(p)|^2 = \Big|\sum_{j=1}^{m-1} \big(\boldsymbol{v}_{T_j}(p) - \boldsymbol{v}_{T_{j+1}}(p)\big)\Big|^2 \lesssim \sum_{j=1}^{m-1} \big|[\boldsymbol{v}]_{\sigma_j}(p)\big|^2. \tag{2.5}$$

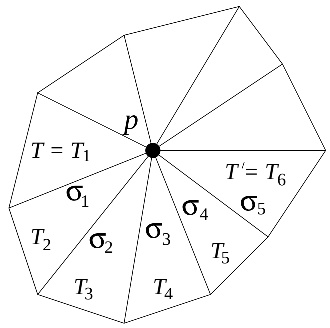

FIGURE 2. A chain of triangles connecting T and T'

The estimate (2.3) follows from (2.4) and (2.5). □

We can now prove a generalized Korn's inequality for functions in $V_{\mathcal{T}}$.

Lemma 2.2. *Let $\Phi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow \mathbb{R}$ be a seminorm such that*

$$|\Phi(\boldsymbol{w})| \le C_\Phi \|\boldsymbol{w}\|_{H^1(\Omega)} \qquad \forall\, \boldsymbol{w} \in [H^1(\Omega)]^d, \tag{2.6}$$

$$\big(\Phi(\boldsymbol{v} - E\boldsymbol{v})\big)^2 \lesssim \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \quad \forall\, \boldsymbol{v} \in V_{\mathcal{T}}, \tag{2.7}$$

where $\mathcal{V}(\sigma)$ is the set of the vertices of σ, and

$$\Phi(\boldsymbol{m}) = 0 \quad \text{and} \quad \boldsymbol{m} \in \mathbf{RM}(\Omega) \iff \boldsymbol{m} = \text{a constant vector}. \tag{2.8}$$

Then the following estimate holds:

$$|\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{v})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{v})\big)^2 + \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \tag{2.9}$$

for all $\boldsymbol{v} \in V_{\mathcal{T}}$, where $\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{v})\big|_T = \boldsymbol{\epsilon}(\boldsymbol{v}_T)$ for all $T \in \mathcal{T}$.

Proof. Observe first that (2.2), (2.3) and a standard finite element estimate for $|\cdot|_{H^1(T)}$ (cf. [4], [3]) imply

$$\begin{aligned} |\boldsymbol{v} - E\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} &\lesssim \sum_{T \in \mathcal{T}} (\operatorname{diam} T)^{d-2} \sum_{p \in \mathcal{V}(T)} |(\boldsymbol{v}_T - E\boldsymbol{v})(p)|^2 \\ &\lesssim \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \qquad \forall\, \boldsymbol{v} \in V_{\mathcal{T}}, \end{aligned} \tag{2.10}$$

where we have also used the relation

$$\operatorname{diam} T \approx \operatorname{diam} \sigma \qquad \forall\, T \in \chi_p,\ \sigma \in \Xi_p \quad \text{and} \quad p \in \mathcal{V}(\mathcal{T}). \tag{2.11}$$

From (1.6), (2.7) and (2.10) we then find, for arbitrary $\boldsymbol{v} \in V_{\mathcal{T}}$,

$$\begin{aligned} |\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} &\lesssim |E\boldsymbol{v}|^2_{H^1(\Omega)} + |\boldsymbol{v} - E\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \\ &\lesssim \|\boldsymbol{\epsilon}(E\boldsymbol{v})\|^2_{L_2(\Omega)} + \big(\Phi(E\boldsymbol{v})\big)^2 + |\boldsymbol{v} - E\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \\ &\lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{v})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{v})\big)^2 + \big(\Phi(\boldsymbol{v} - E\boldsymbol{v})\big)^2 + |\boldsymbol{v} - E\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \\ &\lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{v})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{v})\big)^2 + \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2. \end{aligned}$$

□

The following are examples of Φ that satisfy conditions (2.6)–(2.8). The validity of (2.6) and (2.8) is obvious in all three examples.

Example 2.3. Let $\Phi_1 : [H^1(\Omega,\mathcal{T})]^d \longrightarrow \mathbb{R}$ be defined by

$$\Phi_1(\boldsymbol{v}) = \|Q\boldsymbol{v}\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{v} \in [H^1(\Omega,\mathcal{T})]^d,$$

where Q is the orthogonal projection from $[L_2(\Omega)]^d$ onto the orthogonal complement of the constant vector fields. Condition (2.7) can be verified as follows:

$$\begin{aligned}\big(\Phi_1(\boldsymbol{v}-E\boldsymbol{v})\big)^2 \le \|\boldsymbol{v}-E\boldsymbol{v}\|^2_{L_2(\Omega)} &\lesssim \sum_{T\in\mathcal{T}} (\,\mathrm{diam}\ T)^d \sum_{p\in\mathcal{V}(T)} |(\boldsymbol{v}_{_T} - E\boldsymbol{v})(p)|^2 \\ &\lesssim \sum_{\sigma\in S(\mathcal{T},\Omega)} (\,\mathrm{diam}\ \sigma)^d \sum_{p\in\mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \quad \forall\, \boldsymbol{v}\in V_{\mathcal{T}},\end{aligned}$$

where we have used (2.2), (2.3), (2.11) and a standard finite element estimate for the L_2-norm.

Example 2.4. Let $\Phi_2 : [H^1(\Omega,\mathcal{T})] \longrightarrow \mathbb{R}$ be defined by

$$\Phi_2(\boldsymbol{v}) = \sup_{\substack{\boldsymbol{m}\in\mathbf{RM}(\Omega)\\ \|\boldsymbol{m}\|_{L_2(\Gamma)}=1,\, \int_\Gamma \boldsymbol{m}\,ds=\mathbf{0}}} \int_\Gamma \boldsymbol{v}\cdot\boldsymbol{m}\,ds \qquad \forall\,\boldsymbol{v}\in[H^1(\Omega,\mathcal{T})]^d,$$

where Γ is a measurable subset of $\partial\Omega$ with a positive $(d-1)$-dimensional volume. Using (2.2), (2.3), (2.11) and a standard finite element estimate for the L_2-norm, condition (2.7) can be verified as follows:

$$\begin{aligned}\big(\Phi_2(\boldsymbol{v}-E\boldsymbol{v})\big)^2 &\le \|\boldsymbol{v}-E\boldsymbol{v}\|^2_{L_2(\partial\Omega)} \\ &\lesssim \sum_{\substack{T\in\mathcal{T}\\ \partial T\cap\partial\Omega\neq\emptyset}} (\,\mathrm{diam}\ T)^{d-1}\Big(\sum_{p\in\mathcal{V}(T)} |(\boldsymbol{v}_{_T}-E\boldsymbol{v})(p)|\Big)^2 \\ &\lesssim \sum_{\sigma\in S(\mathcal{T},\Omega)} (\,\mathrm{diam}\ \sigma)^{d-1} \sum_{p\in\mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \qquad \forall\,\boldsymbol{v}\in V_{\mathcal{T}}.\end{aligned}$$

Example 2.5. Let $\Phi_3 : [H^1(\Omega,\mathcal{T})]^d \longrightarrow \mathbb{R}$ be defined by

$$\Phi_3(\boldsymbol{v}) = \Big|\sum_{T\in\mathcal{T}} \int_T \nabla\times\boldsymbol{v}\,dx\Big| \qquad \forall\,\boldsymbol{v}\in[H^1(\Omega,\mathcal{T})]^d.$$

Using (2.10), we can easily verify condition (2.7):

$$\begin{aligned}\big(\Phi_3(\boldsymbol{v}-E\boldsymbol{v})\big)^2 &\lesssim |\boldsymbol{v}-E\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \\ &\lesssim \sum_{\sigma\in S(\mathcal{T},\Omega)} (\,\mathrm{diam}\ \sigma)^{d-2} \sum_{p\in\mathcal{V}(\sigma)} \big|[\boldsymbol{v}]_\sigma(p)\big|^2 \qquad \forall\,\boldsymbol{v}\in V_{\mathcal{T}}.\end{aligned}$$

Remark 2.6. The definitions of the seminorms Φ_1, Φ_2 and Φ_3 can be extended to $[H^1(\Omega,\mathcal{P})]^d$ for a general partition $\mathcal{P}$ of Ω. In fact, if we denote by $\mathfrak{P}_\Omega$ the set of all the partitions of Ω, then Φ_j is a well-defined function on $\bigcup_{\mathcal{P}\in\mathfrak{P}_\Omega}[H^1(\Omega,\mathcal{P})]^d$ for $1\le j\le 3$.

