Koszul Cohomology and Algebraic Geometry

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Preface

The systematic use of Koszul cohomology computations in algebraic geometry can be traced back to the foundational paper [**Gre84a**] by M. Green. In this paper, Green introduced the Koszul cohomology groups $K_{p,q}(X, L)$ associated to a line bundle L on a smooth, projective variety X, and studied the basic properties of these groups. Green noted that a number of classical results concerning the generators and relations of the (saturated) ideal of a projective variety can be rephrased naturally in terms of vanishing theorems for Koszul cohomology, and extended these results using his newly developed techniques. In a remarkable series of papers, Green and Lazarsfeld further pursued this approach. Much of their work in the late 80's centers around the shape of the minimal resolution of the ideal of a projective variety; see [**Gre89**], [**La89**] for an overview of the results obtained during this period.

Green and Lazarsfeld also stated two conjectures that relate the Koszul cohomology of algebraic curves to two numerical invariants of the curve, the *Clifford index* and the *gonality*. These conjectures became an important guideline for future research. They were solved in a number of special cases, but the solution of the general problem remained elusive. C. Voisin achieved a major breakthrough by proving the Green conjecture for general curves in [**V02**] and [**V05**]. This result soon led to a proof of the conjecture of Green–Lazarsfeld for general curves [**AV03**], [**Ap04**].

Since the appearance of Green's paper there has been a growing interaction between Koszul cohomology and algebraic geometry. Green and Voisin applied Koszul cohomology to a number of Hodge–theoretic problems, with remarkable success. This work culminated in Nori's proof of his connectivity theorem [**No93**]. In recent years, Koszul cohomology has been linked to the geometry of Hilbert schemes (via the geometric description of Koszul cohomology used by Voisin in her work on the Green conjecture) and moduli spaces of curves.

Since there already exists an excellent introduction to the subject [**Ei06**], this book is devoted to more advanced results. Our main goal was to cover the recent developments in the subject (Voisin's proof of the generic Green conjecture, and subsequent refinements) and to discuss the geometric aspects of the theory, including a number of concrete applications of Koszul cohomology to problems in algebraic geometry. The relationship between Koszul cohomology and minimal resolutions will not be treated at length, although it is important for historical reasons and provides a way to compute Koszul cohomology by computer calculations.

Outline of contents. The first two chapters contain a review of a number of basic definitions and results, which are mainly included to fix the notation and to obtain a reasonably self–contained presentation. Chapter 3 is devoted to the theory of syzygy schemes. The aim of this theory is to study Koszul cohomology classes in the groups $K_{p,1}(X, L)$ by associating a geometric object to them. This chapter includes a proof of one of the fundamental results in the subject, Green's $K_{p,1}$–theorem. In Chapter 4 we recall a number of results from Brill–Noether theory that will be needed in the sequel and state the conjectures of Green and Green–Lazarsfeld.

Chapters 5–7 form the heart of the book. Chapter 5 is devoted to Voisin's description of the Koszul cohomology groups $K_{p,q}(X, L)$ in terms of the Hilbert scheme of zero–dimensional subschemes of X. This description yields a method to prove vanishing theorems for Koszul cohomology by base change, which is used in Voisin's proof of the generic Green conjecture. In Chapter 6 we present Voisin's proof for curves of even genus, and outline the main steps of the proof for curves of odd genus; this case is technically more complicated. Chapter 7 contains a number of refinements of Voisin's result; in particular, we have included a proof of the conjectures of Green and Green–Lazarsfeld for the general curve in a given gonality stratum of the moduli space of curves. In the final chapter we discuss geometric applications of Koszul cohomology to Hodge theory and the geometry of the moduli space of curves.

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CHAPTER 1

Basic definitions

1.1. The Koszul complex

Let V be a vector space of dimension $r+1$ over a field k. Given a nonzero element $x \in V^\vee$, the corresponding map $\langle x, - \rangle : V \to k$ extends uniquely to an anti–derivation

$$\iota_x : \bigwedge^* V \to \bigwedge^* V$$

of the exterior algebra of degree -1. This derivation is defined inductively by putting $\iota_x|_V = \langle x, - \rangle : V \to k$ and

$$\iota_x(v \wedge v_1 \wedge \ldots \wedge v_{p-1}) = \langle x, v \rangle . v_1 \wedge \ldots \wedge v_{p-1} - v \wedge \iota_x(v_1 \wedge \ldots \wedge v_{p-1}).$$

The resulting maps

$$\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V$$

are called *contraction* (or *inner product*) maps; they are dual to the exterior product maps

$$\lambda_x : \bigwedge^{p-1} V^\vee \xrightarrow{\wedge x} \bigwedge^p V^\vee$$

and satisfy $\iota_x \circ \iota_x = 0$. Hence we obtain a complex

$$K_\bullet(x) : (0 \to \bigwedge^{r+1} V \to \ldots \to \bigwedge^p V \xrightarrow{\iota_x} \bigwedge^{p-1} V \xrightarrow{\iota_x} \bigwedge^{p-2} V \to \ldots \to k \to 0)$$

called the *Koszul complex.*

Note that for any $\alpha \in k^*$, the complexes $K_\bullet(x)$ and $K_\bullet(\alpha x)$ are isomorphic; hence the Koszul complex depends only on the point $[x] \in \mathbb{P}(V^\vee)$.

Lemma 1.1. *Given nonzero elements $\xi \in V$, $x \in V^\vee$, let $\lambda_\xi : \bigwedge^{p-1} V \xrightarrow{\wedge \xi} \bigwedge^p V$ be the map given by wedge product with ξ. We have*

$$\iota_x \circ \lambda_\xi + \lambda_\xi \circ \iota_x = \langle x, \xi \rangle . \operatorname{id}.$$

Proof: It suffices to verify the statement on decomposable elements. To this end, we compute

$$\begin{aligned}
(\lambda_\xi \circ \iota_x)(v_1 \wedge \ldots \wedge v_p) &= \sum_i (-1)^i \langle x, v_i \rangle . \xi \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p \\
(\iota_x \circ \lambda_\xi)(v_1 \wedge \ldots \wedge v_p) &= \langle x, \xi \rangle . v_1 \wedge \ldots \wedge v_p \\
&\quad + \sum_i (-1)^{i-1} \langle x, v_i \rangle . \xi \wedge v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_p
\end{aligned}$$

and the statement follows. □

Corollary 1.2. *For every nonzero element $x \in V^\vee$, the Koszul complex $K_\bullet(x)$ is an exact complex of k-vector spaces.*

Proof: Choose $\xi \in V$ such that $\langle x, \xi \rangle = 1$ and apply Lemma 1.1. □

REMARK 1.3. Put $W_x = \ker(\langle x, -\rangle : V \to k)$. Taking exterior powers in the resulting short exact sequence

$$0 \to W_x \to V \to k \to 0$$

we obtain exact sequences

$$0 \to \bigwedge^p W_x \to \bigwedge^p V \to \bigwedge^{p-1} W_x \to 0$$

for all $p \geq 1$. Using these exact sequences, one checks by induction on p that

$$\bigwedge^p W_x = \ker(\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V) = \mathrm{im}(\iota_x : \bigwedge^{p+1} V \to \bigwedge^p V)$$

for all $p \geq 1$. Hence the contraction map $\iota_x : \bigwedge^p V \to \bigwedge^{p-1} V$ factors through $\bigwedge^{p-1} W_x$.

1.2. Definitions in the algebraic context

Let M be a graded module of finite type over the symmetric algebra S^*V. Let $\iota : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V$ be the dual of the wedge product map $\lambda : \bigwedge^{p-1} V^\vee \otimes V^\vee \to \bigwedge^p V^\vee$. Note that we have the identification

$$\begin{array}{rcl} \bigwedge^p V & \xrightarrow{\iota} & \bigwedge^{p-1} V \otimes V \cong \mathrm{Hom}(V^\vee, \bigwedge^{p-1} V) \\ v_1 \wedge \ldots \wedge v_p & \mapsto & (x \mapsto \iota_x(v_1 \wedge \ldots \wedge v_p)). \end{array}$$

The graded S^*V–module structure of M induces maps $\mu : V \otimes M_q \to M_{q+1}$ for all q. Define a map

$$\delta : \bigwedge^p V \otimes M_q \to \bigwedge^{p-1} V \otimes M_{q+1}$$

by the composition

$$\begin{tikzcd} \bigwedge^p V \otimes M_q \arrow[r, "\iota \otimes \mathrm{id}"] \arrow[dr, "\delta"'] & \bigwedge^{p-1} V \otimes V \otimes M_q \arrow[d, "\mathrm{id} \otimes \mu"] \\ & \bigwedge^{p-1} V \otimes M_{q+1}. \end{tikzcd}$$

DEFINITION 1.4. The *Koszul cohomology group* $K_{p,q}(M, V)$ is the cohomology at the middle term of the complex

$$\bigwedge^{p+1} V \otimes M_{q-1} \xrightarrow{\delta} \bigwedge^p V \otimes M_q \xrightarrow{\delta} \bigwedge^{p-1} V \otimes M_{q+1}. \tag{1.1}$$

An element $x \in V^\vee$ induces a derivation

$$\partial_x : S^*V \to S^*V$$

of degree -1 on the symmetric algebra, which is defined inductively by the rule

$$\partial_x(v.v_1 \ldots v_{p-1}) = \partial_x(v).v_1 \ldots v_{p-1} + v.\partial_x(v_1 \ldots v_{p-1}).$$

If we choose coordinates $X_0, \ldots, X_r$ on V, with duals $x_i \in V^\vee$, the resulting map

$$\partial_{x_k} : S^p V \to S^{p-1} V$$

sends a homogeneous polynomial f of degree p to the partial derivative $\frac{\partial f}{\partial X_k}$. Using the natural map

$$\begin{array}{rcl} S^{q+1} V & \to & S^q V \otimes V \cong \mathrm{Hom}(V^\vee, S^q V) \\ f & \mapsto & (x \mapsto \iota_x(f)) \end{array}$$

and the wedge product map $\lambda : \bigwedge^{p-1} V \otimes V \to \bigwedge^p V$ we define the map

$$D : \bigwedge^{p-1} V \otimes S^{q+1} V \to \bigwedge^p V \otimes S^q V$$

as the composition

$$\begin{tikzcd} \bigwedge^{p-1} V \otimes S^{q+1} V \arrow[r, "\mathrm{id}\otimes\iota"] \arrow[dr, "D"'] & \bigwedge^{p-1} V \otimes V \otimes S^q V \arrow[d, "\lambda\otimes\mathrm{id}"] \\ & \bigwedge^p V \otimes S^q V. \end{tikzcd}$$

Proposition 1.5. *We have $K_{0,0}(S^*V, V) \cong \mathbb{C}$, and $K_{p,q}(S^*V, V) = 0$ for all $(p,q) \neq (0,0)$.*

Proof: The first part follows from the definition. To prove the second part, choose coordinates $X_0, \dots, X_r$ on V and note that

$$D : \bigwedge^p V \otimes S^{q+1} V \to \bigwedge^{p+1} V \otimes S^q V$$

is given by

$$X_{i_1} \wedge \cdots \wedge X_{i_p} \otimes f \mapsto \sum_{k=0}^{r} X_k \wedge X_{i_1} \wedge \cdots \wedge X_{i_p} \otimes \frac{\partial f}{\partial X_k}.$$

The Euler formula

$$\sum_{k=0}^{r} X_k \frac{\partial f}{\partial X_k} = q.f$$

implies that

$$D \circ \delta + \delta \circ D = (p+q).\mathrm{id},$$

hence the Koszul complex is exact. □

Corollary 1.6. *We have an exact complex of graded S^*V–modules*

$$\begin{aligned} K_\bullet(k) : (0 \to \bigwedge^{r+1} V \otimes S^*V(-r-1) \to \bigwedge^r V \otimes S^*V(-r) \to \dots \\ \dots \to \bigwedge^2 V \otimes S^*V(-2) \to V \otimes S^*V(-1) \to S^*V \to k \to 0). \end{aligned} \tag{1.2}$$

Proof: Put together the Koszul complexes for S^*V with various degree shifts. □

1.3. Minimal resolutions

Let $M = \oplus_q M_q$ be a graded S^*V–module of finite type. Let $\mathfrak{m} = \oplus_{d\geq 1} S^d V \subset S^*V$ be the irrelevant ideal of $S^*V = S$.

Definition 1.7. A graded free resolution

$$\cdots \to F_{p+1} \xrightarrow{\varphi_{p+1}} F_p \to \cdots \to F_0 \to M \to 0 \tag{1.3}$$

is called *minimal* if $\varphi_{p+1}(F_{p+1}) \subset \mathfrak{m}.F_p$ for all p.

Remark 1.8. Put $\overline{F}_i = F_i \otimes k$, $\overline{\varphi}_i = \varphi \otimes \mathrm{id} : \overline{F}_i \to \overline{F}_{i-1}$. The resolution $F_\bullet$ is minimal if and only if $\overline{\varphi}_i = 0$ for all i.

Proposition 1.9. *Every finitely generated graded S^*V–module admits a minimal free resolution. The minimal free resolution is unique up to isomorphism.*

Proof: The statement is proved in [**Ei95**, §20.1] for the case of local rings. The proof in the graded case is analogous. □

REMARK 1.10. Bruns and Herzog [**BH93**] note that the statement of Proposition 1.9 extends to so–called *–local rings, a class of rings that includes local rings and symmetric algebras.

DEFINITION 1.11. Let $F_\bullet \to M$ be a minimal graded free resolution. Write $F_i = \oplus_j S(-j)^{\beta_{i,j}} = \oplus_j M_{i,j} \otimes S(-j)$, where $M_{i,j}$ is a k–vector space of dimension $\beta_{i,j}$.

(i) The numbers $\beta_{i,j}$ are called the *graded Betti numbers* of the module M;
(ii) The vector space $M_{0,q}$ is called the space of generators of M of degree q. If $p \geq 1$ then $M_{p,q}$ is called the space of *syzygies* of order p and degree q of the module M.

PROPOSITION 1.12. *We have*

$$K_{p,q}(M,V) \cong M_{p,p+q}.$$

Proof: The symmetry property of the Tor functor implies that we can calculate $\mathrm{Tor}_p^S(M,k)$ via a free resolution of one of the two factors. The Koszul complex $K_\bullet(k)$ provides a free resolution of k by Corollary 1.6. Hence

$$\mathrm{Tor}_p^S(M,k)_{p+q} = H_p(K_\bullet(k) \otimes M)_{p+q} = K_{p,q}(M,V).$$

If we compute $\mathrm{Tor}_p^S(M,k)$ via the free resolution $F_\bullet$ of M, we find

$$\mathrm{Tor}_p^S(M,k)_{p+q} = H_p(F_\bullet \otimes k)_{p+q} = M_{p,p+q}$$

by Remark 1.8, and the statement follows. □

COROLLARY 1.13. *Put* $\kappa_{p,q} = \dim_k K_{p,q}(M,V)$. *We have* $\kappa_{p,q} = \beta_{p,p+q}$.

COROLLARY 1.14 (Hilbert syzygy theorem). *Any graded S^*V module M of finite type has a graded free resolution $F_\bullet \to M$ of length at most* $\dim V$.

Proof: Put $r = \dim \mathbb{P}(V^\vee)$. Clearly $\kappa_{p,q} = 0$ for $p \geq r+2$, hence $M_{p,q} = 0$ for $p \geq r+2$ by Proposition 1.12. □

DEFINITION 1.15. The *Betti table* associated to a graded S^*V–module M is given by

$$\begin{matrix} \beta_{0,0} & \beta_{1,1} & \beta_{2,2} & \cdots \\ \beta_{0,1} & \beta_{1,2} & \beta_{2,3} & \cdots \\ \beta_{0,2} & \beta_{1,3} & \beta_{2,4} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \quad = \quad \begin{matrix} \kappa_{0,0} & \kappa_{1,0} & \kappa_{2,0} & \cdots \\ \kappa_{0,1} & \kappa_{1,1} & \kappa_{2,1} & \cdots \\ \kappa_{0,2} & \kappa_{1,2} & \kappa_{2,2} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

REMARK 1.16. The number of columns with non-zero elements in the Betti diagram equals the *projective dimension* of the module M. See Remark 2.38 for the meaning of the number of rows in the Betti diagram.

EXAMPLE 1.17. The coordinate ring S_X of the twisted cubic $X \subset \mathbb{P}^3$ admits a resolution

$$0 \to S(-3)^2 \to S(-2)^3 \to S \to S_X \to 0.$$

The associated Betti diagram is (we put a dash '−' if the corresponding entry is zero)

$$\begin{matrix} 1 & - & - \\ - & 3 & 2. \end{matrix}$$

1.4. Definitions in the geometric context

Let X be a projective variety over $\mathbb{C}$, and let L be a holomorphic line bundle on X. Put $V = H^0(X, L)$.

Definition 1.18. The Koszul cohomology group $K_{p,q}(X, L)$ is the Koszul cohomology of the graded S^*V–module

$$R(L) = \bigoplus_q H^0(X, L^q).$$

Concretely, this means that $K_{p,q}(X, L)$ is the cohomology at the middle term of the complex

$$\bigwedge^{p+1} V \otimes H^0(X, L^{q-1}) \xrightarrow{\delta} \bigwedge^p V \otimes H^0(X, L^q) \xrightarrow{\delta} \bigwedge^{p-1} V \otimes H^0(X, L^{q+1}),$$

where the differential

$$\bigwedge^{p+1} V \otimes H^0(X, L^{q-1}) \xrightarrow{\delta} \bigwedge^p V \otimes H^0(X, L^q)$$

is given by

$$\delta(v_1 \wedge \ldots \wedge v_{p+1} \otimes s) = \sum_i (-1)^i v_1 \wedge \ldots \wedge \widehat{v_i} \wedge \ldots \wedge v_{p+1} \otimes (v_i.s).$$

More generally, if $V \subset H^0(X, L)$, and $\mathcal{F}$ is a coherent sheaf on X, then we define

$$\begin{aligned} R(\mathcal{F}, L) &= \bigoplus_q H^0(X, \mathcal{F} \otimes L^q) \\ K_{p,q}(X; \mathcal{F}, L, V) &= K_{p,q}(R(\mathcal{F}, L), V). \end{aligned}$$

If $\mathcal{F} = \mathcal{O}_X$ we write $K_{p,q}(X, L, V)$.

Remark 1.19. The above definition admits an obvious generalization to higher cohomology groups. Consider the graded S^*V–module

$$R^i(\mathcal{F}, L) = \bigoplus_q H^i(X, \mathcal{F} \otimes L^q)$$

and put $K^i_{p,q}(X; \mathcal{F}, L) = K_{p,q}(R^i(\mathcal{F}, L), V)$. For technical reasons it is sometimes useful to study these groups; cf. [**Gre84a**].

Remark 1.20. Note that we may shift the second index in the Koszul cohomology groups by changing the coherent sheaf:

$$\begin{aligned} K_{p,q}(X; \mathcal{F}, L) &= K_{p,q-1}(X; \mathcal{F} \otimes L, L) = \ldots \\ &= K_{p,1}(X; \mathcal{F} \otimes L^{q-1}, L) = K_{p,0}(X; \mathcal{F} \otimes L^q, L). \end{aligned}$$

Given a vector bundle E on X and a section $\sigma \in H^0(X, E^\vee)$, the construction of Section 1.1 generalizes and gives a Koszul complex of vector bundles

$$\mathcal{K}_\bullet(\sigma) : (\bigwedge^r E \to \ldots \to \bigwedge^p E \xrightarrow{\iota_\sigma} \bigwedge^{p-1} E \to \ldots \to \mathcal{O}_X).$$

If σ is a *regular section*, i.e. the rank of E coincides with the codimension of the zero locus $Z = V(\sigma)$, this complex provides a resolution of the ideal sheaf $\mathcal{I}_Z$ of Z; cf. [**FL85**]. More generally, given a line bundle L and a homomorphism of vector bundles $\sigma : E \to L$, we obtain a complex of vector bundles

$$\mathcal{K}_\bullet(\sigma) : (\ldots \to \bigwedge^p E \otimes L^q \xrightarrow{\delta} \bigwedge^{p-1} E \otimes L^{q+1} \to \ldots)$$

where δ is defined as the composition

$$\bigwedge^p E \otimes L^q \xrightarrow{\iota\otimes \mathrm{id}} \bigwedge^{p-1} E \otimes E \otimes L^q \xrightarrow{\mathrm{id}\otimes(\mu\circ\sigma)} \bigwedge^{p-1} E \otimes L^{q+1}.$$

Given a line bundle L, put $V = H^0(X, L)$. Applying the above construction to the evaluation map $\mathrm{ev} : V \otimes \mathcal{O}_X \to L$, we obtain a Koszul complex of vector bundles

$$\mathcal{K}_\bullet(X, L) : (\ldots \to \bigwedge^p V \otimes L^q \xrightarrow{\delta} \bigwedge^{p-1} V \otimes L^{q+1} \to \ldots).$$

The Koszul complex $K_\bullet(X, L)$ associated to the S^*V–module $R(L)$ is obtained from this complex by taking global sections.

Example 1.21. Let V be a k–vector space of dimension $r + 1$. Applying the above construction to the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ on $\mathbb{P}(V^\vee)$, we obtain an exact complex of locally free sheaves

$$(1.4)\quad 0 \to \bigwedge^{r+1} V \otimes \mathcal{O}_{\mathbb{P}}(-r-1) \to \ldots \to \bigwedge^2 V \otimes \mathcal{O}_{\mathbb{P}}(-2) \to V \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}} \to 0.$$

Note that complex (1.4) coincides with the sheafification of the complex of graded S^*V–modules appearing in Corollary 1.6. (And conversely, this complex can be recovered from the complex of sheaves (1.4) by taking global sections.)

1.5. Functorial properties

1.5.1. Algebraic case. Consider the category $\mathcal{C}$ whose objects are pairs (M, V), where V is a finite–dimensional k–vector space and M i s a graded S^*V–module. Given a homomorphism $f : M \to M'$ of graded S^*V–modules and a linear map $g : V \to V'$, let

$$S(g) : S^*V \to S^*V'$$

be the induced homomorphism of symmetric algebras. We say that $(f, g) : (M, V) \to (M', V')$ is a morphism in $\mathcal{C}$ if

$$f(\lambda.x) = S(g).f(x) \quad \forall \lambda \in S^*V, \quad \forall x \in M.$$

Given a morphism $\varphi = (f, g)$ as above, we obtain induced maps of Koszul groups

$$\varphi_* : K_{p,q}(M, V) \to K_{p,q}(M', V')$$

for all p, q. Hence Koszul cohomology defines a covariant functor

$$\begin{array}{rcl} K_{\mathrm{alg}} : \mathcal{C} & \to & (BiGr - Vect)_k \\ (M, V) & \mapsto & K_{*,*}(M, V) \end{array}$$

from $\mathcal{C}$ to the category of bigraded k–vector spaces.

Lemma 1.22. *An exact sequence* $0 \to A \to B \to C \to 0$ *of graded* S^*V*–modules induces a long exact sequence of Koszul groups*

$$K_{p+1,q-1}(C, V) \to K_{p,q}(A, V) \to K_{p,q}(B, V) \to K_{p,q}(C, V) \to K_{p-1,q+1}(A, V).$$

Proof: Let

$$K^\bullet(M) = (\ldots \to \bigwedge^{p+1} V \otimes M_{q-1} \to \bigwedge^p V \otimes M_q \to \bigwedge^{p-1} V \otimes M_{q+1} \to \ldots)$$

be the Koszul complex associated to a graded S^*V–module M. The result follows by taking the long exact homology sequence associated to the short exact sequence

$$0 \to K^\bullet(A) \to K^\bullet(B) \to K^\bullet(C) \to 0.$$

□

This leads to the following easy but useful corollary.

COROLLARY 1.23. *Let $f : A \to B$ be a homomorphism of graded S^*V–modules*

(i) *If f is injective then*
  (a) $K_{p,q}(A,V) \to K_{p,q}(B,V)$ *is injective if* $A_{q-1} = B_{q-1}$*;*
  (b) $K_{p,q}(A,V) \to K_{p,q}(B,V)$ *is surjective if* $A_q = B_q$*.*

(ii) *If f is surjective then*
  (a) $K_{p,q}(A,V) \to K_{p,q}(B,V)$ *is injective if* $A_q = B_q$*;*
  (b) $K_{p,q}(A,V) \to K_{p,q}(B,V)$ *is surjective if* $A_{q+1} = B_{q+1}$*.*

Proof: To prove (i), put $C = \ker f$, apply Lemma 1.22 and note that $C_i = 0$ implies $K_{p,i}(C,V) = 0$. For (ii), apply the same reasoning with $C = \mathrm{coker}\ f$.

□

1.5.2. Geometric case. Let $\mathcal{V}$ be the category whose objects are pairs (X, L), with X a projective variety defined over $\mathbb{C}$ and L a holomorphic line bundle on X. A morphism in $\mathcal{V}$ is a pair $(f, f^{\#}) : (X, L) \to (Y, M)$ with $f : X \to Y$ a morphism of k–varieties and $f^{\#} : M \to f_*L$ a homomorphism of line bundles. By adjunction, $f^{\#}$ corresponds to a homomorphism (still denoted $f^{\#}$) $f^*M \xrightarrow{f^{\#}} L$. Given a morphism $\varphi = (f, f^{\#})$ as above, we obtain maps

$$H^0(Y, M^q) \xrightarrow{f^*} H^0(X, f^*M^q) \xrightarrow{f^{\#}} H^0(X, L^q).$$

Put $V = H^0(X, L)$, $W = H^0(Y, M)$. Given a morphism $\varphi = (f, f^{\#})$ as above, we obtain an induced morphism $(R(M), W) \to (R(L), V)$ in $\mathcal{C}$. Hence the preceding construction defines contravariant functors

$$\begin{array}{rcl} R : \mathcal{V} & \to & \mathcal{C} \\ (X, L) & \mapsto & (R(L), V). \end{array}$$

and

$$\begin{array}{rcl} K_{\mathrm{geom}} : \mathcal{V} & \to & (BiGr - Vect)_k \\ (X, L) & \mapsto & K_{*;*}(X, L). \end{array}$$

Note that $K_{\mathrm{geom}} = K_{\mathrm{alg}} \circ R$.

REMARK 1.24. There exists an obvious extension of the functors R and K_{geom} to the category $\widetilde{\mathcal{V}}$ of quadruples $(X; \mathcal{F}, L, V)$ with $\mathcal{F} \in \mathrm{Coh}(X)$, $L \in \mathrm{Pic}(X)$, $V \subseteq H^0(X, L)$. A morphism in this category is a 3–tuple $(f, f^{\#}, f^{\#\#}) : (X, L, \mathcal{F}, V) \to (Y, M, \mathcal{G}, W)$ with

$$f : X \to Y, \quad f^{\#} : M \to f_*L, \quad f^{\#\#} : \mathcal{G} \to f_*\mathcal{F}$$

such that f is proper and such that the induced map $H^0(Y, M) \to H^0(X, L)$ induces a linear map $W \to V$.

DEFINITION 1.25. Given a pair $(X, L) \in \mathcal{V}$ there is a natural homomorphism of graded S^*V–modules $\pi : S^*V \to R(L)$. Put

$$S(X) = \mathrm{im}(\pi), \quad I = \ker(\pi).$$

REMARK 1.26. If L is globally generated with associated morphism $\varphi_L : X \to \mathbb{P}H^0(X, L)^\vee$, then $S(X)$ is the coordinate ring of $\varphi_L(X)$ and I its ideal.

Proposition 1.27. *Notation as above. The inclusion $S(X) \subset R(L)$ induces a homomorphism*

$$K_{p,q}(S(X), V) \to K_{p,q}(R(L), V) = K_{p,q}(X, L).$$

We have

(i) $K_{p,1}(S(X), W) \cong K_{p,1}(X, L, W)$ *for every linear subspace* $W \subset V$*;*
(ii) $K_{p,q}(S(X), V) \cong K_{p-1,q+1}(I, V)$ *for all* $q \geq 1$.

Proof: The first isomorphism follows by applying Corollary 1.23 to the inclusion $S(X) \subset R(L)$, using the equalities $S_0(X) = R_0(X) = \mathbb{C}$, $S_1(X) = R_1(L) = V$. The second statement follows from the long exact sequence of Koszul groups associated to $0 \to I \to S^*V \to S(X) \to 0$ and the vanishing $K_{p,q}(S^*V, V) = 0$ for all $q \geq 1$ (Proposition 1.5). □

Corollary 1.28. *Suppose X and Y are subvarieties of $\mathbb{P}(V^\vee)$ such that $Y \subset X$. If Y is nondegenerate then $K_{p,1}(S(X), V) \subset K_{p,1}(S(Y), V)$.*

Proof: By hypothesis $I_1(X) = I_1(Y) = 0$, hence

$$K_{p-1,2}(I(X), V) \subset K_{p-1,2}(I(Y), V)$$

by Corollary 1.23. The result then follows from Proposition 1.27. □

Let $f : X \to S$ be a projective morphism of schemes, and let $L \in \mathrm{Pic}(X/S)$. Recall [**HAG77**, II, Ex. 5.16] that given a sheaf $\mathcal{F}$ of $\mathcal{O}_S$–modules, its p–th exterior power $\bigwedge^p \mathcal{F}$ is defined as the sheaf associated to the presheaf

$$U \mapsto \bigwedge^p \mathcal{F}(U).$$

In particular, there exist natural wedge product and (by duality) contraction maps

$$\lambda : \bigwedge^{p-1} \mathcal{F} \otimes \mathcal{F} \to \bigwedge^p \mathcal{F}, \quad \iota : \bigwedge^p \mathcal{F} \to \bigwedge^{p-1} \mathcal{F} \otimes \mathcal{F}.$$

Proposition 1.29. *Let $f : X \to S$ be a flat, projective morphism such that S is integral. There exist a coherent sheaf $\mathcal{K}_{p,q}(X/S, L)$ on S and a Zariski open subset $U \subset S$ such that*

$$\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$$

for all $s \in U$.

Proof: Put $\mathcal{E} = f_*L$. Using the natural maps

$$\iota : \bigwedge^p \mathcal{E} \to \bigwedge^{p-1} \mathcal{E} \otimes \mathcal{E}, \quad \mu : f_*L \otimes f_*(L^q) \to f_*(L^{q+1}),$$

we obtain a homomorphism of $\mathcal{O}_S$–modules

$$\delta : \bigwedge^p \mathcal{E} \otimes f_*(L^q) \to \bigwedge^{p-1} \mathcal{E} \otimes f_*(L^{q+1}).$$

The sheaf $\mathcal{K}_{p,q}(X/S, L)$ is defined as the cohomology sheaf at the middle term of the resulting complex of sheaves of $\mathcal{O}_S$–modules

$$\bigwedge^{p+1} \mathcal{E} \otimes f_*(L^{q-1}) \xrightarrow{\delta} \bigwedge^p \mathcal{E} \otimes f_*(L^q) \xrightarrow{\delta} \bigwedge^{p-1} \mathcal{E} \otimes f_*(L^{q+1}). \tag{1.5}$$

As f is proper, the sheaves $f_*(L^q)$ are coherent for all $q \geq 0$. Hence the cohomology sheaf $\mathcal{K}_{p,q}(X/S, L)$ is a coherent sheaf of $\mathcal{O}_S$–modules.

Since f is flat and L is locally free, L^q is flat over S for all $q \geq 0$. Hence the function

$$F_q : s \mapsto \dim_{k(s)} H^0(X_s, L_s^q)$$

is upper semicontinuous for all $q \geq 0$ [**HAG77**, III, Thm. 12.8]. The function F_q takes its minimal value on a nonempty Zariski open subset $U_q \subset S$. Put

$$U = U_0 \cap U_{q-1} \cap U_q \cap U_{q+1}.$$

Since the functions F_0, F_{q-1}, F_q and F_{q+1} are constant on U, the maps

$$f_*(L^k) \otimes k(s) \to H^0(X_s, L_s^k) \tag{1.6}$$

are isomorphisms for $k \in \{0, q-1, q, q+1\}$ by Grauert's theorem; cf. [**HAG77**, III, Cor. 12.9]. Let $i_s : \operatorname{Spec} k(s) \hookrightarrow S$ be the inclusion of a point. Since the functor $\mathcal{F} \mapsto \mathcal{F} \otimes k(s) = i_s^* \mathcal{F}$ is exact, we have

$$H^p(i_s^* \mathcal{F}^\bullet) \cong i_s^* \mathcal{H}^p(\mathcal{F}^\bullet)$$

for every complex $\mathcal{F}^\bullet$ of coherent $\mathcal{O}_S$–modules. Put $V = H^0(X_s, L_s)$ and apply the functor $i_s^* = \_ \otimes k(s)$ to the complex (1.5). Using the isomorphisms (1.6) we obtain a complex

$$\bigwedge^{p+1} V \otimes H^0(X_s, L_s^{q-1}) \xrightarrow{\delta_s} \bigwedge^p V \otimes H^0(X_s, L_s^q) \xrightarrow{\delta_s} \bigwedge^{p-1} V \otimes H^0(X_s, L_s^{q+1})$$

for all $s \in U$. By construction δ_s is the usual Koszul differential. Hence

$$\mathcal{K}_{p,q}(X/S, L) \otimes k(s) \cong K_{p,q}(X_s, L_s)$$

for all $s \in U$. □

Remark 1.30. The construction of the sheaf $\mathcal{K}_{p,q}(X/S, L)$ appears in [**V93**] in the case that $\mathcal{E}$ is locally free. More generally, if $\mathcal{F}$ is a coherent sheaf on X that is flat over S one can define coherent sheaves $\mathcal{K}^i_{p,q}(X/S; \mathcal{F}, L)$ such that

$$\mathcal{K}^i_{p,q}(X/S; \mathcal{F}, L) \otimes k(s) \cong K^i_{p,q}(X_s; \mathcal{F}_s, L_s)$$

for all s belonging to a suitable Zariski open subset of S.

Corollary 1.31. *Notation as above. The function*

$$s \mapsto \kappa_{p,q}(X_s, L_s)$$

is upper semicontinuous on the Zariski open subset $U \subset S$.

Proof: Nakayama's lemma implies that for every coherent sheaf $\mathcal{F}$ on S, the function

$$s \mapsto \dim_{k(s)}(\mathcal{F} \otimes k(s))$$

is upper semicontinuous; cf. [**HAG77**, III, Example 12.7.2]. □

1.6. Notes and comments

The Koszul complex was originally defined as a graded differential algebra. The first example emerged from to the following situation, see [**Ko50**] and [**Hal87**]. Consider $G \to E \to X$ be a (real) principal principal fibre bundle over a real-analytic manifolds X (a typical example is obtained for a Lie group E, when G is a Lie subgroup, and $X = E/G$ is a homogeneous space). If P_G denotes the subspace of primitive elements, then the graded algebra of differential forms $A^*(E)$ on E is isomorphic to the graded algebra $A^*(M) \otimes \bigwedge^* P_G$. Under this isomorphism, the deRham differential corresponds to an explicit differential operator on $A^*(M) \otimes \bigwedge^* P_G$. Hence, the deRham cohomology of E can be explicitly recovered from the given Koszul complex.

The general algebraic situation is the following. Consider A be a commutative ring, n a positive integer, and endow the tensor algebra $A \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n$ the natural graduation, [**Ei95**], obtained giving to any element of the canonical basis $\{e_i\}_i$ of $\mathbb{Z}^n$ the degree one. Choosing an element $a = (a_1, \ldots, a_n)$ of A^n, on the ring $A \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n \cong \bigwedge^*(A^n)$ one defines a differential d_A by mapping A to 0 and any e_i to a_i; note that this map shifts the degrees by -1. This differentiation coincides with the contraction i_a, where a is seen as an element of the dual module $(A^n)^\vee$. The new graded differential algebra is denoted by $K_\bullet(a)$. One readily sees that we have an isomorphism of graded differential algebras $K_\bullet(a) \cong K_\bullet(a_1) \otimes \ldots \otimes K_\bullet(a_n)$, where the structures on the latter algebra are defined naturally.

If M is an A-module, we have an induced differentiation d_M on the $K_\bullet(a)$-graded modules $K_\bullet(a, M) = M \otimes_{\mathbb{Z}} \bigwedge^* \mathbb{Z}^n$, respectively $K^\bullet(a, M) = \mathrm{Hom}_A(K_\bullet(a), M)$ and versions are available for differential graded modules, [**EGA 3-1**].

For other applications of the Koszul cohomology we refer to [**Hal87**].

CHAPTER 2

Basic results

2.1. Kernel bundles

DEFINITION 2.1. Suppose that L is globally generated. The kernel M_L of the surjective map

$$\mathrm{ev} : H^0(X,L) \otimes \mathcal{O}_X \to L$$

is called the *kernel bundle* of L. It is a vector bundle of rank $h^0(X,L) - 1$. If $W \subset H^0(X,L)$ is a linear subspace such that $\mathrm{ev}\,|_W : W \otimes \mathcal{O}_X \to L$ is surjective, we put $M_W = \ker(\mathrm{ev}\,|_W)$.

REMARK 2.2. The vector bundles M_L were extensively used in the work of Green and Lazarsfeld; cf. [**La89**].

REMARK 2.3. The short exact sequence

$$0 \to M_L \to H^0(X,L) \otimes \mathcal{O}_X \to L \to 0$$

is the pull-back via the morphism defined by L of the Euler sequence

$$0 \to \Omega^1_{\mathbb{P}}(1) \to H^0(X,L) \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(1) \to 0$$

on the projective space $\mathbb{P} = \mathbb{P}H^0(X,L)^\vee$.

PROPOSITION 2.4. *Suppose that L is a globally generated line bundle on X. Then*

$$\begin{aligned} K_{p,q}(X,L) &\cong \operatorname{coker}\left(\textstyle\bigwedge^{p+1} V \otimes H^0(X,L^{q-1}) \to H^0(X, \textstyle\bigwedge^p M_L \otimes L^q)\right) \\ &\cong \ker(H^1(X, \textstyle\bigwedge^{p+1} M_L \otimes L^{q-1}) \to \textstyle\bigwedge^{p+1} V \otimes H^1(X, L^{q-1})). \end{aligned}$$

Proof: Taking exterior powers in the short exact sequence

$$0 \to M_L \to V \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}} L \to 0$$

we obtain exact sequences

$$0 \to \textstyle\bigwedge^p M_L \to \textstyle\bigwedge^p V \otimes \mathcal{O}_X \to \textstyle\bigwedge^{p-1} M_L \otimes L \to 0$$

for all $p \geq 1$. As in Remark 1.3 one shows that the map

$$\delta : \textstyle\bigwedge^p V \otimes L^q \to \textstyle\bigwedge^{p-1} V \otimes L^{q+1}$$

factors through $\bigwedge^{p-1} M_L \otimes L^{q+1}$. Hence the kernel and image of the Koszul differential

$$\delta : \textstyle\bigwedge^p V \otimes H^0(X,L^q) \to \textstyle\bigwedge^{p-1} V \otimes H^0(X,L^{q+1})$$

are given by

$$\begin{aligned} \ker\delta &\cong H^0(X, \textstyle\bigwedge^p M_L \otimes L^q) \\ \operatorname{im}\delta &\cong \operatorname{im}(\textstyle\bigwedge^p V \otimes H^0(X,L^q) \to H^0(X, \textstyle\bigwedge^{p-1} M_L \otimes L^{q+1})) \end{aligned}$$

and the first description follows. The second description is obtained from the long exact cohomology sequence associated to the exact sequence

$$0 \to \bigwedge^{p+1} M_L \otimes L^{q-1} \to \bigwedge^{p+1} V \otimes L^{q-1} \to \bigwedge^{p} M_L \otimes L^{q} \to 0.$$

□

REMARK 2.5. From the proof, one easily deduces the following vanishing result

$$H^0(X, \bigwedge^p M_L) = 0, \tag{2.1}$$

for all $p \geq 1$.

REMARK 2.6. More generally, if $W \subset H^0(X, L)$ generates L and $\mathcal{F}$ is a coherent sheaf on X then $K_{p,q}(X; \mathcal{F}, L, W)$ is isomorphic to

$$\operatorname{coker}\left(\bigwedge^{p+1} W \otimes H^0(X, \mathcal{F} \otimes L^{q-1}) \to H^0(X, \mathcal{F} \otimes \bigwedge^p M_W \otimes L^q)\right)$$

and to

$$\ker(H^1(X, \bigwedge^{p+1} M_W \otimes \mathcal{F} \otimes L^{q-1}) \to \bigwedge^{p+1} W \otimes H^0(X, \mathcal{F} \otimes L^{q-1})).$$

PROPOSITION 2.7. *Suppose that L is a very ample line bundle on X. Put $\mathbb{P} = \mathbb{P}H^0(X, L)^*$, and denote $M_{\mathbb{P}} = \Omega^1_{\mathbb{P}}(1)$ the kernel of the evaluation map, and $\mathcal{I}_X$ the ideal sheaf of X in $\mathbb{P}$. Then*

$$\begin{aligned} K_{p,1}(X, L) &\cong H^0(\mathbb{P}, \bigwedge^{p-1} M_{\mathbb{P}} \otimes \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}}(2)) \\ &\cong \ker(H^0(\mathbb{P}, \bigwedge^{p-1} M_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(2)) \to H^0(X, \bigwedge^{p-1} M_L \otimes L^2)). \end{aligned}$$

Proof: The exact sequence of sheaves

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0$$

induces a long exact sequence of Koszul cohomology spaces:

$$K_{p,1}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) \to K_{p,1}(X, L) \to K_{p-1,2}(\mathbb{P}, \mathcal{I}_X, \mathcal{O}_{\mathbb{P}}(1)) \to K_{p-1,2}(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$$

The vanishing of Koszul cohomology on projective spaces (Proposition 1.5) shows that

$$K_{p,1}(X, L) \cong K_{p-1,2}(\mathbb{P}, \mathcal{I}_X, \mathcal{O}_{\mathbb{P}}(1)).$$

As in the proof of the Proposition 2.4 above, we observe that the latter space is isomorphic to $H^0(\mathbb{P}, \bigwedge^{p-1} M_{\mathbb{P}} \otimes \mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}}(2))$.

2.2. Projections and linear sections

Let A and B be finite–dimensional k–vector spaces. Recall the following two basic constructions.

(i) Given a surjective map $p : A \to B$ and a projective variety $X = V(I) \subset \mathbb{P}(A^\vee)$ we obtain an induced map $S(p) : S^*A \to S^*B$. Let $J \subset S^*B$ be the ideal generated by the image of I, and put $Y = V(J) \subset \mathbb{P}(B^\vee)$. Then $Y = X \cap \mathbb{P}(B^\vee)$ is a linear section of X.

(ii) Given an injective map $i : B \to A$ and a projective variety $Y = V(J) \subset \mathbb{P}(B^\vee)$, let I be the ideal generated by the image of J under the induced map $S(i) : S^*B \to S^*A$ and put $X = V(I) \subset \mathbb{P}(A^\vee)$. Then X is the cone over Y with vertex $\mathbb{P}(C^\vee)$, $C = \operatorname{coker} i$.

In the next two subsections we shall consider the effect of these operations on Koszul cohomology. It suffices to treat projections from a point and sections with a hyperplane, since general projections and linear sections are obtained by a repeated application of these operations.

2.2.1. Projection and evaluation maps. Let X be a projective variety over $\mathbb{C}$, and let L be a line bundle on X. Given a linear subspace $V \subseteq H^0(X, L)$ and an element $0 \neq x \in V^\vee$, the map $\mathrm{ev}_x = \langle x, -\rangle$ induces an exact sequence

$$0 \to W_x \to V \xrightarrow{\mathrm{ev}_x} \mathbb{C} \to 0.$$

Taking exterior powers, we obtain an exact sequence

$$0 \to \bigwedge^p W_x \to \bigwedge^p V \to \bigwedge^{p-1} W_x \to 0.$$

As we have seen in Remark 1.3, the map $\bigwedge^p V \to \bigwedge^{p-1} W_x$ is given by contraction with x, i.e., it is given by the formula

$$\iota_x(v_1 \wedge \ldots \wedge v_p) = \sum_i (-1)^i v_1 \wedge \ldots \widehat{v_i} \wedge \ldots \wedge v_p \otimes \mathrm{ev}_x(v_i).$$

The factorization

$$\bigwedge^p V \xrightarrow{\iota_x} \bigwedge^{p-1} W_x \hookrightarrow \bigwedge^{p-1} V$$

induces a commutative diagram

$$\begin{tikzcd} K_{p,q}(M,V) \arrow[r, "\mathrm{pr}_x"] \arrow[dr, "\mathrm{ev}_x"'] & K_{p-1,q}(M,W_x) \arrow[d] \\ & K_{p-1,q}(M,V). \end{tikzcd}$$

The maps

$$\begin{aligned} \mathrm{pr}_x &: K_{p,q}(M,V) \to K_{p-1,q}(M,W_x) \\ \mathrm{ev}_x &: K_{p,q}(M,V) \to K_{p-1,q}(M,V) \end{aligned}$$

are called the *projection* and *evaluation* maps on Koszul cohomology.

