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KPZ EQUATION LIMIT OF HIGHER-SPIN EXCLUSION PROCESSES

BY IVAN CORWIN*,†,‡,1 AND LI-CHENG TSAI§,2

*Columbia University**, *Clay Mathematics Institute*†, *Institut Henri Poincaré*‡ *and Stanford University*§

We prove that under a particular *weak scaling*, the 4-parameter interacting particle system introduced by Corwin and Petrov [*Comm. Math. Phys.* **343** (2016) 651–700] converges to the Kardar–Parisi–Zhang (KPZ) equation. This expands the relatively small number of systems for which *weak universality* of the KPZ equation has been demonstrated.

1. Introduction. This paper demonstrates how the KPZ equation [18] arises as a scaling limit of a 4-parameter interacting particle system introduced in [10] [called here the Higher Spin Exclusion Process (HSEP)] under fairly general choices of three parameters ($\nu \in [0, 1)$, $\alpha > 0$, $J \in \mathbb{Z}_{>0}$) and special tuning of the remaining paremeter ($q \to 1$). This system, through various specializations, and limit procedures includes all known integrable models in the KPZ universality class. It is closely connected to the study of higher-spin vertex models within quantum integrable systems, and hence enjoys a number of nice algebraic properties, some of which play important roles in our convergence proof.

The KPZ equation is a paradigmatic continuum model for a randomly growing interface with local dynamics subject to smoothing, lateral growth and space–time noise (for more background, see the review [8]). Its spatial derivative solves the stochastic Burgers equation with conservative noise, and its exponential (Hopf–Cole transform) satisfies the Stochastic Heat Equation (SHE) with multiplicative white-noise. The connection to stochastic Burgers equation suggests a relation to interacting particle systems while the connection to the SHE suggests a relation to directed polymer models (whose partition functions satisfy discrete versions of the SHE).

The KPZ equation is written as

$$\partial_\tau \mathcal{H}(\tau, r) = \frac{1}{2}\delta\partial_r^2\mathcal{H}(\tau, r) + \frac{1}{2}\kappa\big(\partial_r\mathcal{H}(\tau, r)\big)^2 + \sqrt{D}\eta(\tau, r), \tag{1.1}$$

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where η is space–time white noise, $\delta, \kappa \in \mathbb{R}$, and $D > 0$. Care is needed in making sense of the above equation, and the proper notion of solution is that of the *Hopf–Cole solution to the KPZ equation* which is defined by setting $\mathcal{H}(\tau, r) = \frac{\delta}{\kappa} \log \mathcal{Z}(\tau, r)$ where $\mathcal{Z}$ solves the well-posed SHE

$$\partial_\tau \mathcal{Z}(\tau, r) = \frac{1}{2}\delta \partial_r^2 \mathcal{Z}(\tau, r) + \frac{\kappa}{\delta}\sqrt{D}\mathcal{Z}(\tau, r)\eta(\tau, r). \tag{1.2}$$

To understand how a microscopic system might scale to the KPZ equation, it helps to understand how the KPZ equation itself scales. For real b, z define $\mathcal{H}^\varepsilon(\tau, r) := \varepsilon^b \mathcal{H}(\varepsilon^{-z}\tau, \varepsilon^{-1} r)$. Then $\mathcal{H}^\varepsilon$ satisfies the scaled equation

$$\begin{aligned}\partial_\tau \mathcal{H}^\varepsilon(\tau, r) = \varepsilon^{2-z}\frac{1}{2}\delta \partial_r^2 \mathcal{H}^\varepsilon(\tau, r) + \varepsilon^{2-z-b}\frac{1}{2}\kappa\big(\partial_r \mathcal{H}^\varepsilon(\tau, r)\big)^2 \\ + \varepsilon^{b-\frac{z}{2}+\frac{1}{2}}\sqrt{D}\eta(\tau, r).\end{aligned}$$

There exists no choice of b, z for which the coefficients of the scaled equation remain unchanged. However, if one simultaneously changes the values of some of the δ, κ, D parameters as ε changes, the KPZ equation may scale to itself. If the KPZ equation remains invariant under such a scaling, it stands to reason that a microscopic model with similar properties may converge to the equation under a similar type of scaling and tuning of parameters. Such scalings are generally called *weak scalings* since they involve taking some of the δ, κ, D parameters to zero with ε. It is thus a goal to show the *weak universality* of the KPZ equation by demonstrating how under these scalings, the equation arises from a variety of different microscopic models. Weak universality should be distinguished from *KPZ universality* which holds that without any tuning of parameters, a variety of different systems will converge under the choice of $b = 1/2$ and $z = 3/2$ to a universal limit called the *KPZ fixed-point* [11].

There are very few proved instances of weak universality of the KPZ equation. The first result was in the context of the Asymmetric Simple Exclusion Process (ASEP) [3] for near equilibrium initial condition (see also [2] for step initial condition). The ASEP result came under *weak asymmetry* scaling through which $b = 1/2, z = 2$ and $\kappa \mapsto \varepsilon^{1/2}\kappa$ (δ and D remain unscaled). This result was extended in [12] to certain nonnearest neighbor (and nonexactly solvable) exclusion processes. The only other weak universality result [1] was in the context of discrete directed polymers with arbitrary disordered distributions. This result came under *weak noise* scaling through which $b = 0, z = 2$ and $D \mapsto \varepsilon D$ (δ and κ remain unscaled).

Owing to the round-about Hopf–Cole definition of the KPZ equation, in order to prove that a system converges to the KPZ equation, one must transform it microscopically into an approximate SHE. The work of [16] provides direct meaning to the KPZ equation (though for r on the torus, not the full real line). As of yet, this approach has not yielded weak universality results for the KPZ equation. The work of [14] defines an *energy solution* for the KPZ equation and shows tightness of a certain class of interacting particle systems at equilibrium and that all limit points are energy solutions. The recent work of [15] establishes the uniqueness of equilibrium energy solution on the torus (under the time-reversal symmetry assumption enjoyed by Markov processes at equilibrium).

The partition function for a directed polymer model naturally solves a microscopic SHE with a simple noise. On the other hand, the ASEP result relies heavily on the existence of a microscopic Hopf–Cole transform (known as the Gärtner transform [13]), and the resulting SHE has a much more complicated noise. This renders the associated analysis quite challenging. The work of [12] also relies on an approximate form of the Gärtner transform. Microscopic Hopf–Cole transforms are hard to come by. For the model considered herein, this transform is achieved in Proposition 2.6. The first indication that such a transform should exist came from the Markov duality enjoyed by the model; see Remark 2.7 for more discussions.

The particular choice of weak scalings present in our result is new. It corresponds to the KPZ equation scaling given by $b = 1, z = 3, \delta \mapsto \varepsilon\delta, \kappa \mapsto \varepsilon^2\kappa$ and D remaining unchanged. In terms of the scaling of the microscopic model, we have $b = 1, z = 3$ and $q = e^{-\varepsilon}$, while $\nu \in [0, 1), \alpha > 0$ and $J \in \mathbb{Z}_{>0}$ remain fixed. One sees that microscopically, these choices of parameters corresponds to the above weak scaling. As there are many parameters at play, it is likely that there exist other weak scalings of the system which realized the same KPZ equation limit.

There are a number of degenerations of the HSEP, including the discrete time Bernoulli q-TASEP [4] and (through a limit transition) the continuous time q-TASEP [5, 7]. Strictly speaking, our results do not immediately apply to the continuous time q-TASEP. The restriction on parameters $\nu \in [0, 1), \alpha > 0$ and $J \in \mathbb{Z}_{>0}$ does not allow us to probe all of the degenerations of the system introduced in [10]. For instance, the stochastic six-vertex model [6] (a discrete time version of ASEP) arises through a different choice of specialization, as does the q-Hahn TASEP [9]. These systems likewise enjoy dualities and one may hope to prove their weak universality. We leave this for future work.

Outline. In Section 2, we introduce the 4-parameter particle system and then proceed to state our main results. These are stated in terms of the SHE as Theorems 2.9 and 2.10 (though Corollary 2.15 provides the equivalent statements in terms of the KPZ equation). Section 3 provides the discrete Hopf–Cole transform satisfied by the system. Section 4 provides moment estimates necessary to show tightness as $\varepsilon \to 0$. Section 5 demonstrates how the limit points satisfy the martingale problem for the SHE.

2. Definition of the model and results. We begin by recalling the definition of the HSEP. Let $\mathbb{N} := \mathbb{Z}_{\geq 0}$, $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$ and $\mathbb{Z}^* := \mathbb{Z} \cup \{\infty\}$. Define the space of right-finite particle configurations

$$\mathbb{X}_m := \{\vec{x} = (\infty = \cdots = x_{m-2} = x_{m-1} > x_m > x_{m+1} > \cdots) \in \mathbb{Z}^*\},$$

where imaginary particles are placed at ∞ for the convenience of notation, and define the space of infinite particle configurations $\mathbb{X}_\infty := \{\vec{x} = (\cdots > x_{-1} > x_0 > x_1 > \cdots) \in \mathbb{Z}^{\mathbb{Z}}\}$, with the corresponding spaces of gap configurations

$$\mathbb{G}_m := \{\vec{g} = (g_n) : g_n = \infty, \forall n < m; g_n \in \mathbb{N}, \forall n \geq m\}, \qquad \mathbb{G}_\infty := \mathbb{N}^{\mathbb{Z}}.$$

Fixing $J \in \mathbb{Z}_{>0}$, we let $\operatorname{mod}_J(s) := s - \lfloor s/J \rfloor J$, or more explicitly, $(\operatorname{mod}_J(0), \operatorname{mod}_J(1), \ldots) = (0, 1, \ldots, J-1, 0, \ldots, J-1, \ldots)$. Fixing $q, \nu \in [0, 1)$ and $\alpha > 0$, we let $\alpha_j := \alpha q^j$ and $\alpha(s) := \alpha_{\operatorname{mod}_J(s)}$, and equip our probability space with independent Bernoulli random variables

$$B_n(s, g) \sim \operatorname{Ber}\left(\frac{\alpha(s)(1-q^g)}{1+\alpha(s)}\right), \qquad B'_n(s, g) \sim \operatorname{Ber}\left(\frac{\alpha(s)+\nu q^g}{1+\alpha(s)}\right),$$

indexed by $(s, g, n) \in (\mathbb{N}, \mathbb{N}^*, \mathbb{Z})$, with the corresponding filtration $\mathscr{F}(t) := \sigma(B_n(s, g), B'_n(s, g) : (n, g) \in \mathbb{N} \times \mathbb{N}^*, s = 0, \ldots, t-1)$. Recall from [10] the following definition of the HSEP.

DEFINITION 2.1. Given $\vec{x}(0) \in \mathbb{X}_m$, a right-finite particle configuration, we define an $\mathbb{X}_m$-valued Markov chain $\{\vec{y}(s)\}_{s\in\mathbb{N}}$ by setting $\vec{y}(0) := \vec{x}(0)$, and update $\vec{y}(s)$ as follows. We update $\vec{y}(s)$ sequentially, starting from m, by letting

$$y_m(s+1) = y_m(s) + B_m(s, \infty), \tag{2.1}$$

and letting, for $n > m$,

$$y_n(s+1) = \begin{cases} y_n(s) + B'_n(s, g_n(s)) & \text{if } y_{n-1}(s+1) > y_{n-1}(s), \\ y_n(s) + B_n(s, g_n(s)) & \text{if } y_{n-1}(s+1) = y_{n-1}(s), \end{cases} \tag{2.2}$$

where $g_n(s) := y_{n-1}(s) - y_n(s) - 1$ the nth gap of $\vec{y}(s)$. Namely, we move $y_n(s)$ one step to the right with probability $\frac{\alpha(s)}{1+\alpha(s)}$, and subsequently, we move $y_n(s)$ depending on how $y_{n-1}(s)$ was updated: if $y_{n-1}(s)$ did not move, we then move $y_n(s)$ one step to the right with probability $\frac{\alpha(s)(1-q^{g_n(s)})}{1+\alpha(s)}$, otherwise we move $x_n(s)$ one step to the right with probability $\frac{\alpha(s)+\nu q^{g_n(s)}}{1+\alpha(s)}$.

The HSEP $\{\vec{x}(t)\}_{t\in\mathbb{N}}$ is then defined as the $\mathbb{X}_m$-valued Markov chain $\vec{x}(t) := \vec{y}(Jt)$.

REMARK 2.2. The Markov chain $\vec{y}(t)$ was defined through a local sequential update of particles. Taking $J > 1$ and considering $\vec{x}(t)$, a priori one might think this property is lost. It was shown in [10], Section 3, that, in fact, $\vec{x}(t)$ can be updated through a local sequential update (just like for $J = 1$). In this case, each particle may move a distance between 0 to J sites to the right, and the jump probabilities depend on the length of the previous particle's jump as well as the length of the gap. The explicit form of this probability is somewhat more involved and given in [10], Theorem 3.15. In particular, given a gap g and a jump of the previous particle by h, a particle jumps by h' according to the probability given by $R^{(J)}_\alpha(g, h; g + h - h', h')$ where a concise formulas for $R^{(J)}_\alpha$ is given in [10], Theorem 3.15, and there is also a dependence on q, ν which is suppressed in the notation. For the purposes of this paper, it suffices to study the $\vec{y}(t)$ process and prove convergence of it to the KPZ equation. It follows then immediately that the $\vec{x}(t)$ process likewise converges.

The process introduced in Definition 2.1 is defined by the *sequential* update of (2.1)–(2.2), which is inconvenient for our purpose. We now recast the definition as a *parallel* update, and, as a byproduct, extend Definition 2.1 to the space $\mathbb{X} := \bigcup_{n\in\mathbb{Z}^*} \mathbb{X}_n$ of possibly infinite particle configurations. To this end, we require the following lemma, which we prove in Section 3.