3. Korn's inequalities for piecewise H^1 vector fields with respect to simplicial triangulations

Let $\mathcal{T}$ be a simplicial triangulation of Ω. First we define on each $T \in \mathcal{T}$ an interpolation operator Π_T from $[H^1(T)]^d$ onto $\mathbf{RM}(T)$ (the space of the rigid motions restricted to T) by the following conditions:

$$\Big| \int_T (\boldsymbol{v} - \Pi_T \boldsymbol{v})\, dx \Big| = 0 \qquad \forall\, \boldsymbol{v} \in [H^1(T)]^d, \tag{3.1}$$

$$\Big| \int_T \nabla \times (\boldsymbol{v} - \Pi_T \boldsymbol{v})\, dx \Big| = 0 \qquad \forall\, \boldsymbol{v} \in [H^1(T)]^d. \tag{3.2}$$

These conditions determine Π_T because

$$\boldsymbol{m} \in \mathbf{RM}(T) \quad \text{and} \quad \Big| \int_T \boldsymbol{m}\, dx \Big| = \Big| \int_T \nabla \times \boldsymbol{m}\, dx \Big| = 0 \quad \Longleftrightarrow \quad \boldsymbol{m} = \mathbf{0}.$$

Note that (1.4), (3.2) and Corollary A.3 in the appendix imply

$$|\boldsymbol{v} - \Pi_T \boldsymbol{v}|_{H^1(T)} \lesssim \|\boldsymbol{\epsilon}(\boldsymbol{v} - \Pi_T \boldsymbol{v})\|_{L_2(T)} = \|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L_2(T)} \tag{3.3}$$

for all $T \in \mathcal{T}$, $\boldsymbol{v} \in [H^1(T)]^d$, and (3.1) together with the classical Poincaré-Friedrichs inequality (with scaling) yields

$$\|\boldsymbol{v} - \Pi_T \boldsymbol{v}\|_{L_2(T)} \lesssim (\operatorname{diam} T) |\boldsymbol{v} - \Pi_T \boldsymbol{v}|_{H^1(T)} \qquad \forall\, T \in \mathcal{T},\ \boldsymbol{v} \in [H^1(T)]^d. \tag{3.4}$$

Let $\Pi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow V_{\mathcal{T}}$, the space of piecewise linear vector fields with respect to $\mathcal{T}$, be defined by

$$(\Pi \boldsymbol{u})\big|_T = \Pi_T \boldsymbol{u}_T \qquad \forall\, T \in \mathcal{T}. \tag{3.5}$$

We can now prove a generalized Korn's inequality for functions in $[H^1(\Omega, \mathcal{T})]^d$.

Theorem 3.1. *Let $\Phi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow \mathbb{R}$ be a seminorm satisfying conditions (2.6)–(2.8) and, in addition, the condition that*

$$\Phi(\boldsymbol{u} - \Pi \boldsymbol{u}) \lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|_{L_2(\Omega)} \qquad \forall\, \boldsymbol{u} \in [H^1(\Omega, \mathcal{T})]^d, \tag{3.6}$$

where $\Pi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow V_{\mathcal{T}}$ is defined by (3.5). Then the following estimate holds:

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T})} \le \kappa(\theta_{\mathcal{T}}) \Big(&\|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 \\ &+ \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{-1} \big\| \pi_\sigma [\boldsymbol{u}]_\sigma \big\|^2_{L_2(\sigma)} \Big) \qquad \forall \boldsymbol{u} \in [H^1(\Omega, \mathcal{T})]^d, \end{aligned} \tag{3.7}$$

where $\kappa : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous function independent of $\mathcal{T}$.

Proof. Let $\boldsymbol{u} \in [H^1(\Omega, \mathcal{T})]^d$ be arbitrary. From (1.4), (2.9) and (3.3) we have

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T})} &\lesssim |\boldsymbol{u} - \Pi \boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T})} + |\Pi \boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T})} \\ &\lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\Pi \boldsymbol{u})\big)^2 + \sum_{\sigma \in S(\mathcal{T},\Omega)} (\operatorname{diam} \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\Pi \boldsymbol{u}]_\sigma(p)\big|^2. \end{aligned} \tag{3.8}$$

Using condition (3.6), we immediately find

$$\Phi(\Pi \boldsymbol{u}) \le \Phi(\boldsymbol{u} - \Pi \boldsymbol{u}) + \Phi(\boldsymbol{u}) \lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Phi(\boldsymbol{u}). \tag{3.9}$$

Let $\sigma \in S(\mathcal{T}, \Omega)$ be arbitrary and $p \in \mathcal{V}(\sigma)$. We have, by a standard inverse estimate (cf. [4], [3]),

$$
\begin{aligned}
(3.10) \qquad \big|[\Pi \boldsymbol{u}]_\sigma(p)\big|^2 &\lesssim \big|\pi_\sigma[\boldsymbol{u}]_\sigma(p)\big|^2 + \big|\pi_\sigma[\boldsymbol{u} - \Pi\boldsymbol{u}]_\sigma(p)\big|^2 \\
&\lesssim (\,\mathrm{diam}\ \sigma)^{1-d}\big(\|\pi_\sigma[\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)} + \|\pi_\sigma[\boldsymbol{u} - \Pi\boldsymbol{u}]_\sigma\|^2_{L_2(\sigma)}\big),
\end{aligned}
$$

where we have also used the fact that $\pi_\sigma[\Pi\boldsymbol{u}]_\sigma = [\Pi\boldsymbol{u}]_\sigma$ since $[\Pi\boldsymbol{u}]_\sigma \in [P_1(\sigma)]^d$.

Let $\mathfrak{T}_\sigma$ be the set of the two simplexes in $\mathcal{T}$ sharing σ as a common side. It follows from (3.3), (3.4) and the trace theorem (with scaling) that

$$
\begin{aligned}
\big\|\pi_\sigma[\boldsymbol{u} - \Pi\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)} &\le \big\|[\boldsymbol{u} - \Pi\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)} \\
(3.11) \qquad &\lesssim \sum_{T \in \mathfrak{T}_\sigma} \Big((\,\mathrm{diam}\ T)|\boldsymbol{u}_T - \Pi_T\boldsymbol{u}_T|^2_{H^1(T)} \\
&\qquad\qquad + (\,\mathrm{diam}\ T)^{-1}\|\boldsymbol{u}_T - \Pi_T\boldsymbol{u}_T\|^2_{L_2(T)}\Big) \\
&\lesssim \sum_{T \in \mathfrak{T}_\sigma} (\,\mathrm{diam}\ T)\|\boldsymbol{\epsilon}(\boldsymbol{u}_T)\|^2_{L_2(T)}.
\end{aligned}
$$