LEMMA 2.8. *The projection map* pr_x *fits into a long exact sequence of Koszul cohomology groups*

$$K_{p,q}(M,W_x) \to K_{p,q}(M,V) \xrightarrow{\mathrm{pr}_x} K_{p-1,q}(M,W_x) \to K_{p-1,q+1}(M,W_x). \tag{2.2}$$

Proof: Apply Lemma 1.22 to the exact sequence of S^*V–modules

$$0 \to \bigwedge^p W_x \otimes M \to \bigwedge^p V \otimes M \to \bigwedge^{p-1} W_x \otimes M \to 0.$$

□

COROLLARY 2.9. *If* $K_{p,0}(M,V) = 0$*, then* $K_{p,1}(M,W_x)$ *injects into* $K_{p,1}(M,V)$.

As x varies, the maps ev_x glue together to a homomorphism of vector bundles on $\mathbb{P} = \mathbb{P}(V^\vee)$

$$\mathrm{ev} : \mathcal{O}_{\mathbb{P}} \otimes K_{p,q}(M,V) \to \mathcal{O}_{\mathbb{P}}(1) \otimes K_{p-1,q}(M,V).$$

The homomorphism ev is obtained in the following way. Given $\ell \in \mathbb{Z}$, we put

$$\mathcal{K}^{p,q}(\ell) = \bigwedge^{l-p-q} V \otimes \mathcal{O}_{\mathbf{P}}(p) \otimes M_q.$$

This defines a double complex of vector bundles on $\mathbb{P}(V^\vee)$; the horizontal differential is the differential of the sheafified Koszul complex (1.4), and the vertical differential is the differential of the Koszul complex of M. As the the Koszul complex (1.4) is exact, the rows of this double complex are exact. Hence the associated spectral sequence

$$\mathcal{E}_1^{p,q}(\ell) = H^p(\mathcal{K}^{\bullet,q}(\ell)) = \mathcal{O}_{\mathbb{P}}(p) \otimes K_{l-p-q,q}(M,V)$$

converges to zero. The desired homomorphism is then obtained as the differential

$$\begin{array}{ccc} \mathcal{E}_1^{0,1}(\ell) & \xrightarrow{d_1} & \mathcal{E}_1^{1,1}(\ell) \\ \| & & \| \\ K_{\ell-1,1}(M,V)\otimes\mathcal{O}_{\mathbb{P}} & \to & K_{\ell-2,1}(M,V)\otimes\mathcal{O}_{\mathbb{P}}(1). \end{array}$$

PROPOSITION 2.10. *Let M be a graded S^*V–module such that*

$$M_q = 0 \ \text{ for all } q < 0 \text{ and } K_{p,0}(M,V) = 0 \text{ for all } p \geq 1. \tag{2.3}$$

Then the map

$$H^0(\mathrm{ev}) : K_{p+1,1}(M,V) \longrightarrow V \otimes K_{p,1}(M,V)$$

is injective for all $p \geq 1$.

Proof: We take global sections in the double complex $\mathcal{K}^{p,q}(\ell)$ to obtain a double complex with general term

$$K^{p,q}(\ell) = \begin{cases} \bigwedge^{\ell-p-q} V \otimes S^pV \otimes K_{\ell-p-q,q}(M,V) & p \geq 0 \\ 0 & p < 0 \end{cases}$$

The assumptions on M imply that the associated spectral sequence

$$E_1^{p,q}(\ell) = S^pV \otimes K_{\ell-p-q,q}(M,V)$$

is a first quadrant spectral sequence, i.e., $E_1^{p,q}(\ell) = 0$ if $p < 0$ or $q < 0$. We have

$$E_\infty^{p,q}(\ell) = \begin{cases} 0 & (p,q) \neq (0,\ell) \\ M_\ell & (p,q) = (0,\ell). \end{cases}$$

The map $H^0(\mathrm{ev})$ is the differential $d_1 : E_1^{0,1}(\ell) \to E_1^{1,1}(\ell)$. The assumptions on M and p imply that $\ell \geq 3$ and $E_1^{0,2}(\ell) = K_{\ell-2,0}(M,V) = 0$. Hence there are no incoming or outgoing differentials at position $(0,1)$ and

$$\ker H^0(\mathrm{ev}) = E_2^{0,1}(\ell) = E_\infty^{0,1}(\ell) = 0.$$

□

REMARK 2.11. If L is a line bundle on X such that $L \not\cong \mathcal{O}_X$ and $H^0(X,L) \neq 0$, the hypotheses of the Proposition are satisfied for the module $R(L)$.

REMARK 2.12. If M is a graded S^*V–module such that $M_{-1} = 0$, $M_0 = \mathbb{C}$, $M_1 = V$, and $W \subset V$ is a linear subspace, then $K_{p,0}(M,W) = 0$ for all $p > 0$. Indeed, the assumptions imply that

$$K_{p,0}(M,W) = \ker(\textstyle\bigwedge^p W \xrightarrow{\delta} \bigwedge^{p-1} W \otimes V).$$

The dual map $\delta^\vee$ is surjective, since it factors as

$$\textstyle\bigwedge^{p-1} W^\vee \otimes V^\vee \to \bigwedge^{p-1} W^\vee \otimes W^\vee \xrightarrow{\wedge} \bigwedge^p W^\vee.$$

Corollary 2.13. *If M satisfies the condition (2.3) then*

$$K_{p,1}(M,V)=0 \Longrightarrow K_{p+1,1}(M,V)=0$$

for all $p>0$.

Proposition 2.14. *Let M be a graded S^*V–module that satisfies (2.3), and let $\alpha \in K_{p+1,1}(M,V)$ be a non-zero element. There exists a proper linear subspace $\Lambda \subsetneq \mathbb{P}(V^\vee)$ such that $pr_x(\alpha) \neq 0$ for all $[x] \notin \Lambda$.*

Proof: As $H^0(\mathrm{ev})$ is injective by Proposition 2.10, we have $H^0(\mathrm{ev})(\alpha) \neq 0$. Write

$$H^0(\mathrm{ev})(\alpha)=\sum_{i=1}^{k} v_i \otimes \alpha_i,$$

with $\alpha_i \in K_{p,1}(M,V)$ linearly independent and $v_i \in V$. Let $H_i \subset V^\vee$ be the hyperplane dual to v_i. Since

$$\begin{aligned} \mathrm{ev}_x(\alpha)=0 &\iff v_i \in W_x \text{ for all } i \\ &\iff x \in H_i \text{ for all } i \end{aligned}$$

the assertion follows by taking $\Lambda = \cap_{i=1}^{k}\mathbb{P}(H_i)$. □

Proposition 2.15. *Let $W \subset V$ be a linear subspace. Then $K_{p,q}(S^*V,W)=0$ for all $p>0$.*

Proof: We argue by induction on $c=\mathrm{codim}\,(W)$. If $c=0$ the result follows from Proposition 1.5. For the induction step, we assume that the result is known for linear subspaces of codimension $\leq c-1$. Choose a flag of linear subspaces

$$W=W_c \subset W_{c-1} \subset \ldots \subset W_1 \subset W_0 = V$$

such that $\mathrm{codim}\,(W_i)=i$. As there exists $x \in W_{c-1}^\vee$ such that $W_c=\ker(\mathrm{ev}_x) \subset W_{c-1}$, we have an exact sequence

$$K_{p,q-1}(S^*V,W_c) \to K_{p,q}(S^*V,W_c) \to K_{p,q}(S^*V,W_{c-1}).$$

The induction hypothesis implies that the map

$$K_{p,q-1}(S^*V,W_c) \to K_{p,q}(S^*V,W_c)$$

is surjective for all $q \geq 1$. Hence $K_{p,q}(S^*V,W)$ is a quotient of $K_{p,0}(S^*V,W)$, and the result follows from Remark 2.12. □

Corollary 2.16. *Put $I=\ker(S^*V \to R(L))$. If $p \geq 1$ then*

$$K_{p,1}(S(X),W) \cong K_{p-1,2}(I,W)$$

for every linear subspace $W \subset V$.

Remark 2.17. In the geometric case, the projection map can be defined using kernel bundles. For simplicity, consider X be a curve, L be a line bundle on X, and suppose $x \in X$ is a point which is not a base point of L. As before, we have an induced short exact sequence

$$0 \to W_x \to H^0(X,L) \to \mathbb{C} \to 0,$$

where $W_x = H^0(X,L(-x))$. From the restricted Euler sequences corresponding to L, and $L(-x)$ respectively, we obtain an exact sequence:

$$0 \to M_{L(-x)} \to M_L \to \mathcal{O}_X(-x),$$

and further

$$0 \to \bigwedge^{p+1} M_{L(-x)} \otimes L \to \bigwedge^{p+1} M_L \otimes L \to \bigwedge^{p} M_{L(-x)} \otimes L(-x),$$

for any positive integer p. The exact sequence of global sections, together with the natural sequence

$$0 \to \bigwedge^{p+2} W_x \to \bigwedge^{p+2} H^0(X,L) \to \bigwedge^{p+1} W_x \to 0,$$

induce an exact sequence:

$$0 \to K_{p+1,1}(X,L,W_x) \to K_{p+1,1}(X,L) \xrightarrow{\mathrm{pr}_x} K_{p,1}(X,L(-x)),$$

where pr_x is the projection map centered in x.

In the case of varieties X of higher dimension, one has to blow up the point x.

2.2.2. Lefschetz theorems. Let $X \subset \mathbb{P}(V^\vee)$ be a nondegenerate variety, and let $H = V(t) \subset \mathbb{P}(V^\vee)$ be a hyperplane such that no irreducible component of X is contained in H. Put $Y = X \cap H$, $W = V/\mathbb{C}.t$. Let $I(Y)$ be the saturated ideal defining Y, and put

$$I'(Y) = I(X) + (t)/(t) \subseteq I(Y), \quad S'(Y) = S^*W/I'(Y).$$

LEMMA 2.18. *We have $K_{p,q}(S(X),V) \cong K_{p,q}(S'(Y),W)$ for all p, q.*

Proof: The choice of a splitting $V \cong W \oplus \mathbb{C}.t$ induces isomorphisms

$$K_{p,q}(S'(Y),V) \cong K_{p,q}(S'(Y),W) \oplus K_{p-1,q}(S'(Y),W).$$

The hypothesis of the theorem implies that $t \in S(X)$ is not a zero divisor. The resulting exact sequence

$$0 \to S(X)(-1) \xrightarrow{\cdot t} S(X) \to S'(Y) \to 0$$

gives rise to a long exact sequence

$$K_{p,q-1}(S(X),V) \xrightarrow{\cdot t} K_{p,q}(S(X),V) \to K_{p,q}(S'(Y),V) \to K_{p-1,q}(S(X),V)$$

of S^*V–modules. The commutative diagram

$$\begin{tikzcd}
 & \bigwedge^{p+1} V \otimes S_{q-1}(X) \arrow[d, "\delta"] \\
\bigwedge^p V \otimes S_{q-1}(X) \otimes V \arrow[ur] \arrow[r] & \bigwedge^p V \otimes S_q(X)
\end{tikzcd}$$

shows that the map

$$K_{p,q-1}(S(X),V) \xrightarrow{\cdot t} K_{p,q}(S(X),V)$$

is zero. Hence we obtain short exact sequences

$$0 \to K_{p,q}(S(X),V) \to K_{p,q}(S'(Y),W) \oplus K_{p-1,q}(S'(Y),W) \to K_{p-1,q}(S(X),V) \to 0.$$

By induction on p (the case $p = 0$ being obvious) we find that

$$K_{p,q}(S(X),V) \cong K_{p,q}(S'(Y),W).$$

$\square$

THEOREM 2.19. *Notation as before.*

(i) *The restriction map* $K_{p,1}(S(X), V) \to K_{p,1}(S(Y), W)$ *is injective for all* $p \geq 1$;

(ii) *If the restriction map* $I_2(X) \to I_2(Y)$ *is surjective then* $K_{p,1}(S(X), V) \cong K_{p,1}(S(Y), W)$;

(iii) $K_{p,q}(S(X), V) \cong K_{p,q}(S(Y), W)$ *if* $I_k(X) \to I_k(Y)$ *is surjective for all* $k \in \{q-1, q, q+1\}$.

Proof: To prove (i), note that $I_1(Y) = I'_1(Y) = 0$, hence $S_1(Y) = S'_1(Y)$ and $K_{p,1}(S'(Y), W) \subseteq K_{p,1}(S(Y), W)$ by Corollary 1.23 (ii). The result then follows from Lemma 2.18. To obtain the statements (ii) and (iii), we apply Corollary 1.23 to the inclusion $I(X) \subset I(Y)$ and use the isomorphism of Proposition 1.27 (ii). $\square$

THEOREM 2.20. *Let* X *be an irreducible projective variety, and let* L *be a line bundle on* X. *Given a connected divisor* $Y \in |L|$, *we write* $L_Y = L \otimes \mathcal{O}_Y$.

(i) *Let* W *be the image of the restriction map* $H^0(X, L) \to H^0(Y, L_Y)$. *The map* $K_{p,1}(X, L) \to K_{p,1}(Y, L_Y)$ *induces an inclusion*
$$K_{p,1}(X, L) \subset K_{p,1}(Y, L_Y, W)$$
for all $p \geq 1$;

(ii) *If* $H^1(X, \mathcal{O}_X) = 0$ *then* $K_{p,1}(X, L) \to K_{p,1}(Y, L_Y)$ *is an isomorphism for all* $p \geq 1$;

(iii) *If* $H^1(X, L^q) = 0$ *for all* $q \geq 0$ *then* $K_{p,q}(X, L) \cong K_{p,q}(Y, L_Y)$ *for all* p, q.

Proof: Since Y is connected we have an exact sequence
$$0 \to \mathbb{C} \to H^0(X, L) \to H^0(Y, L_Y) \to 0.$$

As $S_0(Y) = S'_0(L_Y) = \mathbb{C}$, Corollary 1.23 implies that $K_{p,1}(S'(Y), W) \subset K_{p,1}(S(Y), W)$. We then apply Theorem 2.19 (i) and Proposition 1.27 (i). To prove the statements (ii) and (iii) put $R'(L_Y) = \operatorname{im}(R(L) \to R(L_Y))$ and argue as in the proof of part (ii) and (iii) of Theorem 2.19, replacing $S(X)$, $S(Y)$ and $S'(Y)$ by $R(L)$, $R(L_Y)$ and $R'(L_Y)$. $\square$

REMARK 2.21. Using Green's base change spectral sequence, one can prove a more general version of theorem 2.20 valid for divisors Y that do not necessarily belong to $|L|$; cf. [**AN04**].

REMARK 2.22. The condition $H^1(X, \mathcal{O}_X) = 0$ in *(ii)* can be replaced by the surjectivity of the map $H^0(X, L) \to H^0(Y, L_Y)$. Similarly in *(iii)* it suffices to assume that the restriction maps $H^0(X, L^q) \to H^0(Y, L_Y^q)$ are surjective for all q.

2.3. Duality

LEMMA 2.23. *Given a long exact sequence of coherent sheaves*
$$0 \to \mathcal{F} \to \mathcal{I}^1 \to \ldots \to \mathcal{I}^\ell \to \mathcal{G} \to 0$$
we have $\ker((H^\ell(X, \mathcal{F}) \to H^\ell(X, \mathcal{I}^1)) \cong \operatorname{coker}(H^0(X, \mathcal{I}^\ell) \to H^0(X, \mathcal{G}))$ *if*
$$H^{\ell-p}(X, \mathcal{I}^p) = H^{\ell-p}(X, \mathcal{I}^{p+1}) = 0, \quad p = 1, \ldots, \ell - 1.$$

Proof: Break up the long exact sequence into short exact sequences

$$0 \to \mathcal{F} \to \mathcal{I}^1 \to \mathcal{Q}^1 \to 0, \quad 0 \to \mathcal{Q}^1 \to \mathcal{I}^2 \to \mathcal{Q}^2 \to 0, \ldots$$
$$\ldots, 0 \to \mathcal{Q}^{\ell-2} \to \mathcal{I}^{\ell-1} \to \mathcal{Q}^{\ell-1} \to 0, \quad 0 \to \mathcal{Q}^{\ell-1} \to \mathcal{I}^{\ell} \to \mathcal{G} \to 0$$

and note that the hypotheses of the Lemma imply that

$$\ker((H^\ell(X,\mathcal{F}) \to H^\ell(X,\mathcal{I}^1)) \cong H^{\ell-1}(\mathcal{Q}^1) \cong H^{\ell-2}(\mathcal{Q}^2) \cong \ldots$$
$$\ldots \cong H^1(X,\mathcal{Q}^{\ell-1}) \cong \mathrm{coker}\,(H^0(X,\mathcal{I}^\ell) \to H^0(X,\mathcal{G})).$$

(Alternatively, put $\mathcal{I}^0 = \mathcal{F}$, $\mathcal{I}^{\ell+1} = \mathcal{G}$ and use that the spectral sequence $E_1^{p,q} = H^q(X,\mathcal{I}^p)$, which converges to zero since the complex is exact.) $\square$

Theorem 2.24. *Let L be a globally generated line bundle on a smooth projective variety X of dimension n. Put $r = \dim |L|$. If*

$$H^i(X,L^{q-i}) = H^i(X,L^{q-i+1}) = 0, \quad i = 1,\ldots,n-1$$

then

$$K_{p,q}(X,L)^\vee \cong K_{r-n-p,n+1-q}(X,K_X,L).$$

Proof: By Proposition 2.4 and Serre duality we have

$$\begin{aligned} K_{p,q}(X,L) &\cong \mathrm{coker}\,(\textstyle\bigwedge^{p+1} V \otimes H^0(X,L^{q-1}) \to H^0(X,\bigwedge^p M_L \otimes L^q)) \\ &\cong \ker(H^n(X,K_X \otimes \textstyle\bigwedge^p M_L^\vee \otimes L^{-q}) \to \bigwedge^{p+1} V^\vee \otimes H^n(X,K_X \otimes L^{1-q})). \end{aligned}$$

As $\mathrm{rank}\,(M_L) = r$ and $\det M_L \cong L^{-1}$, the latter group is isomorphic to

$$\ker(H^n(X,K_X \otimes \textstyle\bigwedge^{r-p} M_L \otimes L^{1-q}) \to \bigwedge^{r-p} V \otimes H^n(X,K_X \otimes L^{1-q})).$$

Take exterior powers in the exact sequence defining the kernel bundle M_L to obtain a long exact sequence

$$0 \to \textstyle\bigwedge^{r-p} M_L \otimes K_X \otimes L^{1-q} \to \bigwedge^{r-p} V \otimes K_X \otimes L^{1-q} \to$$
$$\to \textstyle\bigwedge^{r-p-1} V \otimes K_X \otimes L^{2-q} \to \ldots$$
$$\ldots \to \textstyle\bigwedge^{r-p-n+1} V \otimes K_X \otimes L^{n-q} \to \bigwedge^{r-p-n} M_L \otimes K_X \otimes L^{n+1-q} \to 0.$$

Applying Lemma 2.23 with $\ell = n$ we find that

$$K_{p,q}(X,L)^\vee \cong K_{r-n-p,n+1-q}(X,K_X,L)$$

if

$$H^{n-1}(X,K_X \otimes L^{1-q}) = \ldots = H^1(X,K_X \otimes L^{n-1-q}) = 0,$$
$$H^{n-1}(X,K_X \otimes L^{2-q}) = \ldots = H^1(X,K_X \otimes L^{n-q}) = 0.$$

By Serre duality, these conditions are equivalent to the hypothesis of the theorem. $\square$

Remark 2.25. A similar argument proves the following slightly more general statement. Let X be a projective manifold of dimension n, L be a line bundle, and E be a vector bundle on X. Suppose that $V \subset H^0(X,L)$ is a base-point-free linear system of dimension $r+1$, and assume that

(1) $H^i(X,E \otimes L^{q-i}) = 0$ for all $i = 1,\ldots,n-1$;
(2) $H^i(X,E \otimes L^{q-i-1}) = 0$ for all $i = 1,\ldots,n-1$.

Then

$$K_{p,q}(X, E, L, V)^\vee \cong K_{r-n-p,n+1-q}(X, K_X \otimes E^\vee, L, V). \tag{2.4}$$

REMARK 2.26. On a curve, the assumptions in the statement above are empty.

2.4. Koszul cohomology versus usual cohomology

We have seen that if V generates L, then the groups $K_{p,q}(X; \mathcal{F}, L, V)$ can be computed with help of the usual cohomology. It is natural to ask if a converse of this description is also true: can usual cohomology of a coherent sheaf be recovered from its Koszul cohomology with values in some line bundle ? The aim of this section is to give a positive answer to this question.

THEOREM 2.27. *Let X be a connected complex compact manifold, let r and q be two positive integers, $\mathcal{F}$ be a coherent sheaf on X, L be a line bundle on X, and V be an $(r+1)$-dimensional subspace of global sections of L which generates L. Suppose that*

(i) $H^{q-i}(X, \mathcal{F} \otimes L^i) = 0$, $i = 1, \ldots, q-1$;
(ii) $H^{q-i}(X, \mathcal{F} \otimes L^{i+1}) = 0$, $i = 0, \ldots, q-1$.

Then

$$H^q(X, \mathcal{F}) \cong K_{r-q,q+1}(X, \mathcal{F}, L, V).$$

Proof: Dualise the exact sequence

$$0 \to M_V \to V \otimes \mathcal{O}_X \to L \to 0 \tag{2.5}$$

and twist it by $\mathcal{F} \otimes L$ to obtain the exact sequence

$$0 \to \mathcal{F} \to V^\vee \otimes \mathcal{F} \otimes L \to M_V^\vee \otimes \mathcal{F} \otimes L \to 0.$$

As rank $(M_V) = r$ and $\det(M_V) = L^\vee$, we can rewrite this exact sequence in the form

$$0 \to \mathcal{F} \to \bigwedge^r V \otimes \mathcal{F} \otimes L \to \bigwedge^{r-1} M_V \otimes \mathcal{F} \otimes L^2 \to 0. \tag{2.6}$$

Proposition 2.4 shows that

$$H^1(X, \mathcal{F}) \cong K_{r-1,2}(X, \mathcal{F}, L, V)$$

if $H^1(X, \mathcal{F} \otimes L) = 0$; this solves the case $q = 1$.

We may thus assume $q \geq 2$. In that case, since $H^q(X, \mathcal{F} \otimes L) = 0$ and $H^{q-1}(X, \mathcal{F} \otimes L) = 0$, the long cohomology sequence associated to the sequence (2.6) shows that

$$H^q(X, \mathcal{F}) \cong H^{q-1}(X, \bigwedge^{r-1} M_V \otimes \mathcal{F} \otimes L^2).$$

Taking exterior powers in (2.5) we obtain a long exact sequence

$$0 \to \bigwedge^{r-1} M_L \otimes \mathcal{F} \otimes L^2 \to \bigwedge^{r-1} V \otimes \mathcal{F} \otimes L^2 \to \bigwedge^{r-2} V \otimes \mathcal{F} \otimes L^3 \to \ldots$$
$$\ldots \to \bigwedge^{r-q+1} V \otimes \mathcal{F} \otimes L^{q+1} \to \bigwedge^{r-q} M_L \otimes \mathcal{F} \otimes L^{q+1} \to 0.$$

The result then follows from Lemma 2.23 and Proposition 2.4. ☐

REMARK 2.28. One can give a more conceptual proof of Theorem 2.27 using the hypercohomology spectral sequence associated to the complex of sheaves

$$0 \to \bigwedge^{r+1} V \otimes \mathcal{F}(-r-1) \to \bigwedge^{r} V \otimes \mathcal{F}(-r) \to \dots$$
$$\dots \to \bigwedge^{2} V \otimes \mathcal{F}(-2) \to V \otimes \mathcal{F}(-1) \to \mathcal{F} \to 0.$$

REMARK 2.29. Under the hypotheses of Theorem 2.27, the Koszul cohomology groups $K_{r-q,q+1}(X;\mathcal{F},L,V)$ do not depend on the choice of V.

DEFINITION 2.30. A property (P) is said to hold for a *sufficiently ample* line bundle if there exists a line bundle L_0 such that property (P) holds for every line bundle L such that $L \otimes L_0^{-1}$ ample.

Let $\mathcal{F}$ be a coherent sheaf on X. An argument due to M. Green (cf. [**Pard98**, Lemma 5.2]) shows that by taking L is sufficiently ample, we may assume that L is globally generated and

$$H^j(X,\mathcal{F}\otimes L^k) = 0 \text{ if } \begin{cases} j = 0, & k < 0 \\ 1 \le j \le \dim X - 1, & k \neq 0 \\ j = \dim X, & k > 0. \end{cases}$$

COROLLARY 2.31. *Let X be an integral projective variety let $\mathcal{F}$ be a coherent sheaf on X. If L is sufficiently ample, then*

$$H^q(X,\mathcal{F}) \cong K_{h^0(X,L)-q-1,q+1}(X,\mathcal{F},L)$$

for all $q \ge 1$.

Corollary 2.31 is a generalization of a previous result due to Green [**Gre84a**, Theorem (4.f.1)], where this statement was proved for the case $\mathcal{F} = \Omega^p_X$. Our proof is nevertheless different from that of Green.

Theorem 2.27 shows that Koszul cohomology can be seen as an extension of classical sheaf cohomology. However, from the practical point of view, the rôle of Koszul cohomology is to complete the classical theory rather than to replace it.

REMARK 2.32. There are a number of immediate consequences of Theorem 2.27, obtained for different particular choices of the sheaf $\mathcal{F}$. For example, if $H^1(X,L) = 0$ then

$$H^1(X,\mathcal{O}_X) \cong K_{r-1,2}(X,L,V).$$

This result shows in particular that the topological genus of a curve can be read off from Koszul cohomology of X with values in any line bundle L with $H^1(X,L) = 0$, see also [**Ei06**], [**Tei07**]. This explains the interest for studying these Koszul cohomology groups in relation with the intrinsic geometry of curves; see [**Gre84a**], [**GL84**]. There exists a similar interpretation for $H^q(X,\mathcal{O}_X)$, $q \ge 2$.

When we apply Theorem 2.27 to the canonical bundle, we obtain an alternative proof of [**Gre84a**, Theorem (2.c.1)]:

THEOREM 2.33 (Green, 1984). *Let X be a connected complex projective manifold of dimension $n \ge 2$, L be a line bundle on X, and V be an $(r+1)$-dimensional subspace of global sections of L ($r \ge 1$) that generates L. Suppose that*

$$H^i(X,L^{-i}) = H^i(X,L^{-i-1}) = 0 \quad \textit{for all } 1 \le i \le n-1.$$

Then

$$K_{r-n,n+1}(X,K_X,L,V)\cong \mathbf{C}.$$

Proof: Serre duality implies that

$$H^{n-i}(X,K_X\otimes L^{i+1})\cong H^i(X,L^{-i-1})^*=0,\ \text{for all } 1\le i\le n-1,$$

and

$$H^{n-i-1}(X,K_X\otimes L^{i+1})\cong H^{i+1}(X,L^{-i-1})^*=0,\ \text{for all } 1\le i\le n-2.$$

Since L is globally generated, it follows that $H^0(X,L^{-1})=0$, hence

$$H^n(X,K_X\otimes L)=0$$

and the hypotheses of Theorem 2.27 are satisfied. □

2.5. Sheaf regularity.

Definition 2.34. Let L be a globally generated ample line bundle on a projective variety X, and let $\mathcal{F}$ be a coherent sheaf on X. We say that $\mathcal{F}$ is *m–regular* with respect to L if

$$H^i(X,\mathcal{F}\otimes L^{m-i})=0 \text{ for all } i>0.$$

Mumford has shown that if $\mathcal{F}$ is m–regular, then $\mathcal{F}$ is $(m+k)$–regular for all $k\ge 0$; cf. [**La04**]. This motivates the following definition.

Definition 2.35. The *Castelnuovo–Mumford regularity* of a coherent sheaf $\mathcal{F}$ with respect to a globally generated ample line bundle L on X is

$$\mathrm{reg}_L(\mathcal{F})=\min\{m\mid \mathcal{F} \text{ is m–regular w.r.t. L}\}$$

Remark 2.36. The conditions of Theorem 2.27 can be rewritten in the form

$$\begin{aligned} H^j(X,\mathcal{F}\otimes L^{q-j}) &= 0,\quad j=1,\dots,q-1\\ H^j(X,\mathcal{F}\otimes L^{q+1-j}) &= 0,\quad j=1,\dots,q.\end{aligned}$$

These conditions can be thought of as partial regularity conditions. In particular, the above conditions hold if $\mathcal{F}$ is q–regular with respect to L.

It is well-known that the regularity of a projective variety can be read off from its graded Betti numbers. This phenomenon can be generalized as follows.

Proposition 2.37. *Let $\mathcal{F}$ be a coherent sheaf on the complex projective manifold X, and L be a globally generated ample line bundle. Let m be a positive integer, and suppose that $\mathcal{F}$ is $(m+1)$-regular with respect to L. Then $\mathcal{F}$ is m-regular if and only if $K_{p,m+1}(X;\mathcal{F},L)=0$ for all p. In particular, the regularity of $\mathcal{F}$ is computed by the formula*

$$reg_L(\mathcal{F})=\min\{m\mid K_{p,m+1}(X,\mathcal{F},L)=0 \text{ for all } p\}.$$

Proof:

Since $\mathcal{F}$ is $(m+1)$-regular, we have

$$H^i(X,\mathcal{F}\otimes L^{m+1-i})=0 \text{ for } i>0.$$

For the *"if"* part, write $r=h^0(X,L)-1$ and let p be an integer. If $p>r$, then $K_{p,m+1}(X;\mathcal{F},L)=0$ by Proposition 2.4 since $\bigwedge^{p+1}M_L=0$. So we may assume that $p < r$; put $q = r - p > 0$. We can write $K_{p,m+1}(X;\mathcal{F},L) = K_{r-q,q+1}(X;\mathcal{F}\otimes L^{m-q},L)$. Since $\mathcal{F}$ is m–regular, the sheaf $\mathcal{F}\otimes L^{m-q}$ is q–regular. Hence the hypotheses of Theorem 2.27 are satisfied for the sheaf $\mathcal{F}\otimes L^{m-q}$ by Remark 2.36, and we obtain $K_{p,m+1}(X;\mathcal{F},L) \cong H^q(X,\mathcal{F}\otimes L^{m-q})$. This last group vanishes as well, by m-regularity.

For the *"only if"* part, we prove by induction on i that

$$H^i(X,\mathcal{F}\otimes L^{m-i}) = 0 \text{ for all } i > 0.$$

If $i = 1$, then we apply Theorem 2.27 to the 1–regular sheaf $\mathcal{F}\otimes L^{m-1}$ to obtain

$$H^1(X,\mathcal{F}\otimes L^{m-1}) \cong K_{r-1,2}(X;\mathcal{F}\otimes L^{m-1},L) = K_{r-1,m+1}(X,L) = 0.$$

For the induction step, suppose that

$$H^j(X,\mathcal{F}\otimes L^{m-j}) = 0 \text{ for } j \leq i-1.$$

Applying Theorem 2.27 to the i–regular sheaf $\mathcal{F}\otimes L^{m-i}$, we find

$$H^i(X,\mathcal{F}\otimes L^{m-i}) \cong K_{r-i,i+1}(X;\mathcal{F}\otimes L^{m-i},L) = K_{r-i,m+1}(X,L) = 0.$$

□

Remark 2.38. Proposition 2.37 says that the Castelnuovo–Mumford regularity $\mathrm{reg}_L(\mathcal{F})$ equals the number of rows in the Betti diagram of the graded module $R(\mathcal{F},L) = \bigoplus_q H^0(X,\mathcal{F}\otimes L^q)$.

2.6. Vanishing theorems

Theorem 2.39 (Green). *Let $W \subset H^0(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}}(d))$ be a base–point free subspace of codimension c. We have $K_{p,q}(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}}(k),\mathcal{O}_{\mathbb{P}}(d);W) = 0$ if $k + (q-1)d \geq p + c$.*

Proof: Choose a flag of linear subspaces

$$W = W_c \subset W_{c-1} \subset \ldots \subset \ldots \subset W_1 \subset W_0 = H^0(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}}(d))$$

such that dim $W_i/W_{i+1} = 1$ for all i, and let M_i be the kernel of the evaluation map

$$W_i \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(d).$$

Using the exact sequence

$$0 \to M_0 \to H^0(\mathbb{P}^r,\mathcal{O}_{\mathbb{P}}(d))\otimes\mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(d) \to 0$$

one easily shows that M_0 is 1–regular in the sense of Castelnuovo–Mumford, hence $\bigwedge^p M_0$ is p–regular (see e.g. [**La04**]). From the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & M_i & \to & W_i\otimes\mathcal{O}_{\mathbb{P}} & \to & \mathcal{O}_{\mathbb{P}}(d) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & M_{i-1} & \to & W_{i-1}\otimes\mathcal{O}_{\mathbb{P}} & \to & \mathcal{O}_{\mathbb{P}}(d) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_{\mathbb{P}} & = & \mathcal{O}_{\mathbb{P}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

we obtain a short exact sequence

$$0 \to M_i \to M_{i-1} \to \mathcal{O}_{\mathbb{P}} \to 0.$$

Taking exterior powers in this exact sequence and using induction, one shows that $\bigwedge^p M_i$ is $(p+i)$–regular for all i; see [**Gre89**, Theorem 4.1] for details. By Proposition 2.4 it suffices to show that

$$H^1(\mathbb{P}^r, \bigwedge^{p+1} M_c \otimes \mathcal{O}_{\mathbb{P}^r}(k) \otimes \mathcal{O}_{\mathbb{P}^r}(d)^{\otimes(q-1)}) = 0.$$

Hence the desired vanishing holds if

$$k + (q-1)d + 1 \geq p + c + 1.$$

$\square$

COROLLARY 2.40. If $W \subset H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ is a base–point free linear subspace of codimension c, then the Koszul complex

$$\bigwedge^{p+1} W \otimes H^0(\mathcal{O}_{\mathbb{P}}(k-d)) \to \bigwedge^p W \otimes H^0(\mathcal{O}_{\mathbb{P}}(k)) \to \bigwedge^{p-1} W \otimes H^0(\mathcal{O}_{\mathbb{P}}(k+d))$$

is exact at the middle term if $k \geq d + p + c$.

Proof: Apply the previous result with $q = 0$. $\square$

COROLLARY 2.41. *Let $\Sigma = \{p_1, \dots, p_d\}$ be a finite set of points on a rational normal curve $\Gamma \subset \mathbb{P}^r = \mathbb{P}(V^\vee)$. Then $K_{p,1}(S(\Gamma), V) \cong K_{p,1}(S(\Sigma), V)$ if $d \geq 2r - p + 2$.*

Proof: Put $L = \mathcal{O}_\Gamma(r)$ and consider the S^*V–modules

$$\begin{aligned} A &= R(L),\ A' = \ker(R(L) \to R(L_\Sigma)) = \bigoplus\nolimits_q H^0(\Gamma, L^q(-d)) \\ A'' &= \operatorname{im}(R(L) \to R(L_\Sigma)) = S(\Sigma). \end{aligned}$$

The exact sequence of S^*V–modules

$$0 \to A' \to A \to A'' \to 0$$

induces a long exact sequence of Koszul groups

$$K_{p,1}(A', V) \to K_{p,1}(A, V) \to K_{p,1}(A'', V) \to K_{p-1,2}(A', V).$$

By definition we have $K_{p,q}(A', V) \cong K_{p,q}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-d), \mathcal{O}_{\mathbb{P}}(r))$. Hence it suffices to show that

$$K_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-d), \mathcal{O}_{\mathbb{P}}(r)) = K_{p-1,2}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(-d), \mathcal{O}_{\mathbb{P}}(r)) = 0.$$

By duality, this is equivalent to

$$K_{r-1-p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d-2), \mathcal{O}_{\mathbb{P}}(r)) = K_{r-p,0}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d-2), \mathcal{O}_{\mathbb{P}}(r)) = 0.$$

By Proposition 2.39 these conditions hold if $d \geq 2r - p + 2$. $\square$

Proposition 2.42. *Let* $X \subset \mathbb{P}H^0(X,L)^\vee$ *be a nondegenerate projective variety of dimension* n. *If* $p > r-n$ *then* $K_{p,1}(X,L) = 0$.

Proof: We intersect X with $n-1$ general hyperplanes to obtain a curve $C = X \cap H_1 \ldots \cap H_{n-1} \subset \mathbb{P}^{r-n+1}$. Let W be the image of the restriction map $H^0(X,L) \to H^0(C, L_C)$. By Theorem 2.20 (i) we have an injective map

$$K_{p,1}(X,L) \to K_{p,1}(C, L_C, W).$$

Note that W generates L_C and $\dim W = r-n+2$. By Proposition 2.4 we have an inclusion

$$K_{p,1}(C, L_C, W) \subset H^1(C, \textstyle\bigwedge^{p+1} M_W).$$

The desired statement then follows, since $\operatorname{rank}(M_W) = \dim W - 1 = r-n+1$. □

Remark 2.43. Many important conjectures on Koszul cohomology center around the relationship between vanishing/nonvanishing of Koszul cohomology groups and the geometry of projective varieties; cf. [**Gre89**]. Proposition 2.42 can be improved by taking into account the geometry of the variety X. This leads to the so–called $K_{p,1}$–theorem, which we shall discuss in Chapter 3.

CHAPTER 3

Syzygy schemes

In this chapter, we present a geometric approach to the study of Koszul cohomology groups $K_{p,1}(X,L)$. In [**Gre82**, (1.1)], M. Green defined a geometric object, which is nowadays called the *syzygy scheme*, associated to a Koszul class in $K_{p,1}(X,L)$. He also proved a strong result, the so-called $K_{p,1}$-Theorem, which gives a description of these schemes in the cases $p = h^0(X,L) - \dim(X) - 1$ and $p = h^0(X,L) - \dim(X) - 2$. F.-O. Schreyer and his students further developed this theory; see e.g. [**vB07a**, Introduction].

3.1. Basic definitions

Let X be a projective variety over $\mathbb{C}$, and let L be a holomorphic line bundle on X.

Lemma 3.1. *Let $[\gamma] \in K_{p,1}(X,L)$ be a Koszul class. Put $V = H^0(X,L)$. If $p \geq 2$ there exists a unique linear subspace $W \subset V$ of minimal dimension such that $[\gamma] \in K_{p,1}(X,L,W)$.*

Proof: Using the isomorphism $K_{p,1}(X,L) \cong K_{p-1,2}(I,V)$ of Proposition 1.27 we represent $[\gamma]$ by an element of $\bigwedge^{p-1} V \otimes I_2(X)$. Clearly there exists a linear subspace $W \subseteq V$ such that $\gamma \in \bigwedge^{p-1} W \otimes S^2 V$. Given two linear subspaces W_1, $W_2 \subset V$ we have

$$\bigwedge^{p-1} W_1 \cap \bigwedge^{p-1} W_2 = \bigwedge^{p-1} (W_1 \cap W_2)$$

if $p \geq 2$. Hence there exists a unique minimal linear subspace W such that $\gamma \in \bigwedge^{p-1} W \otimes I_2(X)$. □

The following definition is taken from [**vB07a**, Definition 2.2].

Definition 3.2. The *rank* of a Koszul class $[\gamma] \in K_{p,1}(X,L)$ is the dimension of the minimal linear subspace $W \subset V$ such that $[\gamma] \in K_{p,1}(X,L,W)$.

Let $[\gamma] \in K_{p,1}(X,L)$ be a Koszul class represented by an element

$$\gamma \in \bigwedge^p V \otimes V.$$

Definition 3.3. An element $\gamma \in \bigwedge^p V \otimes V$ is *supported* on a variety Y if

$$\delta(\gamma) \in \bigwedge^{p-1} V \otimes I_2(Y).$$

Remark 3.4. Note that if $[\gamma] \in K_{p,1}(X,L)$, then γ is supported on Y if and only if $[\gamma]$ belongs to the image of the natural map

$$K_{p,1}(S(Y),V) \to K_{p,1}(S(X),V) = K_{p,1}(X,L).$$

In order to get a better understanding of Koszul classes, one could ask whether it is possible to find a variety Y containing X such that $[\gamma]$ is supported on Y. The advantage of this approach is that since the ideal of Y is smaller, one expects to have a better control over the syzygies and Koszul cohomology of Y. As before, put $I = \ker(S^*V \to R(L))$ and use the isomorphism $K_{p,1}(X, L) \cong K_{p-1,2}(I, V)$ of Proposition 1.27 to represent γ by an element $\widetilde{\gamma} = \sum_{[J|=p-1} v_J \otimes Q_J \in K_{p-1,2}(I, V)$. The commutative diagram

$$\begin{array}{ccc} \bigwedge^{p-1} V \otimes I_2(Y) & \to & \bigwedge^{p-3} V \otimes I_3(Y) \\ \downarrow & & \downarrow \\ \bigwedge^{p-1} V \otimes I_2(X) & \to & \bigwedge^{p-3} V \otimes I_3(X) \end{array}$$

shows that

$$\gamma \in \operatorname{im}(K_{p,1}(S(Y), V) \to K_{p,1}(S(X), V) \Longleftrightarrow Q_J \in I_2(Y) \text{ for all } J.$$

Definition 3.5 (M. Green). Let $[\gamma] \in K_{p,1}(X, L)$ be a Koszul class represented by an element $\gamma \in \bigwedge^p V \otimes V$, and let

$$\tilde{\gamma} = \delta(\gamma) = \sum_{|J|=p-1} v_J \otimes Q_J$$

be the corresponding element of $\bigwedge^{p-1} V \otimes I_2$. The *syzygy ideal* of $[\gamma]$ is the ideal generated by the quadrics Q_J, $|J| = p-1$. The *syzygy scheme* of $[\gamma]$ is the subscheme $\operatorname{Syz}(\gamma) \subset \mathbb{P}(V^\vee)$ defined by the syzygy ideal of γ.

Remark 3.6. The above discussion shows that γ is supported on $Y \subset \mathbb{P}(V^\vee)$ if and only if $X \subseteq Y \subseteq \operatorname{Syz}(\gamma)$. Note that the syzygy ideal of γ is the ideal generated by the image of the map

$$\delta(\gamma) : \bigwedge^{p-1} V^\vee \to S^2 V.$$

Lemma 3.7 (M. Green). *Given* $\gamma \in \operatorname{Hom}(V^\vee, \bigwedge^p V)$ *we have the following set–theoretic description of* $\operatorname{Syz}(\gamma)$*:*

$$\operatorname{Syz}(\gamma) = \{[x] \in \mathbb{P}(V^\vee) \mid i_x(\gamma(x)) = 0\}.$$

Proof: By definition, the Koszul differential $\delta : \bigwedge^p V \otimes V \to \bigwedge^{p-1} V \otimes S^2 V$ factors as

$$\bigwedge^p V \otimes V \xrightarrow{\iota \otimes \mathrm{id}} \bigwedge^{p-1} V \otimes V \otimes \otimes V \xrightarrow{\mathrm{id} \otimes \mu} \bigwedge^{p-1} V \otimes S^2 V$$

where $\iota : \bigwedge^p V \to \bigwedge^{p-1} V \otimes V \cong \operatorname{Hom}(V^\vee, \bigwedge^{p-1} V)$ sends $\lambda \in \bigwedge^p V$ to the map f_λ defined by $f_\lambda(x) = i_x(\lambda)$. Using the identifications

$$\begin{array}{ccc} \bigwedge^p V \otimes V & \to & \bigwedge^{p-1} V \otimes V \otimes V \\ \downarrow \cong & & \downarrow \cong \\ \operatorname{Hom}(V^\vee, \bigwedge^p V) & \to & \operatorname{Hom}(V^\vee, \operatorname{Hom}(V^\vee, \bigwedge^{p-1} V)) \end{array}$$

we find that $\iota \otimes \mathrm{id}$ sends $\gamma \in \operatorname{Hom}(V^\vee, \bigwedge^p V)$ to the map

$$f_\gamma \in \operatorname{Hom}(V^\vee, \operatorname{Hom}(V^\vee, \bigwedge^{p-1} V))$$

defined by $f_\gamma(x)(y) = i_y(\gamma(x))$. The image of this element in $\bigwedge^{p-1} V \otimes S^2 V \cong \operatorname{Hom}(\bigwedge^{p-1} V^\vee, S^2 V)$ is the map g_γ that sends $\varphi \in \bigwedge^{p-1} V^\vee$ to the symmetric bilinear form Q_φ defined by

$$Q_\varphi(x, y) = \langle \varphi, i_y(\gamma(x)) \rangle.$$

The associated quadratic form is $q_\varphi(x) = Q_\varphi(x,x)$. By the definition of the syzygy scheme, we have

$$\begin{aligned} x \in \mathrm{Syz}(\gamma) &\iff \forall\varphi, q_\varphi(x) = 0 \\ &\iff \forall\varphi, \langle\varphi, i_x(\gamma(x))\rangle = 0 \\ &\iff i_x(\gamma(x)) = 0. \end{aligned}$$

□

EXAMPLE 3.8. We consider two basic examples of syzygy schemes.