LEMMA 2.3. *For any fixed $\vec{g} \in (\mathbb{N}^*)^{\mathbb{Z}}$, $s \in \mathbb{N}$, $m \le n \in \mathbb{Z}$, letting*

$$I_{n,m}(s, \vec{g}) := \Bigl(\prod_{n\ge i>m} \bigl(B'_i(s, g_i) - B_i(s, g_i)\bigr)\Bigr) B_m(s, g_m), \tag{2.3}$$

we have

$$K_n(s, \vec{g}) := \sum_{m:n\ge m} I_{n,m} \in \{0, 1\}, \tag{2.4}$$

where the series converges in L^k for all $k \ge 1$, and hence almost surely. Further,

$$K_n(s, \vec{g}) = K_{n-1}(s, \vec{g}) B'_n(s, g_n) + \bigl(1 - K_{n-1}(s, \vec{g})\bigr) B_n(s, g_n). \tag{2.5}$$

DEFINITION 2.4. Fix $m \in \mathbb{N}^*$ and $\vec{x}(0) \in \mathbb{X}_m$. Letting $\vec{g}(\vec{y}) := (y_{n-1} - y_n - 1)_{n\in\mathbb{Z}}$ and $\vec{K}(s, \vec{g}) := (K_n(s, \vec{g}))_{n\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ (by Lemma 2.3), we define a stochastic map

$$\mathcal{T}(s) : \mathbb{X}_m \longrightarrow \mathbb{X}_m, \qquad \vec{y} \longmapsto \vec{y} + \vec{K}\bigl(s, \vec{g}(\vec{y})\bigr), \tag{2.6}$$

and define the $\mathbb{X}_m$-valued Markov chain $\{\vec{x}(t)\}$ and $\{\vec{y}(t)\}$ by letting $\vec{y}(s) := \mathcal{T}(s - 1) \circ \mathcal{T}(s - 2) \circ \cdots \circ \mathcal{T}(0)(\vec{x}(0))$ and $\vec{x}(t) := \vec{y}(tJ)$.

REMARK 2.5. Under the map $\mathcal{T}(s)$, we have that $K_n(s, \vec{g}(\vec{y}(s))) = \mathbf{1}_{\{y_n(s+1)>y_n(s)\}}$, whereby (2.5) becomes

$$\mathbf{1}_{\{y_n(s+1)>y_n(s)\}} = \mathbf{1}_{\{y_{n-1}(s+1)>y_{n-1}(s)\}} B'_n\bigl(s, g_n(s)\bigr) + \mathbf{1}_{\{y_{n-1}(s+1)=y_{n-1}(s)\}} B_n\bigl(s, g_n(s)\bigr).$$

This is equivalent to (2.2), which reduces to (2.1) when $g_{m-1} = \infty$. It is thus easy to see that Definition 2.4 is equivalent to Definition 2.1 when restricting to $\vec{x}(0) \in \mathbb{X}_m$, $m \in \mathbb{N}$.

Our main result is the convergence to the SHE of a certain exponential transform of the process $\vec{y}(t)$. Recall that we say a process $\mathcal{Z}$ on $\mathbb{R}_+ \times \mathbb{R}$ is a mild solution of the SHE starting from the initial condition $\mathcal{Z}^{\mathrm{ic}}$ if

$$\begin{aligned} \mathcal{Z}(\tau, r) = &\int_{\mathbb{R}} P_\tau(r - r')\mathcal{Z}^{\mathrm{ic}}(r')\,dr' \\ &+ \int_0^\tau \int_{\mathbb{R}} P_{\tau-\tau'}(r - r')\mathcal{Z}(\tau', r')\eta(dr', d\tau'), \end{aligned} \tag{2.7}$$

where $P_\tau(r) := \exp[-r^2/(2\tau)](2\pi\tau)^{-1/2}$ denotes the standard heat kernel, and η denotes the space–time white noise. For the existence, uniqueness, continuity, and positivity of solutions of (2.7), see [8], Proposition 2.5.

The key step of showing the convergence to the SHE is finding a discrete SHE. To state it, we fix a parameter $\rho \in (0, 1)$, measuring the limiting density (see Remark 2.11), and set

$$\gamma := \frac{1-\rho}{1-\nu\rho}, \qquad a_j := \frac{\alpha_j\gamma}{1+\alpha_j\gamma}, \qquad b := \frac{\gamma}{1-\gamma}, \qquad b' := \frac{\nu\gamma}{1-\nu\gamma}, \tag{2.8}$$

$$\mu(t) := (a_{\mathrm{mod}_J(t)} - a_{\mathrm{mod}_J(t)+1})(b - b')^{-1}, \tag{2.9}$$

$$\begin{aligned} &\lambda(t) := \frac{1+\alpha(t)\gamma}{1+q\alpha(t)\gamma}, \\ &\widehat{\mu}(t) := \sum_{s=0}^{t-1} \mu(s), \qquad \widehat{\lambda}(t) := \prod_{s=0}^{t-1} \lambda(s), \end{aligned} \tag{2.10}$$

with the conventions $\widehat{\mu}(0) := 0$ and $\widehat{\lambda}(0) := 1$. Letting $Q_n(t) := q^{y_n(t)+n}$ denote the one-particle duality function (see [10] for the definition of duality functions), we define the exponential transform

$$Z(t, \xi) := \widehat{\lambda}(t)\rho^{\xi+\widehat{\mu}(t)} Q_{\xi+\widehat{\mu}(t)}(t), \tag{2.11}$$

for $t \in \mathbb{N}$ and $\xi \in \Xi(t) := (\mathbb{Z} - \widehat{\mu}(t))$. The discrete SHE is expressed in terms of a certain random walk $R(0) + \cdots + R(t-1)$ on $\mathbb{R}$. Here, $R(s) \in (\mathbb{N} - \mu(s))$, $s \in \mathbb{N}$, are independent random variables introduced in (3.10), with zero mean and variance as in (3.11). Let $\Xi(t_2, t_1) := \mathbb{N} + (\widehat{\mu}(t_1) - \widehat{\mu}(t_2))$. For $t_1 \le t_2$, $\zeta \in \Xi(t_2, t_1)$, let

$$p(t_2, t_1, \zeta) := \mathbf{P}(R(t_1) + R(t_1+1) + \cdots + R(t_2-1) = \zeta) \tag{2.12}$$

denotes the corresponding semigroup. We use the shorthand notation $[p(t_2, t_1) * f(t_1)](\xi) := \sum_{\zeta\in\Xi(t_1)} p(t_2, t_1, \xi - \zeta) f(t_1, \zeta)$ to denote convolution. Let $\overline{K}_n(s) := K_n(s) - \mathbf{E}(K_n(s)|\mathscr{F}(s))$ and let

$$W(t, \xi) := \lambda(t)(q-1)\overline{K}_{\xi+\widehat{\mu}(t)}(t), \tag{2.13}$$

representing the discrete analog of the space–time white noise.

PROPOSITION 2.6. *For all $t_1 \le t_2 \in \mathbb{N}$ and $\xi \in \Xi(t_2)$, we have the following discrete SHE*

$$Z(t_2,\xi) = Z_{\mathrm{dr}}(t_2,t_1,\xi) + Z_{\mathrm{mg}}(t_2,t_1,\xi), \tag{2.14}$$

where

$$Z_{\mathrm{dr}}(t_2,t_1,\xi) := \big[p(t_2,t_1) * Z(t_1)\big](\xi), \tag{2.15}$$

$$Z_{\mathrm{mg}}(t_2,t_1,\xi) := \sum_{s=t_1}^{t_2-1} \big[p(t_2,s+1) * \big(Z(s)W(s)\big)\big]\big(\xi+\mu(s)\big). \tag{2.16}$$

Further, for all $\xi_1,\xi_2 \in \Xi(t)$,

$$\begin{aligned} &Z(t,\xi_1)Z(t,\xi_2)\mathbf{E}\big[W(t,\xi_1)W(t,\xi_2)|\mathscr{F}(t)\big] \\ &\quad = \left(\frac{(\nu+\alpha(t))\rho}{1+\alpha(t)}\right)^{|\xi_1-\xi_2|} \Theta_1(t,\xi_1\wedge\xi_2)\Theta_2(t,\xi_1\wedge\xi_2), \end{aligned} \tag{2.17}$$

where

$$\Theta_1(t,\xi) := q\lambda(t)Z(t,\xi) - \big[p(t+1,t) * Z(t)\big]\big(\xi-\mu(t)\big), \tag{2.18}$$

$$\Theta_2(t,\xi) := -\lambda(t)Z(t,\xi) + \big[p(t+1,t) * Z(t)\big]\big(\xi-\mu(t)\big). \tag{2.19}$$

REMARK 2.7. The first indication that a microscopic Hopf–Cole transform as in Proposition 2.6 should exist came from the $k=1$ version of the Markov duality enjoyed by the model, given in [10], Theorem 2.19. This result shows that $\mathbf{E}(q^{y_n(t)+n})$ satisfies the Kolmogorov backward equation in the t and n variables, more explicitly

$$\mathbf{E}\big(q^{y_n(t+1)+n}\big) = \sum_{m\in\mathbb{Z}} p'(t+1,t,n-m)\mathbf{E}\big(q^{y_m(t)+m}\big).$$

Here, $p'(t+1,t,m)$ is the transition probability of a certain (time inhomogeneous) random walk $\{X'(t)\}_{t\in\mathbb{N}}$, defined as in (3.2), which corresponds to the one-particle version of the Higher Spin Zero Range Process, defined in [10], Definition 2.6. The existence of a nice martingale as in (2.17) and finding the correct centering and tiling as in (2.11) require further work given here in Proposition 2.6.

Proceeding to our main result, we consider the weak noise scaling $q = q_\varepsilon := e^{-\varepsilon}$, $\varepsilon \to 0$. Hereafter, throughout the paper, we *fix* $\alpha > 0$, $\nu \in [0,1)$, and $\rho \in (0,1)$, and scale only the parameter $q_\varepsilon \to 1$. To indicate this scaling, we denote *parameters* such as α_j and $\alpha(s)$ by α_j^ε and $\alpha_\varepsilon(s)$, but for *processes* such as $\vec{x}(t)$, $B_n(s,g)$, we often omit the dependence on ε to simplify notation. Under this scaling, to the first order (2.9)–(2.10) read

$$\mu_\varepsilon(t) = \varepsilon\alpha\gamma(1+\alpha\gamma)^{-2}\big(b-b'\big)^{-1} + O\big(\varepsilon^2\big), \tag{2.20}$$

$$\lambda_\varepsilon(t) = 1 + \varepsilon\alpha\gamma(1+\alpha\gamma)^{-1} + O\big(\varepsilon^2\big). \tag{2.21}$$

Let $r_* := (b - b')^{-1}$,

$$\tau_*^\varepsilon := \big[(a_0)^2 - (a_J^\varepsilon)^2 + (a_0 - a_J^\varepsilon)(b + b')\big]^{-1} \tag{2.22}$$

$$= (1 + \alpha\gamma)^2 (J\alpha\gamma)^{-1} (2a_0 + b + b')^{-1} + O(\varepsilon). \tag{2.23}$$

Note that here $a_0 = \alpha\gamma(1 + \alpha\gamma)^{-1}$ is independent of ε. We extend the process $Z(t, \xi)$, defined for $t \in \mathbb{N}$ and $\xi \in \Xi(t)$, to a continuous process on $\mathbb{R}_+ \times \mathbb{R}$ by first linearly interpolate in ξ and then linearly interpolate in t, and then we introduce the scaled process

$$Z_\varepsilon(\tau, r) := Z\big(\varepsilon^{-3}\tau_*^\varepsilon J\tau, \varepsilon^{-1} r_* r\big), \tag{2.24}$$

or, equivalently $Z_\varepsilon(\tau, r) = \exp(H_\varepsilon(\tau, r))$, where

$$H_\varepsilon(\tau, r) := -\varepsilon y_{n_\varepsilon(\tau,r)}\big(t_\varepsilon(\tau)\big) + (\log\rho - \varepsilon) n_\varepsilon(\tau, r) + \log\widehat{\lambda}_\varepsilon\big(t_\varepsilon(\tau)\big), \tag{2.25}$$

$t_\varepsilon(\tau) := \varepsilon^{-3}\tau_*^\varepsilon J\tau$ and $n_\varepsilon(\tau, r) := \varepsilon^{-1} r_* r + \widehat{\mu}_\varepsilon(t_\varepsilon(\tau))$. Following [3], we consider *near equilibrium* initial conditions.

DEFINITION 2.8. Let $Z_\varepsilon(0, \xi)$ be the exponential transform [given as in (2.24)] associated with $\{\vec{x}^\varepsilon(0)\}_\varepsilon \subset \mathbb{X}$. We say $\{\vec{x}^\varepsilon(0)\}_\varepsilon \subset \mathbb{X}$ is *near equilibrium* if, given any $k \in \mathbb{Z}_{>0}$ and $v \in (0, 1/2)$, there exists $u = u(k, v)$, $C = C(k, v) < \infty$ such that

$$\|Z_\varepsilon(0, \xi)\|_k := \big(\mathbf{E}\big(Z_\varepsilon(0, \xi)^k\big)\big)^{1/k} \le C e^{u|\xi|}, \tag{2.26}$$

$$\|Z_\varepsilon(0, \xi) - Z_\varepsilon\big(0, \xi'\big)\|_k \le C|\xi - \xi'|^v e^{u(|\xi| + |\xi'|)}, \tag{2.27}$$

for all $\xi, \xi' \in \varepsilon(r_*)^{-1}\mathbb{Z}$ and $\varepsilon > 0$ small enough.

Hereafter, we endow the spaces $C(\mathbb{R})$, $C(\mathbb{R}_+ \times \mathbb{R})$ and $C((0, \infty) \times \mathbb{R})$ the topology of uniform convergences on compact subsets, and use $\Rightarrow$ to denote weak convergence of probability laws. The following is our main result.

THEOREM 2.9. *Let $\mathcal{Z}$ be the unique $C(\mathbb{R}_+ \times \mathbb{R})$-valued solution of SHE starting from a $C(\mathbb{R})$-valued process $\mathcal{Z}^{\mathrm{ic}}$, and let $Z_\varepsilon(\tau, r) \in C(\mathbb{R}_+ \times \mathbb{R})$ be as in* (2.24), *with some near equilibrium initial condition $\{\vec{x}^\varepsilon(0)\}_\varepsilon$. If $Z_\varepsilon(0, \cdot) \Rightarrow \mathcal{Z}^{\mathrm{ic}}(\cdot)$, then*

$$Z_\varepsilon(\cdot, \cdot) \Rightarrow \mathcal{Z}(\cdot, \cdot) \qquad \text{on } C(\mathbb{R}_+ \times \mathbb{R}), \qquad \text{as } \varepsilon \to 0.$$

Definition 2.8 (and, therefore, Theorem 2.9) leaves out an important initial condition, that is, the step initial condition: $x_n(0) := -n$ for $n \in \mathbb{N}$ and $x_n = \infty$ for $n \in \mathbb{Z}_{<0}$. Following [2], we generalize Theorem 2.9 to the following.