Combining (2.11), (3.10) and (3.11), we find

$$
\begin{aligned}
(3.12) \qquad &\sum_{\sigma \in S(\mathcal{T},\Omega)} (\,\mathrm{diam}\ \sigma)^{d-2} \sum_{p \in \mathcal{V}(\sigma)} \big|[\Pi\boldsymbol{u}]_\sigma(p)\big|^2 \\
&\qquad \lesssim \sum_{\sigma \in S(\mathcal{T},\Omega)} (\,\mathrm{diam}\ \sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)} + \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|^2_{L_2(\Omega)}.
\end{aligned}
$$

The estimate (3.7) follows from (3.8), (3.9) and (3.12). □

Finally we observe that the seminorms in Examples 2.3–2.5 satisfy the condition (3.6). In view of (3.2), this is trivial for Φ_3. Using (3.3) and (3.4), the case of Φ_1 can be established as follows:

$$
\begin{aligned}
\big(\Phi_1(\boldsymbol{u} - \Pi\boldsymbol{u})\big)^2 &\le \sum_{T \in \mathcal{T}} \|\boldsymbol{u}_T - \Pi_T\boldsymbol{u}_T\|^2_{L_2(T)} \\
&\lesssim \sum_{T \in \mathcal{T}} (\,\mathrm{diam}\ T)^2\|\boldsymbol{\epsilon}(\boldsymbol{u}_T)\|^2_{L_2(T)} \lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|^2_{L_2(\Omega)}.
\end{aligned}
$$

For the case of Φ_2, we apply (3.3), (3.4) and the trace theorem to obtain

$$
\begin{aligned}
\big(\Phi_2(\boldsymbol{u} - \Pi\boldsymbol{u})\big)^2 &\le \|\boldsymbol{u} - \Pi\boldsymbol{u}\|^2_{L_2(\partial\Omega)} \\
&\lesssim \sum_{T \in \mathcal{T},\, \partial T \cap \partial\Omega \ne \emptyset} (\,\mathrm{diam}\ T)\|\boldsymbol{\epsilon}(\boldsymbol{u}_T)\|^2_{L_2(T)} \lesssim \|\boldsymbol{\epsilon}_{\mathcal{T}}(\boldsymbol{u})\|^2_{L_2(\Omega)}.
\end{aligned}
$$

Remark 3.2. From here on we assume that $\Phi : \bigcup_{\mathcal{P} \in \mathfrak{P}_\Omega} [H^1(\Omega, \mathcal{P})]^d \longrightarrow \mathbb{R}$ is a seminorm for every $\mathcal{P} \in \mathfrak{P}_\Omega$ (cf. Remark 2.6) and that it satisfies the conditions (2.6)–(2.8) and (3.6) for every $\mathcal{T} \in \mathfrak{P}_\Omega$.

Remark 3.3. By choosing Φ to be Φ_1, Φ_2 or Φ_3, we immediately obtain Korn's inequalities (1.11)–(1.13) in the case where $\mathcal{P}$ is a simplicial triangulation. Similar remarks apply in the next three sections.

4. Korn's inequalities for $[H^1(\Omega, \mathcal{P})]^2$ on a two-dimensional Ω

First we need a precise definition of the set $S(\mathcal{P}, \Omega)$ of interior (open) edges for a general partition $\mathcal{P}$, which in turn requires the concept of a vertex of $\mathcal{P}$. We define a vertex of $\mathcal{P}$ to be a vertex of any of the subdomains in $\mathcal{P}$. (For example, the partition of the square in Figure 1 has 14 vertices.) We then define an open edge of $\mathcal{P}$ to be an open line segment on the boundary of a subdomain in $\mathcal{P}$ bounded between two of the vertices of $\mathcal{P}$. The set $S(\mathcal{P}, \Omega)$ consists of the open edges of $\mathcal{P}$ that are common to the boundaries of two subdomains in $\mathcal{P}$.

Remark 4.1. The concept of an edge of a polygon $D \in \mathcal{P}$ and the concept of an edge of $\mathcal{P}$ on ∂D are different. For example, a square always has 4 edges while there are 5 edges of the two-dimensional partition in Figure 1 on the boundary of the square at the lower right corner.

In order to apply Theorem 3.1 we introduce the set

$$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{T} : \mathcal{T} \text{ is a simplicial triangulation of } \Omega \text{ and each member of } S(\mathcal{P}, \Omega) \text{ is also an edge of } \mathcal{T}\}. \tag{4.1}$$

By definition (4.1), $[H^1(\Omega, \mathcal{P})]^2$ is a subspace of $[H^1(\Omega, \mathcal{T})]^2$ for every $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$. Since functions in $[H^1(\Omega, \mathcal{P})]^2$ are continuous on the edges of $\mathcal{T}$ that are not in $S(\mathcal{P}, \Omega)$, the following result is an immediate consequence of Theorem 3.1.

Theorem 4.2. *Let Φ be as in Remark 3.2. Then we have*

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le & \Big(\inf_{\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}} \kappa(\theta_{\mathcal{T}}) \Big) \\ & \times \Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma \in S(\mathcal{P},\Omega)} (\operatorname{diam} \sigma)^{-1} \big\| \pi_\sigma [\boldsymbol{u}]_\sigma \big\|^2_{L_2(\sigma)} \Big) \end{aligned} \tag{4.2}$$

for all $\boldsymbol{u} \in [H^1(\Omega, \mathcal{P})]^2$, where $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function independent of $\mathcal{P}$.

The set $\{\theta_{\mathcal{T}} : \mathcal{T} \in \mathfrak{T}_{\mathcal{P}}\}$ provides an abstract measure of the shape regularity of the partition $\mathcal{P}$ and the number $\inf_{\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}} \kappa(\theta_{\mathcal{T}})$ can be viewed as a constant depending on the shape regularity of $\mathcal{P}$. However, in applications one may want to relate the abstract estimate (4.2) to a concrete description of the shape regularity of $\mathcal{P}$ given in terms of (i) the shape regularity of individual subdomains and (ii) the relative positions of subdomains that share a common edge of $\mathcal{P}$.

We can measure the shape regularity of a polygon (or a polyhedron in 3D) by using an affine homeomorphism between D and a reference domain and by using the aspect ratio of D defined by (diameter of D)/(radius of the largest disc (or ball) in the closure of D).

The relative positions between subdomains sharing a common edge of $\mathcal{P}$ can be measured in terms of the quantity

$$\rho(\mathcal{P}) = \max\{|\partial D|/|\sigma| : \sigma \in S(\mathcal{P}, \Omega),\ D \in \mathcal{P} \text{ and } \sigma \subset \partial D\}. \tag{4.3}$$

The following corollary gives an application of Theorem 4.2 to a fairly general class of two-dimensional partitions.

Corollary 4.3. *Let Φ be as in Remark 3.2 and let $\{\mathcal{P}_i : i \in I\}$ be a family of partitions of Ω such that*

(i) *the polygons appearing in all the partitions* $\mathcal{P}_i$ *are affine homeomorphic to a fixed finite set of reference polygons and the aspect ratios of the polygons in all the* $\mathcal{P}_i$*'s are uniformly bounded,*

(ii) *the set* $\{\rho(\mathcal{P}_i) : i \in I\}$ *is bounded.*

Then there exists a positive constant C*, independent of* $i \in I$*, such that*

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P}_i)} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}_i}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma\in S(\mathcal{P}_i,\Omega)} (\,\mathrm{diam}\ \sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big) \tag{4.4}$$

for any $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P}_i)]^2$ *and* $i \in I$*.*

Proof. It suffices to show that under the assumptions on the family of partitions we can construct one partition $\mathcal{T}_i \in \mathfrak{T}_{\mathcal{P}_i}$ for each $i \in I$ such that $\inf\{\theta_{\mathcal{T}_i} : i \in I\} \ge \theta_0 > 0$. Then the estimate (4.4) follows from (4.2) if we take C to be an upper bound of the bounded set $\{\kappa(\theta_{\mathcal{T}_i}) : i \in I\}$.