(i) Let $X \subset \mathbb{P}^3$ be the twisted cubic. Its ideal is generated by the 2×2 minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}.$$

Put

$$q_0 = \begin{vmatrix} x_1 & x_2 \\ x_2 & x_3 \end{vmatrix}, \quad q_1 = -\begin{vmatrix} x_0 & x_2 \\ x_1 & x_3 \end{vmatrix}, \quad q_3 = \begin{vmatrix} x_0 & x_1 \\ x_1 & x_2 \end{vmatrix}.$$

As

$$\begin{vmatrix} x_0 & x_1 & x_2 \\ x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{vmatrix} = 0$$

we obtain the relation

$$x_0.q_0 + x_1.q_1 + x_2.q_2 = 0.$$

Hence the element

$$\gamma = x_0 \otimes q_0 + x_1 \otimes q_1 + x_2 \otimes q_2 \in \ker(V \otimes I_2(X) \to I_3(X))$$

defines a class $[\gamma] \in K_{2,1}(X, \mathcal{O}_X(1))$ whose syzygy scheme is

$$\mathrm{Syz}(\gamma) = V(q_0, q_1, q_2) = X.$$

(ii) Let $X \subset \mathbb{P}^3$ be a canonical curve of genus 4. As $X = Q \cap F$ is the complete intersection of a quadric Q and a cubic F, we obtain a resolution

$$0 \to S(-5) \to S(-3) \oplus S(-2) \to S \to S(X) \to 0$$

with associated Betti diagram

$$\begin{array}{cccc} 1 & - & - & - \\ - & 1 & - & - \\ - & 1 & - & - \\ - & - & 1 & -. \end{array}$$

In particular we have $\kappa_{1,1} = 1$. Let γ be a generator of $K_{1,1}(X, \mathcal{O}_X(1))$. Since Q is the only quadric containing X, we obtain $\mathrm{Syz}(\gamma) = Q$.

To obtain a better understanding of the geometry of syzygy schemes it is useful to introduce an additional tool, the so–called *generic syzygy scheme*. Before giving the general definition, we discuss the construction of this scheme for Koszul classes $\gamma \in K_{2,1}(X, L)$ following [**ES94**].

LEMMA 3.9 (Schreyer). *Let $q_1, \dots, q_n$ be quadratic forms in $\mathbb{C}[x_0, \dots, x_s]$, $n \le s$. Suppose that there exists linear forms $\ell_1, \dots, \ell_n$ such that*

$$\ell_1 q_1 + \ldots + \ell_n q_n = 0.$$

Then there exists a skew–symmetric $n \times n$ matrix $A = (a_{ij})$ of linear forms such that

$$q_j = \sum_i \ell_i a_{ij}, \quad j = 1, \dots, n.$$

Proof: Define $X = V(q_1, \dots, q_n) \subset \mathbb{P}^s = \mathbb{P}(V^\vee)$, and put $W = \langle \ell_1, \dots, \ell_n \rangle \subset V$. The linear relation between the quadrics defines an element

$$\widetilde{\gamma} = \sum_i \ell_i \otimes q_i \in K_{1,2}(I(X), W).$$

As $K_{1,2}(I(X), W) \cong K_{2,1}(S(X), W)$ by Corollary 2.16, there exists a class $[\gamma] \in K_{2,1}(S(X), W)$ such that $\widetilde{\gamma} = \delta(\gamma)$. Write

$$\gamma = \sum_{i<j} \ell_i \wedge \ell_j \otimes a_{ij} \in \bigwedge\nolimits^2 W \otimes V$$

and put $A = (a_{ij})$. We then compute

$$\begin{aligned} \delta(\gamma) &= \sum_{i<j} \ell_j \otimes \ell_i a_{ij} - \ell_i \otimes \ell_j a_{ij} \\ &= \sum_k \ell_k \otimes (\sum_{i \neq k} \ell_i a_{ik}) \end{aligned}$$

to obtain the desired result. □

Given a Koszul class γ as above, Schreyer's idea is to study the geometry of the syzygy scheme $\mathrm{Syz}(\gamma) = V(q_1, \dots, q_n)$ by passing to the generic situation. This means that one treats the entries a_{ij} of the matrix A and the linear forms ℓ_i as variables in a polynomial ring; to avoid confusion, we denote these new variables by L_i and A_{ij}. The polynomial ring

$$R = \mathbb{C}[L_1, \dots, L_n, A_{12}, \dots, A_{n-1,n}].$$

has $N = n + n(n-1)/2 = n(n+1)/2$ variables. One then puts

$$Q_j = \sum_i L_i A_{ij}, \quad j = 1, \dots, n$$

and defines the *generic syzygy scheme*

$$\mathrm{Gensyz}(\gamma) = V(Q_1, \dots, Q_n) \subset \mathbb{P}^{N-1}.$$

The syzygy scheme $\mathrm{Syz}(\gamma) \subset \mathbb{P}(V^\vee)$ is obtained from $\mathrm{Gensyz}(\gamma) \subset \mathbb{P}^{N-1}$ in the following way. We first introduce the necessary relations among the variables A_{ij} and L_k, i.e., we pass to a quotient Γ of the vector space

$$\langle L_1, \dots, L_n, A_{12}, \dots, A_{n-1,n} \rangle$$

by a number of linear relations. We then view Γ as a subspace of V, and pass from a subscheme of $\mathbb{P}(\Gamma^\vee)$ to the subscheme of $\mathbb{P}(V^\vee)$ defined by the same equations. Geometrically, this means that $\mathrm{Syz}(\gamma)$ is a cone over a linear section of $\mathrm{Gensyz}(\gamma)$.

A coordinate–free description of the preceding construction is obtained as follows. The skew–symmetric matrix A of linear forms corresponds to a linear map

$\bigwedge^2 W^\vee \to V$. This is simply the image of $\gamma \in \bigwedge^2 W \otimes V$ under the isomorphism $\bigwedge^2 W \otimes V \cong \mathrm{Hom}(\bigwedge^2 W^\vee, V)$. Taking the direct sum of this map with the inclusion $W \hookrightarrow V$, we obtain a map

$$\gamma' : W \oplus \bigwedge^2 W^\vee \to V.$$

The quadrics defining the generic syzygy scheme are obtained as follows. Recall that dim $W = n$. Consider the map

$$\iota : \bigwedge^{n-1} W \to W \otimes \bigwedge^{n-2} W$$

By definition,

$$\begin{aligned}\iota(L_1 \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge L_n) = &\sum_{i<j} (-1)^i L_i \otimes (L_1 \wedge \ldots \wedge \widehat{L_i} \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge L_n) \\ &+ \sum_{i>j} (-1)^{i-1} L_i \otimes (L_1 \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge \widehat{L_i} \wedge \ldots \wedge L_n).\end{aligned}$$

Let $\{\Lambda_i\}$ be the dual basis of $W^\vee$. The isomorphism $\bigwedge^{n-2} W \cong \bigwedge^2 W^\vee$ defined by the duality pairing maps the element

$$L_1 \wedge \ldots \wedge \widehat{L_i} \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge L_n$$

to $(-1)^{i+j-1} \Lambda_i \wedge \Lambda_j$. Writing $A_{ij} = \Lambda_i \wedge \Lambda_j$, we obtain

$$\begin{aligned}\iota(L_1 \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge L_n) &= \sum_{i<j} (-1)^{j-1} L_i \otimes A_{ij} + \sum_{i>j} (-1)^j L_i \otimes A_{ji} \\ &= (-1)^{j-1} \sum_i L_i \otimes A_{ij}.\end{aligned}$$

Hence the map

$$W \otimes \bigwedge^2 W^\vee \to S^2(W \oplus \bigwedge^2 W^\vee)$$

sends $\iota((-1)^{j-1} L_1 \wedge \ldots \wedge \widehat{L_j} \wedge \ldots \wedge L_n)$ to the quadric

$$Q_j = \sum_i L_i A_{ij}.$$

We conclude that the ideal of $\mathrm{Gensyz}(\gamma) \subset \mathbb{P}(W^\vee \oplus \bigwedge^2 W)$ is generated by the image of the composed map

$$\bigwedge^{n-1} W \to \bigwedge^{n-2} W \otimes W \cong \bigwedge^2 W^\vee \otimes W \to S^2(W \oplus \bigwedge^2 W^\vee).$$

The preceding discussion generalizes in an obvious way. Consider a Koszul class $[\gamma] \in K_{p,1}(X, L, W)$ represented by $\gamma \in \bigwedge^p W \otimes V$. Write dim $W = p + r$, and let $i : W \to V$ be the inclusion map. Put

$$\gamma' = (i, \gamma) : W \oplus \bigwedge^r W \to V$$

and define $\Gamma = \mathrm{im}(\gamma') \subset V$. Consider the natural map

$$\psi : \bigwedge^{r+1} W \to W \otimes \bigwedge^r W \hookrightarrow S^2(W \oplus \bigwedge^r W).$$

Definition 3.10. Given a linear subspace $W \subset V$ of dimension $p + r$, the *generic syzygy ideal* $I_{\mathrm{Gensyz}}(W)$ is the ideal in $S^*(W \oplus \bigwedge^r W)$ generated by the image of ψ. The *generic syzygy scheme* of W is the subscheme

$$\mathrm{Gensyz}_r(W) \subset \mathbb{P}(W^\vee \oplus \bigwedge^r W^\vee)$$

defined by $I_{\mathrm{Gensyz}(W)}$.

Remark 3.11. We have adopted the notation $\mathrm{Gensyz}_r(W)$ in stead of $\mathrm{Gensyz}(\gamma)$. This is justified since W is uniquely determined by γ, and for fixed p the geometry of the generic syzygy scheme is determined essentially by the integer $r = \mathrm{rank}\,(\gamma) - p$; see the examples in the next section.

Remark 3.12. The relationship between $\mathrm{Gensyz}_r(W)$ and $\mathrm{Syz}(\gamma)$ is given by the commutative diagram

$$\begin{tikzcd} \bigwedge^{r+1} W \arrow[r, "\psi"] \arrow[d, equal] & S^2(W \oplus \bigwedge^r W) \arrow[d] \\ \bigwedge^{p-1} W^\vee \arrow[r, "\delta(\gamma)"] & S^2 V. \end{tikzcd}$$

The factorization

$$\bigwedge^{r+1} W \to \Gamma \hookrightarrow V$$

then shows that if W is the minimal linear subspace associated to a Koszul class $[\gamma] \in K_{p,1}(X, L)$ then $\mathrm{Syz}(\gamma) \subset \mathbb{P}(V^\vee)$ is a cone over a subscheme $Y \subset \mathbb{P}(\Gamma^\vee)$ which is a linear section of $\mathrm{Gensyz}_r(W) \subset \mathbb{P}(W^\vee \oplus \bigwedge^r W^\vee)$.

Remark 3.13. Given $x \in X$, put $\gamma'_x = \mathrm{ev}_x \circ \gamma' : W \oplus \bigwedge^r W \to k$. We obtain a rational map

$$X - - > \mathbb{P}(W^\vee \oplus \bigwedge^r W^\vee)$$

that sends x to $[\gamma'_x]$. By construction, the (closure of the) image of this map is contained in $\mathrm{Gensyz}_r(W)$.

Example 3.14. Let us determine the generic syzygy scheme associated to the class

$$\gamma = \sum_{i=0}^{2} \ell_i \otimes q_i$$

defined in Example 3.8 (i). As $\ell_i = x_i$ $(i = 0, 1, 2)$ and

$$q_0 = x_1x_3 - x_2^2, \quad q_1 = x_1x_2 - x_0x_3, \quad q_2 = x_0x_2 - x_1^2$$

the relations

$$q_j = \sum_i x_i a_{ij}$$

show that

$$A = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ x_2 & x_1 & 0 \end{pmatrix}.$$

The generic syzygy scheme lies in the projective space $\mathbb{P}^5$ with coordinates $(X_0 : X_1 : X_2 : A_{10} : A_{20} : A_{21})$. The change of variables

$$A_{10} = Y_2, \quad A_{20} = -Y_1, \quad A_{21} = Y_0$$

realizes the generic syzygy scheme $\mathrm{Gensyz}(\gamma) = \mathrm{Gensyz}_1(W)$ as the subscheme of $\mathbb{P}^5$ defined by the 2×2 minors of the matrix

$$B = \begin{pmatrix} X_0 & X_1 & X_2 \\ Y_0 & Y_1 & Y_2 \end{pmatrix}.$$

Hence $\mathrm{Gensyz}_1(W)$ is the Segre threefold, the image of the Segre map $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^5$ [**Ha92**]. Note that $\mathrm{Gensyz}_1(W)$ is a rational normal scroll of degree 3 and codimension 2; the syzygy scheme $\mathrm{Syz}(\gamma)$ (which is a twisted cubic) is a linear section of $\mathrm{Gensyz}_1(W)$. See Lemmas 3.20 and 3.21 for a generalization of this example.

PROPOSITION 3.15. *Let L be a globally generated line bundle on a projective variety X. Put $V = H^0(X, L)$. Given $[x] \in \mathbb{P}(V^\vee)$, let $W_x \subset V$ be the kernel of the map $\mathrm{ev}_x = \langle x, .\rangle : V \to \mathbb{C}$ and let $Y \subset \mathbb{P}(W_x^\vee)$ be the image of the rational map*

$$X - - > \mathbb{P}(W_x^\vee)$$

given by projection with center x. For all $[\gamma] \in K_{p,1}(X, L) = K_{p,1}(S(X), V)$ we have

$$x \in \mathrm{Syz}(\gamma) \Longleftrightarrow pr_x(\gamma) \in K_{p-1,1}(S(Y), W_x).$$

Proof: By Lemma 3.7 we have

$$\mathrm{Syz}(\gamma) = \{[x] \in \mathbb{P}(V^\vee \mid i_x(\gamma(x)) = 0\}.$$

Note that

$$\mathrm{pr}_x(\gamma) \in K_{p-1,1}(S(Y), W_x) \Longleftrightarrow \exists \gamma' : W_x^\vee \to \textstyle\bigwedge^{p-1} W_x \text{ such that the diagram}$$

$$\begin{tikzcd}
\mathbb{C}.x \arrow[d] & & \\
V^\vee \arrow[r, "\gamma"] \arrow[d, "\iota"] & \bigwedge^p V \arrow[r, "\mathrm{pr}_x"] & \bigwedge^{p-1} W_x \\
W_x^\vee \arrow[urr, "\exists\gamma'"] & &
\end{tikzcd}$$

commutes. The latter condition is satisfied if and only if $\mathrm{pr}_x \circ \gamma$ factors through ι, which is equivalent to $\mathrm{pr}_x \circ \gamma|_{\mathbb{C}.x} \equiv 0$. Hence

$$\begin{aligned}
\mathrm{pr}_x(\gamma) \in K_{p-1,1}(S(Y), W_x) &\Longleftrightarrow \mathrm{pr}_x(\gamma(x)) = 0 \\
&\Longleftrightarrow i_x(\gamma(x)) = 0 \text{ (cf. 3.7)} \\
&\Longleftrightarrow x \in \mathrm{Syz}(\gamma).
\end{aligned}$$

□

REMARK 3.16. A precursor of this result was proved in [**Ehb94**]. Ehbauer showed that

$$x \in X \Rightarrow \mathrm{pr}_x(\gamma) \in K_{p-1,1}(S(Y), W_x).$$

LEMMA 3.17. *Let X be a projective variety, and let $\gamma \in K_{p,1}(X, L)$ be a Koszul class.*

(i) *We have $\mathrm{Syz}(\gamma) \cap H \subseteq \mathrm{Syz}(\gamma|_H)$ for every hyperplane $H \subset \mathbb{P}^r$;*
(ii) *We have $\mathrm{pr}_x(\mathrm{Syz}(\gamma)) \subseteq \mathrm{Syz}(\mathrm{pr}_x(\gamma))$ for every point $x \in \mathbb{P}^r$.*

Proof: For the first statement, write

$$\delta(\gamma) = \sum_{|I|=p-1} x_I \otimes Q_I.$$

By a suitable choice of coordinates we may assume that $H = V(X_0)$. We then have

$$\delta(\gamma|_H) = \sum_{0 \notin I} x_I \otimes Q_I.$$

Hence we obtain

$$\mathrm{Syz}(\gamma) \cap H = \cap_I V(Q_I|_H) \subseteq \cap_{0 \notin I} V(Q_I|_H) = \mathrm{Syz}(\gamma_H).$$

To prove (ii), recall that $\mathrm{Syz}(\gamma) = \{[y] \in \mathbb{P}(V^\vee \mid i_y(\gamma(y)) = 0\}$ (3.7). Put $W_x = \ker(\mathrm{ev}_x) \subset V$, and let $z \in W_x^\vee$ be the image of $y \in V^\vee$. The commutative diagram

$$\begin{array}{ccccc} V^\vee & \xrightarrow{\gamma} & \bigwedge^p V & \xrightarrow{i_y} & \bigwedge^{p-1} V \\ \downarrow & & \uparrow & & \uparrow \\ W_x^\vee & \xrightarrow{\mathrm{pr}_x(\gamma)} & \bigwedge^p W_x & \xrightarrow{i_z} & \bigwedge^{p-1} W_x \end{array}$$

shows that $i_y(\gamma(y)) = 0 \Rightarrow i_z(\mathrm{pr}_x(\gamma)(z)) = 0$, hence $\mathrm{pr}_x(\mathrm{Syz}(\gamma)) \subseteq \mathrm{Syz}(\mathrm{pr}_x(\gamma))$. $\square$

Remark 3.18. The inclusion of Lemma 3.17 (i) need not be an equality; for instance, if dim $X = 0$ equality fails for trivial reasons. It is possible to show that equality holds under additional hypotheses. Specifically, one can prove the following result using the ideas of [**NP94**].

Proposition 3.19. *Given $\gamma \in K_{p,1}(X, L) = K_{p,1}(S(X), V)$ and a hyperplane H defined by $t \in S(V)$, write $Y = X \cap H$ and let $\overline{\gamma} \in K_{p,1}(S(Y), W)$ be the restriction of γ. Put*

$$\Sigma = \mathrm{Syz}(\gamma), \quad \overline{\Sigma} = \mathrm{Syz}(\overline{\gamma}).$$

Suppose that there exists a linear subspace Λ' such that, writing $W' = V/\Lambda'$, the restriction maps

$$\begin{aligned} f: &\quad K_{p-1,1}(S(\overline{\Sigma}), W') \to K_{p-1,1}(S(\overline{\Sigma} \cap \Lambda'), W') \\ g: &\quad K_{p-1,1}(S(\overline{\Sigma} \cap \Lambda'), W') \to K_{p-1,1}(S(Y \cap \Lambda'), W') \end{aligned}$$

induced by the inclusions $Y \cap \Lambda' \subset \overline{\Sigma} \cap \Lambda' \subset \overline{\Sigma}$ are isomorphisms. Then $\Sigma \cap H = \overline{\Sigma}$. More precisely, $I(\overline{\Sigma}) = I(\Sigma) + (t)/(t)$.

3.2. Koszul classes of low rank

We discuss the geometry of generic syzygy schemes for Koszul classes of low rank. Let $[\gamma] \in K_{p,1}(X, L, W)$ be a Koszul class of rank $p + r$.

The case $r = 0$. Let $t \in V$ be the image of γ under the isomorphism $\bigwedge^p W \otimes V \cong V$. The ideal of $\mathrm{Gensyz}_0(W)$ is generated by the image of the map

$$W \xrightarrow{\sim} W \otimes \mathbb{C} \hookrightarrow S^2(W \oplus \mathbb{C}) = S^2 W \oplus W \otimes \mathbb{C},$$

hence it consists of reducible quadrics of the form $t.w$, $w \in W$, and

$$Gensyz_0(W) = \mathbb{P}(W^\vee) \cup [t] \subset \mathbb{P}(W^\vee \oplus \mathbb{C})$$

is the union of a hyperplane and a point. To describe $\mathrm{Syz}(\gamma)$, let $\Gamma \subset V$ be the image of the map

$$\gamma' : W \oplus \mathbb{C} \to V.$$

We have either $\Gamma = W$ (if $t \in W$) or $\Gamma = W \oplus \mathbb{C}$ (if $t \notin W$). Put $Y = \mathrm{Gensyz}(W) \cap \mathbb{P}(\Gamma^\vee)$. As the syzygy scheme $\mathrm{Syz}(\gamma)$ is a cone with vertex $\mathbb{P}(V/\Gamma)^\vee$ over Y, we find that

$$\mathrm{Syz}(\gamma) = V(t) \cup \mathbb{P}(V/W)^\vee$$

is the union of a hyperplane and a linear subspace of codimension p.

The case $r = 1$. In this case we can view $\gamma \in \bigwedge^p W \otimes V \cong W^\vee \otimes V$ as a homomorphism $\gamma : W \to V$. The ideal of $\mathrm{Gensyz}_1(W)$ is generated by the image of the map

$$\bigwedge^2 W \to W \otimes W \hookrightarrow S^2(W \oplus W).$$

As the first map is injective, we obtain $I_{\mathrm{Gensyz}} = \langle \bigwedge^2 W \rangle \subset S^*(W \oplus W)$.

LEMMA 3.20. *Under the identification $W^\vee \oplus W^\vee \cong W^\vee \otimes \mathbb{C}^2$, $\mathrm{Gensyz}_1(W)$ is isomorphic to the image of the Segre embedding $\mathbb{P}(W^\vee) \times \mathbb{P}^1 \hookrightarrow \mathbb{P}(W^\vee \otimes \mathbb{C}^2)$.*

Proof: The injective map $\bigwedge^2 W \hookrightarrow S^2(W \oplus W) = S^2W \oplus W \otimes W \oplus S^2 W$ sends $u \wedge v$ to the element $u \otimes v - v \otimes u$ in the middle component. Hence the ideal generated by $\bigwedge^2 W$ is the ideal of 2×2 minors of the corresponding $2 \times (p+2)$ matrix, which is the ideal defining the space of decomposable tensors, i.e., the Segre variety; cf. [**Ha92**]. $\square$

The following result is classical and appears in several sources; see e.g. [**Sch86**] or [**vB07a**, Corollary 5.2], [**Ei06**].

LEMMA 3.21. *The syzygy scheme $\mathrm{Syz}(\gamma)$ of a nonzero Koszul class of rank $p+1$ is a rational normal scroll of codimension p and degree $p+1$.*

Proof: Note that $\mathrm{Gensyz}_1(W)$ is a rational normal scroll. By the classification of varieties of minimal degree [**EH87a**], it suffices to show that $\mathbb{P}(\Gamma^\vee)$ intersects every fiber F of the morphism $\pi : \mathrm{Gensyz}_1(W) \to \mathbb{P}^1$ transversely. The isomorphism $W \oplus W \to W \otimes \mathbb{C}^2$ sends (w_1, w_2) to the element $w \otimes (\lambda, \mu)$ determined by the conditions $w_1 = \lambda.w$, $w_2 = \mu.w$. Hence the intersection of $\mathbb{P}(\Gamma^\vee)$ with the fiber F over $(\lambda : \mu) \in \mathbb{P}^1$ is the projectivisation of the kernel of the map

$$f_{\lambda,\mu} : W \to V$$

defined by $f_{\lambda,\mu}(w) = \lambda.w + \mu.\gamma(w)$, and

$$\begin{aligned} \mathbb{P}(\Gamma^\vee) \cap F = F \quad &\Leftrightarrow \quad f_{\lambda,\mu}|_W \equiv 0 \\ &\Leftrightarrow \quad \exists \nu \in \mathbb{C},\ \gamma(w) = \nu.w\ \forall w \in W. \end{aligned}$$

By definition, the latter condition is equivalent to the vanishing of the Koszul class γ. This finishes the proof. $\square$

DEFINITION 3.22. A Koszul class is called *of scrollar type* if it is supported on a rational normal scroll.

By Lemma 3.21, $[\gamma] \in K_{p,1}(X, L)$ is of scrollar type if and only if it is of rank $p+1$.

The case $r = 2$. Let $[\gamma] \in K_{p,1}(X, L, W)$ be a Koszul class of rank $p+2$. Put $T = \mathbb{C} \oplus W$. By definition $\mathrm{Gensyz}_2(W) \subset \mathbb{P}(W^\vee \oplus \bigwedge^2 W^\vee) = \mathbb{P}(\bigwedge^2 T^\vee)$. The projective space $\mathbb{P}(\bigwedge^2 T^\vee)$ contains the subschemes $G(2, T)$ (the Plücker embedded Grassmannian of 2–dimensional quotients of T) and $\mathbb{P}(\bigwedge^2 W^\vee)$.

The following result is taken from [**vB07a**, Theorem 6.1].

THEOREM 3.23 (von Bothmer).

$$\mathrm{Gensyz}_2(W) = G(2, T) \cup \mathbb{P}(\bigwedge^2 W^\vee).$$

Proof: Put $G = G(2,T)$, $\mathbb{P} = \mathbb{P}(\bigwedge^2 W^\vee)$. In the symmetric algebra $S^*(\bigwedge^2 T)$ we have [**FH91**, p. 228]

$$I_G = \langle \bigwedge^4 T \rangle, \quad I_{\mathbb{P}} = \langle W \rangle.$$

By definition, the ideal of $\mathrm{Gensyz}_2(W)$ is generated by the image of the map

$$\bigwedge^3 W \to W \otimes \bigwedge^2 W \to S^2(W \oplus \bigwedge^2 W).$$

Hence $I_{\mathrm{Gensyz}} = \langle \bigwedge^3 W \rangle \subset S^*(\bigwedge^2 T)$.

The decomposition $W \otimes (W \oplus \bigwedge^2 W) = S^2 W \oplus \bigwedge^2 W \oplus \bigwedge^3 W \oplus S_{2,1} W$ shows that $I_{\mathrm{Gensyz}} \subset I_{\mathbb{P}}$, hence $\mathbb{P} \subset \mathrm{Gensyz}_2(W)$. As $\langle \bigwedge^4 T \rangle = \langle \bigwedge^3 W \rangle + \langle \bigwedge^4 W \rangle$, we have $I_{\mathrm{Gensyz}} \subset I_G$ and $G \subset \mathrm{Gensyz}_2(W)$. The union $G \cup \mathbb{P} \subset \mathbb{P}(\bigwedge^2 T^\vee)$ is defined by the ideal

$$\begin{aligned} I_G \cap I_{\mathbb{P}} &= \langle W \rangle \cap \langle \bigwedge^3 W, \bigwedge^4 W \rangle \\ &= \langle W \rangle \cap \langle \bigwedge^3 W \rangle + \langle W \rangle \cap \langle \bigwedge^4 W \rangle. \end{aligned}$$

The decomposition of $W \otimes \bigwedge^2 T$ shows that $\langle W \rangle \cap \langle \bigwedge^3 W \rangle = \langle \bigwedge^3 W \rangle$. For the second intersection we have

$$\langle W \rangle \cap \langle \bigwedge^4 W \rangle = \langle W \rangle . \langle \bigwedge^4 W \rangle = \langle W \otimes \bigwedge^4 W \rangle.$$

The decompositions ([**FH91**])

$$\begin{aligned} W \otimes \bigwedge^4 W &= \bigwedge^5 W \oplus S_{4,1} W \\ \bigwedge^3 W \otimes (W \oplus \bigwedge^2 W) &= \bigwedge^4 W \oplus S_{3,1} W \oplus \bigwedge^5 W \oplus S_{4,1} W \oplus S_{3,2} W \end{aligned}$$

imply that $\langle W \rangle \cap \langle \bigwedge^4 W \rangle \subseteq \langle \bigwedge^3 W \rangle$. Hence we obtain

$$I_G \cap I_{\mathbb{P}} = \langle \bigwedge^3 W \rangle = I_{\mathrm{Gensyz}}$$

and $\mathrm{Gensyz}_2(W) = G \cup \mathbb{P}$. □

3.3. The $K_{p,1}$ theorem

Proposition 3.24. *Let $X \subset \mathbb{P}^n$ be a nondegenerate irreducible variety. Given $0 \neq \gamma \in K_{p,1}(X, L)$, the image of the map $\delta(\gamma) : \bigwedge^{p-1} V^\vee \to S^2 V$ has dimension at least $\binom{p+1}{2}$.*

Proof: Choose a linear subspace $\Lambda \subset \mathbb{P}^n$ such that $C = X \cap \Lambda$ is a smooth curve. Let

$$I = \{(x, \xi) \in C \times (\mathbb{P}^n)^\vee \mid x \in H_\xi\}$$

be the incidence correspondence, and let $U \subset (\mathbb{P}^n)^\vee$ be the Zariski open subset of hyperplanes that intersect X transversely. By the uniform position principle [**ACGH85**] the monodromy group of $I \to U$ is the full symmetric group. Given $t \in U$, the set $X \cap H_t$ contains a subset $\Sigma_t = \{p_0, \ldots, p_n\} \subset X$ of $n+1$ points in general position. Choose coordinates $(X_0 : \ldots : X_n)$ on $\mathbb{P}^n$ such that

$$\{p_0, \ldots, p_n\} = \{(1 : 0 : \ldots : 0), \ldots, (0 : \ldots : 0 : 1)\}$$

We then have

$$\delta(\gamma) = \sum_{|I| = p-1} \lambda_I \otimes Q_I$$

with

$$Q_I = \sum_{i,j \notin I} a_{ij} X_i X_j.$$

Given $c \in \pi_1(U,t)$, let $\Phi = \rho(c) : \Sigma \to \Sigma$ be the corresponding monodromy transformation. Let $i_t : \Sigma_t \hookrightarrow X$ be the inclusion map. The commutative diagram

$$\begin{tikzcd} \Sigma_t \arrow[r, "\Phi"] \arrow[dr, "i_t"'] & \Sigma_t \arrow[d, "i_t"] \\ & X \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} K_{p,1}(S(\Sigma_t), W_t) \arrow[r, "\Phi^*"] & K_{p,1}(S(\Sigma_t), W_t) \\ K_{p,1}(S(X), V). \arrow[u, "i_t^*"] \arrow[ur, "i_t^*"'] & \end{tikzcd}$$

As i_t^* is injective (Theorem 2.19), we can view an element $\gamma \in K_{p,1}(S(X), V)$ as an element of $K_{p,1}(S(\Sigma_t), W_t)$ that is invariant under the action of the monodromy group. This implies that the symmetric group permutes the quadrics Q_I, $|I| = p-1$, via the action

$$Q_I \mapsto \sigma(Q_I) = \sum_{i,j \notin I} a_{ij} X_{\sigma(i)} X_{\sigma(j)}.$$

In particular we have $\sigma(Q_I) = Q_{\sigma(I)}$ for all I.

Note that $Q_I \not\equiv 0$ for all I. Indeed, suppose that there exists I such that $Q_I \equiv 0$. Then $Q_{\sigma(I)} \equiv 0$ for all $\sigma \in S_{n+1}$ and $\gamma = 0$, contradiction.

We claim that in the expression

$$Q_I = \sum_{i,j \notin I} a_{i,j} X_i X_j$$

all the coefficients $a_{i,j}$ are nonzero. Indeed, $a_{i,j} = 0$ if and only if the restriction of Q_I to the line $\ell_{ij} = \overline{p_i p_j}$ vanishes identically. If there exists a pair (i,j) such that $a_{i,j} = 0$ we obtain

$$Q_I|_{\sigma(\ell_{ij})} = \sigma(Q_I)|_{\ell_{ij}} \equiv 0$$

for all $\sigma \in S_{n+1}$, hence $Q \equiv 0$. This contradicts the previous step.

Fix a set I_0 of cardinality $p+1$, for example $I_0 = \{0, \dots, p\}$, and consider the subsets $I(\lambda, \mu) = I_0 \setminus \{\lambda, \mu\}$. As $X_\lambda X_\mu$ occurs with nonzero coefficient in $Q_{I(\lambda,\mu)}$, and does not occur in the quadrics $Q_{I(\lambda',\mu')}$ with $(\lambda', \mu') \neq (\lambda, \mu)$ we obtain $\binom{p+1}{2}$ linearly independent quadrics in the ideal of $\mathrm{Syz}(\gamma)$. $\square$

Corollary 3.25. *Let $C \subset \mathbb{P}^r$ be a rational normal curve. For all $0 \neq \gamma \in K_{r-1,1}(C, \mathcal{O}_C(1))$ we have* $\mathrm{Syz}(\gamma) = C$.

Proof: As the ideal of C is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} X_0 & \dots & X_{r-1} \\ X_1 & \dots & X_r \end{pmatrix}$$

we have $\dim I_2(C) \leq \binom{r}{2}$. The result then follows from the inclusion $I_{\mathrm{Syz}(\gamma)} \subset I_2(C)$ and the inequality of Proposition 3.24. $\square$

Lemma 3.26. *Let $\xi \subset \mathbb{P}^r$ be a finite set of points. Then*

$$\xi \text{ is contained in a rational normal curve } \iff K_{r-1,1}(\xi) \neq 0.$$

If $\deg \xi \geq r+3$ then $\mathrm{Syz}(\gamma)$ is a rational normal curve for every nonzero Koszul class $\gamma \in K_{r-1,1}(\xi)$.

Proof: If ξ is contained in a rational normal curve Γ, we obtain an inclusion $K_{r-1,1}(\Gamma) \to K_{r-1,1}(\xi)$. then $K_{r-1,1}(\xi) \neq 0$ since $K_{r-1,1}(\Gamma) \neq 0$. More precisely, Proposition 2.4 implies that $K_{r-1,1}(\Gamma, \mathcal{O}_\Gamma(1)) \cong H^1(\mathbb{P}^1, \bigwedge^r M)$ where M is the kernel of the evaluation map $V \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(r)$. As rank $(M) = r$ the latter group is isomorphic to $H^1(\mathbb{P}^1, \mathcal{O}_\mathbb{P}(-r)) = H^0(\mathbb{P}^1, \mathcal{O}_\mathbb{P}(r-2)) = \mathbb{C}^{r-1}$.

The converse statement is classically known if $\deg \xi \leq r+3$. If $\deg \xi \geq r+3$, we choose a subset $\xi' = \{p_1, \ldots, p_{r+3}\} \subset \xi$. There exists a unique rational normal curve Γ such that $\xi' \subset \xi$. By Corollary 2.41 we have $K_{r-1,1}(\xi') \cong K_{r-1,1}(\Gamma, \mathcal{O}_\Gamma(1))$. Combining this isomorphism with the inclusion $K_{r-1,1}(\xi) \subset K_{r-1,1}(\xi')$, we obtain an injective map from $K_{r-1,1}(\xi)$ to $K_{r-1,1}(\Gamma, \mathcal{O}_\Gamma(1))$. The statement then follows from Corollary 3.25. □

Corollary 3.27 (strong Castelnuovo lemma). *Let $X \subset \mathbb{P}^r = \mathbb{P}(V^\vee)$ be a projective variety. Suppose that X contains a set of $d \geq r+3$ points in general position. Then*

$$K_{r-1,1}(S(X), V) \neq 0 \iff X \text{ is contained in a rational normal curve.}$$

Proof: We may assume that X is nondegenerate. If X is contained in a rational normal curve Γ we obtain an inclusion $K_{r-1,1}(S(\Gamma), V) \to K_{r-1,1}(S(X), V)$. The result then follows since $K_{r-1,1}(S(\Gamma), V) = K_{r-1,1}(\Gamma, \mathcal{O}_\Gamma(1)) \neq 0$.

For the converse, note the map $K_{r-1,1}(S(X), V) \to K_{r-1,1}(\Sigma)$ is injective and apply Lemma 3.26 and Proposition 2.42. □

We have seen that if $X \subset \mathbb{P}H^0(X, L)^\vee$ is a subvariety of dimension n, then $K_{p,1}(X, L) = 0$ for all $p > r-n$; see Proposition 2.42. The following result, known as the $K_{p,1}$–theorem, classifies the varieties X such that $K_{r-n,1}(X, L) \neq 0$ or $K_{r-n-1,1}(X, L) \neq 0$.

Theorem 3.28 (Green). *Let $X \subset \mathbb{P}^r = \mathbb{P}H^0(X, L)^\vee$ be a nondegenerate, irreducible subvariety of dimension n.*

(i) *If $K_{r-n,1}(X, L) \neq 0$ then X is a variety of minimal degree;*
(ii) *If $K_{r-n-1,1}(X, L) \neq 0$ and $\deg X \geq r-n+3$ then there exists a variety Y of minimal degree such that $X \subset Y$ is a subvariety of codimension one.*

Proof: Put $c = \mathrm{codim}\,(X) = r-n$, and note that $\deg X \geq c+1$. If $K_{c,1}(X, L) \neq 0$, then $X \cap \Lambda$ is contained in a rational normal curve for a general linear subspace $\Lambda \cong \mathbb{P}^c$ by Lemma 3.26. Hence $\deg X \leq c+1$ and X has minimal degree.

To treat the second case, we need the following auxiliary result.

Sublemma. Let $\gamma \in K_{c-1,1}(X, L)$ be a nonzero Koszul class, and let $H_0, \ldots, H_n \subset \mathbb{P}^r$ be general hyperplanes. Then

$$\mathrm{Syz}(\gamma|_{H_0 \cap \ldots \cap \widehat{H_i} \ldots \cap H_n}) \cap H_i = \mathrm{Syz}(\gamma|_{H_0 \cap \ldots \cap \widehat{H_j} \ldots \cap H_n}) \cap H_j$$

for all i, j.

To prove the sublemma, put

$$\Lambda_i = H_0 \cap \ldots \cap \widehat{H_i} \ldots \cap H_n \cong \mathbb{P}^c, \Lambda = H_0 \cap \ldots \cap H_n \cong \mathbb{P}^{c-1}.$$

Since Λ_i is general, we have $\gamma|_{\Lambda_i} \neq 0$ by Theorem 2.19. As $\deg X \geq c+3$, Lemma 3.26 shows that that $\mathrm{Syz}(\gamma|_{\Lambda_i})$ is a rational normal curve. Hence

$$\xi_i = \mathrm{Syz}(\gamma|_{\Lambda_i}) \cap H_i \subset \Lambda$$

is a set of c points in general position.

By a suitable choice of coordinates $(x_0 : \ldots : x_r)$, we may assume that $H_i = V(x_i)$, $i = 0, \ldots, n$. As before, we write

$$\delta(\gamma) = \sum_{|I|=c-2} x_I \otimes Q_I.$$

Put $\Sigma = \cap_{I\cap\{0,\ldots,n\}=\emptyset} V(Q_I) \cap \Lambda$. By definition we have $\xi_i \subseteq \Sigma$ for all i. To prove that $\xi_i = \Sigma$, it suffices to show that the inclusion $I_2(\Sigma) \subset I_2(\xi_i)$ is an equality, since the ideals of both sets are generated by quadrics. Put $J = \{n+1, \ldots, r\}$, and write $I(\lambda, \mu) = J \setminus \{\lambda, \mu\}$. The monodromy argument of Proposition 3.24 shows that the quadrics $Q_{I(\lambda,\mu)}|_\Lambda$ are linearly independent. Hence dim $I_2(\Sigma) \geq \binom{c}{2}$. On the other hand, the number of quadrics in $\Lambda \cong \mathbb{P}^{c-1}$ containing c points in general position is $\binom{c+1}{2} - c = \binom{c}{2}$. Hence the desired equality of ideals follows.

We now finish the proof of part (ii). Consider n general pencils of hyperplanes $\{H_{\lambda_i}\}_{\lambda_i \in \mathbb{P}^1}$. Let U be the open subset of the n–fold product of $\mathbb{P}^1$ corresponding to the hyperplanes H_{λ_i} such that $H_{\lambda_1} \cap \ldots \cap H_{\lambda_n}$ is in general position with respect to X. Using the natural map

$$(\lambda_1, \ldots, \lambda_n) \mapsto [\mathrm{Syz}(\gamma|_{H_{\lambda_1}\cap\ldots\cap H_{\lambda_n}})]$$

we identify U with an irreducible subset of the Hilbert scheme H of rational curves in $\mathbb{P}^r$ of degree c. Consider the incidence correspondence

$$\begin{array}{ccc} I = \{(x, C) \in \mathbb{P}^r \times H | x \in C\} & \xrightarrow{q} & H \\ \Big\downarrow{\scriptstyle p} & & \\ \mathbb{P}^r & & \end{array}$$

and let $Y \subset \mathbb{P}^r$ be the Zariski closure of $p(q^{-1}U)$. Since the fibers of q are irreducible, Y is irreducible. To show that Y is a variety of minimal degree containing X as a subvariety of codimension one, we consider the Zariski open subset

$$V = \cup_{(\lambda_1,\ldots,\lambda_n)\in U} H_{\lambda_1} \cap \ldots \cap H_{\lambda_n} \subset \mathbb{P}^r.$$

By Lemma 3.17 we have

$$\mathrm{Syz}(\gamma|_{H_{\lambda_1}\cap\ldots\cap H_{\lambda_n}}) \supset \mathrm{Syz}(\gamma) \cap H_{\lambda_1} \cap \ldots \cap H_{\lambda_n}.$$

Hence we obtain $Y \supset Y \cap V \supset \mathrm{Syz}(\gamma) \cap V \supset X \cap V$. Since X is irreducible, this implies that $Y \supset X$.

Let $H_1, \ldots, H_n$ be general hyperplanes. Using the sublemma and induction on n, we obtain

$$\mathrm{Syz}(\gamma|_{H_{\lambda_1}\cap\ldots\cap H_{\lambda_n}}) \cap H_1 \cap \ldots \cap H_n = \mathrm{Syz}(\gamma|_{H_1\cap\ldots\cap H_n})$$

for all $(\lambda_1, \ldots, \lambda_n) \in U$. Hence

$$Y \cap H_1 \cap \ldots \cap H_n \cap V = \mathrm{Syz}(\gamma|_{H_1\cap\ldots\cap H_n}) \cap V,$$

and we obtain $Y \cap H_1 \ldots \cap H_n = \mathrm{Syz}(\gamma|_{H_1 \cap \ldots \cap H_n})$ since both $Y \cap H_1 \ldots \cap H_n$ and the rational normal curve $\mathrm{Syz}(\gamma|_{H_1 \cap \ldots \cap H_n})$ are irreducible. Hence Y is a variety of minimal degree and dimension $n + 1 = \dim X + 1$. □

3.4. Rank-2 bundles and Koszul classes

The aim of this Section is to give a more geometric approach to the problem of describing Koszul classes of low rank.

3.4.1. The method of Voisin. The starting point is the following construction due to Voisin.

Let E be a rank two vector bundle on a smooth projective variety X defined over an algebraically closed field k of characteristic zero. Write $L = \det E$ and $V = H^0(X, L)$, and let

$$d : \bigwedge^2 H^0(X, E) \to V$$

be the determinant map. Given $t \in H^0(X, E)$, define a linear map

$$d_t : H^0(X, E) \to V$$

by $d_t(u) = d(t \wedge u)$, and choose a subspace $U \subset H^0(X, E)$ with $U \cap \ker(d_t) = 0$. Suppose that $\dim(U) = p+2$ with $p \geq 1$, and put $W = d_t(U) \cong U$. The restriction of d to $\bigwedge^2 U$ defines a map $\bigwedge^2 U \to V$, which we can view as an element of

$$\bigwedge^2 U^\vee \otimes V \cong \bigwedge^p U \otimes V.$$

Let

$$\gamma \in \bigwedge^p W \otimes V \subset \bigwedge^p V \otimes V$$

be the image of this element under the map d_t.

Following Voisin [**V05**, (22)], we prove that γ defines a Koszul class in $K_{p,1}(X, L)$. To this end, we make the previous construction explicit using coordinates. If we choose a basis $\{e_1, \ldots, e_{p+3}\}$ of $\langle t \rangle \oplus U \subset H^0(X, E)$ such that $e_1 = t$, we have

$$\gamma = \sum_{i<j} (-1)^{i+j} d(t \wedge e_2) \wedge \ldots \widehat{i} \ldots \widehat{j} \ldots \wedge d(t \wedge e_{p+3}) \otimes d(e_i \wedge e_j). \tag{3.1}$$

As in [**V05**] one shows that the image of the γ by the Koszul differential

$$\delta : \bigwedge^p V \otimes H^0(X, L) \to \bigwedge^{p-1} V \otimes S^2 H^0(X, L)$$

equals

$$\begin{aligned} &\sum_{i<j<k} (-1)^{i+j+k} d(t \wedge e_2) \wedge \ldots \widehat{i} \ldots \widehat{j} \ldots \widehat{k} \ldots \wedge d(t \wedge e_{p+3}) \\ &\otimes \{ d(t \wedge e_i) d(e_j \wedge e_k) - d(t \wedge e_j) d(e_i \wedge e_k) + d(t \wedge e_k) d(e_i \wedge e_j) \}. \end{aligned} \tag{3.2}$$

Lemma 3.29 (Voisin). *Given four elements w_1, w_2, w_3, $w \in H^0(X, E)$ we have the relation*

$$d(w \wedge w_1) d(w_2 \wedge w_3) - d(w \wedge w_2) d(w_1 \wedge w_3) + d(w \wedge w_3) d(w_1 \wedge w_2) = 0$$

in $H^0(X, L^2)$.