THEOREM 2.10. *Let $\widetilde{\mathcal{Z}}(\cdot, \cdot)$ be the unique solution of SHE starting from the delta measure $\delta(\cdot)$, let $\{\vec{y}(t)\}_t \in \mathbb{X}_0$ be the process starting from the step initial condition, and let $\widetilde{Z}_\varepsilon(\tau, r) := \varepsilon^{-1}(1 - \rho) r_* Z_\varepsilon(\tau, r)$. Then*

$$\widetilde{Z}_\varepsilon(\cdot, \cdot) \Rightarrow \widetilde{\mathcal{Z}}(\cdot, \cdot) \qquad \text{on } C\big((0, \infty) \times \mathbb{R}\big), \qquad \text{as } \varepsilon \to 0.$$

REMARK 2.11. By Theorem 2.9, we have that $H_\varepsilon(\tau,r) - H_\varepsilon(\tau, r+\varepsilon/r_*) \to 0$ in probability. Plugging this in (2.25), a posteriori we find that, $y_{n_\varepsilon(\tau,r)}(t_\varepsilon(\tau)) - y_{n_\varepsilon(\tau,r)+1}(t_\varepsilon(\tau)) \approx \varepsilon^{-1}\log(1/\rho)$, or equivalently the limiting density is $\varepsilon/\log(1/\rho)$.

Hereafter, we adapt the convention that $m, n, i, j, k \in \mathbb{Z}$; $s, t \in \mathbb{N}$; $\tau, \tau' \in \mathbb{R}_+$; and $r \in \mathbb{R}$. To simplify notation, we let $\vec{g}(t) := \vec{g}(\vec{y}(t))$, $B_n(t) := B_n(t, \vec{g}(t))$, $B'_n(t) := B'_n(t, \vec{g}(t))$, $K_n(t) := K_n(t, \vec{g}(t))$ and $I_{n,m}(t) := I_{n,m}(t, \vec{g}(t))$, with the consensus that an underlying process $\vec{y}(t)$ has been fixed. We will specify explicitly when a result applies only for near equilibrium initial conditions or the step initial condition, and without specification the result holds for any initial condition $\vec{x}(0) \in \mathbb{X}$.

PROOF OF THEOREM 2.9. This is an immediate consequence of the following propositions, which we establish in Sections 4 and 5, respectively.

PROPOSITION 2.12. *For near equilibrium initial conditions, the collection of processes* $\{Z_\varepsilon\}_\varepsilon$ *is tight in* $C(\mathbb{R}_+ \times \mathbb{R})$.

PROPOSITION 2.13. *For near equilibrium initial conditions, any limiting point* $\mathcal{Z}$ *of* $\{Z_\varepsilon\}_\varepsilon$ *solves the SHE.* □

PROOF OF THEOREM 2.10. We let $\widetilde{Z}(\tau,r) := r_*\varepsilon^{-1}(1-\rho)Z(\tau,r)$ so that $\widetilde{Z}_\varepsilon(\tau,r) = \widetilde{Z}(\varepsilon^{-3}\tau_*^\varepsilon\tau, \varepsilon^{-1}r_*r)$. The pre-factor of $\widetilde{Z}(\tau,r)$ is choose so that

$$\varepsilon r_*^{-1} \sum_{\xi \in \Xi(0)} \widetilde{Z}(0,\xi) = 1. \tag{2.28}$$

Further, using the exponential decay (in $|\xi|$) of $\widetilde{Z}(0,\xi)$, one easily obtains $\widetilde{Z}_\varepsilon(0,\cdot) \Rightarrow \delta(\cdot)$. With this and Theorem 2.9, following the argument of [2], Section 3, Theorem 2.10 is an immediate consequence of the following moment estimates of $\widetilde{Z}(\tau,r)$, which we establish in Section 4.

PROPOSITION 2.14. *For the step initial condition, for any* $T > 0$, $k \geq 1$ *and* $v \in (0, 1/2)$, *there exists* $C = C(T,k,v) < \infty$ *such that*

$$\|\widetilde{Z}(\tau,r)\|_{2k} \leq C(\varepsilon^3\tau)^{-1/2}, \tag{2.29}$$

$$\|\widetilde{Z}(\tau,r) - \widetilde{\mathcal{Z}}(\tau,r')\|_{2k} \leq C(\varepsilon|r-r'|)^v(\varepsilon^3\tau)^{-(1+v)/2}, \tag{2.30}$$

for all $\tau \in (0, \varepsilon^{-3}T]$ *and* $r, r' \in \mathbb{R}$. □

With $\vec{x}(t) = \vec{y}(tJ)$, from Theorems 2.9–2.10 we immediately obtain the following corollary on the convergence of $\vec{x}(t)$. More precisely, letting $\mu_\varepsilon :=$

$\sum_{s=0}^{J-1}\mu_\varepsilon(s)$ and $\lambda_\varepsilon := \sum_{s=0}^{J-1}\lambda_\varepsilon(s)$, we define

$$H_\varepsilon^J(\tau,r) := -\varepsilon x_{\varepsilon^{-1}r_* r+\varepsilon^{-3}\mu_\varepsilon\tau_*^\varepsilon\tau}\big(\varepsilon^{-3}\tau_*^\varepsilon\tau\big) + (\log\rho-\varepsilon)\big(\varepsilon^{-1}r_* r+\varepsilon^{-3}\mu_\varepsilon\tau_*^\varepsilon\tau\big) + \log\big(\varepsilon^{-3}\lambda_\varepsilon\tau_*^\varepsilon\tau\big)$$

(which is defined on $\mathbb{R}_+\times\mathbb{R}$ by the aforementioned linear interpolation). With $H_\varepsilon(\tau,r)$ as in (2.25), we have that $H_\varepsilon^J(\tau,r)=H_\varepsilon(\tau,r)$, for all $\tau\in\varepsilon^3\tau_*^{\varepsilon-1}\mathbb{N}$ and $r\in\mathbb{R}$. From this, Theorems 2.9–2.10 immediately imply

COROLLARY 2.15. (a) *Let $\mathcal{Z}(\tau,r)$ and $\mathcal{Z}^{\mathrm{ic}}(r)$ be as in Theorem 2.9 so that $\mathcal{H}(\tau,r):=\log\mathcal{Z}(\tau,r)$ is the unique solution of the KPZ equation starting from $\log\mathcal{Z}^{\mathrm{ic}}$, and let $\{\vec{x}^\varepsilon(0)\}_\varepsilon$ be a collection of near equilibrium initial conditions. If $Z_\varepsilon(0,\cdot)\Rightarrow\mathcal{Z}^{\mathrm{ic}}(\cdot)$, we have*

$$H_\varepsilon^J \Rightarrow \mathcal{H} \qquad \text{on } C\big((0,\infty)\times\mathbb{R}\big), \qquad \text{as } \varepsilon\to 0. \tag{2.31}$$

(b) *Let $\widetilde{\mathcal{Z}}(\tau,r)$ be as in Theorem 2.10, let $\{\vec{x}^\varepsilon(t)\}_\varepsilon$ be started from the step initial condition, and let $\widetilde{H}_\varepsilon^J(\tau,x):=H_\varepsilon^J(\tau,x)+\log(\varepsilon^{-1}(1-\rho)r_*)$. We have*

$$\widetilde{H}_\varepsilon^J \Rightarrow \widetilde{\mathcal{H}} \qquad \text{on } C\big((0,\infty)\times\mathbb{R}\big), \qquad \text{as } \varepsilon\to 0.$$

REMARK 2.16. (a) In (2.31), the convergence does not include $\tau=0$ as we do not assume $\mathcal{Z}^{\mathrm{ic}}(r)>0$.

(b) From Theorems 2.9 and 2.10, one also easily obtains corresponding convergence results for $Z_\varepsilon(\tau J,r)$, the centered scaled exponential transform of $\vec{x}(t)$, but we do not state the results explicitly here.

3. Discrete SHE, proof of Proposition 2.6.

PROOF OF LEMMA 2.3. Fixing $s\in\mathbb{N}$ and $\vec{g}\in(\mathbb{N}^*)^\infty$, we let K_n and $I_{n,i}$ denote $K_n(s,\vec{g})$ and $I_{n,i}(s,\vec{g})$, respectively, and let $K_{n,i}:=\sum_{n\ge i'\ge i}I_{n,i'}$ denote the ith partial sum of (2.4). With $B_k(s,g)$ and $B_k'(s,g)$ defined as in the preceding, we have

$$\mathbf{E}\big|B_k'(s,g)-B_k(s,g)\big| \le 1-\frac{1}{1+\alpha}\frac{1-\nu}{1+\alpha}<1, \qquad \mathbf{E}B_k(s,g)\le\frac{\alpha}{1+\alpha}<1.$$

Consequently, $K_{n,i}\to K_n$ (as $i\to-\infty$) in L^k for all $k\ge 1$.

To show $K_n\in\{0,1\}$, first we use the identity $I_{n,i}=(B_n'-B_n)I_{n-1,i}$ [which follows from (2.3)] to obtain

$$K_{n,i-1}(s,\vec{g}) = K_{n-1,i-1}(s,\vec{g})B_n'(s,g_n)+\big(1-K_{n-1,i-1}(s,\vec{g})\big)B_n(s,g_n). \tag{3.1}$$

We now show that, in fact, $K_{n,i}\in\{0,1\}$ for all $n\ge i$. Indeed, $K_{n,n}=B_n\in\{0,1\}$. The general case then follows by induction on $n-i\in\mathbb{N}$ using (3.1). Consequently, $K_{n,i}\to K_n\in\{0,1\}$.

The identity (2.5) follows directly by letting $i\to-\infty$ in (3.1). □

Turning to proving Proposition 2.6, as this does not involve the scaling $\varepsilon \to 0$, throughout this section we *suppress* the dependence of parameters on ε. We begin by deriving an equation for $Q_n(t)$. Consider the time-inhomogeneous random walk $X'(t+1) := R'(0) + R'(1) + \cdots + R'(t)$, where $R'(s)$, $s \in \mathbb{N}$, are $\mathbb{N}$-valued, independent, with distribution

$$\begin{aligned} &\mathbf{P}\big(R'(t) = n\big) \\ &\quad := p'(t+1,t,n) \\ &\quad := \begin{cases} \dfrac{\alpha(t)(1-q)}{1+\alpha(t)}\Big(\dfrac{\nu+\alpha(t)}{1+\alpha(t)}\Big)^{n-1}\Big(1-\dfrac{\nu+\alpha(t)}{1+\alpha(t)}\Big), & \text{for } n>0, \\ 1-\dfrac{\alpha(t)(1-q)}{1+\alpha(t)}, & \text{for } n=0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.2}$$

Let $[p'(t+1,t) * Q(t)]_n := \sum_{m\in\mathbb{Z}} p'(t+1,t,n-m)Q_m(t)$ denote convolution.

PROPOSITION 3.1. *For any $t \in \mathbb{N}$ and $n \in \mathbb{Z}$, we have*

$$Q_n(t+1) = \big[p'(t+1,t) * Q(t)\big]_n + Q_n(t)W'_n(t), \tag{3.3}$$

where $W'_n(t) := (q-1)\overline{K}_n(t)$. Further, for any $n_1, n_2 \in \mathbb{Z}$,

$$\begin{aligned} &Q_{n_1}(t)Q_{n_2}(t)\mathbf{E}\big(W'_{n_1}(t)W'_{n_2}(t)|\mathscr{F}(t)\big) \\ &\quad = \Big(\frac{\nu+\alpha(t)}{1+\alpha(t)}\Big)^{|n_1-n_2|}\Theta'_1(t, n_1\wedge n_2)\Theta'_2(t, n_1\wedge n_2), \end{aligned} \tag{3.4}$$

where $\Theta'_1(t,n) := qQ_n(t) - [p'(t+1,t) * Q(t)]_n$ and $\Theta'_2(t,n) := [p'(t+1,t) * Q(t)]_n - Q_n(t)$.

PROOF. Fixing $t \in \mathbb{N}$, to simplify notation we let $\mathbf{E}'(\cdot)$ denote $\mathbf{E}(\cdot|\mathscr{F}(t))$. We begin by proving (3.3). With $Q_n(t) := q^{y_n(t)+n}$, a generic jump $y_n(t) \mapsto y_n(t)+1$ of particles decreases $Q_n(t)$ by $(1-q)Q_n(t)$. Consequently,

$$\begin{aligned} Q_n(t+1) - Q_n(t) &= (q-1)Q_n(t)K_n(t) \\ &= (q-1)Q_n(t)\mathbf{E}'\big(K_n(t)\big) + Q_n(t)W'_n(t). \end{aligned}$$

With $K_n(t)$ as in (2.4), we have

$$\begin{aligned} \mathbf{E}'\big(K_n(t)\big) = \sum_{m:n\geq m} &\frac{\nu+\alpha(t)}{1+\alpha(t)}q^{g_n(t)} \cdots \frac{\nu+\alpha(t)}{1+\alpha(t)}q^{g_{m+1}(t)} \\ &\times \frac{\alpha(t)}{1+\alpha(t)}\big(1-q^{g_m(t)}\big). \end{aligned} \tag{3.5}$$

Multiplying both sides by $(q-1)Q_n(t)$, and then using the readily verify identity

$$Q_n(t)q^{g_n(t)+g_{n-1}(t)+\cdots+g_{m'+1}(t)} = Q_{m'}(t), \tag{3.6}$$

we obtain

$$(q-1)Q_n(t)\mathbf{E}'\big(K_n(t)\big) = \big[p'(t+1,t) * Q(t)\big]_n - Q_n(t), \tag{3.7}$$

whereby (3.3) follows.