First we construct a simplicial triangulation on each reference polygon so that each edge of the reference polygon is also an edge of the triangulation and each triangle in the triangulation can have at most one edge on the boundary of the reference polygon.

Let $D \in \mathcal{P}_i$. We can induce a triangulation $\mathcal{T}_D$ on D using the triangulation on a reference polygon and the corresponding affine homeomorphism. Let $p \in \partial D$ be a vertex of $\mathcal{P}$ which is not a vertex of D. Then p belongs to an edge of D which is an edge of a triangle $T \in \mathcal{T}_D$, and we connect p to the vertex of T not on ∂D by a straight line. In this way we have created a triangulation $\mathcal{T}_i \in \mathfrak{T}_{\mathcal{P}_i}$. (This construction is carried out in Figure 3 for the two-dimensional partition in Figure 1, where the reference square is triangulated by its two diagonals.)

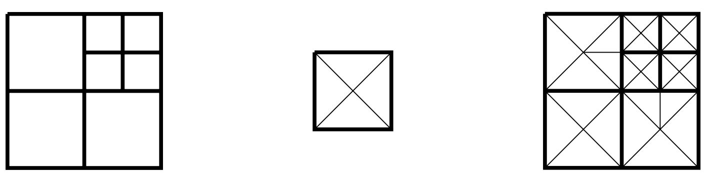

FIGURE 3. An example of the construction of $\mathfrak{T}_{\mathcal{P}_i}$

Let $\hat{D}$ be the reference polygon affine homeomorphic to D and let $\hat{x} \mapsto \alpha(\hat{x}) = B\hat{x} + b$ be the corresponding affine map from $\hat{D}$ to D. The uniform boundedness of the aspect ratios implies (cf. Theorem 3.1.3 in [4]) the existence of a positive constant C_*, independent of $i \in I$, such that

$$\|B\| \le C_*(\mathrm{diam}\, D) \quad \text{and} \quad \|B^{-1}\| \le C_*(\mathrm{diam}\, D)^{-1}, \tag{4.5}$$

where $\|\cdot\|$ is the matrix 2-norm induced by the Euclidean vector norm. Hence we have

$$C_*^{-2}\frac{|\hat{x}_1-\hat{x}_2|}{|\hat{x}_3-\hat{x}_4|} \le \frac{|x_1-x_2|}{|x_3-x_4|} \le C_*^2\frac{|\hat{x}_1-\hat{x}_2|}{|\hat{x}_3-\hat{x}_4|}, \tag{4.6}$$

where $x_j = \alpha(\hat{x}_j)$, and $\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_3$ are any four points such that $\hat{x}_1 \ne \hat{x}_2$ and $\hat{x}_3 \ne \hat{x}_4$.

We conclude from (4.6) and the boundedness of the set $\{\rho(\mathcal{P}_i) : i \in I\}$ that $\theta_{\mathcal{T}_i}$ is bounded away from zero. $\square$

Remark 4.4. If the family of partitions $\{\mathcal{P}_i : i \in I\}$ in Corollary 4.3 is actually a family of triangulations (simplicial or otherwise), then the condition on the boundedness of $\{\rho(\mathcal{P}_i) : i \in I\}$ is redundant.

An example of a family of partitions satisfying the assumptions of Corollary 4.3 is depicted in Figure 4, where a square is being refined successively towards the upper right corner.

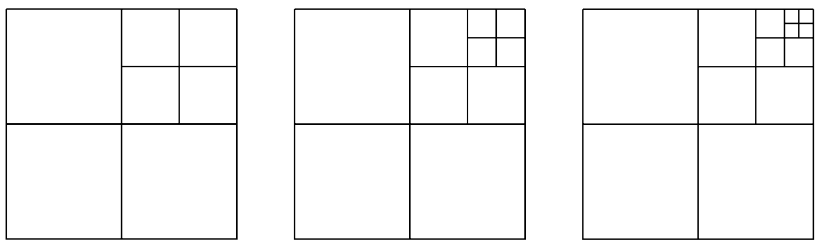

FIGURE 4. A family of partitions of a square

5. KORN'S INEQUALITIES FOR $[H^1(\Omega, \mathcal{P})]^3$ ON A THREE-DIMENSIONAL Ω

In order to give a precise definition of $S(\mathcal{P}, \Omega)$, we first introduce the concept of an edge of $\mathcal{P}$, which is just an edge of any of the subdomains in $\mathcal{P}$. We then define an open face of $\mathcal{P}$ to be an open subset of the boundary of a subdomain in $\mathcal{P}$ enclosed by edges of $\mathcal{P}$. The set $S(\mathcal{P}, \Omega)$ consists of open faces of $\mathcal{P}$ common to the boundaries of two subdomains in $\mathcal{P}$.

Remark 5.1. Again the concept of a face of a polyhedron $D \in \mathcal{P}$ and the concept of a face of $\mathcal{P}$ on ∂D are different. For example, there are always 6 faces on a parallelepiped while there are 9 faces of the three-dimensional partition in Figure 1 on the boundary of the subdomain in the back.

As in the two-dimensional case, we would like to derive a generalized Korn's inequality for partitions from Theorem 3.1. But here the situation is more complicated since the faces in $S(\mathcal{P}, \Omega)$ may not be triangles. Accordingly we introduce the following family of triangulations:

$$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{T} : \mathcal{T} \text{ is a simplicial triangulation of } \Omega \text{ such that each face in } S(\mathcal{P}, \Omega) \text{ is triangulated by the (triangular) faces in } S(\mathcal{T}, \Omega)\}. \tag{5.1}$$

Since a face in $S(\mathcal{P}, \Omega)$ may not be a face in $S(\mathcal{T}, \Omega)$ for $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$, we cannot immediately derive an analog of Theorem 4.2. We need to introduce two more parameters related to the shape regularity of $\mathcal{P}$ in addition to the parameter $\rho(\mathcal{P})$ already defined in (4.3).

Let $\mathcal{T} \in \mathfrak{T}_{\mathcal{P}}$. For $\sigma \in S(\mathcal{P}, \Omega)$ we will denote by $\mathcal{T}_\sigma$ the triangulation of σ by faces of $S(\mathcal{T}, \Omega)$, i.e., $\mathcal{T}_\sigma = \{\tilde{\sigma} \in S(\mathcal{T}, \Omega) : \tilde{\sigma} \subseteq \sigma\}$, and define the parameter

$$\rho(\mathcal{P}, \mathcal{T}) = \max\left\{|\sigma|/|\tilde{\sigma}| : \sigma \in S(\mathcal{P}, \Omega) \text{ and } \tilde{\sigma} \in \mathcal{T}_\sigma\right\} . \tag{5.2}$$

Note the following obvious bound for $|\mathcal{T}_\sigma|$ (the number of elements in $\mathcal{T}_\sigma$):

$$|\mathcal{T}_\sigma| \leq \rho(\mathcal{P},\mathcal{T}) \qquad \forall\, \sigma \in S(\mathcal{P},\Omega)\,. \tag{5.3}$$

Moreover (4.3) and (5.2) imply that

$$\frac{|\partial D|}{|\tilde{\sigma}|} \leq \rho(\mathcal{P})\rho(\mathcal{P},\mathcal{T}) \qquad \text{for any } D \in \mathcal{P},\ \tilde{\sigma} \in \mathcal{T}_\sigma \text{ and } \sigma \subset \partial D\,. \tag{5.4}$$

The other parameter is the smallest number $\lambda(\mathcal{P}) \geq 1$ with the property that

$$\|\boldsymbol{v} - \Pi_D \boldsymbol{v}\|^2_{L_2(\partial D)} \leq \lambda(\mathcal{P})|\partial D|^{1/2}\|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{L_2(D)}, \tag{5.5}$$

where D is any subdomain in $\mathcal{P}$, $\boldsymbol{v}$ is any function in $[H^1(D)]^3$ and $\Pi_D : [H^1(D)]^3 \longrightarrow \mathbf{RM}(D)$ (the space of rigid motions restricted to D) is defined by the conditions

$$\int_D (\boldsymbol{v} - \Pi_D\boldsymbol{v})\, dx = \mathbf{0} = \int_D \nabla \times (\boldsymbol{v} - \Pi_D\boldsymbol{v})\, dx. \tag{5.6}$$

The existence of $\lambda(\mathcal{P})$ is a consequence of (1.4), (5.6), the trace theorem, the Poincaré-Friedrichs inequality and Korn's second inequality (1.9).