Proof: See [**V05**, Lemma 5]. □

The previous lemma shows that γ belongs to the kernel of the Koszul differential

$$\delta_X : \bigwedge^p V \otimes H^0(X, L) \to \bigwedge^{p-1} V \otimes H^0(X, L^2),$$

hence γ defines a class in the Koszul cohomology group.

Definition 3.30. The Koszul class $[\gamma] \in K_{p,1}(X, L, W) \subseteq K_{p,1}(X, L)$ associated to the triple (E, t, U) as before is denoted by $\gamma(U, t)$.

In general, the given class will depend on the choice of the lifting U. This dependence appears specifically if $\ker(d_t)$ contains elements other than multiples of t. The ambiguity disappears if we map the class to an appropriate Koszul group.

First we introduce some notation. Let $\mathcal{O}_X \overset{.t}{\hookrightarrow} E$ be the morphisms induced by t, and let B be the divisorial component of the zero scheme of t. Then there exist a codimension-2 subscheme ξ of X such that E lies in a short exact sequence:

$$0 \to \mathcal{O}_X(B) \xrightarrow{.t} E \xrightarrow{d_t} L(-B) \otimes \mathcal{I}_\xi \to 0,$$

hence $\ker(d_t) = H^0(X, \mathcal{O}_X(B))$.

Proposition 3.31. *Notation as above. There exists a natural surjective map*

$$\pi : K_{p,1}(X, L, W) \to K_{p,1}(X, \mathcal{O}_X(B), L(-B), W).$$

The image of $\gamma(U, t)$ under this map is independent of the choice of the lifting U of W; we denote it by $\gamma(W, t)$. Moreover, any lifting of a given class $\gamma(W, t) \in K_{p,1}(X, \mathcal{O}_X(B), L(-B), W)$ is of the form $\gamma(U', t) \in K_{p,1}(X, L, W)$ for some $U' \subset H^0(X, E)$ with $d_t(U') = W$.

Proof: The commutative diagram

$$\begin{array}{ccccc} \bigwedge^{p+1} W \otimes H^0(\mathcal{O}_X(B)) & \to & \bigwedge^p W \otimes H^0(L) & \to & \bigwedge^{p-1} W \otimes H^0(L^2(-B)) \\ \uparrow .t & & \| & & \downarrow \\ \bigwedge^{p+1} W & \to & \bigwedge^p W \otimes H^0(L) & \to & \bigwedge^{p-1} W \otimes H^0(L^2) \end{array}$$

induces a short exact sequence

$$0 \to \bigwedge^{p+1} W \otimes \frac{\ker(d_t)}{\mathbb{C}.t} \to K_{p,1}(X, L, W) \xrightarrow{\pi} K_{p,1}(X, \mathcal{O}_X(B), L(-B), W) \to 0.$$

This exact sequence implies the first statement.

For the second statement, let $\gamma \in K_{p,1}(X, L, W)$ be a lifting of $\gamma(W, t)$, and choose a basis $e_1, \ldots, e_{p+2}$ of U. The exact sequence shows that the classes γ and $\gamma(U, t)$ differ by an element

$$\sum_j e_1 \wedge \ldots \wedge \widehat{e_j} \wedge \ldots \wedge e_{p+2} \otimes t_j$$

with $t_j \in \ker(d_t)$, hence γ is of the form $\gamma(U', t)$ with

$$U' = \langle e_1 + t_1, \ldots, e_{p+2} + t_{p+2} \rangle.$$

$\square$

Remark 3.32. If $U' \subset \langle t \rangle \oplus U \subset d_t^{-1}(W)$ is another lifting of W, then $\gamma(U, t) = \gamma(U', t)$. In particular, if $\ker(d_t) = \mathbb{C}.t$ the given class only depends on t and W.

3.4.2. The method of Green–Lazarsfeld. The following nonvanishing result was a source of inspiration for several important conjectures on Koszul cohomology.

THEOREM 3.33 (Green-Lazarsfeld). *Let X be a smooth projective variety, and let L be a line bundle on X that admits a decomposition $L = L_1 \otimes L_2$ with $r_i = \dim |L_i| \geq 1$ for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.*

We shall prove this result in the curve case in section 3.5. For the general case, see [**GL84**] or [**V93**].

The Koszul classes appearing in this result are constructed in the following way. Write $L_i = M_i + F_i$ with M_i the mobile part and F_i the fixed part. Let B be the divisorial part of $F_1 \cap F_2$. It is possible to choose $s_i \in H^0(X, L_i)$ such that $V(s_1, s_2) = B \cup Z$ with codim $(Z) \geq 2$. Set $L = L_1 \otimes L_2$, and put $t = (s_1, s_2) \in H^0(X, L_1 \oplus L_2)$, $W = \operatorname{im}(d_t) \subset H^0(X, L(-B))$. By construction $h^0(X, \mathcal{O}_X(B)) = 1$, hence $\ker(d_t) = \mathbb{C}.t$ and $\dim W = r_1 + r_2 + 1$. By the previous discussion, we obtain a Koszul class $\gamma(W, t) \in K_{r_1+r_2-1,1}(X, L)$. We call such classes *Green–Lazarsfeld classes.*

Note that the rank of a Green–Lazarsfeld class is either $p + 1$ or $p + 2$. It is of scrollar type if and only if it comes from a pencil.

3.4.3. The method of Koh–Stillman. Voisin's method produces syzygies of rank $\leq p + 2$. As we have seen in the previous subsection, rank $p + 1$ syzygies are Green–Lazarsfeld syzygies of scrollar type. Rank $p+2$ syzygies can be obtained in the following way. Suppose that L is a globally generated line bundle on a projective variety X, and let $[\gamma] \in K_{p,1}(X, L)$ be a nonzero class represented by an element $\gamma \in \bigwedge^p W \otimes V$ with $\dim W = p + 2$. We view γ as an element in $\bigwedge^2 W^\vee \otimes V \cong \operatorname{Hom}(\bigwedge^2 W, V)$. As before we consider the map

$$\gamma' : \bigwedge\nolimits^2(\mathbb{C} \oplus W) = W \oplus \bigwedge\nolimits^2 W \to V$$

$W \hookrightarrow V$. If we choose a generator e_1 for the first summand and a basis $\{e_2, \ldots, e_{p+3}\}$ for W, we obtain a skew–symmetric $(p + 3) \times (p + 3)$ matrix A by setting

$$a_{ij} = \gamma'(e_i \wedge e_j).$$

By construction, the inclusion $W \to V$ corresponds to the map $\gamma'(e_1 \wedge -)$. This allows us to identify a_{1j} and e_j, $2 \leq j \leq p + 3$. Let α be the image of γ under the Koszul differential

$$\delta : \bigwedge\nolimits^p V \otimes V \to \bigwedge\nolimits^{p-1} V \otimes S^2 V.$$

Writing this out, we obtain

$$\alpha = \sum_{i<j<k} (-1)^{i+j+k} a_{12} \wedge \ldots \widehat{a_{1,i}} \ldots \widehat{a_{1,j}} \ldots \widehat{a_{1,k}} \ldots \wedge a_{1,p+3} \otimes \operatorname{Pf}_{1ijk}(A) \tag{3.3}$$

where

$$\operatorname{Pf}_{1ijk}(A) = a_{1i}a_{jk} - a_{1j}a_{ik} + a_{1k}a_{ij}$$

is a 4×4 Pfaffian of A. As the elements $\{a_{12}, \ldots, a_{1,p+3}\} = \{e_2, \ldots, e_{p+3}\}$ are linearly independent, this expression is nonzero if and only if at least one of the Pfaffians $\operatorname{Pf}_{1ijk}(A)$ is nonzero. Furthermore, since α maps to zero in $\bigwedge^{p-1} V \otimes H^0(X, L^2)$ the Pfaffians $\operatorname{Pf}_{1ijk}(A)$ have to vanish on the image of X.

The preceding discussion shows that every rank $p+2$ syzygy arises from a skew–symmetric $(p+3)\times(p+3)$ matrix A such that

(i) the elements $\{a_{12},\dots,a_{1,p+3}\}$ are linearly independent;
(ii) there exists a nonzero Pfaffian $\mathrm{Pf}_{1ijk}(A)$;
(iii) the Pfaffians $\mathrm{Pf}_{1ijk}(A)$ vanish on the image of X in $\mathbb{P}(V^\vee)$.

This is exactly the method used by Koh and Stillman to produce syzygies; see [**KS89**, Lemma 1.3].

3.5. The curve case

In this section we give a complete characterization of Koszul classes of rank $p+2$ on a curve. The following two results simplify the presentation of [**AN07**].

LEMMA 3.34. *We have isomorphisms*

$$\begin{aligned} K_{p,1}(X,B,L(-B),W) &\cong \ker(\delta : H^1(X,L^\vee(2B)) \to W^\vee\otimes H^1(X,\mathcal{O}_X(B)))\\ &\cong \mathcal{E}_W=\{\xi\in \mathrm{Ext}^1(L(-B),\mathcal{O}_X(B)) \mid W\subset \ker\delta_\xi\}.\end{aligned}$$

Proof: Consider the exact sequence

$$0\to M_W\to W\otimes\mathcal{O}_X\to L(-B)\to 0. \tag{3.4}$$

By Proposition 2.4 and Remark 2.6 $K_{p,1}(X,B,L(-B),W)$ is isomorphic to

$$\begin{aligned}&\ker(H^1(X,\textstyle\bigwedge^{p+1}M_W\otimes\mathcal{O}_X(B))\to H^1(X,\textstyle\bigwedge^{p+1}W\otimes\mathcal{O}_X(B)))\\ &\quad\cong\ker(\delta : H^1(X,L^\vee(2B))\to W^\vee\otimes H^1(X,\mathcal{O}_X(B))).\end{aligned}$$

For the second statement, note that δ is induced by the multiplication map

$$\mu : W\otimes H^1(X,L^\vee(2B))\to H^1(X,\mathcal{O}_X(B)).$$

The formula

$$\delta_\xi(w)=\mu(w\otimes\xi)$$

shows that

$$\xi\in\ker\delta \iff \mu(w\otimes\xi)=0\ \ \forall w\in W,$$

that is, if and only if $W\subset\ker\delta_\xi$. □

PROPOSITION 3.35. *Let $W\subset H^0(X,L(-B))$ be a base–point free linear subspace of dimension $p+2$, and let $t\in H^0(X,E)$ be the image of the canonical section of $\mathcal{O}_X(B)$. Let*

$$\psi : \mathcal{E}_W\to K_{p,1}(X,B,L(-B),W)$$

be the isomorphism defined in the previous Lemma. We have $\psi(\xi)=\gamma(W,t)$.

Proof: Choose a linear subspace $U\subset H^0(X,E)$ such that $d_t : U \xrightarrow{\sim} W$, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & M_U & \to & U\otimes\mathcal{O}_X & \xrightarrow{\alpha} & L(-B) & \to & 0\\ & & \downarrow{\scriptstyle\beta} & & \downarrow & & \| & & \\ 0 & \to & \mathcal{O}_X(B) & \to & E & \to & L(-B) & \to & 0\end{array} \tag{3.5}$$

where M_U is by definition the kernel of α. If we dualise this diagram and twist by $\mathcal{O}_X(B)$, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L^\vee(2B) & \to & U^\vee(B) & \to & M_U^\vee(B) & \to & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \to & L^\vee(2B) & \to & E^\vee(B) & \to & \mathcal{O}_X & \to & 0. \end{array}$$

By construction, $\psi(\xi)$ is obtained by taking the image of $1 \in H^0(X, \mathcal{O}_X)$ under the map φ in the commutative diagram

$$\begin{array}{ccccc} H^0(X, M_U^\vee(B)) & \to & H^1(X, L^\vee(2B)) & \to & U^\vee \otimes H^1(X, \mathcal{O}_X(B)) \\ \uparrow{\scriptstyle\varphi} & & \| & & \\ H^0(X, \mathcal{O}_X) & \to & H^1(X, L^\vee(2B)). & & \end{array}$$

If we take the second exterior power in the rows of diagram (3.5), we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \bigwedge^2 M_U & \to & \bigwedge^2 U \otimes \mathcal{O}_X & \to & M_U \otimes L(-B) & \to & 0 \\ & & & & \downarrow & & \downarrow{\scriptstyle\beta\otimes\mathrm{id}} & & \\ & & 0 & \to & \bigwedge^2 E & \xrightarrow{\sim} & L & \to & 0. \end{array}$$

Dualising and twisting by L, we obtain a commutative diagram

$$\begin{array}{ccc} M_U^\vee(B) & \to & \bigwedge^2 U^\vee \otimes L \\ \uparrow{\scriptstyle\beta} & & \uparrow \\ \mathcal{O}_X & \to & \bigwedge^2 E^\vee \otimes L. \end{array}$$

After taking global sections, we obtain

$$\begin{array}{ccc} H^0(X, M_U^\vee(B)) & \to & \bigwedge^2 U^\vee \otimes V \cong \mathrm{Hom}(\bigwedge^2 U, V) \\ \uparrow{\scriptstyle\varphi} & & \uparrow \\ \mathbb{C} & \to & H^0(X, \bigwedge^2 E^\vee) \otimes V. \end{array}$$

By construction, the image of $\psi(\xi) = \varphi(1)$ in $\mathrm{Hom}(\bigwedge^2 U, V)$ is the restriction of the determinant map $d|_{\bigwedge^2 U}$. Hence we can identify $\psi(\xi)$ and $\gamma(W, t)$. $\square$

Proposition 3.35 yields a short, geometric proof of the Green–Lazarsfeld non-vanishing theorem for curves.

Corollary 3.36. **(Green–Lazarsfeld)** *Let X be a smooth curve, and let L be a line bundle on X that admits a decomposition $L = L_1 \otimes L_2$ with $r_i = \dim |L_i| \geq 1$ for $i = 1, 2$. Then $K_{r_1+r_2-1,1}(X, L) \neq 0$.*

Proof: We define s_1, s_2, t, W, B and $\gamma(W, t)$ as in section 3.4.2. Let C be the base locus of W, seen as a subspace of $H^0(X, L(-B))$. We prove that $\gamma(W, t) \neq 0$. Suppose that $\gamma(W, t) = 0$. Consider the extension

$$0 \to \mathcal{O}_X(B) \to L_1 \oplus L_2 \to L(-B) \to 0.$$

Pulling back this extension along the injective homomorphism $L(-B - C) \to L(-B)$, we obtain an induced extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B - C) \to 0.$$

Applying Proposition 3.35 to the line bundle $L(-C)$, we find that this extension splits. Hence there exists an injective homomorphism

$$\mathcal{O}_X(B) \oplus L(-B-C) \to L_1 \oplus L_2.$$

In particular there exists $i \in \{1, 2\}$ such that $\mathrm{Hom}(L(-B-C), L_i) \neq 0$. This implies that

$$r_i + 1 = h^0(X, L_i) \geq h^0(X, L(-B-C)) \geq \dim W = r_1 + r_2 + 1,$$

and this is impossible since $r_1 \geq 1$ and $r_2 \geq 1$. □

THEOREM 3.37. *Let X be a smooth curve, and let $\alpha \neq 0 \in K_{p,1}(X, L)$ be a Koszul class of rank $p+2$ represented by an element of $\bigwedge^p W \otimes H^0(X, L)$ with* $\dim W = p+2$. *There exist a rank 2 vector bundle E on X, a section $t \in H^0(X, E)$ and a subspace $W \cong U \subset H^0(X, E)$ such that $\alpha = \gamma(U, t)$.*

Proof: Put $T = \mathbb{C} \oplus W$. By Remark 3.13 and Theorem 3.23 we obtain a rational map

$$\psi : X \dashrightarrow \mathbb{P}(\wedge^2 T^\vee)$$

such that the closure of the image of X is contained in $Gensyz_2(W) = G(2, T) \cup \mathbb{P}(\wedge^2 W^\vee)$. Since X is a curve, we may remove the base–locus C of this map, by replacing L by $L(-C)$. By construction, the resulting morphism ψ is given by the skew-symmetric matrix $A = (a_{ij})$, introduced in section 3.4.3, such that

(a) The linear forms in the first row of A span W;
(b) There exists a nonzero 4×4 Pfaffian of A involving the first row and column;
(c) The 4×4 Pfaffians involving the first row and column of A vanish on the image of X in $\mathbb{P}H^0(X, L)^\vee$.

Put $Y = \psi(X)$. Condition (a) shows that Y is not contained in $\mathbb{P}(\bigwedge^2 W^\vee)$. As Y is irreducible, this implies that Y is contained in $G(2, T)$.

Put $E = \psi^* Q$. Twisting the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_G \to \psi_* \mathcal{O}_X \to 0$$

by the universal quotient bundle Q and taking global sections, we obtain an exact sequence

$$0 \to H^0(G, Q \otimes \mathcal{I}_Y) \to H^0(G, Q) \xrightarrow{\psi^*} H^0(G, \psi_* \mathcal{O}_X \otimes Q) \cong H^0(X, E).$$

Condition (a) implies that Y is not contained in $G(2, W) = G(2, T) \cap \mathbb{P}(\bigwedge^2 W^\vee)$, hence t does not vanish identically on X and defines a global section of E. The zero locus of this section is given by the equations $a_{12} = \cdots = a_{1,p+3} = 0$, hence it coincides with B. Consequently the line bundle E is given by an extension

$$0 \to \mathcal{O}_X(B) \to E \to L(-B) \to 0. \tag{3.6}$$

Consider the commutative diagram

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(G,\mathcal{O}_G) & \xrightarrow{t} & H^0(X,\mathcal{O}_X(B)) \\
\downarrow{\scriptstyle \wedge t} & & \downarrow{\scriptstyle \wedge t} \\
H^0(G,Q) & \xrightarrow{\psi^*} & H^0(X,E) \\
\downarrow & & \downarrow{\scriptstyle d_t} \\
W & \xrightarrow{i} & H^0(X,L(-B)).
\end{array}$$

Note that $\ker i = W \cap H^0(G,\mathcal{O}_G(1)\otimes\mathcal{I}_Y) = 0$ by condition (a). As the map $H^0(G,Q) \to W$ is surjective, we find that W is contained in the image of the map $d_t : H^0(X,E) \to H^0(X,L(-B))$. The embedding $W \subset H^0(G,Q) = \langle t\rangle \oplus W$ composed with ψ^* is a section of d_t. Put $U = \psi^*(W)$. By construction we obtain $\gamma = \gamma(U,t)$. □

Remark 3.38. This result is a refinement of [**vB07a**, Theorem 6.7], where it was shown that a rank $p+2$ syzygy gives rise to a rank 2 vector bundle if L is very ample and the ideal of X is generated by quadrics.

Theorem 3.37 shows that Voisin's method may produce nontrivial Koszul classes that are not contained in the subspace of $K_{p,1}(X,L)$ spanned by Green–Lazarsfeld classes.

Example 3.39. By [**ELMS89**, Theorem 3.6 and Theorem 4.3] there exists a smooth curve of genus 14 and Clifford index 5 (see section 4.2 for the definition) whose Clifford index is computed by a unique line bundle L such that $L^2 = K_X$. The line bundle L embeds X in $\mathbb{P}^4$ as a projectively normal curve of degree 13, and the ideal of X is generated by the 4×4 Pfaffians of a skew–symmetric matrix $(a_{ij})_{1\le i,j\le 5}$ with

$$\deg(a_{ij}) = \begin{cases} 2 \text{ if } i=1 \text{ or } j=1 \\ 1 \text{ if } i\ge 2 \text{ and } j \ge 2 \end{cases}$$

such that the quadric $Q = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}$ has rank 5. The Koszul class $[Q] \in K_{1,1}(X,L)$ has rank 3, since it is represented by the linear subspace $W = \langle a_{23}, a_{24}, a_{25}\rangle$. Hence $[Q]$ comes from Voisin's method by Theorem 3.37.

Suppose that $K_{1,1}(X,L)$ contains a Green–Lazarsfeld class. This class would be of scrollar type, since it necessarily comes from two pencils $|L_1|$, $|L_2|$. The equality $\deg(L_1)+\deg(L_2) = 13$ implies that L_1 and L_2 contribute to the Clifford index. This is impossible, since the previous equality implies that there exists i such that $\deg(L_i) \le 6$, hence $\mathrm{Cliff}(L_i) \le 4$.

Remark 3.40. A more geometric description of a subspace W representing $[Q]$ is the following. A smooth intersection of the quadric $V(Q) \subset \mathbb{P}H^0(X,L)^\vee$ with one of the cubic Pfaffians is a $K3$ surface in $\mathbb{P}H^0(X,L)^\vee$ containing a line ℓ which is disjoint from X by [**ELMS89**, Prop. 4.1]. The line ℓ corresponds to a 3-dimensional linear subspace $W \subset H^0(X,L)$, which is base-point-free since ℓ does not meet X.

One could ask whether the syzygies constructed in section 3.4.1 span $K_{p,1}(X, L)$. In principle it may be possible to obtain higher rank syzygies as linear combinations of rank $p+2$ syzygies. However, if $K_{p,1}(X, L)$ is spanned by a single syzygy of rank $\geq p+3$ this is not possible.

Example 3.41 (Eusen–Schreyer). Eusen and Schreyer [**ES94**, Theorem 1.7 (a)] have constructed a smooth curve $X \subset \mathbb{P}^5$ of genus 7 and Clifford index 3 embedded by the linear system $|K_X(-x)|$ such that $K_{2,1}(X, K_X(-x)) \cong \mathbb{C}$ is spanned by a syzygy s_0. The explicit expression for s_0 given on p.8 of [loc. cit.] shows that s_0 is a rank 5 syzygy. Hence s_0 cannot be obtained by the Green–Lazarsfeld construction or the method of section 3.4.1.

3.6. Notes and comments

As mentioned before, the notion of *syzygy scheme* emerged from Green's $K_{p,1}$ theorem, and was introduced for studying the Arbarello-Sernesi modules

$$\bigoplus_q H^0(X, K_X \otimes L^q).$$

The first application was an improvement of a result of Arbarello-Sernesi, see [**Gre84a**, Theorem (4.b.2)] and [**Gre82**, Theorem 2.14]. The core of the proof is the Strong Castelnuovo Lemma 3.26. It generalizes the classical Castelnuovo Lemma which states that if a subvariety X of $\mathbb{P}^r$ imposes $\leq 2r+1$ conditions on quadrics, then either it is contained in a rational normal curve, or else no more than $2r+2$ points of X are in general position.

In the curve case, the $K_{p,1}$ theorem has another nice consequence. It shows that if L is a nonspecial very ample line bundle on a curve C with $h^0(C, L) = r+1$, and $0 \neq \gamma \in K_{r-2,1}(C, L)$, then C is hyperelliptic, and the g^1_2 is induced by the ruling on $\mathrm{Syz}(\gamma)$. The proof idea is the following. Note that $\mathrm{Syz}(\gamma)$ is two-dimensional, hence its desingularization is a Hirzebruch surface Σ_e. A small cohomological computation using the embedding $C \subset \Sigma_e$, based on the nonspeciality of L, shows that the ruling of Σ_e cannot restrict to a pencil of degree larger than two.

Ehbauer analyzed the further case of a line bundle L on a genus-g curve C with $g \geq 13$, $\deg(L) \geq 2g+3$, $h^0(C, L) = r+1$, and $K_{r-3,1}(C, L) \neq 0$. He shows that the intersection of all the syzygy schemes of classes in $K_{r-3,1}(C, L)$ is a 3-dimensional rational normal scroll, and the ruling planes cut out a g^1_3 on C. The hyperelliptic case that follows from the $K_{p,1}$ theorem, and the trigonal case treated by Ehbauer are two entry cases for a more general conjectural statement that will be thoroughly discussed in the next chapters, see Conjecture 4.21. Ehbauer's proof relies on a generalization of the Strong Castelnuovo Lemma; see [**Ehb94**] for details.

CHAPTER 4

The conjectures of Green and Green–Lazarsfeld

4.1. Brill-Noether theory

We are interested in studying the following stratification of the Jacobian of C

$$\mathrm{Pic}_d(C) \supset W_d^0(C) \supset W_d^1(C) \supset \dots$$

where

$$W_d^r(C) := \{A \in \mathrm{Pic}_d(C),\ h^0(C, A) \geq r + 1\},$$

for all r. This problem only makes sense for $d \leq 2g - 2$, as $h^0(C, A)$ is completely determined by d for any A if $d \geq 2g - 1$.

The strata $W_d^r(C)$ are called *Brill-Noether loci*; they are determinantal subvarieties of $\mathrm{Pic}_d(C)$ i.e. are given locally by the vanishing of minors of suitable matrices of functions. This structure is obtained in the following way [**ACGH85**]. Fix E an effective divisor of degree $m \geq 2g - d - 1$. Then for any element $A \in \mathrm{Pic}_d(C)$, the bundle $A(E)$ is nonspecial, hence we have an exact sequence

$$0 \to H^0(C, A) \to H^0(C, A(E)) \to H^0(E, A(E)|_E) \to H^1(C, A) \to 0. \tag{4.1}$$

Obviously $A \in W_d^r(C)$ if and only if

$$\mathrm{rk}(\ker(H^0(C, A(E)) \to H^0(E, A(E)|_E))) \geq r + 1.$$

Remark that both $n = h^0(C, A(E)) = d + m - g + 1$ and $h^0(E, A(E)|_E) = m$ do not depend on the choice of A. This elementary observation allows the construction of the determinantal structure. Let $\Gamma = E \times \mathrm{Pic}_d(C) \subset C \times \mathrm{Pic}_d(C)$ be the product divisor, and denote by $\mathcal{L}$ a Poincaré bundle on $C \times \mathrm{Pic}_d(C)$ [**ACGH85**], and $\pi : C \times \mathrm{Pic}_d(C) \to \mathrm{Pic}_d(C)$ the projection on the second factor. By the considerations above, the sheaves

$$\pi_*(\mathcal{L}(\Gamma)),\ \text{and}\ \pi_*(\mathcal{L}(\Gamma)|_\Gamma)$$

are locally free of rank n, respectively m. If

$$\gamma : \pi_*(\mathcal{L}(\Gamma)) \to \pi_*(\mathcal{L}(\Gamma)|_\Gamma)$$

denotes the sheaf morphism induced by the sequence (4.1), then $W_d^r(C)$ is the locus where

$$\dim(\ker \gamma) \geq r + 1.$$

By the general theory, we obtain a determinantal structure on $W_d^r(C)$, which is independent on the choice of the Poincaré bundle $\mathcal{L}$, and of the divisor E [**ACGH85**] p. 179. In local coordinates, it is given by the vanishing of the minors of order $k = d + m - g - r$ of a matrix $n \times m$ of functions provided, of course, that $k \leq m$, which is equivalent to

$$g - d + r \geq 0. \tag{4.2}$$

Note that $g-d+r = h^0(C, K_C \otimes A^\vee)$ for any A having precisely $r+1$ global sections. Inequality (4.2) is therefore a necessary condition for the existence of elements in $W^r_d(C) \setminus W^{r+1}_d(C)$. If $W^r_d(C)$ is non-empty, then one can prove that (4.2) is also a sufficient condition, in a stronger sense.

PROPOSITION 4.1. *If $r \geq d-g$ then no component of $W^r_d(C)$ is entirely contained in $W^{r+1}_d(C)$.*

Another numerical restriction is given by the Clifford Theorem.

THEOREM 4.2 (Clifford). *Suppose $r \geq d-g$ and $W^r_d(C) \neq \emptyset$. Then $2r \leq d$.*

If condition (4.2) is satisfied, and if $W^r_d(C)$ is non-empty, then all its components have dimension at least

$$\dim(\mathrm{Pic}_d(C)) - (n-k)(m-k) = g - (r+1)(g-d+r).$$

A complete proof can be found in [**ACGH85**], Ch. II. Section 4. However, the problem of non-emptiness of Brill-Noether loci is a very hard one, see Theorem 4.4 below.

The integer

$$\rho(g,r,d) = g - (r+1)(g-d+r) \tag{4.3}$$

which represents the expected dimension of the Brill-Noether locus $W^r_d(C)$ is called *the Brill-Noether number.* Since $h^0(C,A) = r+1$ and $h^0(C, K_C \otimes A^\vee) = g-d+r$ for any $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, we obtain

$$\rho(g,r,d) = h^0(C, K_C) - h^0(C,A)h^0(C, K_C \otimes A^\vee).$$

The Brill-Noether number is related to the multiplication map

$$\mu_{0,A} : H^0(C,A) \otimes H^0(K_C \otimes A^\vee) \to H^0(C, K_C). \tag{4.4}$$

If $\mu_{0,A}$ is injective, then $\rho(g,r,d) = \operatorname{codim} \mu_{0,A}$. A curve is called *Brill-Noether-Petri generic* if the multiplication maps $\mu_{0,A}$ are injective for all A. This condition is usually referred to as the *Brill-Noether-Petri condition.* Its geometric meaning is given by the following theorem, see [**ACGH85**].

THEOREM 4.3. *Suppose $r \geq d-g$ and $W^r_d(C) \neq \emptyset$. Given $A \in W^r_d(C) \setminus W^{r+1}_d(C)$, the tangent space to $W^r_d(C)$ at A is*

$$T_A(W^r_d(C)) = \mathrm{Ann}(\mathrm{im}(\mu_{0,A}))$$

where $\mathrm{Ann}(\mathrm{im}(\mu_{0,A})) \subset H^0(C, K_C)^\vee$ *denotes the annihilator of the subspace*

$$\mathrm{im}(\mu_{0,A}) \subset H^0(C, K_C).$$

In particular, $W^r_d(C)$ is smooth of dimension $\rho(g,r,d)$ at A if and only if $\mu_{0,A}$ is injective.

All the results mentioned above assume non-emptiness of Brill-Noether loci. This is realized effectively when ρ is non-negative.

THEOREM 4.4 (Existence Theorem). *Let C be a smooth curve of genus g, and d, r be integers such that $d \geq 1$, $r \geq 0$ with $\rho(g,r,d) \geq 0$. Then $W^r_d(C)$ is non-empty.*

Under stronger assumptions, one can prove more.

THEOREM 4.5 (Connectedness Theorem). *Let C be a smooth curve of genus g, and d, r be integers such that $d \geq 1$, $r \geq 0$ with $\rho(g,r,d) \geq 1$. Then $W^r_d(C)$ is connected.*

The importance of the Brill-Noether-Petri condition is made clear by the following fundamental result, [**ACGH85**]

THEOREM 4.6. *If C is Brill-Noether-Petri generic, then for any r and d, with $r \geq d-g$ the variety $W^r_d(C)$ is of expected dimension $\rho(g,r,d)$ and smooth away from $W^{r+1}_d(C)$.*

In particular, for a Brill-Noether-Petri generic curve C the Brill-Noether locus $W^r_d(C)$ is empty if $\rho(g,r,d) < 0$.

In Chapter 7, we shall need upper-bounds for the dimensions of the Brill-Noether loci. Two basic results in this direction are the following

THEOREM 4.7 (Martens). *Let C be a smooth curve of genus $g \geq 3$, d and r be integers with $2 \leq d \leq g-1$, and $0 < 2r \leq d$. If C is hyperelliptic then*

$$\dim\, W^r_d(C) = d - 2r,$$

and if C is non-hyperelliptic

$$\dim\, W^r_d(C) \leq d - 2r - 1.$$

THEOREM 4.8 (Mumford). *Let C be a smooth curve of genus $g \geq 4$, d and r be integers with $2 \leq d \leq g-2$, and $0 < 2r \leq d$. If C is none of the following: hyperelliptic, trigonal, bi-elliptic, smooth plane quintic, then*

$$\dim\, W^r_d(C) \leq d - 2r - 2.$$

Recall that a *trigonal curve* admits a cover of degree three on the projective line (see section 4.2), and a *bielliptic curve*s is a double cover of an elliptic curve.

The next case, $\dim\, W^r_d(C) \leq d - 2r - 3$ was treated by Keem, [**Ke90**].

4.2. Numerical invariants of curves

Linear systems on curves can be used to define natural numerical invariants. Theses invariants measure "how special" the curve is, and yield stratifications of the moduli space of curves.

The first invariant of this type is the gonality:

$$\mathrm{gon}(C) := \min\{\deg(A),\ h^0(C,A) \geq 2\} = \min\{d,\ W^1_d(C) \neq \emptyset\}.$$

Elements in $W^1_d(C)$ are called *pencils*; the gonality computes the minimal degree of a surjective morphism from C to the projective line. By lower semi-continuity of the gonality, we obtain a stratification of the moduli space $\mathcal{M}_g$ of genus-g curves; the resulting strata are irreducible, since they are covered by Hurwitz schemes. The Existence Theorem 4.4 applied to pencils gives the inequality

$$\mathrm{gon}(C) \leq \left[\frac{g+3}{2}\right],$$

and the maximal value is realized on an open set of $\mathcal{M}_g$.

The second important invariant is the *Clifford index*. The origin of this notion is in the proof of Clifford's Theorem 4.2. The proof of the statement (see [**ACGH85**,

p. 108]) uses the Riemann-Roch Theorem, and the addition map of effective divisors (the additive version of the multiplication map of sections):

$$|A| \times |K_C \otimes A^\vee| \to |K_C|.$$

Note that this map is injective, and finite onto its image. Therefore, the codimension of the image equals

$$\dim |K_C| - \dim |A| - \dim |K_C \otimes A^\vee| = d - 2h^0(C, A) + 2 \geq 0.$$

The quantity appearing above is called the *Clifford index of* A:

$$\mathrm{Cliff}(A) = d - 2h^0(C, A) + 2.$$

The addition maps that are significant for the geometry of the curve are the non-trivial ones, and the addition map will be non-trivial only if both linear systems $|A|$ and $|K_C \otimes A^\vee|$ are at least one-dimensional. We arrive at the following definition.

DEFINITION 4.9. A line bundle A is said to *contribute to the Clifford index of* C if $h^0(C, A) \geq 2$ and $h^1(C, A) \geq 2$.

Note that this notion is auto-dual, i.e., a line bundle A contributes to the Clifford index if and only if its residual bundle $K_C \otimes A^\vee$ contributes to the Clifford index, and the Clifford indices of the two bundles coincide. The Clifford index of the curve is obtained by taking the minimum of all Clifford indices that contribute:

DEFINITION 4.10 (Martens).

$$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(A),\ h^0(C, A) \geq 2,\ h^1(C, A) \geq 2\}.$$

For $g \leq 3$ this definition has to be modified slightly, see [**La89**].

A line bundle A that contributes to the Clifford index of the curve, and whose Clifford index is minimal is said to *compute the Clifford index of* C.

By the Clifford Theorem, we have $\mathrm{Cliff}(C) \geq 0$, and equality holds only for hyperelliptic curves. Similarly, $\mathrm{Cliff}(C) = 1$ if and only if C is trigonal or plane quintic, and $\mathrm{Cliff}(C) = 2$ if and only if C is tetragonal, or plane sextic. Martens [**Ma82**] has gone one step further, and proved that $\mathrm{Cliff}(C) = 3$ if and only if C is pentagonal, or a plane septic, or a complete intersection of two cubics in $\mathbb{P}^3$. In the general case, we have the following conjecture made by Eisenbud, Lange, Martens and Schreyer, [**ELMS89**, p. 175].

CONJECTURE 4.11 (Eisenbud, Lange, Martens, Schreyer). $\mathrm{Cliff}(C) = p$ if and only if either

(i) $\mathrm{gon}(C) = p + 2$;
(ii) C is a plane curve of degree $p + 4$;
(iii) p is odd, and C is a half-canonical curve of even genus $2p + 4$ embedded in $\mathbb{P}^{\frac{p+3}{2}}$.

The Existence Theorem 4.4 gives the following upper bound for the Clifford index:

$$\mathrm{Cliff}(C) \leq \left[\frac{g-1}{2}\right].$$

As the Clifford index is lower semi-continuous, it produces a stratification of $\mathcal{M}_g$. The dimensions of linear systems that contribute to the Clifford index yield to a new invariant, the *Clifford dimension*:

$$\text{Cliffdim}(C) := \min\{\dim |L|,\ L \text{ computes the Clifford index of } C\}.$$

The Clifford index is intimately related to the gonality; a straightforward argument gives the inequality

$$\text{gon}(C) \geq \text{Cliff}(C) + 2,$$

and equality holds only for curves of Clifford dimension one. Since there exist curves of arbitrary Clifford dimensions larger than three [**ELMS89**], the inequality above can be strict. However, it was proved by Coppens and Martens [**CM91**] that the two invariants cannot differ too much:

$$\text{Cliff}(C) + 3 \geq \text{gon}(C).$$

By what we have said above, $\text{Cliff}(C) = \text{gon}(C)+3$ if and only if C is of Clifford dimension at least two. For Clifford dimension two we obtain plane curves, hence the hard cases to analyze start with Clifford dimension three. A current research topic is to understand the true meaning of the Clifford index, and of the differences between the Clifford index and the gonality. As we shall see in the next section, Koszul cohomology provides a conjectural tool for computing the Clifford index and gonality of a curve.

4.3. Statement of the conjectures

In this sections we discuss two important conjectures on Koszul cohomology of curves.

4.3.1. Green's conjecture. Given a line bundle L on a smooth projective curve C, put

$$r(L) = \dim |L| = h^0(C, L) - 1.$$

Proposition 4.12. *Let A be a line bundle that contributes to* $\text{Cliff}(C)$. *Then*

$$K_{p,1}(C, K_C) \neq 0 \textit{ for all } p \leq g - \text{Cliff}(A) - 2.$$

Proof: Put $L_1 = A$, $L_2 = K_C \otimes A^\vee$, $r_i = r(L_i)$ $(i = 1, 2)$, $d = \deg(A)$. By Riemann–Roch we have $r_2 - r_1 = g - d - 1$, hence $r_1 + r_2 - 1 = g - d + 2r(A) - 2 = g - \text{Cliff}(A) - 2$. Using Theorem 3.33 we obtain

$$K_{g-\text{Cliff}(A)-2,1}(C, K_C) \neq 0.$$

The assertion then follows from Corollary 2.13. □

The strongest nonvanishing result of this type is obtained by taking the minimal value of $\text{Cliff}(L)$. This gives the implication

$$p \geq g - \text{Cliff}(C) - 2 \Longrightarrow K_{p,1}(C, K_C) \neq 0. \tag{4.5}$$

Green [**Gre84a**] conjectures that the converse of (4.5) holds.

Conjecture 4.13 (Green). Let C be a smooth projective curve. Then

$$K_{p,1}(C, K_C) = 0 \Longleftrightarrow p \geq g - \text{Cliff}(C) - 1.$$

Remark 4.14. In view of Corollary 2.13, Green's conjecture holds if and only if

$$K_{g-\mathrm{Cliff}(C)-1,1}(C, K_C) = 0.$$

In the literature Green's conjecture is sometimes stated in another form; to state it, we need the following definition.

Definition 4.15. A line bundle L on a smooth projective variety X satisfies property (N_p) if

$$K_{i,j}(X, L) = 0 \text{ for all } i \leq p, j \geq 2.$$

Remark 4.16. For curves it is enough to ask for the vanishing of $K_{i,2}(C, L)$ for all $i \leq p$, since the groups $K_{i,j}(C, L)$ automatically vanish if $j \geq 3$ [**Gre84a**].

By the duality theorem 2.24 we have

$$K_{g-c-1,1}(C, K_C)^\vee \cong K_{c-1,1}(C, K_C; K_C) = K_{c-1,2}(C, K_C)$$

for all c. Hence Green's conjecture is equivalent to the following statement.

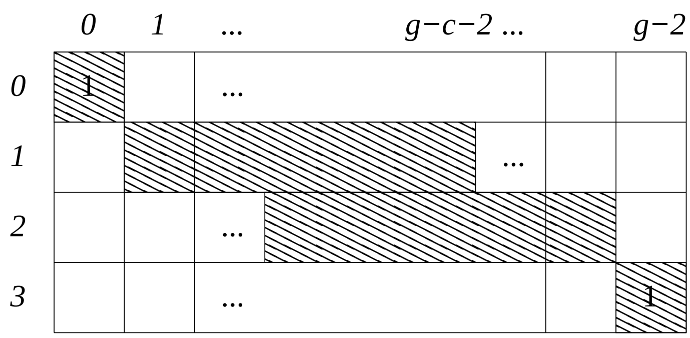
Conjecture 4.17 (Green's conjecture, version II).

$$K_C \text{ satisfies property } (N_p) \iff \mathrm{Cliff}(C) > p.$$

Remark 4.18. If Green's conjecture holds, the Clifford index is computed by the formula

$$\mathrm{Cliff}(C) = \min\{p \mid K_C \text{ does not satisfy } (N_p)\}.$$

Remark 4.19. Put $c = \mathrm{Cliff}(C)$. The Green conjecture predicts the following shape of the Betti table of a canonical curve C, see Definition 1.15:



The only non-zero entries in the Betti table are in in shaded region. Note that this table is symmetric with respect its center; apply Theorem 2.24. For canonical curves of odd genus and maximal Clifford index, the minimal resolution must be *pure*, i.e. on each column there is at most one non-zero entry.

4.3.2. The Green–Lazarsfeld conjecture. Recall that the *gonality* of a curve C is the minimal number k such that C carries a pencil of degree k.

Proposition 4.20. *If C carries a linear system g^1_k, then*

$$K_{p,1}(C, L) \neq 0 \textit{ for all } p \leq r(L) - k$$

for every line bundle L such that $\deg(L) \gg 0$.

Proof: Given a pencil $|M|$ of degree k, put $L_1 = M$ and $L_2 = L \otimes M^{-1}$. If $\deg(L) \gg 0$ then L_2 is nonspecial and $r_2 = r(L) - k$ by Serre's vanishing theorem and Riemann–Roch. The statement then follows from Theorem 3.33. □

The strongest nonvanishing statement of this form is

$$p \leq r(L) - \mathrm{gon}(C) \Longrightarrow K_{p,1}(C,L) \neq 0. \tag{4.6}$$

Again one could ask whether the converse holds.

CONJECTURE 4.21 (Green–Lazarsfeld). Let C be a smooth projective curve. For every line bundle L on C of sufficiently large degree we have

$$K_{p,1}(C,L) = 0 \Longleftrightarrow p \geq r(L) - \mathrm{gon}(C) + 1.$$

REMARK 4.22. In the statement of Proposition 4.20 it suffices to assume that

$$\deg(L) \geq 2g + k - 1$$

or, more generally, that L is k–spanned in the sense of Beltrametti–Sommese (see Definition 6.6). At present it is not clear what conditions should be imposed on $\deg(L)$ for the converse statement.

Again there is an equivalent version of this conjecture, which is stated using the following definition.

DEFINITION 4.23. Let L be a line bundle on a smooth projective curve C. We say that L satisfies property (M_k) if

$$K_{p,1}(C,L) = 0 \text{ for all } p \geq r(L) - k.$$

CONJECTURE 4.24 (Green–Lazarsfeld, version II). For every line bundle L such that $\deg(L) \gg 0$ we have

$$L \text{ satisfies } (M_k) \Longleftrightarrow \mathrm{gon}(C) > k.$$

REMARK 4.25. If the Green–Lazarsfeld conjecture holds, the gonality of a smooth projective curve C is computed by the formula

$$\mathrm{gon}(C) = \min\{k \mid \ (M_k) \text{ fails for every line bundle } L \text{ such that } \deg(L) \gg 0 \ \}.$$

REMARK 4.26. In terms of the property (M_k), Green's conjecture is equivalent to the statement

$$K_C \text{ satisfies } (M_k) \Longleftrightarrow \mathrm{Cliff}(C) \geq k.$$

Hence if Green's conjecture is valid, the Clifford index can be computed by the formula

$$\mathrm{Cliff}(C) = \max\{k \mid K_C \text{ satisfies } (M_k)\}.$$

Even though the statement of the Green-Lazarsfeld conjecture is about *any* line bundle of large degree, one can actually reduce to the case of *one* suitably chosen line bundle.

THEOREM 4.27. *If L is a nonspecial line bundle on a smooth curve C, which satisfies $K_{p,1}(C,L) = 0$, for an integer $p \geq 1$, then, for any effective divisor E of degree $e \geq 1$, one has $K_{p+e,1}(C, L(E)) = 0$.*

Proof: Using induction, we reduce to the case $e=1$; remark that if L is nonspecial, so is $L(E)$ for any effective divisor E.

Consider $x_0 \in C$ a point, and suppose $K_{p+1,1}(C, L\otimes \mathcal{O}_C(x_0)) \neq 0$. Define the following open subset of C

$$U = \{x \in C,\ H^1(C, L\otimes \mathcal{O}_C(x_0 - x)) = 0\};$$

it is non-empty, as $x_0 \in U$.