Turning to (3.4), without lost of generality we assume $n_1 \geq n_2$. With $W_n'(t)$ defined as in the proceeding, we have $\mathbf{E}'(W_{n_1}'(t)W_{n_2}'(t)) = (q-1)^2 \operatorname{Cov}'(K_{n_1}(t), K_{n_2}(t))$, where $\operatorname{Cov}'(K_{n_1}(t), K_{n_2}(t)) := \mathbf{E}'(K_{n_1}(t)K_{n_2}(t)) - \mathbf{E}'(K_{n_1}(t))\mathbf{E}' \times (K_{n_2}(t))$. Letting $\widetilde{I}_{n_1,n_2}(t) := \prod_{n_1 \geq k > n_2}(B_k'(t) - B_k(t))$, with $K_{n_1}(t)$ as in (2.4), we have

$$K_{n_1}(t) = \sum_{n_1 \geq m > n_2} I_{n_1,m}(t) + \widetilde{I}_{n_1,n_2}(t)K_{n_2}(t),$$

for all $n_1 \geq n_2$. Multiply both sides by $K_{n_2}(t)$, using $K_{n_2}(t)^2 = K_{n_2}(t)$, and then take the expectation $\mathbf{E}'(\cdot)$ on both sides. With $\{B_k(s), B_k'(s)\}_k$ being independent, we obtain that

$$\mathbf{E}'\big(K_{n_1}(t)K_{n_2}(t)\big) = \Big(\sum_{n_1 \geq m > n_2} \mathbf{E}'\big(I_{n_1,m}(t)\big) + \mathbf{E}'\big(\widetilde{I}_{n_1,n_2}(t)\big)\Big)\mathbf{E}'\big(K_{n_2}(t)\big).$$

Subtracting $\mathbf{E}'(K_{n_1}(t))\mathbf{E}'(K_{n_2}(t)) = [\sum_{m:n_1 \geq m} \mathbf{E}'(I_{n_1,m}(t))]\mathbf{E}'(K_{n_2}(t))$ from the last expression yields

$$\operatorname{Cov}'\big(K_{n_1}(t)K_{n_2}(t)\big) = \Big(-\sum_{m:n_2 \geq m} \mathbf{E}'\big(I_{n_1,m}(t)\big) + \mathbf{E}'\big(\widetilde{I}_{n_1,n_2}(t)\big)\Big)\mathbf{E}'\big(K_{n_2}(t)\big).$$

Further using $\mathbf{E}'(I_{n_1,m}(t)) = \mathbf{E}'(I_{n_2,m}(t))\mathbf{E}'(\widetilde{I}_{n_1,n_2}(t))$, we arrive at

$$\operatorname{Cov}'\big(K_{n_1}(t)K_{n_2}(t)\big) = \mathbf{E}'\big(\widetilde{I}_{n_1,n_2}(t)\big)\big(-\mathbf{E}'\big(K_{n_2}(t)\big)+1\big)\mathbf{E}'\big(K_{n_2}(t)\big). \tag{3.8}$$

With $\mathbf{E}'(\widetilde{I}_{n_1,n_2}(s)) = (\frac{\nu+\alpha(t)}{1+\alpha(t)})^{n_2-n_1} q^{g_{n_1}(t)+\cdots+g_{n_2+1}(t)}$, multiplying both sides of (3.8) by $(q-1)^2 Q_{n_1}(t)Q_{n_2}(t)$, and then applying (3.6)–(3.7), we conclude (3.4). □

We next introduce a centering to $R'(t)$. Let

$$\sigma(t) := (a_{\operatorname{mod}_J(t)})^2 - (a_{\operatorname{mod}_J(t)+1})^2 + (a_{\operatorname{mod}_J(t)} - a_{\operatorname{mod}_J(t)+1})(b+b'). \tag{3.9}$$

LEMMA 3.2. *For any $t \in \mathbb{N}$, we have $\mathbf{E}(\lambda(t)\rho^{R'(t)}) = 1$, so that*

$$\mathbf{P}\big(R(t)+\mu(t)=n\big) := \lambda(t)\rho^n \mathbf{P}\big(R'(t)=n\big), \qquad n \in \mathbb{N} \tag{3.10}$$

defines an $(\mathbb{N}-\mu(t))$-valued random variable. Further, $\mathbf{E}(R(t)) = 0$ and

$$\mathbf{E}\big(R(t)^2\big) = r_*^2\sigma(t). \tag{3.11}$$

PROOF. Fixing $t \in \mathbb{N}$, we consider the function

$$
\begin{aligned}
f(x) &:= \mathbf{E}\big(\lambda(t)x^{R'(t)}\big) \\
&= \lambda(t)\Bigg[\bigg(1 - \frac{(1-q)\alpha(t)}{1+\alpha(t)}\bigg) \\
&\qquad + \sum_{i=1}^{\infty} x^i \frac{(1-q)\alpha(t)}{1+\alpha(t)}\bigg(\frac{\nu+\alpha(t)}{1+\alpha(t)}\bigg)^{i-1}\frac{1-\nu}{1+\alpha(t)}\Bigg] \\
&= \lambda(t)\frac{1-\nu x + q\alpha(t) - q\alpha(t)x}{1-\nu x+\alpha(t)-\alpha(t)x}.
\end{aligned}
\tag{3.12}
$$

With $\lambda(t)$ defined as in (2.10), specializing (3.12) at $x = \rho$ we obtain $f(\rho) = 1$, thereby concluding $\mathbf{E}(\lambda(t)\rho^{R'(t)}) = 1$. Next, differentiating $f(x)$ yields

$$
\bigg(x\frac{d}{dx}f\bigg)(\rho) = \mathbf{E}\big(\lambda(t)\rho^{R'(t)}R'(t)\big) = \mathbf{E}\big(R(t)+\mu(t)\big),
\tag{3.13}
$$

$$
\bigg(x\frac{d}{dx}\bigg(x\frac{d}{dx}f\bigg)\bigg)(\rho) = \mathbf{E}\big(\lambda(t)\rho^{R'(t)}R'(t)^2\big) = \mathbf{E}\big(\big(R(t)+\mu(t)\big)^2\big).
\tag{3.14}
$$

Plugging (3.12) into the LHS of (3.13)–(3.14) and specializing at $x = \rho$, after some tedious but straightforward calculations, one obtains $(x\frac{df}{dx})(\rho) = \mu(t)$ and $(x\frac{d}{dx}(x\frac{d}{dx}f))(\rho) = \mu(t)^2 + r_*^2\sigma(t)$, thereby concluding $\mathbf{E}(R(t)) = 0$ and (3.11). □

PROOF OF PROPOSITION 2.6. With $[p'(t+1,t) * Q(t)]_n = \mathbf{E}(Q_{n-R'(t)}(t))$ and $[p(t+1,t) * Z(t)](\xi) = \mathbf{E}(Z(t, \xi - R(t)))$, we have the readily verified identity

$$
\begin{aligned}
&\widehat{\lambda}(t+1)\rho^{\xi+\widehat{\mu}(t+1)}\big[p'(t+1,t) * Q(t)\big]_{\xi+\widehat{\mu}(t+1)} \\
&\qquad = \big[p(t+1,t) * Z(t)\big](\xi),
\end{aligned}
\tag{3.15}
$$

for all $\xi \in \Xi(t)$. In (3.3), we set $n = \xi + \widehat{\mu}(t+1)$, and multiply both sides by $\widehat{\lambda}(t+1)\rho^{\xi+\widehat{\mu}(t+1)}$. Using (2.11) and (3.15), we obtain

$$
Z(t+1,\xi) = \big[p(t+1,t) * Z(t)\big](\xi) + Z\big(t, \xi+\mu(t)\big)W\big(t,\xi+\mu(t)\big).
\tag{3.16}
$$

Iterating this equation from $t = t_2 - 1$ until $t = t_1$, we thus conclude (2.14).

To derive (2.17), in (3.4), we set $n_1 = \xi_1 + \widehat{\mu}(t)$ and $n_2 = \xi_2 + \widehat{\mu}(t)$, and multiply both sides by $\widehat{\lambda}(t+1)^2\rho^{\xi_1+\widehat{\mu}(t)}\rho^{\xi_2+\widehat{\mu}(t)}$. Using (2.11) and (3.15) to express the resulting equation in terms of $Z(t,\cdot)$ and $p(t+1,t,\cdot)$, we thus conclude (2.17). □

4. Moment estimates: Proof of Propositions 2.12 and 2.14. Hereafter, we let $C(u_1, u_2, \dots)$ denote a generic finite positive constant that depends only on the designated variables $u_1, u_2, \dots$ and possibly on $\alpha > 0$, $\nu \in [0, 1)$ and $\rho \in (0, 1)$, which are fixed throughout the paper.

LEMMA 4.1. *The function* $\phi(x, \varepsilon; t) := \mathbf{E}(x^{R_\varepsilon(t)})$ *extends analytically in* (x, ε) *to a neighborhood of* $(1, 0) \in \mathbb{C}^2$, *with the Taylor expansion*

$$\phi(x, \varepsilon; t) = 1 + 2^{-1}\big(\partial_{xx\varepsilon}\phi(1, 0; t)\big)\varepsilon(x-1)^2 + O\big(\varepsilon|x-1|^3\big), \tag{4.1}$$

and $\partial_{xx\varepsilon}\phi(1, 0; t) \in (0, \infty)$.

PROOF. Since $R_\varepsilon(t)$ is defined by $R'_\varepsilon(t)$ as in (3.10), we clearly have

$$\phi(x, \varepsilon; t) = \mathbf{E}\big(\lambda_\varepsilon(t)\rho^{R'_\varepsilon(t)} x^{R'_\varepsilon(t)-\mu_\varepsilon(t)}\big). \tag{4.2}$$

By (3.12), the function $\mathbf{E}(\lambda_\varepsilon(t)x^{R'_\varepsilon(t)})$ is analytic in (x, ε) within a neighborhood of $(\rho, 0)$, whereby $\phi(x, \varepsilon; t)$ is analytic within a neighborhood of $(1, 0)$. To obtain the Taylor expansion (4.1), we differentiate (4.2) in x, and then specialized at $x = 1$, thereby obtaining

$$\partial_x\phi(1, \varepsilon; t) = \mathbf{E}\big(\lambda_\varepsilon(t)\rho^{R'_\varepsilon(t)}\big(R'_\varepsilon(t) - \mu_\varepsilon(t)\big)\big) = \mathbf{E}\big(R_\varepsilon(t)\big) = 0, \tag{4.3}$$

$$\partial_{xx}\phi(1, \varepsilon; t) = \mathbf{E}\big(\lambda_\varepsilon(t)\rho^{R'_\varepsilon(t)}\big(R'_\varepsilon(t) - \mu_\varepsilon(t)\big)^2\big) = \mathbf{E}\big(R_\varepsilon(t)^2\big) = r_*^2\sigma_\varepsilon(t). \tag{4.4}$$

With $\sigma_\varepsilon(t)$ defined as in (3.9), we have

$$\sigma_\varepsilon(t) = \varepsilon\alpha\gamma(1+\alpha\gamma)^{-3}\big(2\alpha\gamma + \big(b + b'\big)(1+\alpha\gamma)\big) + O\big(\varepsilon^2\big). \tag{4.5}$$

From (4.3)–(4.5), we conclude that $\partial_x^n\partial_\varepsilon^m\phi(1, 0; t) = 0$, unless $n \ge 2$ and $m \ge 1$, and that $\partial_{xx\varepsilon}\phi(1, 0; t) > 0$, thereby obtaining (4.1). □

Based on Lemma 4.1, we proceed to estimating of the heat kernel.

PROPOSITION 4.2. *Given any* $u, T > 0$ *and* $v \in (0, 1]$, *there exists* $C = C(T, u)$ *such that*

$$\sum_{\zeta\in\Xi(t_2,t_1)} p_\varepsilon(t_2, t_1, \zeta)e^{u\varepsilon|\zeta|} \le C, \tag{4.6}$$

$$\sum_{\zeta\in\Xi(t_2,t_1)} |\zeta|^v p_\varepsilon(t_2, t_1, \zeta)e^{u\varepsilon|\zeta|} \le C\big(\varepsilon|t_2 - t_1|\big)^{v/2}, \tag{4.7}$$

$$p_\varepsilon(t_2, t_1, \xi) \le C\varepsilon^{-1/2}(t_2 - t_1 + 1)^{-1/2}, \tag{4.8}$$

$$\big|p_\varepsilon(t_2, t_1, \xi) - p_\varepsilon\big(t_2, t_1, \xi'\big)\big| \le C\varepsilon^{-(1+v)/2}\big|\xi - \xi'\big|^v(t_2 - t_1 + 1)^{-(1+v)/2}, \tag{4.9}$$

for all $t_1 \le t_2 \in [0, \varepsilon^{-3}T] \cap \mathbb{N}$ *and* $\xi, \xi' \in \Xi(t_2)$.

PROOF. To prove (4.6), we consider $F_1(u') := e^{u'(R_\varepsilon(t_1)+\cdots+R_\varepsilon(t_2-1))}$. With $\mathbf{E}(F_1(u')) = \sum_{\zeta\in\Xi(t_2,t_1)} p_\varepsilon(t_2,t_1,\zeta)e^{u'\zeta}$ and $e^{\varepsilon u|\zeta|} \le e^{\varepsilon u\zeta} + e^{-\varepsilon u\zeta}$, to show (4.6), it suffices to bound the expression $\mathbf{E}(F_1(u')) = \prod_{s=t_1}^{t_2-1}\phi(e^{u'},\varepsilon;s)$, for $u' = \pm u\varepsilon$. This, by (4.1), is bounded by $[1+C\varepsilon(e^{u'}-1)^2]^{t_2-t_1}$. With $t_2-t_1 \le \varepsilon^{-3}T$, the last expression is bounded by $C = C(T,u)$, from which we conclude (4.6).

Turning to showing (4.7), we let $F_2 := R_\varepsilon(s_1)+\cdots+R_\varepsilon(s_2-1)$. Similar to the preceding, it suffices to bound the expression

$$\sum_{\zeta\in\Xi(s_1,s_2)} |\zeta|^v p_\varepsilon(s_1,s_2,\zeta)e^{u'\zeta} = \mathbf{E}\big(F_1(u')|F_2|^v\big) \le \big\|F_1(u')\big\|_{2/(2-v)}\big\||F_2|^v\big\|_{2/v},$$

for $u' = \pm u\varepsilon$, where we used Hölder's inequality in the last inequality. With $(F_1(u'))^{2/(2-v)} = F_1(2u'/(2-v))$, applying (4.6) for $u = 2u/(2-v)$ we obtain $\|F_1(\pm u\varepsilon)\|_{2/(2-v)} \le C$. As for F_2, with $\mathbf{E}(R_\varepsilon(s)) = 0$ and (3.11), we have

$$\begin{aligned}\big\||F_2|^v\big\|_{2/v} &= \big[\mathbf{E}(F_2)^2\big]^{v/2} = \big[\mathbf{E}\big(R_\varepsilon(t_1)^2\big)+\cdots+\mathbf{E}\big(R_\varepsilon(t_2-1)^2\big)\big]^{v/2}\\ &\le C\big[\sigma_\varepsilon(t_1)+\cdots+\sigma_\varepsilon(t_2-1)\big]^{v/2}.\end{aligned}$$

Further using (4.5), we thus obtain $\||f_2|^v\|_{2/v} \le C(\varepsilon|t_2-t_1|)^{v/2}$, thereby concluding (4.7).