We can now state and prove a generalized Korn's inequality.

Theorem 5.2. *Let Φ be as in Remark 3.2. Then we have*

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq & \Big(\inf_{\mathcal{T}\in\mathfrak{T}_\mathcal{P}} K\big(\rho(\mathcal{P}),\lambda(\mathcal{P}),\rho(\mathcal{P},\mathcal{T}),\theta_\mathcal{T}\big)\Big) \\ & \times \Big(\|\boldsymbol{\epsilon}_\mathcal{P}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\operatorname{diam}\,\sigma)^{-1}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big) \end{aligned} \tag{5.7}$$

for all $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3$, where $K : \mathbb{R}^4_+ \longrightarrow \mathbb{R}_+$ is a continuous function independent of $\mathcal{P}$.

Proof. Let $\mathcal{T} \in \mathfrak{T}_\mathcal{P}$ and $\sigma \in S(\mathcal{P},\Omega)$. We have, from (5.3) and the Cauchy-Schwarz inequality,

$$\begin{aligned} (\operatorname{diam}\,\sigma)^2 \leq \Big(\sum_{\tilde{\sigma}\in\mathcal{T}_\sigma} \operatorname{diam}\,\tilde{\sigma}\Big)^2 &\leq \rho(\mathcal{P},\mathcal{T}) \sum_{\tilde{\sigma}\in\mathcal{T}_\sigma} (\operatorname{diam}\,\tilde{\sigma})^2 \\ &\leq \rho(\mathcal{P},\mathcal{T})\kappa_*(\theta_\mathcal{T}) \sum_{\tilde{\sigma}\in\mathcal{T}_\sigma} |\tilde{\sigma}| = \rho(\mathcal{P},\mathcal{T})\kappa_*(\theta_\mathcal{T})|\sigma|, \end{aligned}$$

where $\kappa_* : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function independent of $\mathcal{P}$. Therefore it suffices to show that

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq & \Big(\inf_{\mathcal{T}\in\mathfrak{T}_\mathcal{P}} K_*\big(\rho(\mathcal{P}),\lambda(\mathcal{P}),\rho(\mathcal{P},\mathcal{T}),\theta_\mathcal{T}\big)\Big) \\ & \times \Big(\|\boldsymbol{\epsilon}_\mathcal{P}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} |\sigma|^{-(1/2)}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big) \end{aligned} \tag{5.8}$$

for all $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3$, where $K_* : \mathbb{R}^4_+ \longrightarrow \mathbb{R}_+$ is a continuous function independent of $\mathcal{P}$.

Since definition (5.1) implies that $[H^1(\Omega,\mathcal{P})]^3$ is a subspace of $[H^1(\Omega,\mathcal{T})]^3$ for every $\mathcal{T} \in \mathfrak{T}_\mathcal{P}$, we immediately obtain from (3.7) the estimate

$$\begin{aligned} |\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq \kappa(\theta_\mathcal{T})\Big(& \|\boldsymbol{\epsilon}_\mathcal{P}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 \\ & + \sum_{\sigma\in S(\mathcal{P},\Omega)} \sum_{\tilde{\sigma}\in\mathcal{T}_\sigma} |\tilde{\sigma}|^{-(1/2)}\big\|\pi_{\tilde{\sigma}}[\boldsymbol{u}]_{\tilde{\sigma}}\big\|^2_{L_2(\tilde{\sigma})}\Big) \end{aligned} \tag{5.9}$$

for any $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3$, where $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function independent of $\mathcal{P}$.

Let $\sigma \in S(\mathcal{P},\Omega)$ be arbitrary, let $\tilde{\sigma} \in \mathcal{T}_\sigma$ and let $\mathcal{P}_\sigma$ be the set of the two polyhedra in $\mathcal{P}$ that share σ as a common face. It follows from the Cauchy-Schwarz inequality that, for any $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3$,

$$\begin{aligned}(5.10)\qquad &\big\|\pi_{\tilde{\sigma}}[\boldsymbol{u}]_{\tilde{\sigma}}\big\|^2_{L_2(\tilde{\sigma})}\\ &\le 3\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\tilde{\sigma})} + 3\sum_{D\in\mathcal{P}_\sigma}\big\|\pi_{\tilde{\sigma}}\boldsymbol{u}_D - \pi_\sigma\boldsymbol{u}_D\big\|^2_{L_2(\tilde{\sigma})}\\ &\le 3\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)} + 3\sum_{D\in\mathcal{P}_\sigma}\big\|\pi_{\tilde{\sigma}}(\boldsymbol{u}_D - \Pi_D\boldsymbol{u}_D) - \pi_\sigma(\boldsymbol{u}_D - \Pi_D\boldsymbol{u}_D)\big\|^2_{L_2(\tilde{\sigma})},\end{aligned}$$

since $\pi_{\tilde{\sigma}}(\Pi_D\boldsymbol{u}_D) = \Pi_D\boldsymbol{u}_D = \pi_\sigma(\Pi_D\boldsymbol{u}_D)$ on $\tilde{\sigma}$.

From (5.5) we have

$$\begin{aligned}(5.11)\qquad \big\|\pi_{\tilde{\sigma}}(\boldsymbol{u}_D - \Pi_D\boldsymbol{u}_D) - \pi_\sigma(\boldsymbol{u}_D - \Pi_D\boldsymbol{u}_D)\big\|^2_{L_2(\tilde{\sigma})} &\le 4\|\boldsymbol{u}_D - \Pi_D\boldsymbol{u}_D\|^2_{L_2(\partial D)}\\ &\le 4\lambda(\mathcal{P})|\partial D|^{1/2}\|\boldsymbol{\epsilon}(\boldsymbol{u}_D)\|^2_{L_2(D)}.\end{aligned}$$

Note also that (5.2) and (5.3) imply

$$(5.12)\qquad \sum_{\tilde{\sigma}\in\mathcal{T}_\sigma}\frac{|\sigma|^{1/2}}{|\tilde{\sigma}|^{1/2}} \le |\mathcal{T}_\sigma|\,\rho(\mathcal{P},\mathcal{T})^{1/2} \le \rho(\mathcal{P},\mathcal{T})^{3/2}.$$

Combining (4.3) and (5.10)–(5.12), we find

$$\begin{aligned}(5.13)\qquad &\sum_{\tilde{\sigma}\in\mathcal{T}_\sigma}|\tilde{\sigma}|^{-1/2}\big\|\pi_{\tilde{\sigma}}[\boldsymbol{u}]_{\tilde{\sigma}}\big\|^2_{L_2(\tilde{\sigma})}\\ &\le 3\Big(\sum_{\tilde{\sigma}\in\mathcal{T}_\sigma}\frac{|\sigma|^{1/2}}{|\tilde{\sigma}|^{1/2}}\Big)|\sigma|^{-(1/2)}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\\ &\qquad + 12\lambda(\mathcal{P})\Big(\sum_{\tilde{\sigma}\in\mathcal{T}_\sigma}\frac{|\sigma|^{1/2}}{|\tilde{\sigma}|^{1/2}}\Big)\sum_{D\in\mathcal{P}_\sigma}\frac{|\partial D|^{1/2}}{|\sigma|^{1/2}}\|\boldsymbol{\epsilon}(\boldsymbol{u}_D)\|^2_{L_2(D)}\\ &\le 3\rho(\mathcal{P},\mathcal{T})^{3/2}|\sigma|^{-(1/2)}\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\\ &\qquad + 12\lambda(\mathcal{P})\rho(\mathcal{P})^{1/2}\rho(\mathcal{P},\mathcal{T})^{3/2}\sum_{D\in\mathcal{P}_\sigma}\|\boldsymbol{\epsilon}(\boldsymbol{u}_D)\|^2_{L_2(D)},\end{aligned}$$

for any $\sigma \in S(\mathcal{P},\Omega)$ and $\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3$.