Proposition 2.14 shows that for x a generic point of C, we have

$$K_{p,1}(C, L\otimes \mathcal{O}_C(x_0 - x)) \neq 0.$$

By semi-continuity, Proposition 1.29, and irreducibility of U, we conclude that for any $x \in U$, $K_{p,1}(C, L\otimes \mathcal{O}_C(x_0 - x)) \neq 0$. In particular, for $x = x_0$, we obtain $K_{p,1}(C, L) \neq 0$, contradiction. □

Corollary 4.28. *Suppose that there exists a non-special line bundle L on X such that*

$$K_{h^0(L)-\mathrm{gon}(C),1}(C, L) = 0.$$

Then the Green-Lazarsfeld conjecture holds for X.

4.4. Generalizations of the Green conjecture.

In the sequel, we work on a curve C of genus $g \geq 1$. An obvious generalization of the Green conjecture (see conjecture 4.17) is the following.

Conjecture 4.29. Let $p \geq 0$ be an integer number. Any special very ample line bundle L on the curve C with

$$\deg(L) \geq 2g - 1 + p - \mathrm{Cliff}(C)$$

satisfies property (N_p).

Observe that the inequality in the hypothesis implies $h^1(C, L) = 1$. This will be explained in greater generality in Remark 4.32. For non-hyperelliptic curves, the Green conjecture is a special case of Conjecture 4.29, obtained for $L = K_C$. The two conjectures are actually equivalent, [**KS89**].

Proposition 4.30. *Green's conjecture implies Conjecture 4.29.*

Proof: Using the Duality Theorem 2.24, we obtain

$$K_{i,j}(C, L)^\vee \cong K_{h^0(L)-2-i,2-j}(C, K_C, L).$$

Note that $L^{\otimes n}$ is of degree larger than $2g$ for $n \geq 2$, hence $h^0(K_X \otimes L^{\otimes q}) = 0$ for $q \leq -2$. It implies that

$$K_{p,q}(C, K_C, L) = 0,$$

for $q \leq -2$. Property (N_p) for L will then be equivalent to

$$K_{h^0(L)-2-i,2-j}(C, K_C, L) \text{ for } i \leq p \text{ and } j = 2, \text{ or } j = 3. \tag{4.7}$$

By hypothesis, there exists a non-zero generator of $H^0(C, K_C \otimes L^\vee)$. Using the resulting identification $H^0(C, L)$ with a subspace $W \subset H^0(C, K_C)$, and applying the definition of Koszul cohomology, we obtain canonically induced isomorphisms

$$K_{p,q}(C, K_C, L) \cong K_{p,q+1}(C, K_C, W),$$

for $q = -1$ or $q = 0$. Since $K_{p,q+1}(C, K_C, W) \subset K_{p,q+1}(C, K_C)$ for $q = -1$ or $q = 0$, the condition (4.7) will be implied by

$$K_{h^0(L)-2-i,2-j}(C, K_C) \text{ for } i \leq p \text{ and } j = 1, \text{ or } j = 2.$$

For $j = 2$, the vanishing is straightforward. For $j = 1$ we use the Green conjecture, which predicts that

$$K_{p,1}(C, K_C) = 0, \text{ for all } p \geq g - 1 - \mathrm{Cliff}(C).$$

Simply note that

$$h^0(L) - 2 - i \geq h^0(L) - (p + 2) = \deg(L) - g - p \geq g - 1 - \mathrm{Cliff}(C)$$

to conclude. □

The following more sophisticated generalization of the Green conjecture was introduced by Green and Lazarsfeld; see [**La89**] for a discussion of the origins of this conjecture.

Conjecture 4.31 (Green-Lazarsfeld [**GL86**]). Let L be a very ample line bundle on the curve C such that

$$\deg(L) \geq 2g + 1 + p - 2h^1(L) - \mathrm{Cliff}(C), \tag{4.8}$$

where $p \geq 0$. Then L satisfies property (N_p) unless L embeds C with a $(p+2)$-secant p-plane.

Let us make a few remarks on the conditions appearing in the statement of Conjecture 4.31.

Remark 4.32. A very ample line bundle L satisfying (4.8) will necessarily have $h^1(L) \leq 1$. Indeed, if $h^1(L) \geq 2$, then L contributes to the Clifford index of C, hence $\mathrm{Cliff}(L) \geq \mathrm{Cliff}(C)$, i.e.

$$g + 1 - h^0(L) - h^1(L) \geq \mathrm{Cliff}(C).$$

Applying Riemann-Roch and replacing $h^0(L)$ by $\deg(L) + 1 + h^1(L) - g$, we obtain

$$2g - 2h^1(L) - \mathrm{Cliff}(C) \geq \deg(L) \geq 2g + p + 1 - 2h^1(L) - \mathrm{Cliff}(C).$$

This is clearly impossible.

The condition (4.8) will become either

$$\deg(L) \geq 2g - 1 + p - \mathrm{Cliff}(C),$$

or

$$\deg(L) \geq 2g + 1 + p - \mathrm{Cliff}(C),$$

according to whether L is special or not.

Remark 4.33. The image of C in $\mathbb{P}H^0(L)^\vee$ has a $(p+2)$-secant p-plane if and only if there exists an effective divisor D of degree $(p + 2)$ such that

$$h^0(L(-D)) \geq h^0(L) - (p + 1),$$

the p-plane in question being contained in $(H^0(L)/H^0(L(-D))^\vee$. Hence having no secant $(p + 2)$-secant p-plane means that for any effective divisor D of degree $\leq p + 2$, $h^0(L(-D)) \leq h^0(L) - \deg(D)$. If $p = 0$, this means that L is very ample.

The next result shows that the condition on secant planes is automatically satisfied for special bundles. In fact, we can prove a little bit more.

Proposition 4.34. *Suppose that L is a special line bundle on C with* $\deg(L) \geq 2g+1+p-\mathrm{gon}(C)$*. Then L cannot embed C with a $(p+2)$-secant p-plane.*

Proof: Write $L = K_C(-E)$ with E an effective divisor on C of degree at most $\mathrm{gon}(C)-p-3$; in particular, we have $\mathrm{gon}(C) \geq p+3$. Suppose that there exists another effective divisor D such that $\deg(D) = p+2$ and $h^0(L(-D)) \geq h^0(L) - (p+1)$. The sum $D+E$ is of degree at most $\mathrm{gon}(C)-1$, and applying the definition of the gonality, we obtain $h^0(\mathcal{O}(D+E)) \leq 1$. Then

$$\deg(L(-D)) + 1 - g = h^0(K_C(-D-E)) - h^0(\mathcal{O}_C(D+E)) \geq$$

$$\geq h^0(L) - (p+1) - h^0(\mathcal{O}_C(D+E)) \geq h^0(L) - (p+2),$$

which implies, using $\deg(D) = p+2$

$$h^0(L) - h^1(L) - (p+2) \geq h^0(L) - (p+2),$$

contradicting the fact that L was special. □

Using the inequality $\mathrm{Cliff}(C) \leq \mathrm{gon}(C)-2$, it follows directly from Proposition 4.34 that if L is a special line bundle on C with $\deg(L) \geq 2g-1+p-\mathrm{Cliff}(C)$ i.e. satisfying condition (4.8), then L cannot embed C with a $(p+2)$-secant p-plane, see [**KS89**], Corollary 3.4. Applying Remark 4.32 and Proposition 4.30, we see that the case of special bundles in conjecture 4.31 reduces to the Green conjecture.

For non-special bundles, Conjecture 4.31 becomes

Conjecture 4.35. Let $p \geq 0$ be an integer number. Any non-special very ample line bundle L on the curve C with

$$\deg(L) \geq 2g+1+p-\mathrm{Cliff}(C),$$

satisfies property (N_p), unless L embeds C with a $(p+2)$-secant p-plane.

As in the case of Green's conjecture, one can state a generic version of Conjecture 4.35, in which case $\mathrm{Cliff}(C)$ is replaced by $[(g-3)/2]$.

For the moment, one can prove one direction in Conjecture 4.35; this can be regarded as an analogue of the Green-Lazarsfeld non-vanishing Theorem.

Theorem 4.36 (Koh-Stillman [**KS89**], Green-Lazarsfeld [**GL86**]). *Let L be a non-special line bundle on C with* $\deg(L) \geq 2g+1+p-\mathrm{Cliff}(C)$*. If L embeds C with a $(p+2)$-secant p-plane, then L does not satisfy property (N_p).*

Proof: Applying Remark 4.33, we can find an effective divisor E of degree $\leq p+1$ and a point $p \in C$ ($E+p$ will be a subdivisor of the divisor D in question) such that $L(-E)$ is non-special, and $L(-E-p)$ is special. Hence $h^0(L(-E-p)) = 1$. Put $i = \deg(D) - 1 \leq p$. We prove that

$$K_{h^0(L)-(i+2),0}(C, K_C, L) \neq 0;$$

by the duality Theorem 2.24, it will imply $K_{i,2}(C,L) \neq 0$, which will finish the proof. Note that $K_{h^0(L)-(i+2),0}(C, K_C, L)$ is isomorphic to the kernel of Koszul map, as L is nonspecial, hence we need to construct a non-zero element in the kernel of the Koszul differential defined on

$$\bigwedge^{h^0(L)-(i+2)} H^0(L) \otimes H^0(K_C).$$

Consider a generator t of $H^1(C, L(-D-p))^\vee \cong H^0(C, K_C(D+p)\otimes L^\vee)$, and a non-zero section s of $\mathcal{O}_C(D+p)$ vanishing along $D+p$. We obtain an embedding

$$\bigwedge^{h^0(L)-(i+2)} H^0(L(-D-p))\otimes H^0(L(-D-p)) \overset{s\otimes t}{\hookrightarrow} \bigwedge^{h^0(L)-(i+2)} H^0(L)\otimes H^0(K_C)$$

Since $h^0(L(-D-p)) = h^0(L)-(p+1)$, we have

$$\bigwedge^{h^0(L)-(i+1)} H^0(L(-D-p)) \cong \mathbb{C}.$$

The Koszul differential maps the generator of $\bigwedge^{h^0(L)-(i+1)} H^0(L(-D-p))$ to a non-zero element of

$$\bigwedge^{h^0(L)-(i+2)} H^0(L(-D-p))\otimes H^0(L(-D-p))$$

which is the element we were looking for. $\square$

REMARK 4.37. In the case of a line bundle L of degree $2g+p$, C is embedded with a $(p+2)$-secant p-plane if and only if $h^0(L\otimes K_C^\vee)\neq 0$. Indeed, let us suppose that there exists an effective divisor D of degree $(p+2)$ such that $h^0(L(-D)) \geq h^0(L)-(p+1)$. By Riemann-Roch we have $h^0(L) = g+p+1$, hence $h^0(L(-D)) \geq g$. Again by Riemann-Roch, using $\deg(L(-D)) = 2g-2$, we obtain $h^0(L(-D)) - h^1(L(-D)) = g-1$, implying $h^1(L(-D)) = 1$. It follows that $L(-D) = K_C$, and thus $h^0(L\otimes K_C^\vee) = h^0(\mathcal{O}_C(D)) \neq 0$.

4.5. Notes and comments

In the course of this book, we worked exclusively over the complex number field. Many results quoted here hold in full generality. The Green conjecture, however, is only valid in characteristic zero. In positive characteristic, there are several counter-examples; see [**Sch03**] for a thorough discussion on the subject.

There are several (stronger) versions of the Green conjecture, that imply the original statement. The so-called *geometric Green conjecture* predicts that the Koszul cohomology groups of a canonical curve C are generated by the Green-Lazarsfeld classes; see [**vB07a**] and [**vB07b**] for a discussion and some evidence. Another reformulation was made by Paranjape and Ramanan [**PR88**]; it states that for all $p \leq \mathrm{Cliff}(C)$ the spaces of global sections of all the bundles $\bigwedge^p Q$ are generated by the locally decomposable sections; Q denotes here the dual of the kernel bundle, see section 2.1. Note that the problem to decide whether or not locally decomposable sections generate all the sections of an wedge product is of interest in its own right; Eusen and Schreyer found examples of non-canonical curves where this question has a negative answer; see [**ES94**].

Green and Lazarsfeld observed that normal generation of line bundles can be read off from Koszul cohomology, [**Gre84a**], [**La89**]. Specifically, a globally generated ample line bundle is normally generated if and only if it satisfies property (N_0). Along these lines, the formulation of the Green conjecture in terms of the (N_p) property is very natural. For example, the first case, $p=0$ translates to the following statement: a canonical curve C is projectively normal unless C is hyperelliptic. This classical result was proved by M. Noether [**Noe80**]. Similarly, the next case $p=1$ predicts that the ideal of a non-hyperelliptic canonical curve is generated by quadrics unless the curve is trigonal or plane quintic (in which case the genus is 6). This statement is precisely the Enriques-Petri Theorem [**En19**], [**Pe23**]. The case $p = 2$ of the conjecture indicates that the ideal of a curve is generated by quadrics and the relations between generators are linear, except for the cases: gonality ≤ 4, plane sextic, or curve of genus 9 with a g^3_8. This fact was independently proved by F.-O. Schreyer [**Sch91**] Theorem 4.1 and C. Voisin [**V88a**] (for genus at least 11). Schreyer described the possible Betti diagrams, whereas Voisin used kernel bundles. Kernel bundle techniques were successfully used also by M. Teixidor to prove the Green-Lazarsfeld conjecture for curves of small gonality, see [**Tei07**].

Several other special cases where the Green conjecture is known to be verified are: plane curves [**Lo89**], curves on Hirzebruch surfaces [**Ap02**], curves of genus ≤ 8, [**Sch86**]. There are similar results for the Green-Lazarsfeld conjecture. The most significant cases will be discussed in the next chapters.

CHAPTER 5

Koszul cohomology and the Hilbert scheme

5.1. Voisin's description

LEMMA 5.1. *Let X be a smooth, complex projective variety, let E and F be vector bundles on X with* $\operatorname{rank}(E) = \operatorname{rank}(F)$ *and let $\varphi : E \to F$ be an injective homomorphism of vector bundles such that $D = \operatorname{Supp}(\operatorname{coker} f)$ is a divisor. We have*

$$\det E \cong \det F \otimes \mathcal{O}_X(-D).$$

Proof: Taking determinants, we obtain an injective map $\det E \xrightarrow{\det \varphi} \det F$ whose cokernel is isomorphic to $\det F \otimes \mathcal{O}_D$. Hence the assertion follows. □

Let $X^{[n]}$ be the Hilbert scheme parametrising zero–dimensional length n subschemes of X. Recall that a zero–dimensional subscheme $\xi \subset X$ is called *curvilinear* if for all $x \in X$ there exists a smooth curve $C \subset X$ such that ξ_x is contained in C. Equivalently, this means that $\mathcal{O}_{\xi,x} \cong \mathbb{C}[t]/(t^\ell)$, $\ell = \ell(\xi_x)$ or $\dim T_x\xi \leq 1$. Let $X^{[n]}_{\mathrm{curv}}$ be the open subscheme parametrising curvilinear length n subschemes. Let

$$\Xi_n \subset X^{[n]}_{\mathrm{curv}} \times X$$

be the incidence subscheme. It fits into a diagram

$$\begin{array}{ccc} \Xi_n & \xrightarrow{q} & X^{[n]}_{\mathrm{curv}} \\ \Big\downarrow{\scriptstyle p} & & \\ X. & & \end{array}$$

Let L be a line bundle on X. As q is a flat morphism of degree n, the sheaf

$$L^{[n]} = q_* p^* L$$

is locally free of rank n. Its fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$ is $H^0(\xi, L \otimes \mathcal{O}_\xi)$.

There is a natural evaluation map

$$\mathrm{ev}_n : H^0(X, L) \otimes \mathcal{O}_{X^{[n]}_{\mathrm{curv}}} \to L^{[n]}.$$

On the fiber over $\xi \in X^{[n]}_{\mathrm{curv}}$, this map is given by $s \mapsto s|_\xi$.

The following result appears in several places; cf. [**EGL01**], [**V02**].

LEMMA 5.2. *We have*

$$H^0(X^{[n]}_{\mathrm{curv}}, \det L^{[n]}) \cong \bigwedge^n H^0(X, L).$$

Proof: Let $X^{(n)}$ be the n–fold symmetric product of X. Let $\pi : X^n \to X^{(n)}$ be the quotient map, and let $\rho : X^{[n]}_{\mathrm{curv}} \to X^{(n)}$ be the Hilbert–Chow morphism. Consider the open subset $X^{(n)}_* \subset X^{(n)}$ of zero–cycles of degree n whose support consists of at least $n-1$ points, and put

$$X^{[n]}_* = \rho^{-1}(X^{(n)}_*), \quad X^n_* = \pi^{-1}(X^{(n)}_*), \quad B^n_* = X^{[n]}_* \times_{X^{(n)}_*} X^n_*.$$

Put $\Delta_{i,j} = \{x_i = x_j\} \subset X^n_*$, $\Delta = \cup_{i,j}\Delta_{i,j}$. The scheme B^n_* is the blow-up of X^n_* along Δ, and $X^{[n]}_*$ is the quotient of B^n_* by the action of the symmetric group S_n [**Fo73**, Lemma 4.4]. Consider the commutative diagram

$$\begin{array}{ccc} B^n_* & \xrightarrow{q} & X^n_* \\ \Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \pi} \\ X^{[n]}_* & \xrightarrow{\rho} & X^{(n)}_*. \end{array}$$

There is a natural homomorphism

$$\varphi : p^* L^{[n]} \to q^*(\textstyle\bigoplus_i p_i^* L)$$

whose restriction to the fiber over a point $\underline{x} \in B^n_*$ with $p(\underline{x}) = \xi$, $q(\underline{x}) = (x_1, \ldots, x_n)$ is the restriction map

$$H^0(\xi, L \otimes \mathcal{O}_\xi) \to \textstyle\bigoplus_i H^0(x_i, L \otimes \mathcal{O}_{x_i}).$$

The homomorphism φ is injective (since its restriction to the open subset corresponding to subschemes consisting of n distinct points is obviously injective), and its cokernel is supported on the exceptional divisor $E = q^{-1}(\Delta)$. Hence

$$p^* \det L^{[n]} \cong q^* L^{\boxtimes n}(-E)$$

by Lemma 5.1. As $\mathrm{codim}\,(X^{[n]}_{\mathrm{curv}} \setminus X^{[n]}_*) \geq 2$ we have

$$\begin{array}{rcl} H^0(X^{[n]}_{\mathrm{curv}}, \det L^{[n]}) & \cong & H^0(X^{[n]}_*, \det L^{[n]}) \\ & \cong & H^0(B^n_*, p^* \det L^{[n]})^{S_n} \\ & \cong & H^0(B^n_*, q^* L^{\boxtimes n}(-E))^{S_n}. \end{array}$$

As the S_n–action on $q^* L^{\boxtimes n}$ is induced by the S_n–action on $\oplus_i p_i^* L$ that permutes the factors and passage to the determinant, it is given by

$$\sigma(p_1^* s_1 \otimes \ldots \otimes p_n^* s_n) = \mathrm{sgn}(\sigma).p_1^* s_{\sigma(1)} \otimes \ldots \otimes p_n^* s_{\sigma(n)}.$$

Hence we obtain an injective map

$$H^0(X^{[n]}_{\mathrm{curv}}, \det L^{[n]}) \cong H^0(B^n_*, q^* L^{\boxtimes n}(-E))^{S_n} \overset{i}{\hookrightarrow} H^0(B^n_*, q^* L^{\boxtimes n})^{S_n} = \textstyle\bigwedge^n H^0(X, L).$$

Conversely, the evaluation map

$$H^0(X, L) \otimes \mathcal{O}_{X^{[n]}_{\mathrm{curv}}} \to L^{[n]}$$

induces a map

$$j : \textstyle\bigwedge^n H^0(X, L) \to H^0(X^{[n]}_{\mathrm{curv}}, \det L^{[n]}).$$

Using the definitions of the maps i and j, one checks that $i \circ j = \mathrm{id}$. Hence i induces an isomorphism

$$H^0(X^{[n]}_{\mathrm{curv}}, \det L^{[n]}) \xrightarrow{\sim} \textstyle\bigwedge^n H^0(X, L).$$

□

REMARK 5.3. The definition of curvilinear subschemes shows that subscheme of length n whose support consists of at least $n-1$ points is curvilinear. Hence $X^{[n]}_{\mathrm{curv}} \supset X^{[n]}_{*}$. This implies that $X^{[n]}_{\mathrm{curv}} \subset X^{[n]}$ is a large open subset, i.e., $\operatorname{codim}(X^{[n]} \setminus X^{[n]}_{\mathrm{curv}}) \geq 2$. Furthermore, $X^{[n]}_{\mathrm{curv}}$ is connected if X is connected, since every curvilinear subscheme of length n is a flat limit of zero–dimensional subschemes whose support consists of n distinct points.

As $X^{[n]}_{\mathrm{curv}}$ is a large open subset of $X^{[n]}$ by the previous remark, the conclusion of Lemma 5.2 also holds on $X^{[n]}$. The reason for working with curvilinear subschemes is that every subscheme of a curvilinear subscheme admits a well–defined residual subscheme. In particular, there exists a map

$$\tau : \Xi_{n+1} \to X^{[n]}_{\mathrm{curv}} \times X$$

that sends $(\xi, x) \in \Xi_n$ to (ξ', x), with ξ' the residual subscheme of x in ξ. Consider the open subset

$$U = \{(\xi, x) \in \Xi_n \mid x \text{ is a simple point of } \xi\}.$$

The map τ contracts the divisor $D_\tau = \Xi_{n+1} \setminus U$ to Ξ_n.

LEMMA 5.4. *There is an injective map*

$$H^0(\Xi_{p+1}, \det L^{[p+1]} \boxtimes L^{q-1}) \hookrightarrow \textstyle\bigwedge^p H^0(X, L) \otimes H^0(X, L^q)$$

whose image is isomorphic to the kernel of the Koszul differential δ. This map fits into a commutative diagram

$$\begin{tikzcd}
H^0(X^{[p+1]}_{\mathrm{curv}} \times X, \det L^{[p+1]} \boxtimes L^{q-1}) \arrow[r] \arrow[d, "\cong"] & H^0(\Xi_{p+1}, \det L^{[p+1]} \boxtimes L^{q-1}|_{\Xi_{p+1}}) \arrow[d, hook] \\
\bigwedge^{p+1} H^0(X, L) \otimes H^0(X, L^{q-1}) \arrow[r, "\delta"] & \bigwedge^p H^0(X, L) \otimes H^0(X, L^q).
\end{tikzcd}$$

Proof: Consider the map

$$\psi : q^* L^{[p+1]} \to \tau^*(L^{[p]} \boxplus L)$$

whose restriction to the fiber over a point ξ with $\tau(\xi) = (\xi', x)$ is given by the map

$$H^0(\xi, L \otimes \mathcal{O}_\xi) \to H^0(\xi', L \otimes \mathcal{O}_{\xi'}) \oplus H^0(x, L \otimes \mathcal{O}_x).$$

The map ψ is injective, and its cokernel is supported on D. Hence

$$q^* \det L^{[p+1]} \cong \tau^*(\det L^{[p]} \boxtimes L)(-D_\tau) \tag{5.1}$$

by Lemma 5.1. Taking the tensor product with $p^* L^{q-1}$ and using the equality $p^* = \tau^* p_1^*$, we obtain

$$\det L^{[p+1]} \boxtimes L^{q-1} \cong \tau^*(\det L^{[p]} \boxtimes L^q)(-D_\tau).$$

Hence we have an isomorphism

$$H^0(\Xi_{p+1}, q^* \det L^{[p+1]}) \cong$$
$$\cong \ker\left(H^0(X^{[p]}_{\mathrm{curv}} \times X, \det L^{[p]} \boxtimes L) \to H^0(\Xi_p, \det L^{[p]} \boxtimes L|_{\Xi_p})\right).$$

Using Lemma 5.2 we obtain the desired injection.

For all m and n the S_n–equivariant map

$$\det L^{[n+1]} \boxtimes L^m \to \tau^*(\det L^{[n]} \boxtimes L^{m+1})$$

is given by

$$s_1 \wedge \ldots \wedge s_{n+1}|_\xi \otimes t \mapsto \sum_i (-1)^i s_1 \wedge \ldots \wedge \widehat{s_i} \wedge \ldots \wedge s_{n+1}|_{\xi'} \otimes t \otimes s_i|_x.$$

Hence the induced map

$$\bigwedge\nolimits^{n+1} H^0(X,L) \otimes H^0(X,L^m) \to \bigwedge\nolimits^n H^0(X,L) \otimes H^0(X,L^{m+1})$$

sends $s_1 \wedge \ldots \wedge s_{n+1}$ to $\sum_i (-1)^i s_1 \wedge \ldots \wedge \widehat{s_i} \wedge \ldots \wedge s_{n+1} \otimes (t \otimes s_i)$. Hence the latter map coincides with the Koszul differential. □

Corollary 5.5. *For all integers p and q, the Koszul cohomology $K_{p,q}(X,L)$ is isomorphic to the cokernel of the restriction map*

$$H^0(X^{[p+1]}_{\mathrm{curv}} \times X, \det L^{[p+1]} \boxtimes L^{q-1}) \to H^0(\Xi_{p+1}, \det L^{[p+1]} \boxtimes L^{q-1}|_{\Xi_{p+1}}).$$

In particular,

$$K_{p,1}(X,L) \cong \operatorname{coker}\left(H^0(X^{[p+1]}_{\mathrm{curv}}, \det L^{[p+1]}) \xrightarrow{q^*} H^0(\Xi_{p+1}, q^* \det L^{[p+1]}|_{\Xi_{p+1}})\right).$$

Remark 5.6. The group $K_{p,q}(X;\mathcal{F},L)$ is obtained by replacing L^{q-1} by $\mathcal{F} \otimes L^{q-1}$ in Corollary 5.5.

5.2. Examples

In this section we consider the case where $X = C$ is a smooth curve. As the Hilbert scheme coincides with the symmetric product in this case, the previous description simplifies.

Lemma 5.7. *We have $\Xi_{p+1} \cong C^{(p)} \times C$. Under this isomorphism the projection map $q : \Xi_{p+1} \to C^{(p+1)}$ corresponds to the addition map $\mu : C^{(p)} \times C \to C^{(p+1)}$.*

Proof: The map $\nu : C^{(p)} \times C \to C^{(p+1)} \times C$ defined by $\nu(\xi, x) = (x + \xi, x)$ induces an isomorphism $C^{(p)} \times C \xrightarrow{\sim} \Xi_{p+1}$ that fits into a commutative diagram

$$\begin{array}{ccc} C^{(p)} \times C & \xrightarrow{\nu} & \Xi_{p+1} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle q} \\ C^{(p+1)} & = & C^{(p+1)}. \end{array}$$

□

The symmetric product $C^{(p)}$ carries two natural divisors. Given a base point $x \in C$, we write $D = D_p = x + C^{(p-1)}$. The choice of a base point defines an Abel–Jacobi map $C^{(p)} \to J(C)$. Let F be the pullback of the theta divisor $\Theta \subset J(C)$ under this map.

Lemma 5.8. *The first Chern class of the tautological bundle $L^{[p]}$ belongs to the subgroup $\langle [D], [F] \rangle \subseteq \mathrm{NS}(C^{(p)})$. More precisely, let g be the genus of C. We have*

$$c_1(L^{[p]}) = [F] + (d - p - g + 1)[D].$$

Proof: This follows from the Grothendieck–Riemann–Roch formula; [**ACGH85**, p. 340, Lemma 2.5]. □

LEMMA 5.9. *We have* $q^*D_{p+1} = C^{(p)} \times \{x\} + D_p \times C$.

Proof: This follows immediately from the definition of D and Lemma 5.7. $\square$

5.2.1. Rational normal curves. It is classically known how to compute the numbers $\kappa_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d))$ using the Eagon–Northcott complex; see e.g. [**Sch86**]. As an example, we calculate these numbers using Voisin's method.

PROPOSITION 5.10. *We have*

$$\kappa_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d)) = p.\binom{d}{p+1}.$$

Proof: Put $C = \mathbb{P}^1$, $L = \mathcal{O}_{\mathbb{P}^1}(d)$. Note that $C^{(p)} \cong \mathbb{P}^p$ for all p. Lemma 5.8 implies that

$$\det(L^{[p+1]}) \cong \mathcal{O}_{\mathbb{P}^{p+1}}(d-p).$$

Hence $q^* \det L^{[p+1]} \cong \mathcal{O}_{\mathbb{P}^{p+1}\times\mathbb{P}^1}(d-p, d-p)$ by Lemma 5.9. By Corollary 5.5 we obtain

$$\begin{aligned} \kappa_{p,1}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}}(d)) &= (d-p+1)\binom{d}{p} - \binom{d+1}{p+1} \\ &= \frac{d!}{(p+1)!(d-p-1)!}.\frac{(p+1)(d-p-1)-d-1}{d-p} \\ &= p\binom{d}{p+1}. \end{aligned}$$

$\square$

5.2.2. Elliptic normal curves. Let E be an elliptic curve with origin $u \in E$. Put $D = D_p = u + E^{(p-1)}$. As E is isomorphic to its Jacobian, we have the Abel–Jacobi map $\psi : E^{(p)} \to E$ and $F = \psi^{-1}(u)$.

REMARK 5.11. To distinguish between the sum in the symmetric product and the addition on E, we write elements in the symmetric product in the form $(x_1) + \ldots + (x_r)$. With this notation, we have

$$\psi((x_1) + \ldots + (x_r)) = x_1 + \ldots + x_r.$$

PROPOSITION 5.12 (Ciliberto–Catanese). *Let Z be a divisor on $E^{(p)}$ such that $Z \equiv mD + nF$ ($\equiv$ denotes algebraic equivalence). If $m + np > 0$ and $m \geq 0$ then*

(i) $H^i(E^{(p)}, \mathcal{O}_E(Z)) = 0$ *for all* $i > 0$;
(ii) $h^0(E^{(p)}, \mathcal{O}_E(Z)) = \frac{m+np}{p!}\prod_{i=1}^{p-1}(m+i)$.

Proof: See [**CC93**, Thm. (1.17)]. $\square$

Lemma 5.13. *Write*

$$\Gamma = q^*F = \{((y_1) + \ldots + (y_p), x) \mid x + y_1 + \ldots + y_p = u\}.$$

The projection map $E^{(p)} \times E \to E^{(p)}$ induces an isomorphism $\Gamma \cong E^{(p)}$ and $\mathcal{O}_\Gamma(\Gamma) \cong \mathcal{O}_\Gamma$.

Proof: The first assertion follows from the definition of Γ. Using the adjunction formula and the triviality of ω_E we obtain

$$\omega_{E^{(p)}} \cong \omega_\Gamma \cong \omega_{E^{(p)}\times E}(\Gamma) \times \mathcal{O}_\Gamma \cong \omega_{E^{(p)}} \otimes \mathcal{O}_\Gamma(\Gamma).$$

□

Proposition 5.14. *Put $L = \mathcal{O}_E(d.u)$, $d \geq 3$. We have*

$$\kappa_{p,1}(E, L) = d\binom{d-2}{p} - \binom{d}{p+1}.$$

Proof: As $H^0(E^{(p+1)}, \det L^{[p+1]}) \cong \bigwedge^{p+1} H^0(E, L)$ by Lemma 5.2, we obtain

$$h^0(E^{(p+1)}, \det L^{[p+1]}) = \binom{d}{p+1}.$$

By Lemmas 5.8 and 5.9 we have

$$\begin{aligned} \det L^{[p+1]} &\equiv (d-p-1)D + F \\ q^* \det L^{[p+1]} &\equiv (d-p-1)(E \times \{u\} + D_p \times E) + \Gamma. \end{aligned}$$

To calculate $h^0(E^{(p)} \times E, q^* \det L^{[p+1]})$ we consider the exact sequence

$$0 \to q^* \det L^{[p+1]}(-\Gamma) \to q^* \det L^{[p+1]} \to q^* \det L^{[p+1]} \otimes \mathcal{O}_\Gamma \to 0.$$

Using the Künneth formula and Proposition 5.12 we compute

$$\begin{aligned} h^0(E^{(p)} \times E, q^* \det L^{[p+1]}(-\Gamma)) &= (d-p-1)\frac{d-p-1}{p!}(d-p)\ldots(d-2) \\ &= (d-p-1)\binom{d-2}{p}. \end{aligned}$$

To study $q^* \det L^{[p+1]} \otimes \mathcal{O}_\Gamma$, we determine the intersection of $D_p \times \{u\}$ and $D_p \times E$ with Γ. Using the definitions we obtain

$$\begin{aligned} (E^{(p)} \times \{u\}) \cap \Gamma &= \{((y_1 + \ldots + (y_p), u) \mid y_1 + \ldots + y_p = u\} \\ (D_p \times E) \cap \Gamma &= \{((u) + (y_1) + \ldots + (y_{p-1}), x) \mid x + y_1 + \ldots + y_{p-1} = u\}. \end{aligned}$$

Hence $E^{(p)} \times \{u\} \cap \Gamma \cong F$ via projection to the second factor and

$$(D_p \times E) \cap \Gamma \cong \{u\} + E^{(p-1)} = D.$$

Using Lemma 5.13 and Proposition 5.12 we obtain

$$\begin{aligned} h^0(\Gamma, q^* \det L^{[p+1]} \otimes \mathcal{O}_\Gamma) &= h^0(E^{(p)}, \mathcal{O}((d-p-1)D + (d-p-1)F)) \\ &= \frac{d-p-1+p(d-p-1)}{p!}(d-p)\ldots(d-2) \\ &= (p+1)\binom{d-2}{p}. \end{aligned}$$

As $H^1(E^{(p)} \times E, q^* \det L^{[p+1]}(-\Gamma)) = 0$ by the Künneth formula and Proposition 5.12, we find

$$h^0(E^{(p)} \times E, q^* \det L^{[p+1]}) = d\binom{d-2}{p}$$

and the result follows from Corollary 5.5. $\square$

REMARK 5.15. It is known that

$$\kappa_{p,1}(A, L) = p\binom{d-2}{p+1} + (d-p-2)\binom{d-2}{d-p-1},$$

see for example [**HvB04**, Cor. 8.13], with $\kappa_{p,1} = \beta_{-(p+1),2}$. This coincides with our result; a small computation shows that

$$\begin{aligned} d\binom{d-2}{p} - \binom{d}{p+1} &= p\binom{d-2}{p+1} + (d-p-2)\binom{d-2}{d-p-1} \\ &= \frac{(d-2)!p(d-p-2)d}{(d-p-1)!(p+1)!}. \end{aligned}$$

REMARK 5.16. If $L = \mathcal{O}_C(d.x)$, $x \in C$, then $p^*L \cong \mathcal{O}(D_{p+1})^{\otimes d}$. Hence the same method can be used to compute the numbers

$$\kappa_{p,q}(C, L) = \kappa_{p,1}(C, L^{q-1}, L)$$

for all q if $g(C) \leq 1$.

5.2.3. Projection map on Koszul cohomology of curves. Let C be a curve, L be a line bundle on C, p be an integer, and $x \in C$ be a point. In section 2.2 we have defined the projection map

$$K_{p+1,1}(C, L \otimes \mathcal{O}_C(x)) \to K_{p,1}(C, L)$$

Using the description of Koszul cohomology in terms of symmetric products,

$$K_{p,1}(C, L) \cong H^0(\Xi_{p+1}, q^* \det L^{[p+1]}|_{\Xi_{p+1}})/q^* H^0(C^{(p+1)}, \det L^{[p+1]}).$$

the projection map can be interpreted in the following way. Note first that we have the following identifications

$$\det(L \otimes \mathcal{O}_C(x))^{[p+2]} \cong \det L^{[p+2]} \otimes \mathcal{O}_{C^{(p+2)}}(x + C^{(p+1)})$$

and

$$\left(\det(L \otimes \mathcal{O}_C(x))^{[p+2]}\right)|_{x+C^{(p+1)}} \cong \det L^{[p+1]}. \tag{5.2}$$

The last isomorphism is obtained by exterior product with a section of $L \otimes \mathcal{O}_C(x)$ that does not vanish at x.

Denoting $j_x : C^{(p+1)} \to C^{(p+2)}$ the map $\xi \mapsto \xi + x$, we obtain an exact sequence

$$0 \to \det L^{[p+2]} \to \det(L \otimes \mathcal{O}_C(x))^{[p+2]} \to j_{x,*} \det L^{[p+1]} \to 0 \tag{5.3}$$

The above discussion immediately leads to

PROPOSITION 5.17 (Voisin). *The projection map*

$$K_{p+1,1}(C, L \otimes \mathcal{O}_C(x)) \to K_{p,1}(C, L)$$

identifies with the restriction map from

$$H^0(\Xi_{p+2}, q^* \det(L \otimes \mathcal{O}_C(x))^{[p+2]})/q^* H^0(C^{(p+2)}, \det(L \otimes \mathcal{O}_C(x))^{[p+2]})$$

to

$$H^0(\Xi_{p+1}, q^* \det L_{[p+1]})/q^* H^0(\det L^{[p+1]}),$$

where Ξ_{p+1} *is embedded in* Ξ_{p+2} *as a component of* $q^{-1}(x + C^{(p+1)})$.

5.3. Koszul vanishing via base change

Voisin's description of Koszul cohomology (Corollary 5.5) shows that $K_{k,1}(S, L) = 0$ if the map

$$q^* : H^0(S^{[k+1]}_{\mathrm{curv}}, \det L^{[k]}) \to H^0(\Xi_{k+1}, q^* \det L^{[k+1]})$$

is surjective. The surjectivity of this map can be proved via a suitable base change, as explained below.

The general setup is the following: X and Y are complex algebraic varieties, $f : X \to Y$ is a finite (flat) morphism of degree d, and $\mathcal{L}$ is a line bundle on Y. As the map f is finite and flat, there exists a trace map

$$f_* \quad : \quad H^0(X, f^*\mathcal{L}) \to H^0(Y, \mathcal{L})$$

such that

$$f_* \circ f^* = d. \,\mathrm{id}\,.$$

Since we work over the complex numbers, the pullback map

$$f^* : H^0(Y, \mathcal{L}) \to H^0(X, f^*\mathcal{L})$$

is injective. Moreover, the trace map induces a natural splitting

$$H^0(X, f^*\mathcal{L}) \cong f^* H^0(Y, \mathcal{L}) \oplus \mathrm{coker}\,(f^*).$$

Proposition 5.18. *Suppose there exists a cartesian diagram*

$$\begin{array}{ccc} T = U \times_Y X & \xrightarrow{j} & X \\ \big\downarrow g & & \big\downarrow f \\ U & \xrightarrow{i} & Y \end{array}$$

such that

(i) $j^* : H^0(X, f^*\mathcal{L}) \to H^0(T, j^*q^*\mathcal{L})$ *is injective;*
(ii) $g^* : H^0(U, i^*\mathcal{L}) \to H^0(T, g^*i^*\mathcal{L})$ *is surjective.*

Then f^* *is surjective.*

Proof: As T is a fibered product, the map g is also finite and flat of degree d, and the associated trace map

$$g_* \quad : \quad H^0(T, j^*f^*\mathcal{L}) \to H^0(U, i^*\mathcal{L})$$

satisfies the conditions

$$g_* \circ g^* = d. \,\mathrm{id},$$

and

$$i^* \circ f_* = g_* \circ j^*. \tag{5.4}$$

It suffices to show that

$$\alpha = \frac{1}{d} f^* f_* \alpha$$

for all $\alpha \in H^0(X, f^*\mathcal{L})$. To this end, put

$$\widetilde{\alpha} = \alpha - \frac{1}{d} f^* f_* \alpha.$$

As g^* is surjective, there exists $\beta \in H^0(U, i^*\mathcal{L})$ such that $j^*\alpha = g^*\beta$. Hence

$$\begin{aligned} j^*\widetilde{\alpha} &= j^*\alpha - \frac{1}{d} j^* f^* f_* \alpha \\ &= j^*\alpha - \frac{1}{d} g^* i^* f_* \alpha \\ &= j^*\alpha - \frac{1}{d} g^* g_* j^* \alpha \\ &= j^*\alpha - \frac{1}{d} g^* g_* g^* \beta \\ &= j^*\alpha - g^*\beta = 0. \end{aligned}$$

The result then follows from the injectivity of j^*. □

Remark 5.19. The existence of the trace map shows that f^* is injective. Hence the conditions of Proposition 5.18 imply that f^* is an isomorphism.

Remark 5.20. *Explanation of the proof.* The fact that the diagram considered in the statement was cartesian resulted into the relation (5.4) which eventually shows that the induced j^* behaves naturally on the induced splittings

$$H^0(X, f^*\mathcal{L}) \cong f^* H^0(Y, \mathcal{L}) \oplus \operatorname{coker}(f^*)$$

$$H^0(T, j^* f^*\mathcal{L}) \cong g^* H^0(U, i^*\mathcal{L}) \oplus \operatorname{coker}(g^*),$$

that is $f^*H^0(Y, \mathcal{L})$ is mapped to $H^0(T, j^*f^*\mathcal{L})$ and $\operatorname{coker}(f^*)$ is mapped to $\operatorname{coker}(g^*)$. If j^* is injective, we obtain an induced injective map $\operatorname{coker}(f^*) \to \operatorname{coker}(g^*)$. The other hypothesis $\operatorname{coker}(g^*) = 0$ implies then the vanishing of $\operatorname{coker}(f^*)$.

In the sequel we shall need a refinement of Proposition 5.18. The new setup is the following. We start with X and Y two equidimensional complex algebraic varieties, $f : X \to Y$ is a finite (flat) morphism of degree d, and $\mathcal{L}$ is a line bundle on Y. Consider an equidimensional complex variety U that admits a morphism $h : U \to X$. Put $i = f \circ h$, $T = U \times_Y X$, and let $j : T \to X$ be the induced morphism.

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ U \arrow[r, "i"] \arrow[ur, "h"] & Y \end{tikzcd}$$

By the universal property of the fibered product, for the pair of morphisms $\mathrm{id} : U \to U$ and $h : U \to X$, we obtain a section $i_U : U \to T$ of the morphism g. It is not hard to see that i_U is an closed embedding – since f and g are affine morphisms, we can argue locally, and the embedding will be induced by a natural surjective ring morphisms defined on a tensor product of two algebras.

Let V be the Zariski closure of $T \setminus i_U(U)$. By equidimensionality of V, and by the dimension equality of T and U, we can write $T = U \cup V$, with $\dim(U) = \dim(V)$ (the scheme T will be however equidimensional). By finiteness of g, the induced map $g_V : V \to U$ is finite of degree $d - 1$. Denote by D the scheme-theoretical intersection of U and V inside T – it is defined as the fibered product $D = U \times_T V$.

PROPOSITION 5.21. *Notation as above. If*

$$g_V^* : H^0(U, i^*\mathcal{L}) \to H^0(V, g_V^*\mathcal{L})$$

is surjective and the restriction map

$$r_D : H^0(U, i^*\mathcal{L}) \to H^0(D, i^*\mathcal{L}_{|D})$$

is injective, then the map

$$g^* : H^0(U, i^*\mathcal{L}) \to H^0(T, g^*i^*\mathcal{L})$$

is surjective.

Proof: By hypothesis, coker $(g_V^*) = 0$. Hence

$$H^0(V, g_V^* i^*\mathcal{L}) \cong H^0(U, i^*\mathcal{L}).$$

Note that we can identify $H^0(T, j^*f^*\mathcal{L})$ with the subspace

$$\left\{(\alpha, \beta) \in H^0(U, i^*\mathcal{L}) \oplus H^0(V, g_V^* i^*\mathcal{L}) \mid \alpha_{|D} = \beta_{|D}\right\}.$$

Via this identification, the pullback

$$g^* : H^0(U, i^*\mathcal{L}) \to H^0(T, g^*i^*\mathcal{L})$$

coincides with the diagonal map

$$H^0(U, i^*\mathcal{L}) \to H^0(U, i^*\mathcal{L}) \oplus H^0(U, i^*\mathcal{L}) \cong H^0(V, g_V^* i^*\mathcal{L})$$

If (α, β) is a pair of sections of $i^*\mathcal{L}$ whose restrictions over D coincide, i.e. we are given a global section of $g_V^* i^*\mathcal{L}$, by injectivity of r_D we obtain $\alpha = \beta$. This proves that (α, β) is in the image of the diagonal map, which identified with g^*. □

REMARK 5.22. With some extra work, one can prove that the converse is also true.

COROLLARY 5.23. *Notation as above. If*

(i) g_V^* *is surjective;*
(ii) r_D *is injective;*
(iii) h^* *is injective*

then f^* *is surjective.*

Proof: Applying Proposition 5.21, it follows that g^* is an isomorphism. Hence injectivity of h^* and of j^* are equivalent. We then apply Proposition 5.18 to conclude. □

CHAPTER 6

Koszul cohomology of a $K3$ surface

This chapter is devoted to the discussion of two beautiful results due to C. Voisin [**V02**], [**V05**], which imply the generic Green conjecture.