Proceeding to showing (4.8)–(4.9), first we apply the inversion formula of the characteristic function, $p_\varepsilon(t_2,t_1,\zeta) = \frac{1}{2\pi\mathbf{i}}\int_{-\pi}^{\pi} e^{-\mathbf{i}\zeta r}\prod_{s=t_1}^{t_2-1}\phi(e^{\mathbf{i}r},\varepsilon;s)\,dr$ and the uniform v-Hölder continuity of $x\mapsto e^{\mathbf{i}x}$, $x\in\mathbb{R}$, to obtain

$$p_\varepsilon(t_2,t_1,\xi) \le C\int_{-\pi}^{\pi}\prod_{s=t_1}^{t_2-1}\big|\phi(e^{\mathbf{i}r},\varepsilon;s)\big|\,dr, \tag{4.10}$$

$$\big|p_\varepsilon(t_2,t_1,\xi) - p_\varepsilon\big(s_1,s_2,\xi'\big)\big| \le C\int_{-\pi}^{\pi}\big(|\xi-\xi'|r\big)^v\prod_{s=t_1}^{t_2-1}\big|\phi(e^{\mathbf{i}r},\varepsilon;s)\big|\,dr. \tag{4.11}$$

To further bond these expressions, we apply (4.1) for $x = e^{\mathbf{i}r}$ to obtain

$$\big|\phi\big(e^{\mathbf{i}r},\varepsilon;s\big)\big| \le 1-\varepsilon r^2/C, \qquad \forall s\in\mathbb{N}, \forall r\in\mathbb{R} \text{ with } |r|\le r_0, \tag{4.12}$$

where $r_0 > 0$ is a constant. As for $|r| > r_0$, we let $f(n,\varepsilon;s) := \mathbf{P}(R_\varepsilon(s) = n - \mu_\varepsilon(s))$, whereby $\phi(e^{\mathbf{i}r},\varepsilon;s) = \sum_{n\in\mathbb{N}} e^{\mathbf{i}r(n-\mu_\varepsilon(s))}f(n,\varepsilon;s)$. Expressing the $n=0$ term as the sum of $e^{-\mathbf{i}r\mu_\varepsilon(s)}f(1,\varepsilon;s)$ and $e^{-\mathbf{i}r\mu_\varepsilon(s)}(f(0,\varepsilon;s)-f(1,\varepsilon;s))$, and then combining the former with the $n=1$ term, we obtain

$$\begin{aligned}\phi\big(e^{\mathbf{i}r},\varepsilon;s\big) &= e^{-\mathbf{i}r\mu_\varepsilon(s)}\big(1+e^{\mathbf{i}r}\big)f(1,\varepsilon;s) + e^{-\mathbf{i}r\mu_\varepsilon(s)}\big(f(0,\varepsilon;s)-f(1,\varepsilon;s)\big)\\ &\quad + \sum_{n>1} e^{\mathbf{i}r(n-\mu_\varepsilon(s))}f(n,\varepsilon;s).\end{aligned}$$

Taking the absolute value of this yields

$$\begin{aligned}(4.13)\qquad |\phi(e^{\mathbf{i}r},\varepsilon;s)| \le |1+e^{\mathbf{i}r}| f(1,\varepsilon;s) \\ + |f(0,\varepsilon;s)-f(1,\varepsilon;s)| + \sum_{n>1} f(n,\varepsilon;s).\end{aligned}$$

By (3.2) and (3.10), we find that $f(0,\varepsilon;s) > f(1,\varepsilon;s) > \varepsilon/C$. Using this and $\sum_{n=0}^{\infty} f(n,\varepsilon;s) = 1$ in (4.13), we then obtain the bound

$$\begin{aligned}(4.14)\qquad |\phi(e^{\mathbf{i}r},\varepsilon;s)| \le f(1,\varepsilon;s)|1+e^{\mathbf{i}r}| + (1-2f(1,\varepsilon;s)) \le 1-\varepsilon/C, \\ \forall s\in\mathbb{N}, |r|>r_0.\end{aligned}$$

Now, combining (4.12) and (4.14), we thus obtain $|\phi(e^{\mathbf{i}r},\varepsilon;s)| \le 1-\varepsilon r^2/C$, for all $r\in\mathbb{R}$ and $s\in\mathbb{N}$. Plugging this in (4.10)–(4.11), we conclude $\prod_{s=t_1}^{t_2-1}|\phi(e^{\mathbf{i}r},\varepsilon; s)| \le e^{-\varepsilon r^2(t_2-t_1)/C}$, for all $\varepsilon\in(0,1]$ and $r\in(-\pi,\pi)$. Using this, further integrating over $r\in(-\pi,\pi)$, we conclude (4.8)–(4.9). □

Next, we derive bounds on moments of $Z_{\mathrm{mg}}(t_2,t_1,\xi)$ [as in (2.16)] and $Z_{\nabla,\mathrm{mg}}(t_2,t_1,\xi,\xi') := Z_{\mathrm{mg}}(t_2,t_1,\xi) - Z_{\mathrm{mg}}(t_2,t_1,\xi')$. Hereafter, we adapt the shorthand notation $\xi_{\pm t} := \xi \pm \mu(t)$.

LEMMA 4.3. *For any $T>0$, $k\ge 1$ and $v\in[0,1]$, there exists $C=C(k,T)$ such that for all $t_1\le t_2\in\mathbb{N}\cap[0,\varepsilon^{-3}T]$ and $\xi\in\Xi(t_1)$,*

$$(4.15)\qquad \|Z_{\mathrm{mg}}(t_2,t_1,\xi)\|_{2k}^2 \le C\varepsilon^{3/2}\sum_{s=t_1}^{t_2-1}[\overline{p}^{\varepsilon}(s) * \|Z(s)\|_{2k}^2](\xi),$$

$$\begin{aligned}(4.16)\qquad \|Z_{\nabla,\mathrm{mg}}(t_2,t_1,\xi,\xi')\|_{2k}^2 \le C\varepsilon^{(3-v)/2}|\xi-\xi'|^{v}\sum_{s=t_1}^{t_2-1}\big([\overline{p}_{\nabla}^{\varepsilon}(s) * \|Z(s)\|_{2k}^2](\xi) \\ + [\overline{p}_{\nabla}^{\varepsilon}(s) * \|Z(s)\|_{2k}^2](\xi')\big),\end{aligned}$$

where $\overline{p}^{\varepsilon}(s,\zeta) := (|t_2-s|+1)^{-1/2}p_{\varepsilon}(t_2,s,\zeta)$ and $\overline{p}_{\nabla}^{\varepsilon}(s,\zeta) := (|t_2-s|+1)^{-(1+v)/2}p_{\varepsilon}(t_2,s,\zeta)$.

PROOF. Fix $t_1\le t_2\in\mathbb{N}$ and $\xi,\xi'\in\Xi(t_1)$. To prove (4.15), we estimate the corresponding quadratic variation. To this end, letting $F(s,\zeta) := [p_{\varepsilon}(t_2,s+1) * (Z(s)W(s))](\zeta)$, we consider the discrete time martingales

$$M(t) := \sum_{s=t_1}^{t-1} F(s,\xi_{+s}), \qquad M_{\nabla}(t) := \sum_{s=t_1}^{t-1}\big(F(s,\xi_{+s}) - F(s,\xi'_{+s})\big)$$

with the respectively predictable compensator $V_M(s) := \mathbf{E}[F(s,\xi_{+s})^2|\mathscr{F}(s)]$ and $V_M^\nabla(s) := \mathbf{E}[(F(s,\xi_{+s}) - F(s,\xi'_{+s}))^2|\mathscr{F}(s)]$. With $M(t_2) = Z_{\mathrm{mg}}(t_2,t_1,\xi)$ and $M_\nabla(t_2) = Z_{\nabla,\mathrm{mg}}(t_2,t_1,\xi,\xi')$, applying Burkholder's inequality we obtain

$$
\begin{aligned}
\|Z_{\mathrm{mg}}(t_2,t_1,\xi)\|_{2k}^2 &\le C\sum_{s=t_1}^{t_2-1}\|V_M(s)\|_{2k}, \\
\|Z_{\nabla,\mathrm{mg}}(t_2,t_1,\xi,\xi')\|_{2k}^2 &\le C\sum_{s=t_1}^{t_2-1}\|V_M^\nabla(s)\|_{2k}.
\end{aligned}
\tag{4.17}
$$

Having derived the inequality (4.17), we now proceed to estimating $|V_M(s)|$ and $|V_M^\nabla(s)|$. By (2.17), we have

$$
\begin{aligned}
V_M(s) = \sum_{\zeta_1,\zeta_2\in\Xi(s)} &\widehat{p}^\varepsilon(s,\zeta_1,\zeta_2)\left(\frac{(\nu+\alpha(s))\rho}{1+\alpha(s)}\right)^{|\zeta_1-\zeta_2|} \\
&\times \Theta_1(s,\zeta_1\wedge\zeta_2)\Theta_2(s,\zeta_1\wedge\zeta_2),
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
V_M^\nabla(s) = \sum_{\zeta_1,\zeta_2\in\Xi(s)} &\widehat{p}_\nabla^\varepsilon(s,\zeta_1,\zeta_2)\left(\frac{(\nu+\alpha(s))\rho}{1+\alpha(s)}\right)^{|\zeta_1-\zeta_2|} \\
&\times \Theta_1(s,\zeta_1\wedge\zeta_2)\Theta_2(s,\zeta_1\wedge\zeta_2),
\end{aligned}
\tag{4.19}
$$

where $\widehat{p}^\varepsilon(s,\zeta_1,\zeta_2) := p_\varepsilon(t_2,s,\xi_{+s}-\zeta_1)p_\varepsilon(t_2,s,\xi_{+s}-\zeta_2)$, and

$$
\widehat{p}_\nabla^\varepsilon(s,\zeta_1,\zeta_2) := \prod_{k=1,2}[p_\varepsilon(t_2,s,\xi_{+s}-\zeta_k) - p_\varepsilon(t_2,s,\xi'_{+s}-\zeta_k)].
$$

To bound $V_M(s)$ and $V_M^\nabla(s)$, set $(\zeta,\zeta') := (\zeta_1\wedge\zeta_2,\zeta_1-\zeta_2)$ in (4.18)–(4.19), whereby $\sum_{\zeta_1,\zeta_2\in\Xi(s)} = \sum_{\zeta'\in\mathbb{Z}}\sum_{\zeta\in\Xi(s)}$; take absolute value on both sides of (4.18)–(4.19); then, use (4.8) in (4.18) to bound $|\widehat{p}^\varepsilon(s,\zeta_1,\zeta_2)|$ by $C\varepsilon^{-1/2}\overline{p}^\varepsilon(s+1,\xi_{+s}-\zeta)$; and similarly use (4.9) in (4.19) to bound $|\widehat{p}_\nabla^\varepsilon(s,\zeta_1,\zeta_2)|$ by $C\varepsilon^{-(1+\upsilon)/2}|\xi-\xi'|^\upsilon[\overline{p}_\nabla^\varepsilon(s+1,\xi_{-s}-\zeta)+\overline{p}_\nabla^\varepsilon(s+1,\xi'_{+s}-\zeta)]$. Upon summing over ζ', we obtain

$$
|V_M(s)| \le C\varepsilon^{-1/2}[\overline{p}^\varepsilon(s+1)*(\Theta_1(s)\Theta_2(s))](\xi_{+s}),
\tag{4.20}
$$

$$
\begin{aligned}
|V_M^\nabla(s)| \le C\varepsilon^{-(1+\upsilon)/2}|\xi-\xi'|^\upsilon\big(&[\overline{p}_\nabla^\varepsilon(s+1)*(\Theta_1(s)\Theta_2(s))](\xi_{+s}) \\
&+[\overline{p}_\nabla^\varepsilon(s+1)*(\Theta_1(s)\Theta_2(s))](\xi'_{+s})\big).
\end{aligned}
\tag{4.21}
$$

With (4.20)–(4.21), we now turn to estimating $|\Theta_1(s,\zeta)|$ and $|\Theta_2(s,\zeta)|$. To this end, we let

$$
\widehat{\mathcal{L}}_\varepsilon(s,\zeta) := p_\varepsilon(s+1,s,\zeta) - \mathbf{1}_{\{\zeta=-\mu_\varepsilon(s)\}}, \qquad \zeta\in(\mathbb{N}-\mu_\varepsilon(s))
\tag{4.22}
$$

denote a pseudo generator [as the true generator is $\mathcal{L}_\varepsilon(s,\zeta) := p_\varepsilon(s+1,s,\zeta) - \mathbf{1}_{\{\zeta=0\}}$], and then rewrite $\Theta_1(s,\zeta)$ and $\Theta_2(s,\zeta)$ [as in (2.18)–(2.19)] as

$$\Theta_1(s,\zeta) = \big(q_\varepsilon\lambda_\varepsilon(t) - 1\big)Z(s,\zeta) - \big[\widehat{\mathcal{L}}_\varepsilon(s) * Z(s)\big]\big(\zeta - \mu(s)\big), \tag{4.23}$$

$$\Theta_2(s,\zeta) = \big(1 - \lambda_\varepsilon(t)\big)Z(s,\zeta) + \big[\widehat{\mathcal{L}}_\varepsilon(s) * Z(s)\big]\big(\zeta - \mu(s)\big). \tag{4.24}$$

By using (3.2) and (3.10), with $(1-q_\varepsilon) \le C\varepsilon$, it is not hard to show that

$$\big|\widehat{\mathcal{L}}_\varepsilon(s,\zeta)\big| \le C\varepsilon\bigg(\frac{(\nu+\alpha_\varepsilon(s))\rho}{1+\alpha_\varepsilon(s)}\bigg)^{|\zeta|}, \tag{4.25}$$

for some $C<\infty$. Further, with $Z(s,\zeta')$ defined as in (2.11), using (3.6) we have

$$Z\big(s,\zeta'\big) = \rho^{\zeta'-\zeta} q_\varepsilon^{\sum \text{gaps}} Z(s,\zeta) \le \rho^{-|\zeta-\zeta'|}Z(s,\zeta), \qquad \forall \zeta \ge \zeta'.$$