Finally we observe that the number of faces in $S(\mathcal{P},\Omega)$ that appear on the boundary of any subdomain in $\mathcal{P}$ is less than or equal to $\rho(\mathcal{P})$, and hence

$$(5.14)\qquad \sum_{\sigma\in S(\mathcal{P},\Omega)}\sum_{D\in\mathcal{P}_\sigma}\|\boldsymbol{\epsilon}(\boldsymbol{u}_D)\|^2_{L_2(D)} \le \rho(\mathcal{P})\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} \qquad \forall\,\boldsymbol{u} \in [H^1(\Omega,\mathcal{P})]^3.$$

The inequality (5.8) follows from (5.9), (5.13) and (5.14), with the function K_* given by, for example,

$$K_*\big(\rho(\mathcal{P}),\lambda(\mathcal{P}),\rho(\mathcal{P},\mathcal{T}),\theta_{\mathcal{T}}\big) = 13\lambda(\mathcal{P})\rho(\mathcal{P})^{3/2}\rho(\mathcal{P},\mathcal{T})^{3/2}\kappa(\theta_{\mathcal{T}}).$$

□

The set $\{(\rho(\mathcal{P}), \lambda(\mathcal{P}), \rho(\mathcal{P}, \mathcal{T}), \theta_{\mathcal{T}}) : \mathcal{T} \in \mathfrak{T}_{\mathcal{P}}\}$ provides an abstract measure of the shape regularity of the partition $\mathcal{P}$ and we can think of

$$\inf_{\mathcal{P} \in \mathfrak{T}_{\mathcal{P}}} K\big(\rho(\mathcal{P}), \lambda(\mathcal{P}), \rho(\mathcal{P}, \mathcal{T}), \theta_{\mathcal{T}}\big)$$

as a constant depending on the shape regularity of $\mathcal{P}$. Under appropriate concrete shape regularity assumptions one can also obtain from Theorem 5.2 Korn's inequalities for a family of partitions with a uniform constant. For simplicity we only give an analog of Corollary 4.3 for partitions by *convex* polyhedra.

Since a face of a partition $\mathcal{P}$ consisting of convex polyhedra is a convex polygon, it can be triangulated by connecting its center to the vertices of $\mathcal{P}$ on its boundary by straight lines. Such a triangulation will be referred to as the *canonical triangulation* of the face.

Corollary 5.3. *Let Φ be as in Remark 3.2 and $\{\mathcal{P}_i : i \in I\}$ be a family of partitions of Ω with the following properties*:

(i) *The polyhedra appearing in all the partitions $\mathcal{P}_i$ are affine homeomorphic to a fixed finite set of convex reference polyhedra and the aspect ratios of the polyhedra in all the $\mathcal{P}_i$'s are uniformly bounded.*

(ii) *The set $\{\rho(\mathcal{P}_i) : i \in I\}$ is bounded.*

(iii) *The angles of the triangles in the canonical triangulations of the faces of all the partitions $\mathcal{P}_i$ are bounded below by a positive constant.*

Then there exists a positive constant C, independent of $i \in I$, such that

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P}_i)} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}_i}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma \in S(\mathcal{P}_i,\Omega)} (\operatorname{diam} \sigma)^{-1} \big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big) \tag{5.15}$$

for any $\boldsymbol{u} \in [H^1(\Omega, \mathcal{P}_i)]^3$ and $i \in I$.

Proof. Let $D \in \mathcal{P}_i$ be affine homeomorphic to the reference polyhedron $\hat{D}$ and let $\alpha(\hat{x}) = B\hat{x} + b$ be the corresponding affine map from $\hat{D}$ to D. Then the estimates (4.5) and (4.6) again follow from condition (i).

From (1.4), (5.6), the trace theorem (with scaling), the classical Poincaré-Friedrichs inequality (with scaling), condition (i) and Lemma A.2 in the appendix, we have

$$\|\boldsymbol{v} - \Pi_D \boldsymbol{v}\|^2_{L_2(\partial D)} \le C_\dagger(\hat{D})|\partial D|^{1/2} \|\boldsymbol{\epsilon}(\boldsymbol{v})\|^2_{L_2(D)} \qquad \forall\, \boldsymbol{v} \in [H^1(D)]^3, \tag{5.16}$$

where $C_\dagger(\hat{D})$ is a positive constant depending only on $\hat{D}$. Since there are only finitely many reference polyhedra for the partitions $\mathcal{P}_i$, we conclude from (5.16) that the set

$$\{\lambda(\mathcal{P}_i) : i \in I\} \text{ is bounded.} \tag{5.17}$$

For each $i \in I$ we can construct a triangulation $\mathcal{T}_i \in \mathfrak{T}_{\mathcal{P}_i}$ by first imposing the canonical triangulation on each face of $\mathcal{P}_i$ and then triangulating each subdomain $D \in \mathcal{P}_i$ using its center and the triangles on its faces.

Let σ be a face of $\mathcal{P}_i$. Condition (iii) implies that the number of triangles in the canonical triangulation of σ is uniformly bounded (since these triangles share the center of σ as a common vertex) and the areas of any two triangles in the canonical triangulation are also comparable. It follows that

$$\{\rho(\mathcal{P}_i, \mathcal{T}_i) : i \in I\} \text{ is bounded.} \tag{5.18}$$

Condition (ii) implies that the number of faces of $\mathcal{P}_i$ on the face F of a subdomain $D \in \mathcal{P}_i$ is uniformly bounded, which together with the observation in the previous paragraph implies that the number of triangles of $\mathcal{T}_i$ on F is also uniformly bounded. It then follows from condition (iii) that the triangulation of F by the triangular faces from $\mathcal{T}_i$ is quasi-uniform. Moreover condition (i) implies that the number of faces of D is uniformly bounded and that the sizes of any two faces of D are comparable. Therefore the triangulation of ∂D by the faces from $\mathcal{T}_i$ is also quasi-uniform, which together with (4.6) implies

$$\inf\{\theta_{\mathcal{T}_i} : i \in I\} > 0\,. \tag{5.19}$$

Combining condition (ii) and (5.17)–(5.19), we see that $\{\big(\rho(\mathcal{P}_i), \lambda(\mathcal{P}_i), \rho(\mathcal{P}_i, \mathcal{T}_i), \theta_{\mathcal{T}_i}\big) : i \in I\}$ is a precompact subset of $\mathbb{R}^4_+$. The estimate (5.15) then follows from (5.7) if we take C to be an upper bound of the bounded set $\{K\big(\rho(\mathcal{P}_i), \lambda(\mathcal{P}_i), \rho(\mathcal{P}_i, \mathcal{T}_i), \theta_{\mathcal{T}_i}\big) : i \in I\}$. $\square$

Remark 5.4. If the family of partitions in Corollary 5.3 is actually a family of triangulations (simplicial or otherwise), then conditions (ii) and (iii) are redundant. Moreover, it is not necessary to assume that the subdomains are convex, since there are only finitely many different reference polygons for the faces of the $\mathcal{P}_i$'s.

An example of a family of partitions satisfying the assumptions of Corollary 5.3 is depicted in Figure 5, where a cube is being refined successively towards the upper left front corner.