Sections 6.1 and 6.2 contain some preparatory material for these results. Specifically, we recall the Brill-Noether theory of curves on $K3$ surfaces.

In section 6.3 we discuss the even genus case. We explain how the proof can be reduced to a number of cohomological calculations, which shall not be carried out in detail here.

In the last section we give a short outline of the proof of the generic Green conjecture in the odd genus case; it proceeds along similar lines, but involves additional technical complications for which we refer to Voisin's paper [**V05**].

6.1. The Serre construction, and vector bundles on K3 surfaces

The Serre construction provides an effective method to construct a rank-2 bundle starting from a locally complete intersection (lci) 0-dimensional subscheme of a surface. Throughout this section, we fix the following data:

- a $K3$ surface S;
- a 0-dimensional lci subscheme $\xi \subset S$;
- a line bundle L on S;
- a non-zero element $t \in H^1(S, L \otimes I_\xi)^*$.

By the Grothendieck-Serre duality Theorem, we have

$$H^1(S, L \otimes I_\xi)^* \cong \mathrm{Ext}^1(L \otimes I_\xi, \mathcal{O}_S),$$

hence to each $t \in H^1(S, L \otimes I_\xi)^*$ we may associate a sheaf $\mathcal{E}$ is given by an extension

$$0 \to \mathcal{O}_S \to \mathcal{E} \to L \otimes I_\xi \to 0.$$

Remark that the global section s of $\mathcal{E}$ coming from the inclusion $\mathcal{O}_S \hookrightarrow \mathcal{E}$ vanishes on ξ.

The precise criterion for an extension as above to be locally free is the following:

Theorem 6.1 ([**GH78**]). *There exist a rank two vector bundle E on S with $\det(E) = L$, and a section $s \in H^0(S, E)$ such that $V(s) = \xi$ if and only if every section of L vanishing at all but one of the points in the support of ξ also vanishes at the remaining point.*

An immediate consequence of Theorem 6.1 is the following result.

Corollary 6.2. *Let S be a $K3$ surface, and L a line bundle on S. Then, for any 0-dimensional subscheme ξ of S, such that $h^0(S, L \otimes I_{\xi'}) = 0$, for all $\xi' \subset \xi$ with $\lg(\xi') = \lg(\xi) - 1$, there exists a rank two vector bundle E on S given by an extension*

$$0 \to \mathcal{O}_S \to E \to L \otimes I_\xi \to 0.$$

6.2. Brill-Noether theory of curves on K3 surfaces

For curves suitably embedded in a surface with special geometry, the Brill-Noether theory will be determined by objects globally defined on the surface. We shall discuss in this section curves on $K3$ surfaces. The interest for this situation is extremely high, by reasons that we will try to explain below. To mention one of them, examples of curves on $K3$ surfaces of any Clifford dimension r were constructed in [**ELMS89**], and for large r these are the only concrete examples known so far.

Let S be a (smooth, projective) $K3$ surface, L a globally generated line bundle on S. To any pair (C, A) with and C a smooth curve in the linear system $|L|$, and A a base-point-free line bundle in $W^r_d(C) \setminus W^{r+1}_d(C)$ one associates a vector bundle $E := E(C, A)$ of rank $r+1$ on S, called the *Lazarsfeld-Mukai bundle*; cf. [**La86**] and [**M89**]. This is done in the following way. Define the rank-$(r+1)$ vector bundle $F(C, A)$ as the kernel of the evaluation of sections of A

$$0 \to F(C, A) \to H^0(C, A) \otimes \mathcal{O}_S \stackrel{ev}{\to} A \to 0. \tag{6.1}$$

Dualizing the sequence (6.1) and setting $E := E(C, A) := F(C, A)^\vee$ we obtain the short exact sequence

$$0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E \to K_C \otimes A^\vee \to 0. \tag{6.2}$$

In other words, the bundle E is obtained from the trivial bundle by modifying it along the curve C. The properties of E are summarized in the following:

PROPOSITION 6.3. *One has*

1. $\det(E) = L$;
2. $c_2(E) = d$;
3. $h^0(S, E) = h^0(C, A) + h^1(C, A) = 2r - d + 1 + g(C)$;
4. $h^1(S, E) = h^2(S, E) = 0$;
5. $\chi(S, E \otimes E^\vee) = 2(1 - \rho(g, r, d))$, *where* $g = g(C)$.
6. *E is globally generated outside the base locus of* $K_C \otimes A^\vee$.

Conversely, if E is a rank-$(r+1)$ bundle that is generated outside a finite set, and if $\det(E) = L$, then there is a natural rational map from the Grassmanniann of $(r+1)$-dimensional subspaces of $H^0(S, E)$ to the linear system $|L|$:

$$d_E : \mathrm{Gr}(r+1, H^0(S, E)) \dashrightarrow |L|. \tag{6.3}$$

A generic subspace $\Lambda \in \mathrm{Gr}(r+1, H^0(S, E))$ is mapped to the degeneracy locus of the evaluation map:

$$ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E;$$

note that, generically, this degeneracy locus cannot be the whole surface, since E is generated outside a finite set. For a generic element Λ in the Grassmannian, the image $d_E(\Lambda)$ is a smooth curve C, and the cokernel of ev_Λ is a line bundle $K_C \otimes A^\vee$ of C, where $\deg(A) = c_2(E)$.

Coming back to the original situation $C \in |L|$, A a base-point-free line bundle in $W^r_d(C) \setminus W^{r+1}_d(C)$, and E the associated Lazarsfeld-Mukai bundle, one can prove that the multiplication map

$$\mu_{0,A} : H^0(C, A) \otimes H^0(C, K_C \otimes A^\vee) \to H^0(C, K_C),$$

which plays a key role in the Brill-Noether theory, coincides to the differential of d_E at the point $\Lambda = H^0(C, A)^\vee \subset H^0(C, E)$. Moreover, its kernel can be described explicitly as follows. Put M_A the rank-r vector bundle on C defined by the kernel of the evaluation map of global sections

$$0 \to M_A \to H^0(C, A) \otimes \mathcal{O}_C \xrightarrow{ev} A \to 0. \tag{6.4}$$

Twisting (6.4) with $K_C \otimes A^\vee$, we obtain the following identification:

$$\ker(\mu_{0,A}) = H^0(C, M_A \otimes K_C \otimes A^\vee).$$

Note that there is a natural surjective map from $F(C, A)_{|C}$ to M_A, and, by determinant reason, we have

$$0 \to A \otimes K_C^\vee \to F(C, A)_{|C} \to M_A \to 0. \tag{6.5}$$

The Lazarsfeld-Mukai bundles have proved to be extremely useful in a number of problems concerning curves on $K3$ surfaces. They are key objects, for instance, in the proof of the following modified version of the Harris-Mumford conjecture:

Theorem 6.4 (Green-Lazarsfeld). *For any smooth irreducible curve $D \in |C|$ we have $\mathrm{Cliff}(D) = \mathrm{Cliff}(C)$. Moreover, if $\mathrm{Cliff}(C)$ is different from the maximal value $[(g-1)/2]$, then there exists a line bundle L on S with $h^0(S, L), h^1(S, L) \geq 2$, whose restriction to any smooth $D \in |C|$ computes the Clifford index of D.*

Lazarsfeld's original motivation for considering the bundles $E(C, A)$ was to use them for proving the following cornerstone result:

Theorem 6.5 (Lazarsfeld). *Let C be a smooth curve of genus $g \geq 2$ on a $K3$ surface S, and assume that any divisor in the linear system $|C|$ is smooth and irreducible. Then a generic element in the linear system $|C|$ is Brill-Noether-Petri generic.*

The following special case will be used in Chapter 6. Consider S a $K3$ surface with cyclic Picard group generated by an ample line bundle L, and suppose that $L^2 = 4k - 2$ with k a positive integer. By Theorem 6.5, a generic smooth curve $C \in |L|$ is Brill-Noether-Petri generic, hence it has gonality $k + 1$. Moreover, since the Brill-Noether number is zero, a generic curve C will carry finitely many g^1_{k+1}'s.

Put $E = E(C, A)$ the Lazarsfeld-Mukai bundle, where A is a g^1_{k+1} on C. From Proposition 6.3, it follows that $c_1(E) = L$, $h^0(S, E) = k + 2$, and E is globally generated. The exact sequence (6.2) shows that $h^0(S, E(-L)) = 0$. Since the Picard group is cyclic, this vanishing proves the stability of E. A remarkable fact is that E does not depend on the choice of the pair (C, A). Indeed, if E' is another Lazarfeld-Mukai bundle associated to a pair as above, using $\chi(E, E') = 2$ [**La86**], we conclude that either $\mathrm{Hom}(E, E') \neq 0$ or $\mathrm{Hom}(E', E) \neq 0$. By stability, the two bundles are isomorphic. This shows that the bundle E is rigid (this fact is not surprising: a dimension calculation shows that the moduli space of stable bundles with given invariants is zero). An alternate way to construct the bundle E is via Serre's construction. Note that a zero-dimensional subscheme ξ of length $k + 1$

supported on a smooth curve C is associated to a g^1_{k+1} if and only if the restriction map

$$r_\xi : H^0(S,L) \to H^0(\xi, L|_\xi)$$

is not surjective. By Brill-Noether-Petri genericity, the corank of r_ξ is 1 in this case, and for any subscheme $\xi' \subsetneq \xi$, the map $r_{\xi'}$ is surjective. Applying Theorem 6.1, we obtain a rank 2 bundle E on S given by an extension

$$0 \to \mathcal{O}_S \to E \to L \otimes \mathcal{I}_\xi \to 0.$$

One immediately proves the stability of this bundle and, arguing as before, we conclude that it is isomorphic to the unique Lazarsfeld-Mukai bundle.

The determinant map d_E, see (6.3), is defined everywhere. Indeed, if d_E is not defined at a point in $G(2, H^0(E))$ corresponding to a two dimensional space $\Lambda \subset H^0(S,E)$, then $\mathrm{im}(\mathrm{ev}_\Lambda)$ is of rank one. This contradicts the stability of E.

Definition 6.6 (Beltrametti–Sommese). A line bundle L on S is called *k–very ample* if the restriction map

$$r_\xi : H^0(S,L) \to H^0(\xi, L \otimes \mathcal{O}_\xi)$$

is surjective for all $\xi \in S^{[k+1]}$. If r_ξ is surjective for all $\xi \in S^{[k+1]}_{\mathrm{curv}}$ we say that L is *k–spanned.*

Remark 6.7. Note that L is 0–very ample if and only if L is globally generated, and L is 1–very ample if and only if L is very ample. The preceding discussion shows that if S is a K3 surface whose Picard group is generated by an ample line bundle L such that $L^2 = 4k-2$, then L is $(k-1)$–very ample, but not k–very ample.

The notion of k–very ampleness admits the following geometric interpretation. Let $G(k+1, H^0(S,L))$ be the Grassmann variety of $(k+1)$–dimensional quotients of $H^0(S,L)$, and consider the rational map

$$\varphi_k : S^{[k+1]} - -- > G(k+1, H^0(S,L))$$

defined by $\varphi_k(\xi) = H^0(\xi, L \otimes \mathcal{O}_\xi)$. If L is k–very ample then φ_k is a morphism.

Proposition 6.8. *If L is k–very ample, then* $\det L^{[k+1]}$ *is globally generated.*

Proof: Let

$$\psi_k : S^{[k+1]} \to \mathbb{P}H^0(S, \bigwedge^{k+1} H^0(S,L)^\vee)$$

be the composition of φ_k and the Plücker embedding. As $L^{[k+1]}$ is the pullback of the universal quotient bundle on the Grassmannian, we have

$$\det L^{[k+1]} \cong \psi_k^* \mathcal{O}_{\mathbb{P}}(1).$$

Hence the result follows. □

6.3. Voisin's proof of Green's generic conjecture: even genus

The aim of this section is to prove the following result

THEOREM 6.9 (C. Voisin). *Let S be a $K3$ surface endowed with a ample line bundle L such that L generates* $\mathrm{Pic}(S)$ *and $L^2 = 2g-2$, with $g = 2k$. Then*

$$K_{k,1}(S,L) = 0. \tag{6.6}$$

This result is particularly interesting from the point of view of the Green conjecture.

COROLLARY 6.10. *The Green conjecture holds for generic curves of even genus.*

Proof: Recall that Green's conjecture for a curve C predicts that

$$K_{g-\mathrm{Cliff}(C)-1,1}(C, K_C) = 0.$$

Since the generic curve of genus $g = 2k$ has Clifford index $k-1$, we have to show that $K_{k,1}(C, K_C) = 0$. By semicontinuity (Corollary 1.31) it suffices to produce one smooth curve C with the given invariants that satisfies the desired vanishing. Since the curves appearing in the statement of Theorem 6.9 have this property, it suffices to apply Theorem 2.20. □

To apply the results of section 5.3, we need a suitable cartesian diagram. This diagram is constructed in [**V02**]; we follow the presentation of [**V01**].

Let E be the Lazarsfeld–Mukai bundle on S (section 6.2). There exists a morphism $\mathbb{P}H^0(S,E) \to S^{[k+1]}$ that sends a global section $s \in H^0(S,E)$ to its zero set $Z(s)$. By restriction to a an open subset $\mathbb{P} \subset \mathbb{P}H^0(S,E)$, we obtain a morphism $\mathbb{P} \to S^{[k+1]}_{\mathrm{curv}}$. Define $\mathbb{P}' = \mathbb{P} \times_{S^{[k+1]}_{\mathrm{curv}}} \Xi_{k+1}$. Set–theoretically

$$\mathbb{P}' = \{(Z(s), x) | s \in H^0(S,E), x \in Z(s)\}.$$

Consider the cartesian diagram

$$\begin{array}{ccc} \mathbb{P}' & \to & \Xi_{k+1} \\ \Big\downarrow{\scriptstyle q'} & & \Big\downarrow{\scriptstyle q} \\ \mathbb{P} & \to & S^{[k+1]}_{\mathrm{curv}}. \end{array}$$

The following result implies that the map j^* arising from this diagram is not injective. Hence cannot apply Proposition 5.18 or 5.21 to this diagram.

LEMMA 6.11. *The restriction map*

$$i^* : H^0(S^{[k+1]}_{\mathrm{curv}}, \det L^{[k+1]}) \to H^0(\mathbb{P}, i^* \det L^{[k+1]})$$

is identically zero.

Proof: Recall that $H^0(S^{[k+1]}_{\mathrm{curv}}, \det L^{[k+1]}) \cong \bigwedge^{k+1} H^0(S,L)$, Lemma 5.2. Given $\xi \in \mathbb{P}$, let

$$r_\xi : H^0(S,L) \to H^0(\xi, L \otimes \mathcal{O}_\xi)$$

be the restriction map. Given $s \in \bigwedge^{k+1} H^0(S,L)$, we have

$$i^* s(\xi) = \bigwedge^{k+1} r_\xi(s) \in \bigwedge^{k+1} H^0(\xi, L \otimes \mathcal{O}_\xi).$$

If $\xi \in \mathbb{P}$ then $h^1(S, L \otimes I_\xi) = 1$. Hence, if $\xi \in \mathbb{P}$ then rank $r_\xi \le k$ and $\bigwedge^{k+1} r_\xi$ is identically zero. □

Voisin's idea is to modify the previous construction by considering zero–cycles of the form

$$Z(s) - x + y$$

with $[s] \in \mathbb{P}$, $x \in \mathrm{Supp}(Z(s))$ and $y \in S$. To make this more precise, consider the difference map

$$\tau : \Xi_{k+1} \to S^{[k]}_{\mathrm{curv}} \times S$$

and let $\psi : \mathbb{P}' \to S^{[k]}_{\mathrm{curv}}$ be the composed map

$$\mathbb{P}' \to \Xi_{k+1} \xrightarrow{\tau} S^{[k]}_{\mathrm{curv}} \times S \to S^{[k]}_{\mathrm{curv}}.$$

By the Cayley-Bacharach property, section 6.2, ψ is injective. Set $\psi_S = \psi \times \mathrm{id}_S : \mathbb{P}' \times S \to S^{[k]}_{\mathrm{curv}} \times S$. We have $\psi_S(Z(s), x, y) = (Z(s) - x, y)$. Consider the rational map

$$\mu : S^{[k]}_{\mathrm{curv}} \times S \dashrightarrow \Xi_{k+1}$$

given by addition of zero–cycles. Composing with ψ_S, we obtain a rational map $\mathbb{P}' \times S \dashrightarrow \Xi_{k+1}$. We can resolve the indeterminacy of this map by blowing up along $\psi_S^{-1}(\Xi_k)$ to obtain a scheme U and a morphism $h : U \to \Xi_{k+1}$. This leads to a commutative diagram

$$\begin{tikzcd} U \arrow[r, "h"] \arrow[d, "\varepsilon"] & \Xi_{k+1} \arrow[d, "\tau"] \\ \mathbb{P}' \times S \arrow[r, "\psi_S"] & S^{[k]}_{\mathrm{curv}} \times S. \end{tikzcd}$$

Composing the map $h : U \to \Xi_{k+1}$ with the projection to $S^{[k+1]}_{\mathrm{curv}}$, we obtain a morphism $i : U \to S^{[k+1]}_{\mathrm{curv}}$ that sends $(Z(s), x, y)$ to $Z(s) - x + y$. Put $T = U \times_{S^{[k+1]}_{\mathrm{curv}}} \Xi_{k+1}$. We obtain a cartesian diagram

$$\begin{tikzcd} T \arrow[r, "j"] \arrow[d, "g"] & \Xi_{k+1} \arrow[d, "q"] \\ U \arrow[r, "i"] \arrow[ur, "h"] & S^{[k+1]}_{\mathrm{curv}}. \end{tikzcd}$$

Set–theoretically

$$T = \{((Z(s), x, y), (\xi', x')) \mid x \in Z(s),\ x' \in \xi',\ \xi' = Z(s) - x + y\}.$$

There are two possibilities for the choice of x'.

a) $x' = y$. This component maps isomorphically to U;
b) $x' \in Z(s) - x + y$, $x' \neq y$. This leaves k possibilities for x'. The resulting degree k covering of U is called V.

The intersection D of the two components coincides with the exceptional locus of the blowup $\varepsilon : U \to \mathbb{P}' \times S$. Note that D is the pullback of the divisor $D_\tau \subset \Xi_{k+1}$ under the map $h : U \to \Xi_{k+1}$.

The second component V can be constructed in the following way. Define $\mathbb{P}''$ by the cartesian diagram

$$\begin{tikzcd} \mathbb{P}'' \arrow[r] \arrow[d, "\pi"] & \Xi_k \arrow[d, "q"] \\ \mathbb{P}' \arrow[r, "\psi"] & S^{[k]}_{\mathrm{curv}}. \end{tikzcd}$$

As before, there exists a rational map $\mathbb{P}'' \times S \dashrightarrow \Xi_{k+1}$. Blowing up along the inverse image of Ξ_k in $\mathbb{P}'' \times S$, we obtain a scheme V and a morphism $V \to \Xi_{k+1}$ that fits into a diagram

$$\begin{array}{ccc} V & \to & \Xi_{k+1} \\ \big\downarrow & & \big\downarrow \tau \\ \mathbb{P}'' \times S & \to & S^{[k]}_{\text{curv}} \times S. \end{array}$$

The varieties appearing in the proof and the maps between them are drawn in the following diagrams:

$$\begin{tikzcd}
V \arrow[r, "\text{blowup}"] & \mathbb{P}'' \times S \arrow[d] \arrow[dr, "(\pi,\text{id})"] & \psi_S^{-1}(\Xi_k) \arrow[d, hook] & D \arrow[l] \arrow[d, hook] \\
 & \mathbb{P}'' = \mathbb{P}' \times_{S^{[k]}_{curv}} \Xi_k \arrow[dr, "\pi"] & \mathbb{P}' \times S \arrow[d] \arrow[ddl, "\psi_S"'] & U \arrow[l, "\text{blowup}", "\varepsilon"'] \arrow[d, "h"] \arrow[dd, bend left, "i"] \\
D_\tau \arrow[r, hook] \arrow[d] & \Xi_{k+1} \arrow[d, "\tau"] & \mathbb{P}' \arrow[r, hook] \arrow[d, "q'"] \arrow[ddl, "\psi"'] & \Xi_{k+1} \arrow[d, "q"] \\
\Xi_k \arrow[r, hook] & S^{[k]}_{curv} \times S \arrow[d] & \mathbb{P} \arrow[r, hook] & S^{[k+1]}_{curv} \\
 & S^{[k]}_{curv} & &
\end{tikzcd}$$

$$\begin{tikzcd}
\# \arrow[r] & (Z(s), x, x', y) \arrow[d] \arrow[dr, "(\pi,\text{id})"] & (Z(s), x, x'') \arrow[d] & \# \arrow[l] \arrow[d] \\
 & (Z(s), x, x') \arrow[dr, "\pi"] & (Z(s), x, y) \arrow[d] \arrow[ddl, "\psi_S"'] & \# \arrow[l, "\text{blowup}", "\varepsilon"'] \arrow[d, "h"] \arrow[dd, bend left, "i"] \\
\# \arrow[r] \arrow[d] & (Z(s) - x + y, y) \arrow[d, "\tau"] & (Z(s), x) \arrow[r] \arrow[d, "q'"] \arrow[ddl, "\psi"'] & (Z(s) - x + y, y) \arrow[d, "q"] \\
\# \arrow[r] & (Z(s) - x, y) \arrow[d] & Z(s) \arrow[r] & Z(s) - x + y \\
 & Z(s) - x & &
\end{tikzcd}$$

To prove the vanishing of $K_{k,1}(S, L)$, it suffices to show that the diagram

$$\begin{tikzcd}
T = U \cup V \arrow[r, "j"] \arrow[d, "g"] & \Xi_{k+1} \arrow[d, "q"] \\
U \arrow[ur, "h"] \arrow[r, "i"] & S^{[k+1]}_{\text{curv}}.
\end{tikzcd}$$

satisfies the conditions (i), (ii) and (iii) of Corollary 5.23. Specifically, this means that we have to verify

(i) $g_V^* : H^0(U, i^* \det L^{[k+1]}) \to H^0(U, g_V^* i^* \det L^{[k+1]})$ is surjective;
(ii) $r_D : H^0(U, i^* \det L^{[k+1]}) \to H^0(D, i^* \det L^{[k+1]}|_D)$ is injective;
(iii) $h^* : H^0(\Xi_{k+1}, q^* \det L^{[k+1]}) \to H^0(U, i^* \det L^{[k+1]})$ is injective.

The proof proceeds via a number of reduction steps.

Step 1. We start with condition (ii) of Corollary 5.23, i.e., injectivity of the restriction map r_D. This is obtained by restricting to a smooth curve $C \in |L|$ and using Petri's theorem. Put $\mathcal{L} = i^*(\det L^{[k+1]})$. Recall that a generic element of U is of the form $(Z(s), x, y)$, with $x \in Z(s)$. Given a smooth curve $C \in |L|$, put $\mathbb{P}_C$ be the projectivization of the vector space of global sections in E whose support is contained in C, and put $\mathbb{P}'_C$ be the fibered product of $\mathbb{P}_C$ with the incidence scheme $C^{(k)} \times C$, and $U_C = \mathbb{P}'_C \times C$. Put $D_C = D \cap U_C \subset U_C$, and let $|L|_0 \subset |L|$ be the open subset parametrizing smooth curves that are Brill-Noether-Petri generic.

A dimension count shows that if $k \geq 2$, there exists a smooth curve $C \in |L|$ such that $Z(s) \cup \{y\} \subset C$. Hence the commutative diagram

$$\begin{array}{ccc} H^0(U, \mathcal{L}) & \longrightarrow & \prod_{C \in |L|_0} H^0(U_C, \mathcal{L}|_{U_C}) \\ \downarrow{\scriptstyle r_D} & & \downarrow \\ H^0(D, \mathcal{L}|_D) & \longrightarrow & \prod_{C \in |L|_0} H^0(D_C, \mathcal{L}|_{D_C}) \end{array}$$

shows that r_D is injective if the restriction map

$$H^0(U_C, \mathcal{L}|_{U_C}) \to H^0(D_C, \mathcal{L}|_{D_C})$$

is injective for all $C \in |L|_0$. Fix $C \in |L|_0$. We give a more concrete description of the varieties U_C and D_C. By Brill-Noether-Petri genericity we have $\dim W^1_{k+1}(C) = 0$ if $C \in |L|_0$, [**ACGH85**]. Write $W^1_{k+1}(C) = \{L_1, \ldots, L_N\}$. The map

$$\mathbb{P}_C \to C^{(k+1)},\ s \mapsto Z(s)$$

identifies $\mathbb{P}_C$ with a disjoint union of N projective lines $\mathbb{P}^1_i$, where $\mathbb{P}^1_i \subset C^{(k+1)}$ corresponds to the pencil $|L_i|$.

Similarly, the induced map

$$\psi_C : \mathbb{P}'_C \to C^{(k)}$$

identifies $\mathbb{P}'_C$ with a disjoint union of N curves $C_i \cong C$. The divisor $D_C \subset \mathbb{P}'_C$ is a disjoint union of divisors $D_i \subset C \times C_i$. Let $\varphi_i : C_i \to \mathbb{P}^1_i$ be the map induced by L_i. By construction, the fiber of the map $D_i \to \mathbb{P}^1_i$ over a point $x \in \mathbb{P}^1_i$ is the set

$$\{(\varphi_i^{-1}(x) - y, y),\ y \in \varphi_i^{-1}(x)\}.$$

Hence,

$$D_i = (\varphi_i \times \varphi_i)^{-1}(\mathrm{diag}(\mathbb{P}^1)) - \mathrm{diag}(C),$$

where $\mathrm{diag}(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1_i \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathrm{diag}(C) \subset C \times C_i \cong C \times C$ are the diagonals. Hence

$$\mathcal{O}(D_i) \cong (L_i \boxtimes L_i)(-\mathrm{diag}(C)). \tag{6.7}$$

By the formula (5.1), we have

$$\mathcal{L} = h^*(\tau^*(\det L^{[k]} \boxtimes L))(-D).$$

Write $A_i = \det L^{[k]}|_{C_i}$. Since $L|_C = K_C$, the preceding discussion shows that it suffices to prove that the maps

$$H^0(C \times C_i, (K_C \boxtimes A_i)(-D_i)) \to H^0(D_i, (K_C \boxtimes A_i)(-D_i)|_{D_i})$$

are injective for all i, i.e.

$$H^0(C \times C_i, (K_C \boxtimes A_i)(-2D_i)) = 0,$$

for all i. Rewriting this using (6.7), we reduce to the vanishing of

$$H^0(C \times C_i, \left((K_C \otimes L_i^{-2}) \boxtimes (A_i \otimes L_i^{-2})\right)(2\mathrm{diag}(C))).$$

As C is Brill-Noether-Petri generic, we obtain

$$H^0(C, K_C \otimes L_i^{-2}) = 0, \tag{6.8}$$

using the base-point free pencil trick [**ACGH85**, p. 126]. If $x \in C$ is a general point, the restriction map

$$H^0(C, L_i^2) \to H^0(L_i^2|_{2x})$$

is surjective. Hence, we obtain

$$H^0(C, K_C \otimes L_i^{-2} \otimes \mathcal{O}_C(2x)) = 0,$$

using (6.8). Consider the projection $\mathrm{pr}_2 : C \times C_i \to C$. The previous result implies that the direct image of $\left((K_C \otimes L_i^{-2}) \boxtimes (A_i \otimes L_i^{-2})\right)(2\mathrm{diag}(C))$ is zero on a non-empty open subset of C. Hence, the bundle

$$\left((K_C \otimes L_i^{-2}) \boxtimes (A_i \otimes L_i^{-2})\right)(2\mathrm{diag}(C))$$

has no sections over this inverse image of this set. It implies that

$$\left((K_C \otimes L_i^{-2}) \boxtimes (A_i \otimes L_i^{-2})\right)(2\mathrm{diag}(C))$$

has no sections at all.

Step 2. Condition (iii) is satisfied if the map

$$\psi^* : H^0(S^{[k]}_{\mathrm{curv}}, \det L^{[k]}) \to H^0(\mathbb{P}', \psi^* \det L^{[k]})$$

is injective.

Indeed, let D be the exceptional divisor of $\varepsilon : U \to \mathbb{P}' \times S$. We have

$$q^* \det L^{[k+1]} = \tau^*(\det L^{[k]} \boxtimes L)(-D_\tau)$$

where D_τ is the divisor contracted by $\tau : \Xi_{k+1} \to S^{[k]}_{\mathrm{curv}} \times S$; see the proof of Lemma 5.2. Hence

$$h^* q^* \det L^{[k+1]} = \varepsilon^*(\det L^{[k]} \boxtimes L)(-D).$$

Let $\mathcal{I}$ be the ideal sheaf of $\psi_S^{-1}(\Xi_k) \subset \mathbb{P}' \times S$. By the above result, the map

$$\varepsilon^* : H^0(\mathbb{P}' \times S, \psi_S^*(\det L^{[k]} \boxtimes L) \otimes \mathcal{I}) \to H^0(U, h^* q^* \det L^{[k+1]})$$

is an isomorphism. The commutative diagram

$$\begin{tikzcd}
H^0(\Xi_{k+1}, q^* \det L^{[k+1]}) \arrow[r, "h^*"] & H^0(U, h^* q^* \det L^{[k+1]}) \\
H^0(S^{[k]}_{\mathrm{curv}} \times S, (\det L^{[k]} \boxtimes L) \otimes \mathcal{I}_{\Xi_k}) \arrow[u, "\tau^*"] \arrow[r, "\psi_S^*"] & H^0(\mathbb{P}' \times S, \psi_S^*(\det L^{[k]} \boxtimes L) \otimes \mathcal{I}) \arrow[u, "\varepsilon^*"']
\end{tikzcd}$$

then shows that h^* is injective if ψ_S^* is injective. Since $\psi_S = \psi \times \mathrm{id}$, the injectivity of ψ_S^* follows from the injectivity of ψ^*.

Step 3. Consider the map $q' : \mathbb{P}' \to \mathbb{P}$ obtained by base change from $q : \Xi_{k+1} \to S^{[k+1]}_{\text{curv}}$.

LEMMA 6.12. [**V02**, Lemma 2] *We have*

$$\psi^* \det L^{[k]} \cong (q')^* \mathcal{O}_{\mathbb{P}}(k).$$

Proof: [of Lemma 6.12] By construction, the line bundle L has the following property: for all $\xi \in \mathbb{P}$, the restriction map

$$r_\xi : H^0(S, L) \to H^0(\xi, L|_\xi)$$

is not surjective, whereas $r_{\xi'}$ is surjective for any $\xi' \subsetneq \xi$, Section 6.2. This property implies that given $\zeta \in \mathbb{P}'$, we have

$$H^0(S, L \otimes \mathcal{I}_{\psi(\zeta)}) \cong H^0(S, L \otimes \mathcal{I}_{q'(\zeta)}).$$

Hence the fiber of the vector bundle $\psi^* L^{[k]}$ at $\zeta \in \mathbb{P}'$ is isomorphic to

$$H^0(S, L)/H^0(S, L \otimes \mathcal{I}_{q'(\zeta)}),$$

and $\psi^* \det L^{[k]} \cong (q')^* (\det F)^{-1}$, where F is the coherent sheaf on $\mathbb{P}$ whose fiber over a point $[s] \in \mathbb{P}$ is $H^0(S, L \otimes \mathcal{I}_{Z(s)})$. This isomorphism is obtained from the short exact sequence

$$0 \to (q')^* F \to H^0(S, L) \otimes \mathcal{O}_{\mathbb{P}'} \to \psi^* L^{[k]} \to 0.$$

Recall that for any section $s \in H^0(S, E)$, we have the short exact sequence

$$0 \to \mathcal{O}_S \xrightarrow{s} E \xrightarrow{\wedge s} L \otimes \mathcal{I}_{Z(s)} \to 0,$$

and the associated sequence

$$0 \to \mathbb{C}.s \xrightarrow{s} H^0(S, E) \xrightarrow{\wedge s} H^0(S, L \otimes \mathcal{I}_{Z(s)}) \to 0.$$

Consider the Koszul complex

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \to H^0(E) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \bigwedge^2 H^0(E) \otimes \mathcal{O}_{\mathbb{P}} \to \dots$$

on $\mathbb{P} = \mathbb{P}H^0(S, E)$, and put

$$F' = \operatorname{coker}(\mathcal{O}_{\mathbb{P}}(-2) \to H^0(E) \otimes \mathcal{O}_{\mathbb{P}}(-1)).$$

Taking global sections, we obtain $F'_s \cong F_s$ for all $s \in \mathbb{P}$, hence $F \cong F'$. The resulting exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-2) \to H^0(E) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to F \to 0$$

shows that $\det F \cong \mathcal{O}_{\mathbb{P}}(-k)$. $\square$

By Lemma 6.12, $(q')^*$ induces a natural injective morphism

$$S^k H^0(S, E)^\vee \hookrightarrow H^0(\mathbb{P}', \psi^* \det L^{[k]}).$$

Step 4.

LEMMA 6.13. *The map* $\psi : \mathbb{P}' \to S^{[k]}_{\text{curv}}$ *induces an isomorphism*

$$\psi^* : H^0(S^{[k]}_{\text{curv}}, \det L^{[k]}) = \bigwedge^k H^0(S, L) \xrightarrow{\cong} S^k H^0(S, E)^\vee \subset H^0(\mathbb{P}', \psi^* \det L^{[k]}).$$

Proof: Since the determinant map

$$d_E : \bigwedge^2 H^0(S,E) \to H^0(S,L)$$

does not vanish on decomposable elements, we obtain a morphism

$$d : G(2, H^0(S,E)) \to \mathbb{P}H^0(S,L).$$

Note that d is a finite morphism whose fiber over a point $C \in |L|$ is $W^1_{k+1}(C)$.

Put $G = G(2, H^0(S,E))$ and consider the map

$$d^* : H^0(S,L)^\vee \to H^0(G, \mathcal{O}_G(1)) = \bigwedge^2 H^0(S,E)^\vee.$$

As d is finite, d^* is injective by the existence of the trace map, hence we obtain a base-point free linear subspace $V^\vee \subset H^0(G, \mathcal{O}_G(1))$. Let

$$0 \to \bigwedge^{2k+1} V^\vee \otimes \mathcal{O}_G(-2k-1) \to \ldots \to V^\vee \otimes \mathcal{O}_G(-1) \to \mathcal{O}_G \to 0 \tag{6.9}$$

be the associated exact Koszul complex. Consider the tautological exact sequence

$$0 \to \mathcal{S} \to H^0(S,E) \otimes \mathcal{O}_G \to \mathcal{Q} \to 0$$

and note that

$$H^0(G, S^k\mathcal{S}^\vee) \cong S^k H^0(S,E)^\vee.$$

Twisting the Koszul complex (6.9) by $S^k\mathcal{S}^\vee$ and chasing through the associated spectral sequence of hypercohomology, using suitable vanishing theorems, one obtains an isomorphism

$$d_{k+1} : \bigwedge^k H^0(S,L) \xrightarrow{\cong} S^k H^0(S,E)^\vee.$$

It remains to show that this map coincides (up to scalar) with ψ^*. This is done using representation theory. Consider the map

$$\alpha : \mathbb{P}' \to G(k+1, H^0(S,L))$$

defined by $\alpha(\zeta) = H^0(S, L \otimes \mathcal{I}_{\psi(\zeta)})$. As we have seen in the proof of Lemma 6.12, we have $H^0(S, L \otimes \mathcal{I}_{\psi(\zeta)}) \cong H^0(S, L \otimes \mathcal{I}_{q'(\zeta)})$, hence α factors through a map

$$\beta : \mathbb{P} \to G(k+1, H^0(S,L))$$

defined by

$$\beta(s) = \mathrm{im}(d_s : H^0(S,E) \to H^0(S,L)) = H^0(S, L \otimes \mathcal{I}_{Z(s)}).$$

The factorization $d_s = d_E \circ (- \wedge s)$ shows that β is the composition of the maps

$$\mathbb{P} \xrightarrow{\gamma} G(k+1, \bigwedge^2 H^0(S,E)) \dashrightarrow G(k+1, H^0(S,L))$$

Applying a similar spectral sequence argument to the base-point free linear system $|\mathcal{O}_G(1)|$, we obtain a surjective map

$$D_{k+1} : \bigwedge^{k+1}(\bigwedge^2 H^0(S,E)^\vee) \to S^k H^0(S,E)^\vee,$$

whose restriction to $\bigwedge^{k+1} V^\vee$ equals d_{k+1}. Since ψ^* is a restriction map and since $\det L^{[k]}$ is globally generated by Proposition 6.8 and Remark 6.7, the map ψ^* is not identically zero. Note that the maps D_{k+1}, and γ^* are SL$(k+2)$-equivariant. The desired statement then follows by decomposing the SL$(k+2)$–module $\bigwedge^{k+1}(\bigwedge^2 H^0(S,E)^\vee)$ into irreducible submodules.

□

Steps 2 to 4 show that the condition (iii) is satisfied. It remains to verify condition (i).

Step 5. To verify condition (i), the surjectivity of g_V^*, recall that U is a blowup of $S \times \mathbb{P}'$ and V is a blowup of $S \times \mathbb{P}''$. These blowups are related via the map $\pi : \mathbb{P}'' \to \mathbb{P}'$. It follows that g_V^* is surjective if the map

$$\pi^* : H^0(\mathbb{P}', (q')^* \det L^{[k]}) \to H^0(\mathbb{P}'', \pi^*(q')^* \det L^{[k]})$$

is surjective. The commutative diagram

$$\begin{tikzcd} H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \arrow[r] \arrow[d, "(q')^*"] & H^0(\mathbb{P}'', \pi^*(q')^* \det L^{[k]}) \\ H^0(\mathbb{P}', (q')^* \det L^{[k]}) \arrow[ur, "\pi^*"] & \end{tikzcd}$$

shows that it suffices to prove the surjectivity of the map

$$H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k)) \xrightarrow{\pi^* \circ (q')^*} H^0(\mathbb{P}'', \pi^*(q')^* \det L^{[k]}).$$

Step 6. Let B be the blowup of $S \times S$ along the diagonal. There exist a vector bundle E_2 on B and a global section

$$\sigma \in H^0(B \times \mathbb{P}, E_2 \boxtimes \mathcal{O}_{\mathbb{P}}(1))$$

such that $\mathbb{P}'' = Z(\sigma) \subset B \times \mathbb{P}$. The bundle E_2 is obtained as follows. Let $\widetilde{S \times S}$ be the blowup of $S \times S$ along the diagonal, and consider the commutative diagram

$$\begin{array}{ccc} \widetilde{S \times S} & \xrightarrow{\rho} & S^{[2]} \\ \downarrow & & \downarrow \\ S \times S & \to & S^{(2)}. \end{array}$$

Let $E^{[2]}$ be the tautological rank 4 bundle on $S^{[2]}$ with fiber $H^0(\eta, E \otimes \mathcal{O}_\eta)$ over η, and put $E_2 = \rho^* E^{[2]}$. Under the isomorphism $\mathbb{P}'' \cong Z(\sigma)$, the map $\pi^* \circ (q')^*$ is identified with the restriction map

$$H^0(B \times \mathbb{P}, \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}}(k)) \to H^0(\mathbb{P}'', \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}}(k)|_{\mathbb{P}''}).$$

Hence it suffices to show that

$$H^1(B \times \mathbb{P}, \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}}(k) \otimes \mathcal{I}_{\mathbb{P}''}) = 0. \tag{6.10}$$

Using the Koszul complex

$$0 \to \bigwedge^4 E_2^\vee \boxtimes \mathcal{O}(-4) \to \ldots \to E_2^\vee \boxtimes \mathcal{O}(-1) \to \mathcal{I}_{\mathbb{P}''} \to 0,$$

this vanishing follows from the study of the groups

$$H^i(\widetilde{S \times S} \times \mathbb{P}, \bigwedge^i E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-i)) \cong H^i(\widetilde{S \times S}, \bigwedge^i E_2^\vee) \otimes S^{k-i} H^0(S, E)^\vee$$

for $i = 1, \ldots, 4$. Voisin shows that these groups are zero if i is odd, [**V02**, Proposition 6]. A spectral sequence argument then shows that condition (6.10) is verified if the maps

$$H^2(\widetilde{S \times S} \times \mathbb{P}, \bigwedge^2 E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-2)) \to H^2(\widetilde{S \times S} \times \mathbb{P}, E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-1)) \tag{6.11}$$

and

$$H^4(\widetilde{S \times S} \times \mathbb{P}, \bigwedge^4 E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-4)) \to H^4(\widetilde{S \times S} \times \mathbb{P}, \bigwedge^3 E_2^\vee \boxtimes \mathcal{O}_{\mathbb{P}}(k-3)) \tag{6.12}$$

are injective. We outline the proof of the injectivity of the map (6.11). The injectivity of (6.12) is obtained by similar, but more complicated, calculations. Dualizing and rewriting using [loc.cit, Prop. 6], we reduce to the surjectivity of the map

$$(H^0(E) \oplus H^0(E)) \otimes S^{k-1}H^0(E) \to (H^0(L) \oplus H^0(L)) \otimes S^{k-2}H^0(E).$$

This map is the direct sum of two copies of the map μ defined via the commutative diagram

$$\begin{tikzcd} H^0(E) \otimes S^{k-1}H^0(E) \arrow[r] \arrow[dr, "\mu"] & H^0(E) \otimes H^0(E) \otimes S^{k-2}H^0(E) \arrow[d] \\ & H^0(L) \otimes S^{k-2}H^0(E). \end{tikzcd}$$

Since the morphism $d : G \to \mathbb{P}H^0(L)$ is finite and surjective, the induced map $\bigwedge^2 H^0(E) \to H^0(L)$ is surjective. Hence every element $t \in H^0(L)$ is of the form $t = d_E(s_1 \wedge s_2)$, with $s_1, s_2 \in H^0(E)$. Write $W = \langle s_1, s_2 \rangle \subset H^0(E)$. We have

$$t \otimes S^{k-2}W \in \mathrm{im}(\mu),$$

because the composition

$$W \otimes S^{k-1}W \to W \otimes W \otimes S^{k-2}W \to \bigwedge^2 W \otimes S^{k-2}W$$

is surjective (as one sees by writing out this map in terms of the basis). Hence, it suffices to prove that the family of subspaces

$$\{S^{k-2}W\}_{W \in d_E^{-1}(t)}$$

generates $S^{k-2}H^0(E)$, for general t.

If $t \in H^0(L)$ is general, then $Z(t) = C \in |L|$ is smooth and the fibre of $d_E^{-1}(t)$ consists of N distinct g^1_{k+1}'s $W_1, \ldots, W_N$ on C. After dualizing, it is sufficient to show that the map

$$S^{k-2}H^0(E)^\vee \to \bigoplus_{i=1}^{N} S^{k-2}W_i^\vee$$

is injective. This map is identified with the restriction map

$$H^0(G, S^{k-2}\mathcal{S}^\vee) \to H^0(d_E^{-1}(t), S^{k-2}\mathcal{S}^\vee|_{d_E^{-1}(t)}).$$

Since the finite subscheme $d_E^{-1}(t) \subset G$ is a complete intersection of hyperplanes, we have once more a Koszul resolution for its ideal sheaf. Using this resolution and suitable vanishing theorems on the Grassmannian, we conclude.

6.4. The odd-genus case (outline)

In the odd-genus case, a natural thing to do would be to try and mimic the proof of Theorem 6.9. Consider a $K3$ surface with cyclic Picard group by a very ample line bundle L, with $L^2 = 4k$, where k is a positive integer. By Theorem 6.5, a generic smooth curve $C \in |L|$ is Brill-Noether-Petri generic, of genus $2k+1$ and gonality $k+2$, hence it is a good candidate for verifying the Green conjecture. However, this strategy turns out to be a cul-de-sac. Compared to the even-genus case treated in the previous section, the Lazarsfeld-Mukai bundles are not unique, and depend on the choice of the curve and of the pencil. These bundles are parametrized by another $K3$ surface $\widehat{S}$, which is a moduli space for stable vector bundles on S

[**M84**]. The unicity of the Lazarsfeld-Mukai bundle was a crucial point in the proof of Theorem 6.9, hence it cannot be adapted directly to the odd-genus setup.

Voisin found an ingenious way to circumvent these difficulties and to reduce to the even-genus case. The situation is as follows. We take the genus $g = 2k + 1$. Green's conjecture predicts the vanishing of $K_{k,1}(C, K_C)$ for a generic genus-g curve.