Using this and (4.25) in (4.23)–(4.24), we thus obtain $|\Theta_1(s,\zeta)|, |\Theta_2(s,\zeta)| \le C\varepsilon Z(s,\zeta)$. Plugging this in (4.18)–(4.19), we now arrive at

$$\big|V_M(s)\big| \le C\varepsilon^{3/2}\big[\overline{p}^\varepsilon(s+1) * Z(s)^2\big](\xi_{+s}), \tag{4.26}$$

$$\begin{aligned}\big|V_M^\nabla(s)\big| \le C\varepsilon^{(3-v)/2}\big|\xi-\xi'\big|^v\big(\big[\overline{p}_\nabla^\varepsilon(s+1) * Z(s)^2\big](\xi_{+s}) \\ + \big[\overline{p}_\nabla^\varepsilon(s+1) * Z(s)^2\big](\xi'_{+s})\big).\end{aligned} \tag{4.27}$$

Further, by (4.25), we have $p_\varepsilon(s+1,s,-\mu_\varepsilon(s)) > 1 - C\varepsilon$, whereby $Z(s,\zeta)^2 \le C[p(s+1,s) * Z(s)^2](\zeta_{-s})$. Plugging this in (4.26)–(4.27), using the semigroup property $[p_\varepsilon(t,s+1) * p_\varepsilon(s+1,s)](\zeta) = p_\varepsilon(t,s,\zeta)$, we further obtain

$$\big|V_M(s)\big| \le C\varepsilon^{3/2}\big[\overline{p}^\varepsilon(s) * Z(s)^2\big](\xi),$$

$$\big|V_M^\nabla(s)\big| \le C\varepsilon^{(3-v)/2}\big|\xi-\xi'\big|^v\big(\big[\overline{p}_\nabla^\varepsilon(s) * Z(s)^2\big](\xi) + \big[\overline{p}_\nabla^\varepsilon(s) * Z(s)^2\big](\xi')\big).$$

Now, taking the L^k-norm of both sides, and then combining the result with (4.17), we thus conclude (4.15)–(4.16). □

Based on (4.15), we establish a chaos-series-type bound on the moment of $Z(t,\xi)$.

LEMMA 4.4. *For any $k \ge 1$, $t \in \mathbb{N}$ and $\xi \in \Xi(t)$,*

$$\begin{aligned}\|Z(t,\xi)\|_{2k}^2 \le \big(\big[p_\varepsilon(t,0) * \|Z(0)\|_{2k}\big](\xi)\big) \\ \times \sum_{n=0}^\infty \sum_{\vec{s}\in\Delta_{n+1}(t)} h_\varepsilon(s_1)\cdots h_\varepsilon(s_n)\big(\big[p_\varepsilon(s_{n+1},0) * \|Z(0)\|_{2k}\big](\xi)\big),\end{aligned} \tag{4.28}$$

where $h_\varepsilon(s) := C\varepsilon^{3/2}(s+1)^{-1/2}$, $C = C(k) < \infty$, and $\Delta_n(t) := \{(s_1,\dots,s_n) \in (\mathbb{Z}_{\ge 1})^n : s_1 + \cdots + s_n = t\}$.

REMARK 4.5. Note that $\Delta_n(t) = \varnothing$ for all $n > t$, so the sum over n in (4.28) is in fact finite.

PROOF OF LEMMA 4.4. In (2.14), set $(t_1, t_2) = (0, t)$, applying the elementary inequality $|x + y|^2 \leq 2|x|^2 + 2|y|^2$ and then taking the L^k-norm of both sides, we obtain $\|Z(t, \xi)\|_{2k}^2 \leq 2\|Z_{\mathrm{dr}}(t, 0, \xi)\|_{2k}^2 + 2\|Z_{\mathrm{mg}}(t, 0, \xi)\|_{2k}^2$. For the first term on the RHS, by the triangle inequality we clearly have $\|Z_{\mathrm{dr}}(t, 0, \xi)\|_{2k} \leq [p_\varepsilon(t, 0) * \|Z(0)\|_{2k}](\xi)$, and for second term we apply (4.15). With this, we thus obtain

$$\begin{aligned}\|Z(t, \xi)\|_{2k}^2 \leq{}& 2\big([p_\varepsilon(t, 0) * \|Z(0)\|_{2k}](\xi)\big)^2 \\ &+ C_1 \sum_{s=0}^{t-1} (t - s + 1)^{-1/2} [p_\varepsilon(t, s) * \|Z(s)\|_{2k}^2](\xi),\end{aligned} \tag{4.29}$$

for some $C_1 = C_1(k) < \infty$. Let $h_\varepsilon(s) := 2C_1(s + 1)^{-1/2}$. The bound (4.28) now follows by iterating (4.29), using the semi-group property $[p_\varepsilon(s_3, s_2) * p(s_2, s_1)](\zeta) = [p_\varepsilon(s_3, s_1)](\zeta)$. □

PROOF OF PROPOSITION 2.12. Fix a collection of near equilibrium initial condition, with the corresponding $u = u(k, v)$ (as in Definition 2.8), and fix $T < \infty$, $k \geq 1$ and $v \in [0, 1/2)$. We prove the following moment estimates:

$$\|Z(t, r)\|_{2k} \leq C e^{u\varepsilon|r|}, \tag{4.30}$$

$$\|Z(\tau, r) - Z(\tau, r')\|_{2k} \leq C\big(\varepsilon|r - r'|\big)^v e^{u\varepsilon(|r|+|r'|)}, \tag{4.31}$$

$$\|Z(\tau, r) - Z(\tau', r)\|_{2k} \leq C\big(\varepsilon^3|\tau' - \tau|\big)^{v/2} e^{2u\varepsilon|r|}, \tag{4.32}$$

for some $C = C(T, k, v) < \infty$ and for all $\tau, \tau' \in [0, \varepsilon^{-3}T]$, $r, r' \in \mathbb{R}$ and $\varepsilon > 0$ small enough. These estimates, by the Kolmogorov–Chentsov criterion of tightness [17], Corollary 14.9, immediately imply the tightness of $\{Z_\varepsilon(\cdot, \cdot)\}$ in $C(\mathbb{R}_+ \times \mathbb{R})$.

By definition, $Z(\tau, r)$ is defined on $\mathbb{R}_+ \times \mathbb{R}$ by linear interpolation, so without lost of generality we assume $\tau = t, \tau' = t' \in \mathbb{N} \cap [0, \varepsilon^{-3}T]$ and $r = \xi, r' = \xi' \in \Xi(t)$, and prove (4.30)–(4.32) as follows. □

PROOF OF (4.30). By (2.26), we have $[p_\varepsilon(t, 0) * \|Z(0)\|_{2k}](\xi) \leq \sum_{\zeta \in \Xi(0)} p_\varepsilon(t, 0, \xi - \zeta)(Ce^{u\varepsilon|\zeta|})$. In the last expression, using $e^{u\varepsilon|\zeta|} \leq e^{u\varepsilon|\zeta-\xi|} \times e^{u\varepsilon|\xi|}$ and then using (4.6) to sum over ζ, we obtain $[p_\varepsilon(t, 0) * \|Z(0)\|_{2k}](\xi) \leq C\varepsilon^{u\varepsilon|\xi|}$. Now, plugging this in (4.28), we arrive at

$$\|Z(t, \xi)\|_{2k}^2 \leq Ce^{2u\varepsilon|\xi|} \sum_{n=0}^{\infty} \sum_{\vec{s} \in \Delta_{n+1}(t)} h_\varepsilon(s_1) \cdots h_\varepsilon(s_n). \tag{4.33}$$

For each n, further applying the readily verified inequality

$$\sum_{\vec{s}\in\Delta_{n+1}(t)} h_\varepsilon(s_1)\cdots h_\varepsilon(s_n) \le \int_{\tau_1+\cdots+\tau_n\le t} \prod_{i=1}^{n} (C\varepsilon^{3/2}\tau_i^{-1/2}\,d\tau_i)$$
$$= \frac{[C\Gamma(1/2)(\varepsilon^3 t)^{1/2}]^n}{n\Gamma(n/2)},$$

and then summing over n in (4.33), we thus conclude (4.30). □

PROOF OF (4.31). Let $Z_\nabla(t,\xi,\xi') := Z(t,\xi) - Z(t,\xi')$ and $Z_{\nabla,\mathrm{dr}}(t,\xi,\xi') := Z_{\mathrm{dr}}(t,0,\xi) - Z_{\mathrm{dr}}(t,0,\xi')$. By (2.14), we have $Z_\nabla(t,\xi,\xi') = Z_{\nabla,\mathrm{dr}}(t,\xi,\xi') + Z_{\nabla,\mathrm{mg}}(t,0,\xi,\xi')$, whereby

$$\|Z_\nabla(t,\xi,\xi')\|_{2k}^2 \le 2\|Z_{\nabla,\mathrm{dr}}(t,\xi,\xi')\|_{2k}^2 + 2\|Z_{\nabla,\mathrm{mg}}(t,0,\xi,\xi')\|_{2k}^2. \tag{4.34}$$

Letting $2(Z^*_{\nabla,\mathrm{dr}})^2$ and $2(Z^*_{\nabla,\mathrm{mg}})^2$ denote the respective terms on the RHS, we estimate these terms as follows.

For $Z^*_{\nabla,\mathrm{dr}}$, by the triangle inequality we clearly have $Z^*_{\nabla,\mathrm{dr}} \le \sum_{\zeta\in\Xi(t,0)} p_\varepsilon(t,0,\zeta)\|Z(0,\xi-\zeta) - Z(0,\xi'-\zeta)\|_{2k}$. Using (2.27) in the last expression to replace $\|Z(0,\xi-\zeta) - Z(0,\xi'-\zeta)\|_{2k}$ with $C(\varepsilon|\xi-\xi'|)^v e^{u\varepsilon(|\xi-\zeta|+|\xi'-\zeta|)} \le C(\varepsilon|\xi-\xi'|)^v e^{u\varepsilon(|\xi|+|\xi'|)} e^{2u|\zeta|}$, and then using (4.6) to sum over ζ, we obtain the desired bound $Z^*_{\nabla,\mathrm{dr}} \le C(\varepsilon|\xi-\xi'|)^v e^{u\varepsilon(|\xi|+|\xi'|)}$.

Turning to $Z^*_{\nabla,\mathrm{mg}}$, first we use (4.30) to obtain $[p_\varepsilon(t,s) * \|Z(s)\|_{2k}^2](\xi'') \le \sum_{\zeta\in\Xi(s)} p_\varepsilon(t,s,\xi''-\zeta)(Ce^{2u\varepsilon|\zeta|})$. Further replacing $e^{2u\varepsilon|\zeta|}$ with $e^{2u\varepsilon|\xi''-\zeta|}e^{2u\varepsilon|\xi''|}$, and then using (4.6) to sum over ζ, we obtain

$$[p_\varepsilon(t,s) * \|Z(s)\|_{2k}^2](\xi'') \le Ce^{2u\varepsilon|\xi''|}. \tag{4.35}$$

Now, specializing (4.16) at $(t_1,t_2) = (0,t)$, and combining the result with (4.35) for $\xi'' = \xi$ and $\xi'' = \xi'$, we obtain

$$\begin{aligned}(Z^*_{\nabla,\mathrm{mg}})^2 &\le C(\varepsilon|\xi-\xi'|)^{2v}\varepsilon^{3/2-3v}\sum_{s=0}^{t-1}(t-s+1)^{-(1+2v)/2} \\ &\quad\times \sum_{\zeta\in\Xi(s)} (p_\varepsilon(t,s,\xi-\zeta) + p_\varepsilon(t,s,\xi'-\zeta))e^{2u\varepsilon|\zeta|}.\end{aligned} \tag{4.36}$$

In (4.36), use $e^{2u\varepsilon|\zeta|} \le (e^{u\varepsilon|\xi-\zeta|} + e^{u\varepsilon|\xi'-\zeta|})e^{u\varepsilon(|\xi|+|\xi'|)}$, and then use (4.6) to sum over ζ. With $t \le \varepsilon^{-3}T$, further summing over s we obtain the desired bound $(Z^*_{\nabla,\mathrm{mg}})^2 \le C(\varepsilon|\xi-\xi'|)^{2v}e^{u\varepsilon(|\xi|+|\xi'|)}$. Combining the preceding estimates of $Z^*_{\nabla,\mathrm{dr}}$ and $Z^*_{\nabla,\mathrm{mg}}$ with (4.34), we conclude (4.31). □

PROOF OF (4.32). Without lost of generality, we assume $t' \le t$. By (2.14), we have

$$Z(t,\xi) - Z(t',\xi) = (Z_{\mathrm{dr}}(t,t',\xi) - Z(t',\xi)) + Z_{\mathrm{mg}}(t,t',\xi).$$

Similar to the preceding, we bound separately $Z^*_{\mathrm{dr}} := \|Z_{\mathrm{dr}}(t,t',\xi) - Z(t,\xi)\|_{2k}$ and $Z^*_{\mathrm{mg}} := \|Z_{\mathrm{mg}}(t,t',\xi)\|_{2k}$.

For Z^*_{dr}, with $\sum_{\zeta\in\Xi(t')} p_\varepsilon(t,t',\xi-\zeta) = 1$, we have

$$\big(Z_{\mathrm{dr}}(t,t',\xi) - Z(t',\xi)\big) = \sum_{\zeta\in\Xi(t')} p_\varepsilon(t,t',\xi-\zeta)\big(Z(t',\zeta) - Z(t',\xi)\big).$$

Take the L^{2k}-norm on both sides, and then use (4.31) to replace $\|Z(t',\zeta) - Z(t',\xi)\|_{2k}$ with $C(\varepsilon|\xi-\zeta|)^v e^{u\varepsilon(|\xi|+|\zeta|)} \le C(\varepsilon|\xi-\zeta|)^v e^{u\varepsilon|\xi-\zeta|+2u\varepsilon|\xi|}$. Using (4.7) to sum over ζ, we then obtain the derided bound $Z^*_{\mathrm{dr}} \le C(\varepsilon^3|t-t'|)^{v/2} e^{2u\varepsilon|\xi|}$.