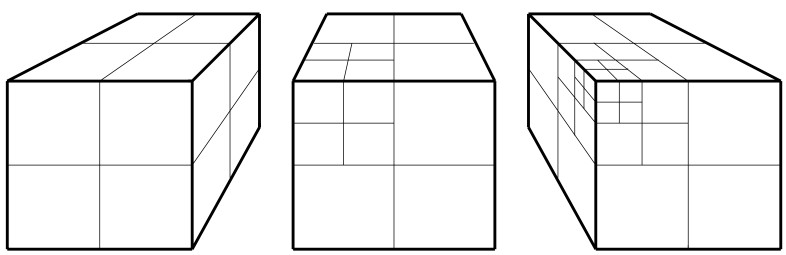

FIGURE 5. A family of partitions of a cube

6. KORN'S INEQUALITIES FOR PIECEWISE POLYNOMIAL VECTOR FIELDS WITH RESPECT TO TRIANGULATIONS BY POLYHEDRAL SUBDOMAINS

Attentive readers may have already noticed that the inequality (2.9) for piecewise linear vector fields is different from the inequalities (3.7), (4.2) and (5.7) for piecewise H^1 vector fields. Since pointwise evaluation is not well defined for functions in $[H^1(T)]^d$ and $d \geq 2$, the formulation of Korn's inequalities in (3.7), (4.2) and (5.7) is the appropriate one for piecewise H^1 vector fields. However, for piecewise polynomial vector fields associated with a triangulation $\mathcal{T}$ (simplicial or otherwise), pointwise evaluation of the jump across $\sigma \in S(\mathcal{T}, \Omega)$ is possible. The following theorem generalizes Lemma 2.2 to such vector fields.

Theorem 6.1. *Let Φ be as in Remark 3.2 and let $\{\mathcal{T}_i : i \in I\}$ be a family of triangulations of Ω by polygons ($d = 2$) or polyhedra ($d = 3$). Assume that the subdomains appearing in all the triangulations $\mathcal{T}_i$ are affine homeomorphic to a fixed finite set of reference domains and that the aspect ratios of the subdomains in all the $\mathcal{T}_i$'s are uniformly bounded. Let $V_i = \{\boldsymbol{v} \in [L_2(\Omega)]^d : \boldsymbol{v}_T = \boldsymbol{v}\big|_T \in [P_n(T)]^d \ \ \forall T \in \mathcal{T}_i\}$ for $i \in I$, where n is a positive integer. Then there exists a positive constant C, independent of $i \in I$, such that*

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T}_i)} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{T}_i}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \big(\Phi(\boldsymbol{u})\big)^2 + \sum_{\sigma\in S(\mathcal{T}_i,\Omega)} (\operatorname{diam}\sigma)^{d-2} \sum_{p\in\mathcal{V}(\sigma)} \big|[\boldsymbol{u}]_\sigma(p)\big|^2\Big) \tag{6.1}$$

for any $\boldsymbol{u} \in V_i$ and $i \in I$.

Proof. We will use C to denote a generic positive constant independent of $i \in I$.

Let $i \in I$ and $\sigma \in S(\mathcal{T}_i, \Omega)$ be arbitrary. Recall that $\mathfrak{T}_\sigma$ is the set of the two simplexes sharing σ as a common face. For $\boldsymbol{u} \in V_i$, we have

$$\big\|\pi_\sigma[\boldsymbol{u}]_\sigma\big\|_{L_2(\sigma)} \leq \big\|[\boldsymbol{u}]^I_\sigma\big\|_{L_2(\sigma)} + \sum_{T\in\mathfrak{T}_\sigma} \|\pi_\sigma \boldsymbol{u}_T - \boldsymbol{u}^I_T\|_{L_2(\sigma)}, \tag{6.2}$$

where $[\boldsymbol{u}]^I_\sigma \in [P_1(\sigma)]^d$ (respectively $\boldsymbol{u}^I_T \in [P_1(T)]^d$) is the linear nodal interpolant of $[\boldsymbol{u}]_\sigma$ (respectively $\boldsymbol{u}_T$).

The trace theorem (with scaling) and the Bramble-Hilbert lemma (cf. [1]) imply that

$$\|\pi_\sigma \boldsymbol{u}_T - \boldsymbol{u}^I_T\|^2_{L_2(\sigma)} \leq C(\operatorname{diam} T)^3 |\boldsymbol{u}_T|^2_{H^2(T)} \quad \text{for } T \in \mathfrak{T}_\sigma. \tag{6.3}$$

Moreover, from the well-known relation (cf. [8])

$$\frac{\partial^2 u_j}{\partial x_k \partial x_l} = \frac{\partial \epsilon_{jl}(\boldsymbol{u})}{\partial x_k} + \frac{\partial \epsilon_{jk}(\boldsymbol{u})}{\partial x_l} - \frac{\partial \epsilon_{kl}(\boldsymbol{u})}{\partial x_j} \quad \text{for } 1 \leq j,k,l \leq d$$

and a standard inverse estimate we have

$$|\boldsymbol{u}_T|_{H^2(T)} \leq C|\boldsymbol{\epsilon}(\boldsymbol{u}_T)|_{H^1(T)} \leq C(\operatorname{diam} T)^{-1}\|\boldsymbol{\epsilon}(\boldsymbol{u}_T)\|_{L_2(T)}. \tag{6.4}$$

Combining (2.11), (6.3) and (6.4), we find

$$(\operatorname{diam}\sigma)^{-1}\|\pi_\sigma \boldsymbol{u}_T - \boldsymbol{u}^I_T\|^2_{L_2(\sigma)} \leq C\|\boldsymbol{\epsilon}(\boldsymbol{u}_T)\|^2_{L_2(T)} \quad \text{for } T \in \mathfrak{T}_\sigma. \tag{6.5}$$

On the other hand we obtain, from a standard finite element estimate for the L_2-norm,

$$(\operatorname{diam}\sigma)^{-1}\big\|[\boldsymbol{u}]^I_\sigma\big\|^2_{L_2(\sigma)} \leq C(\operatorname{diam}\sigma)^{d-2} \sum_{p\in\mathcal{V}(\sigma)} \big|[\boldsymbol{u}]_\sigma(p)\big|^2. \tag{6.6}$$

The inequality (6.1) follows from (6.2), (6.5), (6.6), Corollary 4.3, Corollary 5.3, Remark 4.4 and Remark 5.4. □

Using Theorem 6.1, we can immediately obtain Korn's inequalities (1.14)–(1.16) for Wilson's brick/rectangle (cf. [17], [4], [16], [19]) and other nonconforming quadrilateral elements in [20] which are continuous at the vertices of the triangulation. Note that these elements do not satisfy the weak continuity condition (1.17).

Appendix A. Dependence of the constant in Korn's second inequality on the underlying domain

Let D be a bounded connected open polyhedral domain in $\mathbb{R}^d$ and let $k(D)$ be the smallest positive number such that

$$|\boldsymbol{v}|_{H^1(D)} \leq k(D)\bigg(\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L_2(D)} + (\,\mathrm{diam}\, D)^{1-d}\,\Big|\int_D \nabla\times\boldsymbol{v}\,dx\Big|\bigg) \tag{A.1}$$

for all $\boldsymbol{v} \in [H^1(D)]^d$. In this appendix we briefly discuss the behavior of $k(D)$ under affine homeomorphisms. More precisely, we assume that D is homeomorphic to a reference domain $\hat{D}$ under the affine transformation $\alpha : \hat{D} \longrightarrow D$ defined by $\alpha(\hat{x}) = B\hat{x} + b$, and we consider the dependence of $k(D)$ on $B \in GL(d)$, the Lie group of nonsingular $d \times d$ matrices.