THEOREM 6.14 (C. Voisin). *Consider a smooth projective $K3$ surface S, such that $\mathrm{Pic}(S)$ is isomorphic to $\mathbb{Z}^2$, and is freely generated by L and $\mathcal{O}_S(\Delta)$, where Δ is a smooth rational curve such that deg $L_{|\Delta} = 2$, and L is a very ample line bundle with $L^2 = 2g - 2$, $g = 2k + 1$. Then $K_{k+1,1}(S, L + \Delta) = 0$ and*

$$K_{k,1}(S, L) = 0. \tag{6.13}$$

By Theorem 2.20 and Corollary 1.31, Green's generic conjecture follows from Theorem 6.14. Here, one has to remark that smooth curves in the linear system $|L|$ are Brill-Noether-Petri generic. To reduce to the case of even genus, put $L' = L \otimes \mathcal{O}_S(\Delta)$. A smooth curve in $C' \in |L'|$ has genus $2k + 2$, and does not meet Δ.

The proof of Theorem 6.14 proceeds in several steps.

By a modification of the techniques of [**V02**] outlined in section 6.3, one proves

$$K_{k+1,1}(S, L') = 0.$$

By Green's duality theorem 2.24,

$$K_{k,1}(S, L)^\vee \cong K_{k-1,2}(S, L).$$

To obtain the desired vanishing, consider the multiplication map

$$\mu : K_{k-1,0}(S, L(-\Delta), L) \otimes H^0(S, L') \to K_{k-1,2}(S, L) \cong K_{k-1,0}(S, L^2, L).$$

Voisin first proves that μ is surjective; see [**V05**, Proposition 6]. Put $V = H^0(S, L)$, define

$$K := \ker\left(\delta' : \bigwedge^{k-1} V \otimes H^0(S, L(-\Delta)) \to \bigwedge^{k-2} V \otimes H^0(S, L^2(-\Delta))\right)$$

and note that

$$K_{k-1,0}(S, L(-\Delta), L) \cong K.$$

The next step is to calculate $\dim(K)$. To this end, Voisin [**V05**, Lemma 4] shows that

$$K_{k-i,i-1}(S, L(-\Delta), L) = 0, \quad \text{for all } i \geq 2.$$

An Euler-characteristic calculation [**V05**, Corollary 1] shows that

$$\dim(K) = \binom{2k+1}{k-1}.$$

Consider the Lazarsfeld-Mukai bundle E associated to a smooth curve $C' \in |L'|$. The construction from section 6.3 provides us with a map

$$\varphi : S^{k-1} H^0(S, E) \to K.$$

The proof is now finished by showing that

(i) φ is an isomorphism, and
(ii) $\mathrm{im}(\varphi) \subset \ker(\mu)$.

The main technical difficulty consists in proving (i). Since the spaces $S^{k-1}H^0(S, E)$ and K have the same dimension, it suffices to prove that φ is injective. This is accomplished in [**V05**, Proposition 8].

COROLLARY 6.15. *The Green conjecture is verified for a generic curve of odd genus.*

Proof: The proof proceeds as in Corollary 6.10, with one modification. One has to check that smooth curves in the linear system $|L|$ are Brill-Noether-Petri generic. This is done using Theorem 6.4; see Voisin [**V05**, Proposition 1]. □

When combined with the Hirschowitz-Ramanan result [**HR98**], Corollary 6.15 gives a very strong result that will be used throughout the next chapter; see 7.1.

The next result will be used in section 7.2, and follows from Theorem 2.20.

COROLLARY 6.16. *Notation as in Theorem 6.14. Let $Y \in |L|$ be a singular curve with one node. Then $K_{k,1}(Y, \omega_Y) = 0$.*

Other versions of this result lead to the proof of the conjectures of Green and Green-Lazarsfeld for generic curves of large gonality; see [**AV03**] and [**Ap04**]. In the next chapter we shall discuss some refined versions.

6.5. Notes and comments

Curves on $K3$ surfaces as in the statement of Lazarsfeld's Theorem 6.5 were the first concrete examples of Brill-Noether-Petri generic curves. (Before Lazarsfeld's paper the existence of Brill-Noether-Petri generic curves was proved by degeneration techniques.) The condition appearing in Lazarsfeld's theorem holds, for example, if the Picard group of S is cyclic. Since $K3$ surfaces with cyclic Picard group form a 19-dimensional family, the Gieseker-Petri Theorem follows. An important argument of Lazarsfeld's proof is to show that the associated Lazarsfeld-Mukai bundle is *simple*, i.e. does not have any endomorphisms besides the homotheties; cf. [**La89**], [**P96**]. Then one applies the description of the moduli space of simple bundles, due to Mukai.

CHAPTER 7

Specific versions of the syzygy conjectures

7.1. The specific Green conjecture

In this Section, we discuss a remarkable result that follows from Voisin's Theorem 6.14 and the work [**HR98**].

THEOREM 7.1. *Any smooth curve C of genus $g = 2k+1 \geq 5$ with $K_{k,1}(C, \omega_C) \neq 0$ carries a pencil of degree $k+1$.*

It represents the solution in the odd genus case to the following conjecture.

CONJECTURE 7.2 (specific Green conjecture). Let $C \subset \mathbb{P}^{g-1}$ be a canonically embedded curve of genus g with maximal Clifford index $[(g-1)/2]$, and let Q be the universal quotient bundle on $\mathbb{P}^{g-1}$. Then $H^0(\bigwedge^j Q \otimes \mathcal{I}_C(1))$ vanishes if $j = [(g+1)/2]$.

REMARK 7.3. Proposition 2.7 shows that this statement is indeed the Green conjecture for curves of maximal Clifford index.

Proof: (of Theorem 7.1) Let $\mathcal{M}_g$ be the moduli space of curves of genus g. The statement of the Theorem is equivalent to the equality of the supports of the two subvarieties of $\mathcal{M}_g$:

$$D_{k+1} = \{[C] \in \mathcal{M}_g,\ \text{there exists a } g^1_{k+1} \text{on } C\}$$

and

$$D'_{k+1} = \{[C] \in \mathcal{M}_g,\ K_{k,1}(C, K_C) \neq 0\}.$$

The computations of [**HM82**] show that D_{k+1} is a reduced divisor with respect to the natural structure induced by the Brill-Noether theory. It is moreover irreducible, as is any other gonality stratum; cf. [**AC81a**]. A consequence of the Green-Lazarsfeld nonvanishing Theorem 3.33, is a set-theoretical inclusion

$$D_{k+1} \subset D'_{k+1}.$$

Observe that the locus corresponding to hyperelliptic curves is completely contained in both D_{k+1} (by definition) and D'_{k+1} (by the Green-Lazarsfeld nonvanishing Theorem), Therefore, we can work directly over the complement of the hyperelliptic locus in $\mathcal{M}_g$.

Step 1. We show that D'_{k+1} is of pure codimension one; this will be a consequence of Voisin's Theorem 6.14. To this end, consider a covering

$$\pi : \mathcal{S} \to \mathcal{M}_g$$

on which a universal curve $\mathcal{C} \to \mathcal{S}$ exists (see for example [**Loo94**]). By definition, for any point $x \in \mathcal{S}$, the fibre $\mathcal{C}_x$ is the curve of genus g whose isomorphism class is given by the image of x in $\mathcal{M}_g$. Put ω_π the relative canonical bundle and let E be its direct image on $\mathcal{S}$. Restricting over the non-hyperelliptic locus, the morphism π factors through a natural canonical embedding of $\mathcal{C}$ in the projective bundle $p : \mathbb{P}(E) \to \mathcal{S}$. Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{C}$ in $\mathbb{P}(E)$, and $\mathcal{O}_{\mathbb{P}(E)}(1)$ be the relatively ample (hyperplane) line bundle along the fibres.

Let C be a non-hyperelliptic curve of odd genus $g = 2k + 1$, and denote by $\mathcal{I}_C$ the ideal sheaf of C in $\mathbb{P}^{g-1}$, and by Q the tautological quotient bundle of rank $g-1$ on $\mathbb{P}^{g-1}$. Applying Proposition 2.7, we see that the restriction of D'_{k+1} to the non-hyperelliptic locus coincides with

$$\{[C] \mid \ H^0(\bigwedge^k Q \otimes \mathcal{I}_C(1)) \neq 0\}.$$

Let $\mathcal{Q}$ be the universal quotient bundle on $\mathbb{P}(E)$; it fits into an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}(E)}(1) \to p^*(E)^\vee \to \mathcal{Q} \to 0.$$

Note that $\mathcal{E} = p_*(\bigwedge^l \mathcal{Q}(1))$ is a vector bundle of rank

$$\binom{g}{l} g - \binom{g}{l-1}.$$

Similarly, $\mathcal{F} = p_*(\bigwedge^l \mathcal{Q}(1) \otimes \mathcal{O}_{\mathcal{C}})$ is a vector bundle of rank

$$\binom{g-1}{l}(2l + g - 1)$$

for each $l > 0$. To this end, we apply the Riemann-Roch Theorem, and the Serre duality on each fibre $\mathcal{C}_x$ to the bundle $\bigwedge^l \mathcal{Q}_{\mathcal{C}_x} \otimes \omega_{\mathcal{C}_x}$ to obtain $h^1(\mathcal{C}_x, \bigwedge^l \mathcal{Q}_{\mathcal{C}_x} \otimes \omega_{\mathcal{C}_x}) = h^0(\mathcal{C}_x, \wedge^l \mathcal{Q}^*_{\mathcal{C}_x})$. The latter vanishes, see Remark 2.5.

The restriction from $\mathbb{P}(E)$ to $\mathcal{C}$ yields a morphism $p_*(\bigwedge^{k+1} \mathcal{Q}(1)) \to p_*(\bigwedge^{k+1} \mathcal{Q}(1) \otimes \mathcal{O}_{\mathcal{C}})$. Voisin's Theorem 6.14 implies that this sheaf morphism is injective. Observe next that the two vector bundles have the same rank, namely

$$\binom{2k}{k+1}(4k+2) = \binom{2k+1}{k} 2k,$$

hence the above map defines a degeneracy divisor in $\mathcal{S}$ whose image in $\mathcal{M}_g$ is D'_{k+1}, by definition.

Step 2. We compare two classes of divisors on the moduli space of curves.

It is convenient to work over the large open subvariety $\mathcal{M}_g^0$ of the moduli space $\mathcal{M}_g$ corresponding to smooth curves with trivial automorphism group. Since the complement in $\mathcal{M}_g$ is of codimension at least two, it is clear that if the restrictions of D_{k+1} and D'_{k+1} to $\mathcal{M}_g^0$ coincide set-theoretically, then we obtain equality over the whole moduli space.

A fundamental property of the variety $\mathcal{M}_g^0$ is that its Picard group (and hence the Picard group of $\mathcal{M}_g$, too) is cyclic. The generator is obtained in the following way. Consider the universal curve $\mathcal{C}$ over $\mathcal{M}_g^0$, and $\pi : \mathcal{C} \to \mathcal{M}_g^0$ the natural morphism. By definition, for any point $x \in \mathcal{M}_g^0$, the fibre $\mathcal{C}_x$ is the curve of genus g whose isomorphism class is given by x. Put ω_π the relative canonical bundle and let E be its direct image on $\mathcal{M}_g^0$. Then $\mathrm{Pic}(\mathcal{M}_g^0) \cong \mathbb{Z}.\lambda$, where $\lambda = c_1(E)$ [**Har83**]; cf. [**HM82**]. Moreover, since $\mathcal{M}_g^0$ admits a compactification to a projective variety such that that boundary is of codimension two [**Ar71**], it follows that any effective divisor on $\mathcal{M}_g^0$ which is rationally equivalent to zero is indeed zero.

Curves with trivial automorphism group are clearly non-hyperelliptic, and it follows that the morphism π factors through a natural canonical embedding of $\mathcal{C}$ in the projective bundle $p : \mathbb{P}(E) \to \mathcal{M}_g^0$. Following ad-litteram the argument given in Section 1, we endow (the restriction of) D'_{k+1} (to $\mathcal{M}_g^0$ a natural Cartier divisor structure, as a jump locus.

Now, we compare the rational classes

$$\begin{array}{rcl} v & = & [D'_{k+1}] = c_1(p_*(\bigwedge^l \mathcal{Q}(1) \otimes \mathcal{O}_{\mathcal{C}})) - c_1(p_*(\bigwedge^l \mathcal{Q}(1))) \\ c & = & [D_{k+1}] \end{array}$$

of D'_{k+1} and the $(k+1)$-gonal locus D_{k+1} in the Picard group of $\mathcal{M}_g^0$; cf. [**HM82**]. Put $\lambda = c_1(E)$. Using Grothendieck-Riemann-Roch and the Porteous formula, Hirschowitz and Ramanan showed that

$$v = 6(k+2)k\frac{(2k-2)!}{(k-1)!(k+1)!}\lambda.$$

In [**HM82**, Section 6] it was proved that the class of the $(k+1)$-gonal locus D_{k+1} is

$$c = 6(k+2)\frac{(2k-2)!}{(k-1)!(k+1)!}\lambda.$$

Hence $v = kc$ in $\mathrm{Pic}(\mathcal{M}_g^0)$.

Step 3. To conclude, use the fact that a curve corresponding to a generic point in D_{k+1} satisfies

$$\dim K_{k,1}(C, K_C) \geq k.$$

The idea of proof (that can be found in [**HR98**]) is to show that Green-Lazarsfeld classes generate a vector space of dimension at least k. These classes are scrollar and the Koszul cohomology of the scroll is contained in the Koszul cohomology of the curve. For details, see [**HR98**]. □

7.2. Stable curves with extra-syzygies

The main result discussed in this section is a degenerate version of Theorem 7.1. This result will be obtained via a deformation to the smooth case. For technical reasons, we restrict ourselves to stable curves with trivial automorphism group and very ample canonical bundle; see [**C82**, Theorem F] and [**CFHR99**, Theorem 3.6] for precise criteria of very ampleness.

Hirschowitz and Ramanan used the term *with extra-syzygies* to designate smooth curves C of genus $2k+1$ for which $K_{k,1}(C, K_C) \neq 0$. We adopt this terminology and extend it to singular stable curves with the same property. In the smooth case, extra-syzygies produce pencils. In the singular case, extra-syzygies will produce suitable torsion-free sheaves rather than line bundles, since the Jacobian of a singular curve is not necessarily compact. Recall that a coherent sheaf on a stable curve X is *torsion-free* if it has no non-zero subsheaf with zero-dimensional support [**Se82**].

We study the Koszul cohomology of a stable singular curve X of arithmetic genus $g = 2k+1$, with $k \geq 2$. Recall that *stable curves* are reduced connected curves with finite group of automorphisms, and with only simple double points (nodes) as possible singularities. They have been introduced by Deligne and Mumford with the aim of compactifying the moduli space $\mathcal{M}_g$ of smooth curves of genus g. Singular stable curves of arithmetic genus g lie on a normal-crossing divisor $\Delta_0 \cup \cdots \cup \Delta_{[g/2]}$ in $\overline{\mathcal{M}}_g$, on the boundary of $\mathcal{M}_g$, and the general element in Δ_0 is irreducible, whereas a general element in Δ_i is the union of two curves of genus i and $g-i$ respectively, meeting in one point.

It is known that the Picard group of $\overline{\mathcal{M}}_g$ is generated over $\mathbb{Q}$ by the classes $\delta_i = [\Delta_i]$ and by the Hodge class λ; see, for example [**HaM98**]. Intuitively, λ is the class of the line bundle over $\overline{\mathcal{M}}_g$ whose fibre over $[C]$ is $\bigwedge^g H^0(C, \omega_C)$. The bundle in question is nef and big (see, for example, [**Ar71**], [**HaM90**, Introduction]).

Definition 7.4. The open subspace in $\overline{\mathcal{M}}_g$ of points corresponding to stable curves of genus g with very ample canonical bundle, is denoted by $\mathcal{M}_g^\omega$.

Remark 7.5. It is easy to show that $\mathcal{M}_g^\omega$ is contained in $\mathcal{M}_g \cup \Delta_0$.

The Riemann-Roch Theorem implies that the degree of a line bundle A on a smooth curve C equals $\chi(C,A) - \chi(C, \mathcal{O}_C)$. Given a rank 1 torsion-free sheaf F on a stable curve X, one can view the quantity $\chi(X,F) - \chi(X,\mathcal{O}_X)$ as a substitute for the degree. Hence the degenerate analogue of a g^1_{k+1} will be a sheaf with $\chi(X,F) = 1-k$.

The degenerate version of Theorem 7.1 is the following statement

Theorem 7.6. *Let X be a singular stable curve of genus $g = 2k+1$ with very ample canonical bundle and trivial automorphism group. If*

$$K_{k,1}(X, \omega_X) \neq 0,$$

then there exists a torsion-free, ω_X-semistable sheaf F of rank one on X with $\chi(X,F) = 1-k$ and $h^0(X,F) \geq 2$.

We refer to [**Se82**] for a precise definition of semi-stability.

Proof: (of Theorem 7.6) Put $D^\omega_{k+1} = \overline{D}_{k+1} \cap \mathcal{M}^\omega_g$, and $\Delta^\omega_0 = \Delta_0 \cap \mathcal{M}^\omega_g$.

Step 1. We prove that the locus of curves with extra-syzygies in $\mathcal{M}^\omega_g$ is a divisor.

It amounts to proving that its inverse image on a covering $\mathcal{S}^\omega \to \mathcal{M}^\omega_g$ is a divisor. We choose a smooth $\mathcal{S}^\omega$ on which an universal curve exists.

Let $[X] \in \mathcal{M}^\omega_g$, and denote, for simplicity $\mathbb{P} := \mathbb{P}H^0(X, \omega_X)^\vee$, which contains the image of X, set $Q = T_{\mathbb{P}}(-1)$ the universal quotient bundle, and $Q_X = M^\vee_{\omega_X}$ the restriction of Q to X. As in the smooth case, the Koszul cohomology of X with values in ω_X has the following description

$$K_{p,1}(X,\omega_X) \cong \ker\left(H^0(\mathbb{P}, \wedge^{2k-p+1}Q(1)) \to H^0(X, \wedge^{2k-p+1}Q_X \otimes \omega_X)\right).$$

For the choice $p = k$, and for a smooth curve X, the two spaces appearing in the description above, namely $H^0(\mathbb{P}, \wedge^{k+1}Q(1))$ and $H^0(X, \wedge^{k+1}Q_X \otimes \omega_X)$, have the same dimension, see the proof of Theorem 7.1. Applying the Riemann-Roch Theorem and Serre duality, we observe that the two spaces in question are still of the same dimension if X is a singular stable curve with very ample canonical bundle, since $H^0(X, \wedge^{k+1}Q^\vee_X) = 0$. The vanishing of $H^0(X, \wedge^{m+1}Q^\vee_X)$ for any $m \geq 0$ follows from Proposition 2.4, and from the vanishing of $K_{p,0}(X,\omega_X)$ for $p \geq 1$.

By semi-continuity, the locus of curves in $\mathcal{M}^\omega_g$ with extra-syzygies is closed. Similarly to the proof of Theorem 7.1, the locus of points on $\mathcal{S}^\omega$ corresponding to curves with extra-syzygies is the degeneracy locus of a morphism of vector bundles of the same rank. Then this locus is a divisor, since we know that it is not the whole space by Theorem 6.14.

Step 2. We prove that the condition $K_{k,1}(X, \omega_X) \neq 0$ holds if and only if $[X]$ belongs to the closure $\overline{D}_{k+1}$ in $\mathcal{M}_g^\omega$ of the locus D_{k+1} of $(k+1)$-gonal smooth curves.

Imitating the argument given in the smooth case, we see that on $\mathcal{M}_g^\omega$, the divisor of curves with extra-syzygies is equal to a multiple of D_{k+1}^ω plus, possibly, a multiple of Δ_0^ω (recall that Δ_0 is irreducible, and so is Δ_0^ω). The possibility that the whole Δ_0^ω be contained in the locus of curves with extra-syzygies is ruled out by Corollary 6.16. In particular, it follows that a curve in Δ_0^ω has extra syzygies if and only if it belongs to D_{k+1}^ω; this is what we wanted to prove.

Step 3. We show that the condition of having extra-syzygies yields the existence of the desired torsion-free sheaves.

From Step 1, we know that a singular stable curve X with ω_X very ample and with extra-syzygies lies in a one-dimensional flat family $\mathcal{C} \to T$ of curves such that $\mathcal{C}_{t_0} \cong X$, and $\mathcal{C}_t$ are smooth, and belong to D_{k+1} for $t \neq t_0$. By shrinking T if needed, we can make the same hypothesis for the curves $\mathcal{C}_t$. By the compactification theory of the generalized relative Jacobian, see [**Ca94**] and [**P96**], there exists a family $\mathcal{J}_{1-k}(\mathcal{C}/T)$, flat and proper over T, whose fiber over $t \neq t_0$ is the Jacobian variety of line bundles of degree $k+1$, whereas the fiber over t_0 parametrizes gr-equivalence classes of torsion-free, ω_X-semistable sheaves F of rank one on X with $\chi(X, F) = 1-k$; see [**P96**], [**Se82**] for precise definitions. It follows that the subspace of pairs $\{(\mathcal{F}_t, \mathcal{C}_t) \in \mathcal{J}_{1-k}(\mathcal{C}/T) \times_T \mathcal{C},\ h^0(\mathcal{C}_t, \mathcal{F}_t) \geq 2\}$ is closed in the fibered product, and, since $[\mathcal{C}_t] \in D_{k+1}$ for all $t \neq t_0$, we conclude. $\square$

7.3. Curves with small Brill-Noether loci

In this section we show that the degenerate version of Theorem 7.1 obtained in the previous section implies the conjectures of Green and Green-Lazarsfeld for specific open subsets of every gonality stratum

$$\{[C] \in \mathcal{M}_g,\ \mathrm{gon}(C) = d\}$$

subject to the numerical condition

$$d < \left[\frac{g}{2}\right] + 2. \tag{7.1}$$

Note that this inequality excludes exactly one case, namely the case of odd genus and maximal gonality. In this case, the Green conjecture is known by Theorem 7.1. The Green-Lazarsfeld conjecture for this case will be proved in the next section.

The idea of the construction is as follows. Starting from a smooth curve C of genus g and gonality d satisfying (7.1), we produce a stable curve X to which one can apply Theorem 7.6. This curve X is obtained by repeatedly identifying points on C to create nodes. Before passing to the general case, we illustrate this technique in the simplest case.

Special case: $g = 2k$, $d = k + 1$. In this case, Green's conjecture states that $K_{k,1}(C, K_C) = 0$. Suppose that this is not the case. Choose two general points x and y on C, and let X be the nodal curve obtained by identifying these two points. Let $f : C \to X$ be the normalization map. The morphism f^* induces an isomorphism between $H^0(X, \omega_X)$ and $H^0(C, K_C(x + y))$, hence $H^0(C, K_C)$ can be viewed as a subspace of $H^0(X, \omega_X)$ and

$$K_{p,1}(C, K_C) \subset K_{p,1}(X, \omega_X)$$

for all p. The previous assumption implies that $K_{k,1}(X, \omega_X) \neq 0$. Hence, there exists a torsion-free rank 1 sheaf F on X, such that $h^0(X, F) \geq 2$ and $\chi(X, F) = 1 - k$. We now distinguish two cases:

(1) F is not locally free. Then there exists a line bundle A on C such that $F = f_*A$, by [**Se82**]. This leads to a contradiction, since $\deg(A) = k$, and $h^0(C, A) \geq 2$.
(2) F is a line bundle. Put $A = f^*F$. Then A has the following properties: $h^0(C, A) \geq 2$, $\deg(A) = k + 1$ and it is impossible to separate the points x and y by using sections of A.

Suppose that C satisfies the following additional condition (which is satisfied by a generic curve):

$$\dim W^1_{k+1}(C) = 0. \tag{7.2}$$

Consider the subvariety

$$\{(x, y),\ \text{there exists } g^1_{k+1} \text{ passing through } x + y\} \subset C \times C.$$

The fibers of the first projection are finite since we assumed (7.2), showing that this variety is one-dimensional. Hence we obtain a contradiction by the genericity of x and y.

Since $H^0(X, \omega_X) \cong H^0(C, K_C(x + y))$, the argument given above shows that $K_{k,1}(C, K_C(x + y)) = 0$. This condition implies the Green-Lazarsfeld conjecture for C, see Corollary 4.28.

The crucial ingredients of the above example were condition (7.2), and the vanishing

$$K_{k,1}(C, K_C(x + y)) = 0.$$

This generalizes as follows.

Theorem 7.7. *Let $d \geq 3$ be an integer, and C be a smooth d-gonal curve of genus g with $d < [g/2] + 2$, satisfying the following condition*

$$\dim (W^1_{d+n}(C)) \leq n \text{ for all } 0 \leq n \leq g - 2d + 2. \tag{7.3}$$

Then

$$K_{g-d+1,1}(C, K_C(x + y)) = 0$$

for general points x, y of C.

Proof: Let $\nu \geq 0$ be an integer and let X be the stable curve obtained by identifying $(\nu + 1)$ pairs of generic points (x_i, y_i), with $0 \leq i \leq \nu$, on A. Let $|A|$ be a minimal pencil on C, and $f : C \to X$ be the normalization morphism. The sheaf $F = f_*A$ is torsion-free of rank 1. In order to apply Theorem 7.6, we want to have the arithmetic genus of X equals $2k+1$, and $\chi(X, F) - \chi(X, \mathcal{O}_X) = k+2$ (the idea is that F deforms to a g^1_{k+2} under any smooth flat deformation).

The integers k and ν are determined as follows. Since $\chi(X,F)-\chi(X,\mathcal{O}_X)=d+\nu+1$, and $2k+1=g+\nu+1$ we obtain $k=g-d+1\geq 1$ and $\nu=g-2d+2\geq 0$. The case $\nu=0$ corresponds to the previous example.

By genericity, we can assume that for any choice of $n+1$ pairs (x_{i_j},y_{i_j}), with $0\leq j\leq n$ and $0\leq n\leq \nu$, among the points (x_i,y_i), there exists no $A_n\in W^1_{d+n}(C)$ such that $h^0(X,A_n(-x_{i_j}-y_{i_j}))\geq 1$ for all $0\leq j\leq n$. The $(\nu+1)$-tuple of cycles $(x_0+y_0,\ldots,x_\nu+y_\nu)$ can be chosen to be generic in the space $C^{(2)}\times\cdots\times C^{(2)}$. This is possible since for any n, the incidence variety

$$\{(x_0+y_0,\ldots,x_n+y_n,A_n),\ h^0(A_n(-x_i-y_i))\geq 1\text{ for all }i\}$$

is at most $(2n+1)$-dimensional, whereas dim $\left(C^{(2)}\times\cdots\times C^{(2)}\right)=2n+2$.

Since the cycles x_i+y_i are generic, the curve X has no non-trivial automorphisms and its canonical bundle is very ample (apply [**C82**, Theorem F], [**CFHR99**, Theorem 3.6]).

We prove first that $K_{k,1}(X,\omega_X)=0$. Suppose that $K_{k,1}(X,\omega_X)\neq 0$. From Proposition 7.6, we obtain a torsion-free sheaf F of rank one on X with $\chi(X,F)=1-k$, and $h^0(X,F)\geq 2$. The sheaf F is either a line bundle, or the direct image of a line bundle on a partial normalization of X. Observe that this partial normalization cannot be C itself. Indeed, if $F=f_*A$ with A a line bundle on C, then $\chi(C,A)=\chi(X,F)=1-k$, and $h^0(C,A)=h^0(F)\geq 2$, which means that A is a g^1_{d-1} on C, contradicting the hypothesis. Let us consider then $\varphi:Z\to X$ the normalization of the $(\nu-n)$ points $p_{n+1},\ldots,p_\nu$, for some $0\leq n\leq\nu$. Let furthermore $\psi:C\to Z$ be the normalization of the remaining $(n+1)$ points $p_0,\ldots,p_n$, and suppose $F=\varphi_*A$, for a line bundle A on Z. Under these assumptions, we obtain $\chi(A)=\chi(F)=1-k$, and so $\chi(\psi^*A)=2-k+n$, which implies that $\deg(\psi^*A)=d+n$. Besides, ψ^*A has at least two independent sections. Since for any node p_i with $0\leq i\leq n$ there is a non-zero section of F vanishing at p_i, it follows that $h^0(C,(\psi^*A)(-x_i-y_i))\geq 1$ for all $0\leq i\leq n$, which contradicts the choice we made.

We proved $K_{k,1}(X,\omega_X)=0$. Since $K_{k,1}(C,K_C(x_i+y_i))\subset K_{k,1}(X,\omega_X)$, [**V02**, p. 367], we obtain $K_{k,1}(C,K_C(x_i+y_i))=0$, for all i. ☐

REMARK 7.8. The condition (7.3) is satisfied by the generic d-gonal curve; see [**Ap05**].

COROLLARY 7.9. *Notation as in Theorem 7.7. Then* Cliff$(C)=d-2$*, and* C *satisfies the conjectures of Green, and Green-Lazarsfeld.*

Proof: The statement of Theorem 7.7 is the Green-Lazarsfeld conjecture for the bundle $K_C(x+y)$. Using Corollary 4.28 it follows that the Green-Lazarsfeld conjecture holds for the curve C (and is verified for any line bundle of degree at least $3g$),

It was observed by Voisin that $K_{k,1}(C,K_C)\subset K_{k,1}(C,K_C(x+y))$, [**AV03**]; hence $K_{k,1}(C,K_C)=0$. The vanishing $K_{k,1}(C,K_C)=0$ is the statement of the Green conjecture for C, the fact that Cliff(C) equals $d-2$ being implied by the Green-Lazarsfeld non-vanishing theorem, Corollary 3.36. ☐

For small d, one can use classical results due H. Martens, Mumford and Keem on the dimensions of the Brill-Noether loci, *cf.* [**ACGH85**], [**HMa67**], [**Mu74**], [**Ke90**] to obtain the following.

Corollary 7.10. *Let C be a non-hyperelliptic smooth curve of gonality $d \leq 6$, with $d < [g_C/2] + 2$, and suppose that C is not one of the following: plane curve, bielliptic, triple cover of an elliptic curve, double cover of a curve of genus three, hexagonal curve of genus* 10 *or* 11. *Then* $\mathrm{Cliff}(C) = d - 2$ *and C verifies both Green, and Green-Lazarsfeld conjectures.*

Proof: For a trigonal curve C, one has to prove that $\dim(W^1_{n+3}(C)) \leq n$ for all $0 \leq n \leq g_C - 4$. This follows from [**HMa67**, Theorem 1], as we know that $\dim(W^1_{n+3}(C)) \leq n+1$ and equality is never achieved, since C is non-hyperelliptic. If $d = 4$, one has to prove $\dim(W^1_{n+4}(C)) \leq n$ for all $0 \leq n \leq g_C - 6$. In this case, we apply Mumford's refinement to the Theorem of H. Martens, *cf.* [**Mu74**], which shows that $\dim(W^1_{n+4}(C)) \leq n + 1$, and equality could eventually hold only for trigonal (which we excluded), bielliptic curves or smooth plane quintics. The other cases $d = 5$ and $d = 6$ are similar, and follow from [**Ke90**, Theorem 2.1], and [**Ke90**, Theorem 3.1], respectively.

Remark 7.11. For trigonal curves, Green's conjecture was known to hold by the work of Enriques and Petri, and the Green-Lazarsfeld conjecture was verified by Ehbauer [**Ehb94**]. For tetragonal curves, the Green conjecture was verified by Schreyer [**Sch91**] and Voisin [**V88a**]. All the other cases are new. Plane curves, which were excepted from the statement, also verify the two conjectures, *cf.* [**Lo89**], and [**Ap02**]. Note that in a number of other cases for which the previous result does not apply, Green's conjecture is nonetheless satisfied; for instance, for hexagonal curves of genus 10 and Clifford index 3, complete intersections of two cubics in $\mathbb{P}^3$, *see* [**Lo89**].

For large d we cannot give similar precise results, but we still obtain a number of examples for which Theorem 7.7 can be applied. For instance, curves of even genus which are Brill-Noether-Petri generic satisfy the hypothesis of Theorem 7.7, so they verify the two syzygy conjectures. Other cases are obtained by looking at curves on some surfaces, when the special geometry of the pair $(curve, surface)$ is used, as in the following.

Corollary 7.12. *Let C be a smooth curve of genus $2k$ and maximal Clifford index $k - 1$, with $k \geq 2$ abstractly embedded in a $K3$ surface. Then C verifies the Green conjecture.*

Proof: Since the Clifford index of C is maximal, and Clifford index is constant in the linear system $|C|$, [**GL87**], the gonality is also maximal, and thus constant for smooth curves in $|C|$. Then the hypotheses of [**CP95**, Lemma 3.2 (b)] are verified, which implies that a general smooth curve in the linear system $|C|$ has only finitely many pencils of degree $k+1$. From Theorem 7.7 it follows that the Green conjecture is verified for a general smooth curve $C' \in |C|$, that is $K_{k,1}(C', K_{C'}) = 0$. By applying Green's hyperplane section theorem 2.20 (ii) twice, we obtain $K_{k,1}(C, K_C) = 0$, which means that C satisfies Green's conjecture, too. $\square$

Note that Corollary 7.12 does not apply to the particular curves considered by Voisin in [**V02**], [**V05**], as they are implicitly used in the proof.

7.4. Further evidence for the Green-Lazarsfeld conjecture

In the last section, we verified the conjectures of Green, and Green-Lazarsfeld for generic d-gonal curves, with one exception. In this section, we treat this case, which needs a slightly more complicated degeneration; see Remark 7.14.

THEOREM 7.13. *The Green-Lazarsfeld conjecture is valid for any smooth curve C of genus $g_C = 2k-1$ and gonality $k+1$, with $k \geq 2$.*

Proof: Note that $\dim(W^1_{k+1}(C)) = 1$, *see* [**FHL84**], [**ACGH85**, Lemma IV.(3.3) p. 181 and Ex. VII.C-2, p. 329]. Then one can find three distinct points x, y and z of C which do not belong at the same time to a pencil of degree $k+1$. Since the incidence variety $\{(x+y+z, A) \in C^{(3)} \times W^1_{k+1}(C), h^0(A(-x-y-z)) \geq 1\}$ is two-dimensional, and hence the image of its projection to $C^{(3)}$ is a surface, the cycle $x+y+z$ can be generically chosen in $C^{(3)}$; cf. the proof of Theorem 7.7,.

For these three generic points we prove that $K_{k,1}(C, K_C(x+y+z)) = 0$. This fact, together with Theorem 4.27, shows that the Green-Lazarsfeld conjecture is verified for any line bundle of degree at least $3g_C+1$ on C.

We suppose to the contrary that $K_{k,1}(C, K_C(x+y+z)) \neq 0$, and reach a contradiction. To this end, we introduce a curve X with two irreducible components: the first one is C, and the second one is a smooth rational curve E which passes through the points x, y and z. The curve X is stable, and of arithmetic genus $g = 2k+1$, and $K_{k,1}(X, \omega_X) \cong K_{k,1}(C, K_C(x+y+z))$. As in the proof of Theorem 7.7, from the genericity of the cycle $x+y+z$, we can suppose X is free from non-trivial automorphisms and with very ample canonical bundle.

From Proposition 7.6, we obtain a torsion-free, ω_X-semistable sheaf F of rank one on X with $\chi(F) = 1-k$ and $h^0(X, F) \geq 2$. We show that F yields either to a pencil of degree $k+1$ on C which passes through x, y, and z or to a pencil of degree at most k. Let F_E, and F_C be the torsion-free parts of the restrictions of F to E and C, respectively. It is known that there is a natural injection $F \to F_E \oplus F_C$ whose cokernel is supported at the points among x, y and z where F is invertible, [**Se82**]. We distinguish four cases according to the number of nodes where F is invertible.

Suppose F is invertible at all three x, y, and z. In this case, $F_{|E} = F_E$, $F_{|C} = F_C$, and we have two exact sequences

$$0 \to F_E(-3) \to F \to F_C \to 0, \tag{7.4}$$

and, respectively,

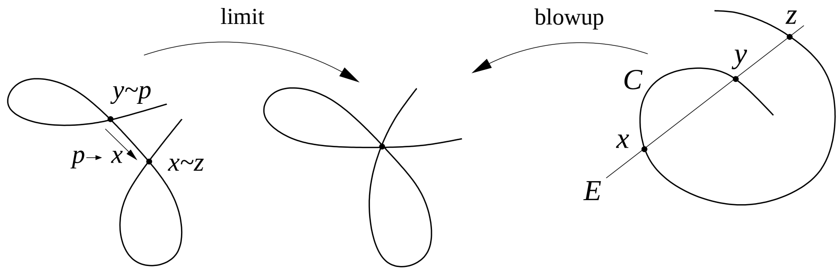
$$0 \to F_C(-x-y-z) \to F \to F_E \to 0. \tag{7.5}$$

The subsheaves $F_E(-3)$ and $F_C(-x-y-z)$ are of multiranks $(1,0)$ and, respectively $(0,1)$, and, since $\deg(\omega_{X|E}) = 1$, and $\deg(\omega_{X|C}) = 2g_C+1$, their ω_X-slopes are equal to

$$\mu(F_E(-3)) = \chi(F_E(-3)) = \deg(F_E) - 2,$$

and, respectively,

$$\mu(F_C(-x-y-z)) = \frac{\chi(F_C(-x-y-z))}{2g_C+1} = \frac{\deg(F_C) - 2 - g_C}{2g_C+1},$$



see, for example [**P96**, Definition 1.1]. The ω_X-slope of F equals

$$\mu(F) = \frac{1-k}{2g_C+2}.$$

From the ω_X-semistability of F, we obtain $\deg(F_E) \leq 1$ and $\deg(F_C) \leq k+2$. The exact sequence (7.4) implies $\chi(F_C) = \chi(F) - \chi(F_E(-3)) = 3 - k - \deg(F_E)$, hence $\deg(F_C) = k + 1 - \deg(F_E)$. These numerical relations force $\deg(F_C) \in \{k, k+1, k+2\}$.

If $\deg(F_C) = k+2$, then $F_E \cong \mathcal{O}_E(-1)$ hence any global section of F vanishes along E. Then any global section of F vanishes at all the three points x, y, and z. Since F has at least two sections, the sublinear system $H^0(F) \subset H^0(F_C)$ on C has x, y, and z as base-points, in particular $h^0(F_C(-x-y-z)) \geq 2$. Then C carries a g^1_{k-1}, fact which contradicts the hypothesis.

If $\deg(F_C) \leq k+1$, from (7.4) we obtain $h^0(C, F_C) \geq h^0(X, F) \geq 2$. Since C does not carry a g^1_k, it follows that $F_E = \mathcal{O}_E$, and F_C is a base-point-free g^1_{k+1} on C. Let σ be a non-zero global section of F_C which vanishes at x; such a σ exists as $h^0(C, F_C) = 2$. Then σ is the restriction of global section σ_0 of F, as the restriction morphism on global sections is in this case an isomorphism. Since $F_E = \mathcal{O}_E$, the restriction of σ_0 to E is a constant function. The section σ_0 vanishes at x, hence it vanishes on the whole E. In particular, σ_0 vanishes at y and z, and hence σ vanishes at y and z as well. This is a contradiction, as we supposed that there was no such a σ.

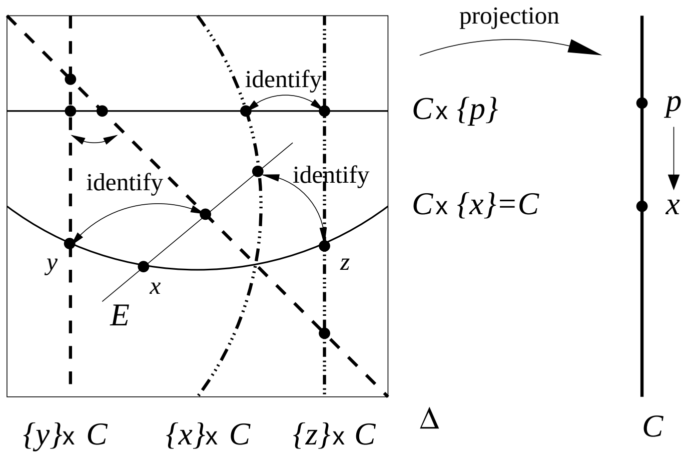
The other three remaining cases (F is invertible at y, and z, and is not invertible at x, or F is invertible at x, and is not invertible at y and z, or $F \cong F_E \oplus F_C$) are solved in a similar manner. The idea is always to use semi-stability, which means briefly that the slopes of the restrictions to the two components cannot differ too much; we refer to [**Ap05**] for a complete proof. $\square$

Remark 7.14. The stable curves considered in this section are limits of irreducible curves with two nodes. The limit is obtained by blowing up the point (x, x) on $C \times C$, and identifying the strict transforms of the diagonal and of $y \times C$ on the one hand, and the strict transforms of $x \times C$ and of $z \times C$ on the other hand (see the figures below). This fact indicates that this case is "more degenerate" than the others.

We obtain directly from Theorem 7.13 a version of the Hirschowitz-Ramanan-Voisin Theorem for syzygies of pluricanonical curves.

Corollary 7.15. *If C is a smooth curve of genus $2k-1$, where $k \geq 2$, with*

$$K_{2(2n-1)(k-1)-(k+1),1}(C, K_C^{\otimes n}) \neq 0$$



for some $n \geq 2$*, then* C *carries a* g^1_k.

7.5. Exceptional curves

It was proved in the previous Sections that Green conjecture is valid for curves with small Brill-Noether loci. It remains to verify the Green conjecture for curves which do not verify condition (7.3). One result in this direction was proved in [**AP06**].

THEOREM 7.16. *Let* S *be a* $K3$ *surface with* $\mathrm{Pic}(S) = \mathbb{Z}.H \oplus \mathbb{Z}\ell$*, with* H *very ample,* $H^2 = 2r - 2 \geq 4$*, and* $H.\ell = 1$*. Then any smooth curve in the linear system* $|2H + \ell|$ *verifies the Green conjecture.*

Smooth curves in the linear system $|2H + \ell|$ count among the few known examples of curves whose Clifford index is not computed by pencils, i.e. $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$, as was shown in [**ELMS89**] (other obvious examples are given by plane curves, for which the Green conjecture was checked in [**Lo89**]). Such curves are the most special ones in the moduli space of curves from the point of view of the Clifford dimension; for this reason, some authors call them *exceptional curves*. Hence, the case of smooth curves in $|2H + \ell|$ may be considered as opposite to that of a generic curve of fixed gonality. Note that these exceptional curves carry a one-parameter family of pencils of minimal degree (see [**ELMS89**]), hence the condition (7.3) of Theorem 7.7 is not satisfied.

The idea of the proof is to look at the family of pairs (C, A), with $C \in |2H|$ smooth and $A \in W^1_{2r-2+n}(C)$, and to give a bound on the dimension of the irreducible components dominating $|2H|$, then apply a version of Theorem 7.7. Thanks to the work of Lazarsfeld and Mukai, to the data (C, A) (for simplicity we assume here that A is a complete and base-point-free pencil) one can attach a rank-2 vector bundle $E(C, A)$ on the surface S. If this bundle is simple, then the original argument of Lazarsfelds [**La86**], or the variant provided by Pareschi [**Pare95**], allows one to determine these dimensions. In the non-simple case a useful lemma (see [**GL87**], [**DM89**] and [**CP95**]), leads to a very concrete description of the parameter space for such bundles. This description, together with the infinitesimal approach of Pareschi [**Pare95**], allows us to conclude.

Another consequence of the argument given is that the Green-Lazarsfeld conjecture holds for the generic curves in the linear system $|2H|$, see [**AP06**, Corollary 4.5].

7.6. Notes and comments

The proof of Theorem 7.1 relies on a comparison of the two divisors D_{k+1} and D'_{k+1} in the moduli space $\mathcal{M}_{2k+1}$. The hardest part is to show that the latter is a genuine divisor, and does not cover the whole moduli space; this is implied by Theorem 6.14.

Starting from Theorems 6.9 and 6.14, C. Voisin had the idea to degenerate a smooth curve on a $K3$ surface to an irreducible nodal curve X in the same linear system, in order to make the Koszul cohomology of X vanish. The aim was to verify the Green conjecture for the normalisation of X; see [**V02**]. This resulted into a very short and elegant solution for the Green conjecture for generic curves C of non-maximal gonality larger than $g_C/3$. A previous result of M. Teixidor [**Tei02**], using completely different methods, implied the Green conjecture for generic d-gonal curves with d bounded from above in the range $g_C/3$. The obstruction to an extension of Voisin's approach is given by the existence of nodal curves with a prescribed number of nodes.

Along the same lines, C. Voisin remarked that the degeneration method to nodal curves leads, in combination with Theorem 4.27, to a solution for the Green-Lazarsfeld conjecture for generic curves of fixed gonality in the same order range; cf.[**AV03**, Theorem 1.3] and [**AV03**, Theorem 1.4]. This strategy did much better than the partial result [**Ap02**, Theorem 3]; see [**AV03**, Remark 2] for some related comments.

At the other end of the spectrum, normalizations of irreducible nodal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ have been used to verify the two conjectures for generic curves C in any stratum of gonality bounded in the order $\sqrt{g_C}$, see [**Sch89**, Theorem], and [**Ap02**, Theorem 4], the vanishing of the Koszul cohomology having been proved, in contrast to Voisin, directly on the normalizations. Further connections between Green and Green-Lazarsfeld conjectures are discussed in [**Ap02**, Appendix and II].