As for Z^*_{mg}, combining (4.15) and (4.35), one obtains $(Z^*_{\mathrm{mg}})^2 \le Ce^{2u\varepsilon|\xi|} \times \sum_{s=t'}^{t-1} \varepsilon^{3/2}(t-s+1)^{-1/2}$. With $t \le \varepsilon^{-3}T$, summing over s we obtain the desired bound $(Z^*_{\mathrm{mg}})^2 \le Ce^{2u\varepsilon|\xi|}(\varepsilon^3|t-t'|)^{1/2} \le Ce^{2u\varepsilon|\xi|}(\varepsilon^3|t-t'|)^v$. □

This completes the proof of Proposition 2.12. We now turn to the proof of Proposition 2.14.

PROOF OF PROPOSITION 2.14. As explained at the beginning of the proof of Proposition 2.12, without lost of generality we let $\tau = t \in \mathbb{N} \cap (0, \varepsilon^{-3}T]$ and $r = \xi, r' = \xi' \in \Xi(t)$.

To show (2.29), multiply both sides of (4.28) by $[r_*\varepsilon^{-1}(1-\rho)]^2$ to obtain

$$\begin{aligned}\|\widetilde{Z}(t,\xi)\|_{2k}^2 &\le \big([p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi)\big)\\ &\quad\times \sum_{n=0}^{\infty}\sum_{\vec{s}\in\Delta_{n+1}(t)} h_\varepsilon(s_1)\cdots h_\varepsilon(s_n)\big([p_\varepsilon(s_{n+1},0) * \widetilde{Z}(0)](\xi)\big).\end{aligned} \tag{4.37}$$

Note that here $\widetilde{Z}(0,\xi)$ is deterministic. By (4.8) and (2.28), we have

$$[p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi) \le C[\varepsilon^3(t+1)]^{-1/2}. \tag{4.38}$$

Applying this to the last term in (4.37), we arrive at

$$\begin{aligned}\|\widetilde{Z}(t,\xi)\|_{2k}^2 &\le C\big([p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi)\big)\\ &\quad\times \sum_{n=0}^{\infty}\sum_{\vec{s}\in\Delta_{n+1}(t)} h_\varepsilon(s_1)\cdots h_\varepsilon(s_n)[\varepsilon^3(s_{n+1}+1)]^{-1/2}.\end{aligned}$$

Further, for each n applying the readily verified inequality

$$\begin{aligned}&\sum_{\vec{s}\in\Delta_{n+1}(t)} h_\varepsilon(s_1)\cdots h_\varepsilon(s_n)[\varepsilon^3(s_{n+1}+1)]^{-1/2}\\ &\quad\le \int_{\tau_1+\cdots+\tau_{n+1}=t} (\varepsilon^3\tau_{n+1})^{-1/2} \prod_{i=1}^{n} (C\varepsilon^{3/2}\tau_i^{-1/2}d\tau_i) = \frac{(C\Gamma(1/2)\varepsilon^3 t)^{n+1}}{\Gamma((n+1)/2)},\end{aligned}$$

and summing over n, with $t \le \varepsilon^{-3}T$, we obtain

$$\|\widetilde{Z}(t,\xi)\|_{2k}^2 \le C\big([p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi)\big)(\varepsilon^3 t)^{-1/2}. \tag{4.39}$$

Using again (4.38), we thus conclude (2.29).

Turning to showing (2.30), we let $\widetilde{Z}_\nabla(t,\xi,\xi')$ denote $\widetilde{Z}(t,\xi) - \widetilde{Z}(t,\xi') = r_*\varepsilon^{-1}(1-\rho)(Z(t,\xi) - Z(t,\xi'))$, and similarly for $\widetilde{Z}_{\nabla,\mathrm{dr}}(t,\xi,\xi')$ and $\widetilde{Z}_{\nabla,\mathrm{mg}}(t,\xi,\xi')$. Multiplying both sides of (4.34) by $(r_*\varepsilon^{-1}(1-\rho))^2$, we obtain

$$\|\widetilde{Z}_\nabla(t,\xi,\xi')\|_{2k}^2 \le 2\widetilde{Z}_{\nabla,\mathrm{dr}}(t,\xi,\xi')^2 + 2\|\widetilde{Z}_{\nabla,\mathrm{mg}}(t,\xi,\xi')\|_{2k}^2. \tag{4.40}$$

Let $2(\widetilde{Z}^*_{\nabla,\mathrm{dr}})^2$ and $2(\widetilde{Z}^*_{\nabla,\mathrm{mg}})^2$ denote the respective terms on the RHS. We estimate them as follows.

For $\widetilde{Z}^*_{\nabla,\mathrm{dr}}$, combining (4.9) for $(s_1,s_2) = (0,t)$ and (2.28) we obtain the desired bound $\widetilde{Z}^*_{\nabla,\mathrm{dr}} \le (\varepsilon|\xi-\xi'|)^v(\varepsilon^3 t)^{-(v+1)/2}$. As for $\widetilde{Z}^*_{\nabla,\mathrm{mg}}$, multiplying both sides of (4.16) by $(r_*\varepsilon^{-1}(1-\rho))^2$, we obtain

$$\begin{aligned}(\widetilde{Z}^*_{\nabla,\mathrm{mg}})^2 &\le C\varepsilon^{(3-6v)/2}(\varepsilon|\xi-\xi'|)^{2v} \\ &\quad \times \sum_{s=0}^{t-1}(t-s+1)^{-(2v+1)/2}\big([p_\varepsilon(t,s) * \|\widetilde{Z}(s)\|_{2k}^2](\xi) \\ &\quad + [p_\varepsilon(t,s) * \|\widetilde{Z}(s)\|_{2k}^2](\xi')\big).\end{aligned} \tag{4.41}$$

Plugging (4.39) in (4.41), and summing over s, with $t \le \varepsilon^{-3}T$, we obtain

$$(\widetilde{Z}^*_{\nabla,\mathrm{mg}})^2 \le C(\varepsilon|\xi-\xi'|)^{2v}\big([p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi) + [p_\varepsilon(t,0) * \widetilde{Z}(0)](\xi')\big).$$

In the last expression, further using (4.38), we obtain the desired bound $(\widetilde{Z}^*_{\nabla,\mathrm{mg}})^2 \le C(\varepsilon|\xi-\xi'|)^{2v}(\varepsilon^3 t)^{-1/2} \le C(\varepsilon|\xi-\xi'|)^{2v}(\varepsilon^3 t)^{-(v+1)}$. Combining the preceding bounds on $\widetilde{Z}^*_{\nabla,\mathrm{dr}}$ and $\widetilde{Z}^*_{\nabla,\mathrm{dr}}$ we conclude (2.30). □

5. The martingale problem: Proof of Proposition 2.13. Hereafter, we use $\mathcal{B}_\varepsilon(s,\vec{\zeta})$ and $\mathcal{E}_\varepsilon(s,\vec{\zeta})$, $\vec{\zeta} = (\zeta_1,\dots,\zeta_n)$, to denote respectively *generic* processes that are uniformly bounded and uniformly vanishing, that is,

$$\begin{aligned}&\sup\{\|\mathcal{B}_\varepsilon(s,\vec{\zeta})\|_k : \vec{\zeta} \in (\varepsilon^{-1}U)^n, s \le T\varepsilon^{-3}, \varepsilon \in (0,1]\} < \infty, \\ &\qquad \sup\{\|\mathcal{E}_\varepsilon(s,\vec{\zeta})\|_k : \vec{\zeta} \in (\varepsilon^{-1}U)^n, s \le T\varepsilon^{-3}\} \longrightarrow 0, \qquad \text{as } \varepsilon \to 0,\end{aligned}$$

for any compact $U \subset \mathbb{R}$, $k \ge 1$ and $T > 0$.

We begin by deriving an approximate expression for the cross variance as in (2.17).

LEMMA 5.1. *For near equilibrium initial conditions, we have*

$$
\begin{aligned}
&Z(s,\zeta_1)Z(s,\zeta_2)\mathbf{E}\big(W(s,\zeta_1)W(s,\zeta_2)|\mathscr{F}(s)\big) \\
&\quad = \varepsilon^2 \frac{\alpha\gamma}{(1+\alpha\gamma)^2}\bigg(\frac{(\nu+\alpha(s))\rho}{1+\alpha(s)}\bigg)^{|\zeta_1-\zeta_2|}\big(Z(s,\zeta_1\wedge\zeta_2)^2+\mathcal{E}_\varepsilon(s,\zeta_1,\zeta_2)\big).
\end{aligned}
\tag{5.1}
$$

PROOF. We prove (5.1) by approximating the identities (4.23)–(4.24), using the moment estimate (4.31). By (2.21) we have $(q_\varepsilon\lambda_\varepsilon(t)-1) = -\varepsilon(1+\alpha\gamma)^{-1} + O(\varepsilon^2)$, so that $(q_\varepsilon\lambda_\varepsilon(t)+1)Z(s,\zeta) = -\varepsilon(1+\alpha\gamma)^{-1}Z(s,\zeta)+\varepsilon\mathcal{E}_\varepsilon(s,\zeta)$, and by (4.31), fixing arbitrary $v\in(0,1/2)$, we have $Z(s,\zeta') = Z(s,\zeta)+|\varepsilon(\zeta'-\zeta)|^v\mathcal{B}_\varepsilon(s,\zeta,\zeta')$. In (4.23), using these approximations we obtain

$$
\begin{aligned}
\Theta_1(s,\zeta) &= -\varepsilon(1+\alpha\gamma)^{-1}Z(s,\zeta)+\varepsilon\mathcal{E}_\varepsilon(t,\zeta) \\
&\quad -\bigg(\sum_{\zeta'\in\Xi(s)}\widehat{\mathcal{L}}_\varepsilon\big(s,\zeta_{-s}-\zeta'\big)\bigg)Z(s,\zeta) \\
&\quad +\sum_{\zeta\in\Xi(s)}\widehat{\mathcal{L}}_\varepsilon\big(s,\zeta_{-s}-\zeta'\big)\big|\varepsilon\big(\zeta'-\zeta\big)\big|^v\mathcal{B}_\varepsilon\big(s,\zeta,\zeta'\big).
\end{aligned}
\tag{5.2}
$$

With $\widehat{\mathcal{L}}_\varepsilon(s,\zeta_{-s}-\zeta')$ as in (4.22), the second last term in (5.2) is zero since $\mathcal{L}_\varepsilon(s,\zeta)$, and the last term is of the form $\varepsilon^{1+v}\mathcal{B}_\varepsilon(s,\zeta)\le\varepsilon\mathcal{E}_\varepsilon(s,\zeta)$ by (4.25). Consequently,

$$
\Theta_1(s,\zeta) = -\varepsilon(1+\alpha\gamma)^{-1}Z(s,\zeta)+\varepsilon\mathcal{E}_\varepsilon(t,\zeta).
\tag{5.3}
$$

Similarly, for $\Theta_2(s,\xi_2)$ we have

$$
\Theta_2(s,\zeta) = -\varepsilon\alpha\gamma(1+\alpha\gamma)^{-1}Z(s,\zeta)+\varepsilon\mathcal{E}_\varepsilon(s,\zeta).
\tag{5.4}
$$

Combining (5.3)–(5.4) with (2.17) yields (5.1). □

We proceed to proving Proposition 2.13. Recall from [3] the following martingale problem of the SHE.

DEFINITION 5.2. Let $\mathcal{Z}$ be a $C([0,\infty),C(\mathbb{R}))$-valued process such that given any $T>0$, there exists $u<\infty$ such that

$$
\sup_{\tau\in[0,T]}\sup_{r\in\mathbb{R}} e^{-u|r|}\mathbf{E}\big(\mathcal{Z}(\tau,r)^2\big)<\infty.
\tag{5.5}
$$

For such $\mathcal{Z}$ and for $\psi\in C_c^\infty(\mathbb{R})$, let $\langle\mathcal{Z}(\tau),\psi\rangle := \int_{\mathbb{R}}\mathcal{Z}(\tau,r)\psi(r)\,dr$. We say $\mathcal{Z}$ solves the martingale problem with initial condition $\mathcal{Z}^{\mathrm{ic}}\in C(\mathbb{R})$ if $\mathcal{Z}(0,\cdot)=\mathcal{Z}^{\mathrm{ic}}(\cdot)$ in distribution, and

$$
\begin{aligned}
&\tau\longmapsto N_\psi(\tau) := \langle\mathcal{Z}(\tau),\psi\rangle-\langle\mathcal{Z}(0),\psi\rangle-\int_0^\tau 2^{-1}\bigg\langle\mathcal{Z}(\tau'),\frac{d^2}{dx^2}\psi\bigg\rangle d\tau', \\
&\tau\longmapsto \widehat{N}_\psi(\tau) := \big(N_\psi(\tau)\big)^2-\int_0^\tau\langle\mathcal{Z}(\tau')^2,\psi^2\rangle d\tau'
\end{aligned}
$$

are local martingales, for any $\psi\in C_c^\infty(\mathbb{R})$.

PROOF OF PROPOSITION 2.13. Recall from [3], Proposition 4.11, that for any initial condition $\mathcal{Z}^{\mathrm{ic}}$ satisfying

$$\|\mathcal{Z}^{\mathrm{ic}}(r)\|_2 \le Ce^{a|r|}, \qquad \text{for some } a > 0, \tag{5.6}$$

the martingale problem of Definition 5.2 has a unique solution, which coincides with the law of the solution of the SHE with initial condition $\mathcal{Z}^{\mathrm{ic}}$. By passing to the relative subsequence, we assume that $Z_\varepsilon \Rightarrow \mathcal{Z}$, which, by (4.30), satisfies the moment condition (5.5). It hence suffices to show that $\mathcal{Z}$ solves the martingale problem in Definition 5.2.