Remark A.1. The estimates (A.3) and (A.4) below are crucial for Lemma 2.2 and Corollary 5.3. These estimates, though elementary, do not seem to be in the literature.

Without loss of generality, we may assume $b = 0$ and write the constant $k(D)$ in (A.1) as $k(B)$. We have,

$$\begin{aligned} k(B) &\\ &= \sup_{\substack{\hat{\boldsymbol{v}}\in[H^1(\hat{D})]^d \\ |\hat{\boldsymbol{v}}|_{H^1(\hat{D})}=1}} \left(\frac{|\hat{\boldsymbol{v}}\circ\alpha^{-1}|_{H^1(D)}}{\|\boldsymbol{\epsilon}(\hat{\boldsymbol{v}}\circ\alpha^{-1})\|_{L_2(D)} + (\,\mathrm{diam}\,D)^{1-d}\,\big|\int_D \nabla\times(\hat{\boldsymbol{v}}\circ\alpha^{-1})\,dx\big|}\right). \end{aligned} \tag{A.2}$$

Observe that, since $|\hat{\boldsymbol{v}}|_{H^1(\hat{D})} = 1$, the quotients on the right-hand side of (A.2) form a family of equicontinuous functions on $GL(d)$. Therefore $k(\cdot)$, defined as the supremum of this equicontinuous family, is continuous on $GL(d)$ (cf. [7]).

Lemma A.2. *Let $\{D_i : i \in I\}$ be a family of domains affine homeomorphic to the reference domain $\hat{D}$. Assume also that the aspect ratios of the D_i's are uniformly bounded. Then there exists a positive constant C, independent of $i \in I$, such that*

$$|\boldsymbol{v}|_{H^1(D_i)} \leq C\bigg(\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L_2(D_i)} + (\,\mathrm{diam}\; D_i)^{1-d}\,\Big|\int_{D_i} \nabla\times\boldsymbol{v}\,dx\Big|\bigg) \tag{A.3}$$

for any $\boldsymbol{v} \in [H^1(D_i)]^d$ and $i \in I$.

Proof. We may assume without loss of generality that $\mathrm{diam}\, D_i = 1$ for all $i \in I$. It follows from the uniform boundedness of the aspect ratios of the D_i's that the norm of the nonsingular matrix B_i in the affine transformation $\alpha_i : \hat{D} \longrightarrow D_i$ and the norm of its inverse are uniformly bounded for all $i \in I$ (cf. (4.5)). Hence $\{B_i : i \in I\}$ is a precompact subset of $GL(d)$ and the boundedness of $\{k(B_i) : i \in I\}$ follows from the continuity of $k(\cdot)$ on $GL(d)$. $\square$

For a simplex T we can also control the constant $k(T)$ in terms of the minimum angle θ_T of T.

Corollary A.3. *There exists a continuous decreasing function $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that, for any simplex T,*

$$|\boldsymbol{v}|_{H^1(T)} \leq \kappa(\theta_T)\bigg(\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L_2(T)} + (\mathrm{diam}\, T)^{1-d}\,\Big|\int_T \nabla\times\boldsymbol{v}\,dx\Big|\bigg) \tag{A.4}$$

for all $\boldsymbol{v} \in [H^1(T)]^d$.

Proof. It follows from Lemma A.2 that, for any $\theta > 0$, the set

$$S_\theta = \{k(T) : T \text{ is a simplex and the minimum angle of } T \text{ is} \geq \theta\} \tag{A.5}$$

is bounded. Then $\eta(\theta) = \sup S_\theta$ defines a nonnegative decreasing function on $\mathbb{R}_+$. In view of (A.1) and (A.5) we have

$$|\boldsymbol{v}|_{H^1(T)} \leq \eta(\theta_T)\left(\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{L_2(T)} + (\operatorname{diam} T)^{1-d}\left|\int_T \nabla \times \boldsymbol{v}\, dx\right|\right) \tag{A.6}$$

for all $\boldsymbol{v} \in [H^1(T)]^d$.

The estimate (A.4) follows from (A.6) if we choose $\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ to be any continuous decreasing function satisfying the condition $\kappa \geq \eta$. (There are infinitely many such functions.) □

References

[1] J.H. Bramble and S.R. Hilbert. Estimation of linear functionals on Sobolev spaces with applications to Fourier transforms and spline interpolation. *SIAM J. Numer. Anal.*, 7:113–124, 1970. MR **41:**7819

[2] S.C. Brenner. Poincaré-Friedrichs inequalities for piecewise H^1 functions. *SIAM J. Numer. Anal.* 41: 306–324, 2003 (electronic).

[3] S.C. Brenner and L.R. Scott. *The Mathematical Theory of Finite Element Methods* (*Second Edition*). Springer-Verlag, New York-Berlin-Heidelberg, 2002. MR **2003a:**65103

[4] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems*. North-Holland, Amsterdam, 1978. MR **58:**25001

[5] P.G. Ciarlet. *Mathematical Elasticity Volume I: Three-Dimensional Elasticity*. North-Holland, Amsterdam, 1988. MR **89e:**73001

[6] B. Cockburn, G.E. Karniadakis, and C.-W. Shu, editors. *Discontinuous Galerkin Methods*. Springer-Verlag, Berlin-Heidelberg, 2000. MR **2002b:**65004

[7] J. Dieudonné. *Foundations of Modern Analysis*. Pure and Applied Mathematics, Vol. X. Academic Press, New York, 1960. MR **22:**11074

[8] G. Duvaut and J.L. Lions. *Inequalities in Mechanics and Physics*. Springer-Verlag, Berlin, 1976. MR **58:**25191

[9] R.S. Falk. Nonconforming finite element methods for the equations of linear elasticity. *Math. Comp.*, 57:529–550, 1991. MR **92a:**65290

[10] M. Fortin. A three-dimensional quadratic nonconforming element. *Numer. Math.*, 46:269–279, 1985. MR **86f:**65192

[11] M. Fortin and M. Soulie. A non-conforming piecewise quadratic finite element on triangles. *Internat. J. Numer. Methods Engrg.*, 19:505–520, 1983. MR **84g:**76004

[12] P. Hansbo and M.G. Larson. Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. *Comput. Methods Appl. Mech. Engrg.*, 191:1895–1908, 2002.

[13] P. Knobloch. On Korn's inequality for nonconforming finite elements. *Technische Mechanik*, 20:205–214 and 375 (Errata), 2000.

[14] J.A. Nitsche. On Korn's second inequality. *RAIRO Anal. Numér.*, 15:237–248, 1981. MR **83a:**35012

[15] Ch. Schwab. *hp*-FEM for fluid flow simulation. In T.J. Barth and H. Deconinck, editors, *High-Order Methods for Computational Physics*, pages 325–438. Springer-Verlag, Berlin-Heidelberg, 1999. MR **2000k:**76093

[16] M. Wang. The generalized Korn inequality on nonconforming finite element spaces. *Chinese J. Numer. Math. Appl.*, 16:91–96, 1994. MR **98a:**65166

[17] E.L. Wilson, R.L. Taylor, W. Doherty, and J. Ghaboussi. Incompatible displacement models. In S.J. Fenves, N. Perrone, A.R. Robinson, and W.C. Schnobrich, editors, *Numerical and Computer Methods in Structural Mechanics*, pages 43–57. Academic Press, New York, 1973.

[18] B.I. Wohlmuth. *Discretization Methods and Iterative Solvers Based on Domain Decomposition*. Springer-Verlag, Heidelberg, 2001. MR **2002c:**65231

[19] X. Xu. A discrete Korn's inequality in two and three dimensions. *Appl. Math. Letters*, 13:99–102, 2000. MR **2000m:**74087

[20] Z. Zhang. Analysis of some quadrilateral nonconforming elements for incompressible elasticity. *SIAM J. Numer. Anal.*, 34:640–663, 1997. MR **98b:**73041

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