The two conjectures seem to be intricately related to each other, altough the precise relation between them remain unclear for the moment. However, section 7.3 gives clear evidence pleading for unity. The idea there was to use degeneration to nodal curves, and to apply Theorem 7.1. Voisin's degenerations on $K3$ surfaces indicate that the right method is to compute the Koszul cohomology directly on nodal curves, instead of normalising them first as in [**Ap02**] and [**Sch89**]. The proof of Theorem 7.7 shows that this is a very natural thing to do.

The case considered in section 7.4 is somewhat different. Two points x and y added to the canonical bundle K_C will never suffice to verify the Green-Lazarsfeld conjecture. The reason is that the Brill-Noether locus of pencils of minimal degree on C is one-dimensional, hence through any two points on the curve passes a minimal pencil. By the Green-Lazarsfeld nonvanishing (Corollary 3.36) we cannot obtain the desired vanishing for $K_C(x+y)$. A similar phenomenon occurs for any curve carrying infinitely many minimal pencils. For any such curves, we need to add at least three points to the canonical bundle in order to verify the Green-Lazarsfeld conjecture.

CHAPTER 8

Applications

8.1. Koszul cohomology and Hodge theory

In this section we discuss the relationship between Koszul cohomology and infinitesimal computations in Hodge theory.

8.1.1. Variations of Hodge structure. We present a brief introduction to the theory of variations of Hodge structure and infinitesimal variations of Hodge structure, including only the basic definitions; see [**CMP**] or [**V02b**] for a more detailed treatment. A short introduction to infinitesimal variations of Hodge structure is given in [**Ha85**].

Let X and S be smooth quasi–projective varieties over $\mathbb{C}$, and let $f : X \to S$ be a smooth, projective morphism of relative dimension n. Given a simply connected open subset $U \subset S$ and a base point $0 \in U$, Ehresmann's fibration theorem provides us with a diffeomorphism $f^{-1}(U) \simeq X_0 \times U$ that allows us to identify $H^n(X_s, \mathbb{Z})$ with $H^n(X_0, \mathbb{Z})$ for all $s \in U$. Taking the Hodge filtration $F^\bullet$ on the cohomology of the fibers, we locally obtain a family of filtrations $\{F^pH^n(X_s, \mathbb{C})\}_{s\in U}$ on a fixed vector space $H^n(X_0, \mathbb{C}) \cong H^n(X_s, \mathbb{C})$. The groups $H^n(X_s, \mathbb{Z})$ $(s \in S)$ glue together to give a locally constant sheaf $R^nf_*\mathbb{Z}$. The associated holomorphic vector bundle

$$\mathcal{H} = R^nf_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_S$$

is called the *Hodge bundle*. It admits a decreasing filtration by holomorphic subbundles

$$\mathcal{H} = \mathcal{F}^0 \supset \mathcal{F}^1 \supset \ldots \supset \mathcal{F}^{n-1} \supset \mathcal{F}^n$$

such that the fiber of $\mathcal{F}^p$ over $s \in S$ is $F^pH^n(X_s, \mathbb{C})$. The Hodge bundle comes equipped with a natural connection, the *Gauss–Manin connection*

$$\nabla : \mathcal{H} \to \Omega^1_S \otimes \mathcal{H}$$

whose flat sections are the sections of the local system $R^nf_*\mathbb{Z}$. This connection satisfies the *Griffiths transversality* property

$$\nabla(\mathcal{F}^p) \subseteq \Omega^1_S \otimes \mathcal{F}^{p-1}.$$

Hence ∇ induces maps

$$\overline{\nabla} : \mathcal{F}^p/\mathcal{F}^{p+1} \to \Omega^1_S \otimes \mathcal{F}^{p-1}/\mathcal{F}^p$$

for all p. These maps are $\mathcal{O}_S$–linear. The data $(\mathcal{H}, \mathcal{F}^\bullet, \nabla)$ is called a *variation of Hodge structure*.

Fix a base point $0 \in S$, and put $H_{\mathbb{Z}} = H^n(X_0, \mathbb{Z})$. The abelian group $H_{\mathbb{Z}}$ carries a Hodge structure of weight n, i.e., $H_{\mathbb{C}} = H_{\mathbb{Z}} \otimes \mathbb{C}$ admits a Hodge decomposition

$$H_{\mathbb{C}} = \bigoplus\nolimits_{p+q=n} H^{p,q}, \quad \overline{H^{p,q}} = H^{q,p}.$$

Cup product defines a polarisation

$$Q : H_{\mathbb{Z}} \times H_{\mathbb{Z}} \to H^{2n}(X_0, \mathbb{Z}) \cong \mathbb{Z}$$

such that $Q_{\mathbb{C}} : H^{p,q} \times H^{n-p,n-q} \to \mathbb{C}$ is a perfect pairing. Let $T = T_0 S$ be the holomorphic tangent space to S at 0. The fiber of $\overline{\nabla}$ at 0 provides us with maps

$$\overline{\nabla}_0 : H^{p,q} \to \mathrm{Hom}(T, H^{p-1,q+1})$$

for all p and q. Hence we obtain a map

$$\delta : T \to \bigoplus_p \mathrm{Hom}(H^{p,q}, H^{p-1,q+1})$$

defined by $\delta(v)(\xi) = \overline{\nabla_0}(v)(\xi)$; this map is called the *differential of the period map.* Griffiths showed that the map δ is given by cup product with the Kodaira–Spencer class, i.e., we have $\delta(v)(\xi) = \kappa(v) \cup \xi$ with

$$\kappa : T \to H^1(X_0, T_{X_0})$$

the Kodaira–Spencer map. Furthermore the operators $\delta(v_1)$ and $\delta(v_2)$ commute for all v_1, $v_2 \in T$ and one has the relation

$$Q(\delta(v)\xi, \eta) = -Q(\xi, \delta(v)\eta) \tag{8.1}$$

for all $v \in T$, $\xi \in H^{p,q}$, $\eta \in H^{n-p+1,n-q+1}$. The data $(H_{\mathbb{Z}}, F^{\bullet}, Q, T, \delta)$ constitute an *infinitesimal variation of Hodge structure* (IVHS). We shall present several examples where calculations with infinitesimal variations of Hodge structures can be reduced to vanishing theorems for Koszul cohomology.

An important ingredient used in the first two examples is Griffiths's description of the primitive cohomology of hypersurfaces in projective space using Jacobi rings [**Gri69**]. Given a smooth hypersurface $X = V(f) \subset \mathbb{P}^{n+1}$ of degree d, consider the *Jacobi ideal* $J = (\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_{n+1}})$ in the polynomial ring $S = \mathbb{C}[x_0, \ldots, x_{n+1}]$ and define $R = S/J$. The ring R is a graded ring called the *Jacobi ring*, and Griffiths proved that

$$H^{n-p,p}_{\mathrm{prim}}(X) \cong R_{(p+1)d-n-2}.$$

8.1.2. Explicit Noether–Lefschetz. Let

$$S_U = \{(x, [F]) \mid F(x) = 0\}$$

be the universal family of smooth surfaces of degree d in $\mathbb{P}^3$, with

$$U \subset H^0(\mathbb{P}^3, \mathcal{O}_P(d))/PGL(4)$$

the open subset parametrising smooth surfaces. Recall that the *Noether–Lefschetz locus* is the subset

$$\mathrm{NL}_d = \{t \in U \mid \mathrm{Pic}(S_t) \neq \mathbb{Z}[\mathcal{O}_S(1)]\} \subset U.$$

It is known to be a countable union of closed subvarieties of U. The Noether–Lefschetz problem asks for which values of d we have $\mathrm{NL}_d \subsetneq U$. By the Lefschetz (1,1)–theorem, t belongs to NL_d if and only if there exists a primitive Hodge class $\lambda_t \in H^{1,1}_{\mathrm{prim}}(S_t) \cap H^2(S_t, \mathbb{Z})$. Let Σ be an irreducible component of NL_d, and let $0 \in \Sigma$ be a base point which is chosen outside the singular locus of Σ. Put $H = H^2_{\mathrm{prim}}(S_0, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, and let $\lambda \in H^{1,1}_{\mathrm{prim}}(S_0) \cap H^2(S_0, \mathbb{Z})$ be a nonzero primitive Hodge class. Let T_0 be the tangent space to U at 0, and consider the map

$$\delta : T_0 \to \mathrm{Hom}(H^{1,1}, H^{0,2})$$

induced by the differential of the period map. The Hodge class λ remains of Hodge type (1,1) under an infinitesimal deformation with direction v if and only if $\delta(v)(\lambda) = 0$. Let Q be the polarisation on H. Using the relation (8.1) we obtain

$$\begin{aligned}\delta(v)(\lambda) = 0 \quad &\iff \quad \forall \mu \in H^{2,0}, Q(\delta(v)(\lambda), \mu) = 0\\ &\iff \quad \forall \mu \in H^{2,0}, Q(\lambda, \delta(v)(\mu)) = 0.\end{aligned}$$

Let $T_0' \subset T_0$ be the tangent space to Σ at 0. The preceding discussion shows that the image of the map

$$\psi : T_0' \otimes H^{2,0} \to H^{1,1}, \quad \psi(v \otimes \mu) = \delta(v)(\mu)$$

is contained in the orthogonal complement $\lambda^\perp$ of λ. In particular, the map ψ cannot be surjective. We now translate this problem into the language of Jacobi rings. It can be shown that $T_0 \cong R_d$ [**Gri69**], hence T_0' can be identified with a subspace of R_d. Under this identification, the map ψ is induced by the multiplication map

$$R_d \otimes R_{d-4} \to R_{2d-4}.$$

Let W be the inverse image of T_0' under the projection from S_d to R_d. Then W is a base–point free linear subspace of S_d (since it contains the Jacobi ideal J, and the surface S_0 is smooth) with the property that the map $W \otimes R_{d-4} \to R_{2d-4}$ is not surjective. It follows that the multiplication map $W \otimes S_{d-4} \to S_{2d-4}$ is not surjective. Hence codim $W \geq d - 3$ by Corollary 2.40, and we obtain the following result due to Green [**Gre84b**], [**Gre88**] and Voisin [**V88b**].

THEOREM 8.1 (Explicit Noether–Lefschetz theorem). *Every irreducible component of* NL_d *has codimension at least* $d - 3$.

This result is a refinement of the classical Noether–Lefschetz theorem. Note that the degree estimate is sharp, since the component of surfaces in $\mathbb{P}^3$ containing a line has codimension $d - 3$.

8.1.3. The image of the Abel-Jacobi map. Let $X = V(f) \subset \mathbb{P}^4$ be a smooth hypersurface of degree d. Griffiths constructed a complex torus

$$J^2(X) = H^2(X, \mathbb{C})/F^2H^3(X, \mathbb{C}) + H^3(X, \mathbb{Z}) \cong F^1H^3(X, \mathbb{C})^\vee/H_3(X, \mathbb{Z}),$$

the *intermediate Jacobian*, and an *Abel–Jacobi map*

$$\mathrm{AJ}_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$$

from the Chow group of homologically trivial codimension two cycles to this complex torus. This map is defined as follows. Given a codimension two cycle Z with zero homology class, choose a topological 3–chain γ such that $\partial\gamma = Z$, and let ψ_Z be the integration current that sends a 3–form ω to $\int_\gamma \omega$. One then shows that the class of ψ_Z is well–defined in $J^2(X)$.

Let $f : X_T \to T$ be a family of hypersurfaces in $\mathbb{P}^4$. To this family we can associate a holomorphic fiber space of complex tori

$$J^2(X_T/T) = \cup_{t \in T} J^2(X_t).$$

Let $\mathcal{H} = R^3 f_* \mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_T$ be the Hodge bundle. The sheaf

$$\mathcal{J} = \mathcal{H}^3/\mathcal{F}^2 + R^3 f_* \mathbb{Z}$$

is the sheaf of holomorphic sections of the fiber space $J^2(X_T/T)$. A holomorphic global section of $\mathcal{J}$ is called a *normal function*. (If T is not projective, one should add growth conditions at infinity; cf. [**Z84**] for a discussion of this issue.) Given a family $\{Z_t\}_{t\in T}$ of homologically trivial codimension two cycles, we obtain a normal function ν of $\mathcal{J}$ by setting $\nu(t) = \mathrm{AJ}(Z_t)$. Using the Griffiths transversality property of the Gauss–Manin connection ∇ on $\mathcal{H}$, we obtain a commutative diagram

$$\begin{array}{ccccc}
\mathcal{F}^2 + R^3 f_*\mathbb{Z} & \xrightarrow{\nabla} & \Omega^1_T \otimes \mathcal{F}^1 & \xrightarrow{\nabla} & \Omega^2_T \otimes \mathcal{F}^0 \\
\downarrow & & \downarrow & & \| \\
\mathcal{H} & \xrightarrow{\nabla} & \Omega^1_T \otimes \mathcal{H} & \xrightarrow{\nabla} & \Omega^2_T \otimes \mathcal{H} \\
\downarrow & & \downarrow & & \\
\mathcal{J} & \xrightarrow{\overline{\nabla}} & \Omega^1_T \otimes \mathcal{H}/\mathcal{F}^1. & &
\end{array}$$

A normal function coming from a family of algebraic cycles has the additional property of being *horizontal.* This means that it is a section of the sheaf

$$\mathcal{J}_{\mathrm{hor}} = \ker(\overline{\nabla} : \mathcal{J} \to \Omega^1_T \otimes \mathcal{H}/\mathcal{F}^1).$$

To a horizontal normal function one can associate an *infinitesimal invariant.* This is done in the following way. Since ∇ is a flat connection, the commutative diagram above defines a short exact sequence of complexes of sheaves

$$0 \to \mathcal{F}^\bullet \to \mathcal{H}^\bullet \to \mathcal{J}^\bullet \to 0.$$

The connecting homomorphism associated to this short exact sequence gives a map from $\mathcal{J}_{\mathrm{hor}}$ to the first cohomology sheaf $\mathcal{H}^1(\mathcal{F}^\bullet)$ of the complex $\mathcal{F}^\bullet$. Taking the induced map on global sections, we obtain a map

$$\delta : H^0(T, \mathcal{J}_{\mathrm{hor}}) \to H^0(T, \mathcal{H}^1(\mathcal{F}^\bullet)).$$

The image of a horizontal normal function ν under this map is called the infinitesimal invariant $\delta\nu$. This invariant enables us to decide whether a normal function has a locally constant lifting to $\mathcal{H}$. Specifically, if $\tilde{\nu}$ is a local lifting of ν to the Hodge bundle $\mathcal{H}$, horizontality of ν implies that $\nabla\tilde{\nu}$ is a section of $\Omega^1_T \otimes \mathcal{F}^1$. If $\delta\nu = 0$, there exists locally a section $\hat{\nu}$ of $\mathcal{F}^2$ such that $\nabla\hat{\nu} = \nabla\tilde{\nu}$, hence $\tilde{\nu} - \hat{\nu}$ is a flat local lifting of ν.

The existence of locally constant liftings implies strong rigidity properties of the normal function. In the case at hand, a monodromy argument due to Voisin [**V94**, Prop. 2.6] shows that if $\delta\nu = 0$ then ν is a torsion section of $\mathcal{J}$. It remains to see when the condition $\delta\nu = 0$ is satisfied. Using semicontinuity and a little homological algebra, one shows that $\delta\nu = 0$ if the complexes

$$H^{2,1}(X_t) \xrightarrow{\overline{\nabla}_t} \Omega^1_{T,t} \otimes H^{1,2}(X_t) \xrightarrow{\overline{\nabla}_t} \Omega^2_{T,t} \otimes H^{0,3}(X_t)$$
$$H^{3,0}(X_t) \xrightarrow{\overline{\nabla}_t} \Omega^1_{T,t} \otimes H^{2,1}(X_t) \xrightarrow{\overline{\nabla}_t} \Omega^2_{T,t} \otimes H^{1,2}(X_t)$$

are exact at the middle term.

Consider the universal family $X_T \to T$ of hypersurfaces in $\mathbb{P}^4$ (more precisely, an étale base change of this family). Dualising the above complexes and rewriting them using the Jacobi ring, we obtain complexes

$$\textstyle\bigwedge^2 R_d \otimes R_{d-5} \to R_d \otimes R_{2d-5} \to R_{3d-5}$$
$$\textstyle\bigwedge^2 R_d \otimes R_{2d-5} \to R_d \otimes R_{3d-5} \to R_{4d-5}.$$

The exactness of these complexes is governed by the following result due to Donagi, with subsequent refinements by Green [**DG84**].

Lemma 8.2 (Symmetrizer Lemma). *The complex*

$$\bigwedge^2 R_d \otimes R_{k-d} \to R_d \otimes R_k \to R_{k+d}$$

is exact at the middle term if $k \geq d+1$.

Proof: A diagram chase shows that it suffices to show that the complex

$$\bigwedge^2 S_d \otimes R_{k-d} \to S_d \otimes R_k \to R_{k+d}$$

at the middle term. The commutative diagram

$$\begin{array}{ccccc} \bigwedge^2 S_d \otimes J_{k-d} & \to & S_d \otimes J_k & \to & J_{k+d} \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge^2 S_d \otimes S_{k-d} & \to & S_d \otimes S_k & \to & S_{k+d} \\ \downarrow & & \downarrow & & \downarrow \\ \bigwedge^2 S_d \otimes R_{k-d} & \to & S_d \otimes R_k & \to & R_{k+d} \end{array}$$

shows that it suffices to have exactness of

$$\bigwedge^2 S_d \otimes S_{k-d} \to S_d \otimes S_k \to S_{k+d}$$

at the middle term and surjectivity of the map $S_d \otimes J_k \to J_{k+d}$. The second condition is satisfied if $k \geq d-1$, since the ideal J is generated in degree $d-1$. Using Theorem 2.39, we find that the first condition holds if $k \geq d+1$. □

Using the symmetrizer Lemma, one obtains the following result due to Green [**Gre89**] and Voisin (unpublished).

Theorem 8.3 (Green–Voisin). *Let $X \subset \mathbb{P}^4$ be a very general hypersurface of degree d. If $d \geq 6$, then the image of the Abel–Jacobi map*

$$\mathrm{AJ}_X : \mathrm{CH}^2_{\mathrm{hom}}(X) \to J^2(X)$$

is contained in the torsion points of $J^2(X)$.

8.1.4. Hodge theory and Green's conjecture. The following reformulation of Green's conjecture in terms of variations of Hodge structure is due to Voisin. Recall that the generic Green conjecture states that

$$K_{k,1}(C, K_C) = 0$$

if C is a general curve of genus $g \in \{2k, 2k+1\}$, conjecture 4.13. The starting point of Voisin's construction is to rewrite the complex

$$\bigwedge^{k+1} H^0(C, K_C) \to \bigwedge^k H^0(C, K_C) \otimes H^0(C, K_C) \to \bigwedge^{k-1} H^0(C, K_C) \otimes H^0(C, K_C^2)$$

in terms of the cohomology of the Jacobian of C. As the Jacobian $J(C)$ is an abelian variety, we have an isomorphism of Hodge structures $H^k(J(C)) \cong \bigwedge^k H^1(C)$ for all $k \geq 0$. Hence

$$\begin{aligned} H^{p,q}(J(C)) &\cong \bigwedge^p H^{1,0}(C) \otimes \bigwedge^q H^{0,1}(C) \\ &\cong \bigwedge^p H^0(C, K_C) \otimes \bigwedge^q H^1(C, \mathcal{O}_C). \end{aligned}$$

Using Serre duality we can rewrite this as

$$H^{p,q}(J(C)) \cong \bigwedge^p H^0(C, K_C) \otimes \bigwedge^{g-q} H^0(C, K_C).$$

In particular, we obtain the isomorphisms

$$\bigwedge^{k+1} H^0(C,K_C) \cong H^{0,g-k-1}(J(C)),\ \ \bigwedge^{k} H^0(C,K_C)\otimes H^0(C,K_C) \cong H^{1,g-k}(J(C))$$

for the first two groups appearing in the Koszul complex. Let $\mathcal{M}_g^0$ be the open subset of the moduli space $\mathcal{M}_g$ parametrising smooth curves of genus g with trivial automorphism group, and consider the variation of Hodge structure associated to the Jacobian fibration $f: \mathcal{J} \to \mathcal{M}_g^0$. Put $\mathcal{H}^k = R^k f_*\mathbb{Z} \otimes_{\mathbb{Z}} \mathcal{O}_{\mathcal{M}_g^0}$, and write

$$\mathcal{H}^{p,q} = \mathcal{F}^p\mathcal{H}^{p+q}/\mathcal{F}^{p+1}\mathcal{H}^{p+q}.$$

By Griffiths transversality, the Gauss–Manin connection induces $\mathcal{O}_{\mathcal{M}_g^0}$–linear maps

$$\overline{\nabla} : \mathcal{H}^{p,q} \to \Omega^1_{\mathcal{M}_g^0} \otimes \mathcal{H}^{p-1,q+1}.$$

Given $[C] \in \mathcal{M}_g^0$, we have $T_{[C]}\mathcal{M}_g \cong H^1(C,T_C) \cong H^0(C,K_C^2)^\vee$. Taking $(p,q) = (1, g-k)$ and restricting to the fiber over $[C]$, we obtain (using the previous identifications) a map

$$\overline{\nabla}_C : H^0(C,K_C) \otimes \bigwedge^k H^0(C,K_C) \to H^0(C,K_C^2) \otimes \bigwedge^{k-1} H^0(C,K_C)$$

which can be identified with the Koszul differential. The map

$$\bigwedge^{k+1} H^0(C,K_C) \to \bigwedge^{k} H^0(C,K_C) \otimes H^0(C,K_C)$$

is identified with the map

$$\theta : H^{0,g-k-1}(J(C)) \to H^{1,g-k}(J(C))$$

given by cup product with the class $\theta \in H^{1,1}(J(C))$ of the theta divisor. The cokernel of this map is isomorphic to the primitive cohomology $H^{1,g-k}_{\mathrm{prim}}(J(C))$. Hence we obtain the following reformulation of the generic Green conjecture in terms of Hodge theory.

PROPOSITION 8.4 (Voisin). *Green's conjecture holds for a general curve C of genus $g \in \{2k, 2k+1\}$ if and only if the map*

$$\overline{\nabla}_C : H^{1,g-k}_{\mathrm{prim}}(J(C)) \to \Omega^1_{\mathcal{M}_g,[C]} \otimes H^{0,g-k+1}_{\mathrm{prim}}(J(C))$$

induced by the Gauss–Manin connection is injective.

8.2. Koszul divisors of small slope on the moduli space.

In this Section we discuss briefly the slope conjecture, and the counter-examples found recently by G. Farkas and M. Popa. We begin by recalling the statement and the motivation behind this conjecture.

Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves. In order to study the birational geometry of $\overline{\mathcal{M}}_g$, one could ask for a description of the cones

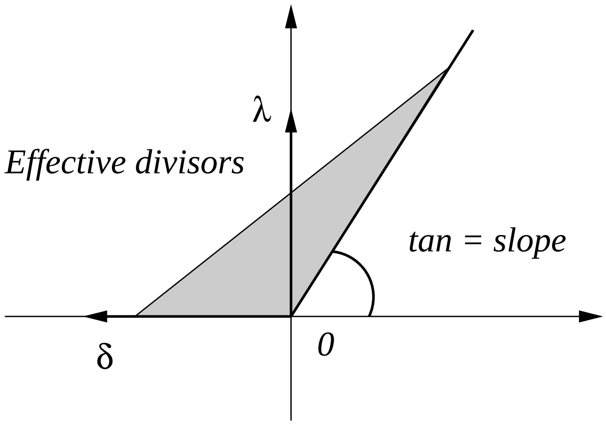
$$\mathrm{Ample}(X) \subset \mathrm{Eff}(X) \subset \mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$$

of ample and effective divisors on $\overline{\mathcal{M}}_g$. This question goes back to Mumford [**Mu77**]. The effective cone describes the *rational contractions* of $\overline{\mathcal{M}}_g$, i.e., rational maps from $\overline{\mathcal{M}}_g$ to projective varieties with connected fibers. Furthermore, it gives information about the Kodaira dimension of $\overline{\mathcal{M}}_g$; see Remark 8.7. The description of the full ample and effective cones turns out to be a difficult problem. (Gibney, Keel and Morrison [**GKM02**] have proposed a conjectural description of the ample cone.) Hence one often pursues a more modest goal, the description of the intersection of these cones with a suitable plane in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$.

It is known that $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ is generated by the Hodge class λ (which is nef and big) and the boundary classes δ_i, $i = 0, \ldots, [\frac{g}{2}]$. Given a class γ which is an effective linear combination of the boundary divisors, let

$$\Lambda_\gamma = \mathbb{R}.\lambda + \mathbb{R}.(-\gamma)$$

be the plane inside $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ spanned by λ and $-\gamma$. The only interesting part of the intersection of $\mathrm{Ample}(X)$ and $\mathrm{Eff}(X)$ with this plane is the first quadrant; cf. [**Ha87**]. Specifically, if we define the *slope* of $a\lambda - b\gamma$ $(a > 0,\ b > 0)$ as the number $s = a/b$, both cones will be bounded from below by a ray of given slope s_{ample} resp. s_{eff} (see the figure).



In practice, one usually works with the plane Λ_δ spanned by λ and the class $\delta = \sum_i \delta_i$ of the boundary divisor $\Delta = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$; we shall exclusively work with this plane in the sequel. Cornalba and Harris [**CH87**] showed that in this case $s_{\mathrm{ample}} = 11$.

The description of $\mathrm{Eff}(X) \cap \Lambda_\delta$ is more delicate. The constant s_{eff} is usually denoted by s_g and is called *the slope of* $\overline{\mathcal{M}}_g$. By what we have said before, we have

$$s_g = \inf\left\{\frac{a}{b} \mid a\lambda - b\delta \text{ is effective }, a, b > 0\right\}.$$

To obtain a lower bound for s_g, consider an effective divisor D whose support does not contain any boundary divisor Δ_i. The class of such a divisor has the form

$$[D] = a\lambda - \sum_i b_i\delta_i$$

with $a \geq 0$, $b_i \geq 0$ for all i. Put $b = \min\left\{b_i | i = 0, \ldots, [\frac{g}{2}]\right\}$, and define *the slope of* D by the formula

$$s(D) = a/b.$$

By construction $a\lambda - b\delta$ belongs to $\mathrm{Eff}(\overline{\mathcal{M}}_g) \cap \Lambda_\delta$, hence $s_g \leq s(D)$.

Consider the *Brill–Noether locus*

$$\mathrm{BN}^r_d = \{[C] \in \mathcal{M}_g \mid C \text{ carries a } g^r_d\}.$$

Eisenbud and Harris [**EH87b**] have shown that for positive r and d with $\rho(g,r,d) = -1$ (this condition implies that $g+1$ is composite) the locus BN^r_d is a divisor, the *Brill–Noether divisor*; they showed that the class of its compactification is given by

$$[\overline{\mathrm{BN}^r_d}] = c\left((g+3)\lambda - \frac{g+1}{6}\delta_0 - \sum_{i\geq 1} i(g-i)\delta_i\right).$$

Hence we obtain the upper bound

$$s_g \leq 6 + \frac{12}{g+1}$$

if $g+1$ is composite. Based on the idea that Brill–Noether divisors should be the divisors of minimal slope, Harris and Morrison [**HaM90**] made the following conjecture.

Conjecture 8.5 (Slope Conjecture). For any $g \geq 3$ we have $s_g \geq 6 + \frac{12}{g+1}$, with equality if and only if $g+1$ is composite.

Remark 8.6. One finds two different versions of this conjecture in the literature: in the work of Farkas and the paper of Harris–Morrison [**HaM90**] the conjecture is stated in the version above, whereas in [**HaM98**] and [**V07**] the conjecture is stated using the δ_0- slope. Both versions of the conjecture should be closely related; see [**FP05**, Thm. 1.4 and Conj. 1.5].

Remark 8.7. The slope of $mK_{\overline{\mathcal{M}}_g}$ equals 13/2 [**HM82**, p. 52] for all m. Note that

$$\frac{13}{2} < 6 + \frac{12}{g+1} \iff g < 23.$$

Hence the slope conjecture would imply that if $g < 23$ the ray spanned by $K_{\overline{\mathcal{M}}_g}$ is ineffective and $\kappa(\overline{\mathcal{M}}_g) = -\infty$.

Farkas and Popa [**FP05**] found a counter-example in genus 10 using the following result.

Proposition 8.8. *Denote*

$$\mathcal{K}_g = \{[C] \in \mathcal{M}_g \mid C \text{ lies on a } K3 \text{ surface}\}.$$

The support of any effective divisor D with $s(D) < 6 + 12/(g+1)$ contains $\mathcal{K}_g$.

In [**FP05**], Farkas and Popa show that $\overline{\mathcal{K}}_{10}$ is a divisor on $\overline{\mathcal{M}}_{10}$ and calculate its class. Their computation shows that $s(\overline{\mathcal{K}}_{10}) = 7 < 6 + \frac{12}{11}$, hence the divisor $\overline{\mathcal{K}}_{10}$ provides a counterexample to the slope conjecture for $g = 10$.

For $g \leq 9$ and $g = 11$ we have $\overline{\mathcal{K}}_g = \overline{\mathcal{M}}_g$, since for these values of g the moduli space $\overline{\mathcal{M}}_g$ is swept out by Lefschetz pencils of curves on K3 surfaces of degree $2g-2$ in $\mathbb{P}^g$ by results of Mukai [**M92**]. Hence the Slope Conjecture holds for $g \leq 9$ and for $g = 11$; this result had been obtained in [**HaM90**], [**Tan98**] using similar methods. For $g \geq 12$ the K3 locus $\overline{\mathcal{K}}_g$ is not a divisor in $\overline{\mathcal{M}}_g$. Proposition 8.8 suggests that one could obtain counterexamples for the slope conjecture by looking for a divisorial condition on curves of genus g that is weaker than the condition of lying on a K3 surface. The starting point is the following characterization of the divisor $\mathcal{K}_{10}$.

THEOREM 8.9 (Farkas-Popa, [**FP05**]). *The divisor $\mathcal{K}_{10}$ admits the following set-theoretical description.*

$$\mathcal{K}_{10} = \{[C] \in \mathcal{M}_{10} \mid \exists L \in W^4_{12}(C),\ S^2H^0(C,L) \to H^0(C,L^2) \textit{ is not surjective}\}.$$

The condition in the statement of Theorem 8.9 is equivalent to the existence of a pencil $A \in W^1_6(C)$ such that the multiplication map

$$S^2H^0(C, K_C \otimes A^\vee) \to H^0(C, (K_C \otimes A^\vee)^2)$$

is not surjective. In terms of Koszul cohomology, Theorem 8.9 says that $[C] \in \mathcal{K}_{10}$ if and only of there exists $A \in W^1_6(C)$ such that the property (N_0) (see Definition 4.15) fails for $K_C \otimes A^\vee$. Starting from this observation, G. Farkas found a series of interesting cycles on the moduli space of curves defined using Koszul cohomology.

A first general series of examples was given in [**Fa06a**]. Fix $g = 6i + 10$ and $d = 3i+6$, and denote by $\sigma : G^1_k \to \mathcal{M}_g$ the Hurwitz scheme of k-sheeted coverings of $\mathbb{P}^1$ of genus g parametrising pairs (C, A) with $C \in \mathcal{M}_g$ and $A \in W^1_k(C)$. For each $i \geq 0$ one introduces the cycle $U_{g,i}$ consisting of pairs $(C, L) \in G^1_k$ such that the Green-Lazarsfeld property (N_i) fails for $K_C \otimes A^\vee$. By taking the push-forward $\sigma_*(U_{g,i})$ and restricting to the open subvariety $\mathcal{M}^0_{g,k}$ of $\mathcal{M}_g$ parametrising k-gonal curves with trivial automorphisms group, one obtains a stratification

$$Z_{g,0} \subset Z_{g,1} \subset \ldots \subset Z_{g,i} \subset \ldots \subset \mathcal{M}^0_{g,k}.$$

In [**Fa06b**] Farkas generalised the previous construction in the following way. Given integers $i \geq 0$ and $s \geq 1$, put

$$g = rs + s, \quad r = 2s + si + i, \quad d = rs + r.$$

Under these numerical assumptions $\rho(g, r, d) = 0$, hence there exists a unique component of G^r_d which dominates $\mathcal{M}_g$, and the restriction of $G^r_d \xrightarrow{\sigma} \mathcal{M}_g$ over that component is generically finite. Define

$$U_{g,i} = \{(C, L) \in G^r_d \mid L \text{ does not satisfy } (N_i)\}.$$

As $K_{p,q}(C, L) = 0$ for all $q \geq 3$ one has

$$U_{g,i} = \{(C, L) \in G^r_d \mid K_{i,2}(C, L) \neq 0\}.$$

LEMMA 8.10. *Let L be a very ample line bundle on C, and let*

$$C \hookrightarrow \mathbb{P} = \mathbb{P}H^0(C, L)^\vee$$

be the associated embedding. Let M_L be the kernel bundle associated to L, and let $M_\mathbb{P}$ be the kernel bundle associated to $\mathcal{O}_\mathbb{P}(1)$. We have $K_{i,2}(C, L) = 0$ if and only if the restriction map

$$H^0(\mathbb{P}, \textstyle\bigwedge^i M_\mathbb{P} \otimes \mathcal{O}_\mathbb{P}(2)) \to H^0(C, \textstyle\bigwedge^i M_L \otimes L^2)$$

is surjective.

Proof: Put $V = H^0(C, L)$, and note that $M_L = M_\mathbb{P} \otimes \mathcal{O}_C$. By Proposition 2.4 $K_{i,2}(C, L) = 0$ if and only if the map

$$\alpha : \textstyle\bigwedge^{i+1} V \otimes H^0(C, L) \to H^0(C, \textstyle\bigwedge^i M_L \otimes L^2)$$

is surjective. Consider the commutative diagram

$$\begin{array}{ccc} \bigwedge^{i+1} V \otimes H^0(C,L) & \xrightarrow{\alpha} & H^0(C, \bigwedge^i M_L \otimes L^2) \\ \uparrow \cong & & \uparrow \beta \\ \bigwedge^{i+1} V \otimes H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1)) & \xrightarrow{\gamma} & H^0(\mathbb{P}, \bigwedge^i M_{\mathbb{P}}(2)). \end{array}$$

The Euler sequence shows that $M_{\mathbb{P}} \cong \Omega^1_{\mathbb{P}}(1)$, hence $H^1(\mathbb{P}, \bigwedge^{i+1} M_{\mathbb{P}}(1)) = 0$ by the Bott vanishing theorem. This implies that γ is surjective, hence

$$\alpha \text{ is surjective } \iff \beta \text{ is surjective.}$$

□

Farkas constructs vector bundles $\mathcal{A}$ and $\mathcal{B}$ over G^r_d whose fibers over (C, L) are given by

$$\mathcal{A}_{(C,L)} = H^0(\mathbb{P}, \bigwedge^i M_{\mathbb{P}}(2)), \quad \mathcal{B}_{(C,L)} = H^0(C, \bigwedge^i M_L \otimes L^2).$$

and a homomorphism of vector bundles $\varphi : \mathcal{A} \to \mathcal{B}$. The previous Lemma shows that

$$U_{g,i} = \{(C, L) \mid \varphi_{(C,L)} \text{ is not surjective}\}$$

is the degeneracy locus of φ. With the given numerical hypotheses, one checks that rank $\mathcal{A}$ = rank $\mathcal{B}$. Hence $U_{g,i}$ is a *virtual divisor*, i.e., its expected codimension is one, and its image $Z_{g,i} = \sigma_* U_{g,i}$ is also a virtual divisor.

In [**Fa06b**] Farkas shows that φ extends to a morphism of torsion free sheaves $\varphi : \mathcal{A} \to \mathcal{B}$ of the same rank over the compactification of G^r_d given by limit linear series. His main result of is the computation of the class $\sigma_* c_1(\mathcal{B} - \mathcal{A})$ of the virtual degeneracy locus of φ; see [**Fa06b**, Theorem 1.1]. One remarkable fact about this result is that for suitable choices of s and i, it specializes to the divisor class calculations carried out in [**HM82**], [**EH87b**], [**Kh06**], [**FP05**] and [**Fa06a**].

If the map φ is generically nondegenerate, then $Z_{g,i}$ is a genuine divisor. As pointed out in [**Fa06b**], the problem of deciding whether the loci $Z_{g,i}$ are genuine divisors is extremely difficult. For example, the statement that $Z_{2i+3,i}$ is a divisor on $\mathcal{M}_{2i+3}$ is essentially Green's Conjecture for a generic curve of odd genus, [**V05**]. Below we list a number of special cases where it is known that $Z_{g,i}$ is a divisor.

- The case $s = 1$. In this case $g = 2i + 3$ and $g^r_d = g^{g-1}_{2g-2} = K_C$ (note that the canonical bundle is the only g^{g-1}_{2g-2} on a curve of genus g). The generic Green conjecture [**V05**], [**HR98**] gives a set-theoretic identification between $Z_{2i+3,i}$ and the locus of $(i+2)$-gonal curves (the two divisors coincide actually up to some factor). In this case [**Fa06b**, Theorem 1.1] provides a new way of determining the class of the compactification of the Brill-Noether divisor first computed by Harris and Mumford [**HM82**].
- The case $s = 2$. Here the numerical invariants are

  $$g = 6i + 10, \quad r = 3i + 4, \quad d = 9i + 12.$$

  In this case we recover the case discussed earlier in this section, since $h^1(K_C \otimes L^\vee) = 2$ and G^r_d is isomorphic to a Hurwitz scheme parametrising covers of $\mathbb{P}^1$. For $i = 0$ one recovers the class of the K3 divisor $\mathcal{K}_{10}$ treated in [**FP05**]. In [**Fa06a**] Farkas has verified that $Z_{6i+10,i}$ is a divisor for $i = 1$ and $i = 2$. In general, one has to show that if $[C] \in \mathcal{M}_{6i+10}$ is a general curve, then one (or, equivalently, all) of the finitely many linear systems $g^{3i+4}_{9i+12} = K_C(-g^1_{3i+6})$ satisfies property (N_i). This is the analogue of the Green-Lazarsfeld Conjecture 4.31 for the case of line bundles L with $h^1(C, L) = 2$.

- The case $i = 0$. This gives the numerical invariants

$$g = s(2s + 1), \quad r = 2s, \quad d = 2s(s + 1).$$

Farkas [**Fa06b**, Thm. 1.5] has shown that $Z_{g,0}$ is a divisor f or all $s \geq 2$. The cases $s = 2$ and $s = 3$ were treated in [**FP05**] and [**Kh06**].

These cases provide infinitely many counterexamples to the Harris-Morrison Slope Conjecture.

8.3. Slopes of fibered surfaces

Let $f : S \to B$ be a fibered smooth projective surface over a smooth projective curve B. Let us suppose that the fibration is relatively minimal (i.e. there are no (-1)–curves contained in the fibers of f), not isotrivial, and that the genus of the fibers is at least 2. Using the relative canonical bundle $K_{S/B}$, one defines two invariants of the fibration. The first one is

$$\chi(f) = \deg f_*(K_{S/B}) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_B)\chi(\mathcal{O}_F),$$

where F is a general fiber of f. It is known that $\chi(f) \geq 0$; see for example [**Xi87**]. The second invariant is defined by

$$\lambda(f) = \frac{K^2_{S/B}}{\chi(f)}.$$

In [**Xi87**], Xiao proved that the following inequalities hold:

$$4 - \frac{4}{g} \leq \lambda(f) \leq 12$$

The equality $\lambda(f) = 4 - 4/g$ holds if and only if the general fiber is hyperelliptic.

Over the years, several authors tried to improve this bound. To this end, one needs another invariant: the *Clifford index of f*, denoted $\mathrm{Cliff}(f)$, which is by definition the Clifford index of a general fibre. For fibrations of Clifford index one, (i.e. the general fiber of f is a trigonal curve, or plane quintic) the lower bound was improved to

$$\lambda(f) \geq 14\frac{g-1}{3g+1}.$$

It is expected that the optimal lower bound depends on the Clifford index of the fibration.

The generic Green conjecture for curves of odd genus has the following consequence; see [**Ko99**] for details.

THEOREM 8.11 (Konno). *Notation as above. Suppose that $g \geq 3$ is odd, and* $\mathrm{Cliff}(f) = (g-1)/2$. *Then*

$$\lambda(f) \geq \frac{6(g-1)}{g+1}.$$

Any non–isotrivial fibration f induces a curve $B \subset \overline{\mathcal{M}}_g$. Applying Theorem 8.11 and arguing as in [**Xi87**, Corollary 1], one obtains the inequality

$$\frac{B \cdot \delta}{B \cdot \lambda} \leq 6 + \frac{12}{g+1}. \tag{8.2}$$

This bound is related to the Slope Conjecture in the following way. If $D \equiv a\lambda - b\delta$ is an effective divisor on $\overline{\mathcal{M}}_g$ and f is a general fibration, then $D \cdot B \geq 0$; hence $s(D) = a/b \geq (B \cdot \delta)/(B \cdot \lambda)$. If there exists a curve B in $\overline{\mathcal{M}}_g$ for which the inequality (8.2) is an equality, then the Slope Conjecture holds; this was the approach used in [**Tan98**].

8.4. Notes and comments

As Green explains in the introduction of [**Gre89**], his original motivation for studying Koszul cohomology comes from Hodge theory; specifically, he cites the paper [**LPW**] on the infinitesimal Torelli problem as a source of inspiration. The theory of infinitesimal variations of Hodge structure was developed by Griffiths and his co–workers in the influential paper [**CGGH83**]. The idea is that by working at the infinitesimal level, certain Hodge–theoretic questions can be rephrased in terms of multilinear algebra; they can then frequently be solved using vanishing theorems for Koszul cohomology.

As we have seen, the explicit Noether–Lefschetz theorem and the theorem of Green–Voisin on the image of the Abel–Jacobi map follow from vanishing theorems for the Koszul groups $K_{0,2}(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}}(-4), \mathcal{O}_{\mathbb{P}}(d); W)$ and $K_{1,2}(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}}(-5), \mathcal{O}_{\mathbb{P}}(d))$. Nori found a beautiful generalization of these results, which provides a Hodge–theoretic interpretation of the vanishing of the Koszul groups $K_{p,2}(\mathbb{P}^n, K_{\mathbb{P}}, \mathcal{O}_{\mathbb{P}}(d))$ in terms of a Lefschetz–type connectivity theorem for the cohomology of the universal family of hypersurfaces in projective space [**No93**]. See [**CMP**], [**V02b**] or [**Gre94**, Lecture 8] for a more detailed discussion of this result. More generally, Nori's theorem is valid for the universal family of complete intersections of sufficiently high multidegree in an arbitrary polarised variety. A proof of this theorem using Koszul cohomology computations was obtained in [**Na02**]; cf. also [**Na04**]. One of the key points is the description of the variable cohomology of such complete intersections using a generalized Jacobi ring. Specifically, if Y is a smooth projective variety, the *variable cohomology* of a smooth divisor $X \in |L|$ of dimension n with inclusion $i : X \hookrightarrow Y$ is defined by

$$H^n_{\mathrm{var}}(X) = \operatorname{coker}(H^n(Y) \xrightarrow{i^*} H^n(X));$$

this coincides with the usual primitive cohomology if Y is a projective space. In [**Gre85**], Green showed that if L is sufficiently ample, $H^{n-p,p}_{\mathrm{var}}(X)$ is isomorphic to the quotient of $H^0(Y, K_Y \otimes L^{p+1})$ by a suitably defined generalized Jacobi ideal. This generalizes Griffiths's description of the primitive cohomology of hypersurfaces in projective space. Using this description, the exactness of the complexes appearing in the infinitesimal study of this problem can be reduced to vanishing theorems for the groups $K_{p,q}(Y, \Omega^i_Y, L)$.

The problem of determining the effective cone of the moduli space $\overline{\mathcal{M}}_g$ remains a hard open problem. Even though the slope conjecture was disproved, we can still ask about the possible lower bounds for slopes. Using inequalities of type $s(D) = a/b \geq (B \cdot \delta)/(B \cdot \lambda)$ (cf. section 8.3), one can try to find good families of curves, parametrized by bases B with suitable $B \cdot \delta / B \cdot \lambda$. Along these lines, it was proved that if D is an effective divisor with $s(D) < 8$ then D contains the hyperelliptic locus, and if $s(D) < 7 + \frac{6}{g}$ then D contains the trigonal locus, [**HaM90**], [**Tan98**]. The question remains to know if the slope of any effective divisor is at least 6, or close to this bound. This question is related to the Schottky problem; cf. [**Mo**].

Moduli spaces of pointed curves are also of high interest. In genus zero, the effective cone has been explicitly described in [**KeM96**]. Their main result shows that in this case the effective cone is generated by the boundary classes; in particular it is simplicial.

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