We begin by deriving the discrete analog of $N_\psi(\tau)$ and $\widehat{N}_\psi(\tau)$. To this end, fixing $\psi \in C_c^\infty(\mathbb{R})$, we consider the discrete approximation

$$\langle Z(t), \psi\rangle_\varepsilon := \varepsilon r_*^{-1} \sum_{\xi \in \Xi(t)} Z(t,\xi)\psi(\varepsilon r_*^{-1}\xi) \tag{5.7}$$

of $\langle Z_\varepsilon(\varepsilon^3(\tau_*^\varepsilon J)^{-1}t), \psi\rangle$, and similarly define

$$M_\psi(t) := \varepsilon r_*^{-1} \sum_{\xi \in \Xi(t+1)} Z(t, \xi + \mu(t)) W(t, \xi + \mu(t)) \psi(\varepsilon r_*^{-1}\xi). \tag{5.8}$$

In (3.16), multiply both sides by $\varepsilon r_*^{-1}\psi(\varepsilon r_*^{-1}\xi)$. Upon summing over $\xi \in \Xi(t+1)$, we obtain $\langle Z(t+1), \psi\rangle_\varepsilon = \langle Z(t), \psi_{p_\varepsilon(t+1,t)}\rangle_\varepsilon + M_\psi(t)$, where

$$\psi_{p_\varepsilon(t+1,t)}(\zeta) := \sum_{\xi \in \Xi(t+1)} p_\varepsilon(t+1, t, \xi - \zeta)\psi(\varepsilon^{-1} r_* \xi).$$

Subtracting $\langle Z(t), \psi\rangle_\varepsilon$ from both sides, we further obtain

$$\langle Z(t+1), \psi\rangle_\varepsilon - \langle Z(t), \psi\rangle_\varepsilon = \langle Z(t), \psi_{\mathcal{L}_\varepsilon(t)}\rangle_\varepsilon + M_\psi(t, \xi), \tag{5.9}$$

where

$$\psi_{\mathcal{L}_\varepsilon(t)}(\zeta) := \sum_{\xi \in \Xi(t+1)} p_\varepsilon(t+1, t, \xi - \zeta)\psi(\varepsilon^{-1} r_* \xi) - \psi(\varepsilon^{-1} r_* \zeta). \tag{5.10}$$

Now, summing (5.9) over $s = 0, \dots, t-1$, we arrive at

$$\langle Z(t), \psi\rangle_\varepsilon - \langle Z(0), \psi\rangle_\varepsilon - \sum_{s=0}^{t-1} \langle Z(s), \psi_{\mathcal{L}_\varepsilon(t)}\rangle_\varepsilon = \sum_{s=0}^{t-1} M_\psi(s) := N_\psi^\varepsilon(t). \tag{5.11}$$

The process $t \mapsto N_\psi^\varepsilon(t)$ is a discrete time martingale of quadratic variation $\sum_{s=0}^{t-1} \mathbf{E}(M_\psi(s)^2 | \mathscr{F}(s))$, so

$$\widehat{N}_\psi^\varepsilon(t) := (N_\psi^\varepsilon(t))^2 - \sum_{s=0}^{t-1} \mathbf{E}(M_\psi(s)^2 | \mathscr{F}(s)) \tag{5.12}$$

is also a discrete time martingale.

With $N^{\varepsilon}_{\psi}(t)$ and $\widehat{N}^{\varepsilon}_{\psi}(t)$ as in the preceding, we proceed to showing that $N_{\psi}(\tau)$ and $\widehat{N}_{\psi}(\tau)$ are local martingales. Since, by (4.32), passing from discrete time to continuous time introduces only a negligible error, it suffices to show that terms in (5.11)–(5.12) converge in distribution to the corresponding terms. More precisely, recalling $t_{\varepsilon}(\tau) := \varepsilon^{-3}\tau^{\varepsilon}_{*} J\tau$, our goal is to show

$$\langle Z(t_{\varepsilon}(\tau)), \psi\rangle_{\varepsilon} \Longrightarrow \langle \mathcal{Z}(\tau), \psi\rangle,$$

$$\sum_{s<t_{\varepsilon}(\tau)} \langle Z(s), \psi_{\mathcal{L}_{\varepsilon}}\rangle_{\varepsilon} \Longrightarrow \int_0^{\tau} 2^{-1}\Big\langle \mathcal{Z}(\tau'), \frac{d^2\psi}{dx^2}\Big\rangle d\tau',$$

$$\sum_{s<t_{\varepsilon}(\tau)} \mathbf{E}\big(M_{\psi}(s)^2 | \mathscr{F}(s)\big) \Longrightarrow \int_0^{\tau} \langle \mathcal{Z}^2(\tau'), \psi(\tau')^2\rangle d\tau'.$$

To this end, since $Z_{\varepsilon} \Rightarrow \mathcal{Z}$, it clearly suffices to show

$$\mathbf{E}\big|\langle Z(t_{\varepsilon}(\tau)), \psi\rangle_{\varepsilon} - \langle Z_{\varepsilon}(\tau), \psi\rangle\big| \longrightarrow 0, \tag{5.13}$$

$$\mathbf{E}\bigg|\sum_{s<t_{\varepsilon}(\tau)} \langle Z(s), \psi_{\mathcal{L}_{\varepsilon}}\rangle_{\varepsilon} - \int_0^{\tau}\Big\langle Z_{\varepsilon}(\tau'), \frac{d^2\psi}{dx^2}\Big\rangle d\tau'\bigg| \longrightarrow 0, \tag{5.14}$$

$$\mathbf{E}\bigg|\sum_{s<t_{\varepsilon}(\tau)} \mathbf{E}\big(M_{\psi}(s)^2 | \mathscr{F}(s)\big) - \int_0^{\tau} \langle Z^2_{\varepsilon}(\tau'), \psi^2\rangle d\tau'\bigg| \longrightarrow 0. \tag{5.15}$$

We prove (5.13)–(5.15) as follows. □

PROOF OF (5.13). This amounts to show that the terms

$$\langle Z_{\varepsilon}(\tau), \psi\rangle = \varepsilon r_*^{-1}\int_{\mathbb{R}} Z\big(t_{\varepsilon}(\tau), r\big)\psi\big(\varepsilon r_*^{-1} r\big)\, dr \tag{5.16}$$

and $\langle Z(t_{\varepsilon}(\tau)), \psi\rangle_{\varepsilon}$ are approximately equal. To this end, fixing arbitrary $\zeta \in \Xi(t)$ and $|r-\zeta| \le 1$, we use the smoothness of ψ and the moment estimates (4.30)–(4.31), for arbitrary $v \in (0, 1/2)$, to obtain $Z(t,\zeta)\psi(\varepsilon r_*^{-1}\zeta) - Z(t,r)\psi(\varepsilon r_*^{-1} r) = \varepsilon^{v}\mathcal{B}_{\varepsilon}(\zeta, r) = \mathcal{E}_{\varepsilon}(t,\zeta,r)$. From this, with $\langle Z_{\varepsilon}(\tau), \psi\rangle$ and $\langle Z(t_{\varepsilon}(\tau)), \psi\rangle_{\varepsilon}$ as in (5.16) and (5.7), we conclude $\langle Z_{\varepsilon}(\tau), \psi\rangle = \langle Z(t_{\varepsilon}(\tau)), \psi\rangle_{\varepsilon} + \mathcal{E}_{\varepsilon}(t)$, whereby (5.13) follows. □

PROOF OF (5.14). Taylor expanding $\psi(\xi\varepsilon r_*^{-1})$ around $\xi = \zeta$ yields

$$\begin{aligned}\psi\big(\varepsilon r_*^{-1}\xi\big) &= \psi\big(\varepsilon r_*^{-1}\zeta\big) + \Big(\frac{d\psi}{dx}\big(\varepsilon r_*^{-1}\zeta\big)\Big)\varepsilon r_*^{-1}(\zeta - \xi)\\ &\quad + \Big(2^{-1}\Big(\frac{d^2\psi}{dx^2}\big(\varepsilon r_*^{-1}\zeta\big)\Big) + \mathcal{E}_{\varepsilon}(\xi,\zeta)\Big)\varepsilon^2 r_*^{-2}(\zeta - \xi)^2.\end{aligned}$$

Plug this in (5.10). With $\sum_{\xi\in\Xi(s+1)} p_\varepsilon(s+1,s,\xi-\zeta)(\xi-\zeta)^k = \mathbf{E}(R_\varepsilon(s)^k)$, using $\mathbf{E}(R(s))=0$, (3.11) and $\sigma_\varepsilon \le C\varepsilon$, we obtain

$$\psi_{\mathcal{L}_\varepsilon(s)}(\zeta) = 2^{-1}\left(\frac{d^2\psi}{dx^2}(\varepsilon r_*^{-1}\zeta)\right)\varepsilon^2\sigma_\varepsilon(s) + \varepsilon^3\mathcal{E}_\varepsilon(s,\zeta).$$

Now, plugging this expression of $\psi_{\mathcal{L}_\varepsilon(s)}(\zeta)$ in the LHS of (5.14), with $t_\varepsilon(\tau)\le \varepsilon^{-3}C$, we obtain

$$\sum_{s<t_\varepsilon(\tau)} \langle Z(s), \psi_{\mathcal{L}_\varepsilon(s)}\rangle_\varepsilon = \sum_{s<t_\varepsilon(\tau)} \varepsilon^2\sigma_\varepsilon(s)\left\langle Z(s), 2^{-1}\frac{d^2\psi}{dx^2}\right\rangle_\varepsilon + \mathcal{E}_\varepsilon.$$

Next, divide the sum on the r.h.s. into sums over the disjoint intervals $T_t := \mathbb{Z}\cap[tJ, tJ+J)$. We neglect the boundary terms of $T_{\tau\varepsilon^{-3}/\tau_*^\varepsilon}\cap[0,t_\varepsilon(\tau))$, since, by (4.30), those terms contribute only $\varepsilon^2\sigma_\varepsilon(s)\mathcal{B}_\varepsilon = \mathcal{E}_\varepsilon$. Within each interval T_t, use (4.32) to replace $\langle Z(s), \frac{d^2\psi}{dx^2}\rangle_\varepsilon$ with $\langle Z(tJ), \frac{d^2\psi}{dx^2}\rangle_\varepsilon + \mathcal{E}_\varepsilon(s)$. Further, with $\sigma_\varepsilon(s)$ and τ_*^ε as in (3.9) and (2.22), we have $\sum_{s\in T_t}\sigma_\varepsilon(s) = \varepsilon(\tau_*^\varepsilon)^{-1}$. Consequently,

$$\sum_{s<t_\varepsilon(\tau)} \langle Z(s), \mathcal{L}_\varepsilon\psi\rangle_\varepsilon = \frac{\varepsilon^3}{\tau_*^\varepsilon}\sum_{t<\varepsilon^{-3}\tau_*^\varepsilon\tau}\left\langle Z(tJ), 2^{-1}\frac{d^2\psi}{dx^2}\right\rangle_\varepsilon + \mathcal{E}_\varepsilon. \tag{5.17}$$

The RHS of (5.17) represents a discrete approximation of $\int_0^\tau\langle Z_\varepsilon(\tau'), 2^{-1}\times\frac{d^2\psi}{dx^2}\rangle\,d\tau'$. In particular, by following the same procedure as in the proof of (5.13), one obtains $\frac{\varepsilon^3}{\tau_*^\varepsilon}\sum_{t<t_\varepsilon(\tau)}\langle Z(tJ), 2^{-1}\frac{d^2\psi}{dx^2}\rangle_\varepsilon = \int_0^\tau\langle Z(\tau'), 2^{-1}\frac{d^2\psi}{dx^2}\rangle\,d\tau' + \mathcal{E}_\varepsilon$, thereby concluding (5.14). □

PROOF OF (5.15). To calculate $\mathbf{E}(M_\psi(s)^2|\mathscr{F}(s))$, we use the expression (5.8) and the approximation (5.1) to obtain

$$\begin{aligned}&\mathbf{E}(M_\psi(s)^2|\mathscr{F}(s))\\ &\quad= \frac{\varepsilon^4\alpha\gamma}{(1+\alpha\gamma)^2 r_*^2}\sum_{\xi\in\Xi(s+1)} Z(s,\xi_{+s})^2\psi(\varepsilon r_*^{-1}\xi)F(s,\xi) + \varepsilon^3\mathcal{E}_\varepsilon(s),\end{aligned} \tag{5.18}$$

where

$$F(s,\xi) := \sum_{n\in\mathbb{Z}}\psi\bigl(\varepsilon r_*^{-1}(\xi+|n|)\bigr)\left(\frac{(\nu+\alpha_\varepsilon(s))\rho}{1+\alpha_\varepsilon(s)}\right)^{|n|}. \tag{5.19}$$

Let $\eta_\varepsilon := \sum_{n\in\mathbb{Z}}(\frac{(\nu+\alpha_\varepsilon(s))\rho}{1+\alpha_\varepsilon(s)})^{|n|}$. In (5.19), using the continuity of ψ at $\varepsilon r_*^{-1}\xi$, we further obtain $F(s,\xi) = \eta_\varepsilon\psi(\varepsilon r_*^{-1}\xi) + \mathcal{E}_\varepsilon(s,\xi)$. Plugging this expression in (5.18), we arrive at

$$\mathbf{E}(M_\psi(s)^2|\mathscr{F}(s)) = \varepsilon^3\alpha\gamma\eta_\varepsilon\bigl((1+\alpha\gamma)^2 r_*\bigr)^{-1}\langle Z^2(s), \psi^2\rangle_\varepsilon + \varepsilon^3\mathcal{E}_\varepsilon(s). \tag{5.20}$$

Calculating η_ε to the first order we have $\eta_\varepsilon = \frac{1+\alpha+\nu\rho+\alpha\rho}{1+\alpha-\nu\rho-\alpha\rho} + O(\varepsilon)$. Using this and (2.23), a tedious but straightforward calculation shows that $\alpha\gamma\eta_\varepsilon((1+\alpha\gamma)^2 r_*)^{-1} = (J\tau_*^\varepsilon)^{-1} + O(\varepsilon)$. Consequently, summing (5.20) over $s < t_\varepsilon(\tau)$ yields

$$\sum_{s<t_\varepsilon(\tau)} \mathbf{E}\big(M_\psi(s)^2 \big| \mathscr{F}(s)\big) = \frac{\varepsilon^3}{\tau_*^\varepsilon J} \sum_{s<t_\varepsilon(\tau)} \big\langle Z^2(s), \psi^2\big\rangle_\varepsilon + \mathcal{E}_\varepsilon(s).$$

The RHS represents a discrete approximation of $\int_0^\tau \langle Z_\varepsilon(\sigma)^2, \psi^2\rangle \, d\sigma$, so following the same procedure as in the proof of (5.13), one concludes (5.15). □

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DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
2990 BROADWAY
NEW YORK, NEW YORK 10027
USA
AND
CLAY MATHEMATICS INSTITUTE
10 MEMORIAL BLVD. SUITE 902
PROVIDENCE, RHODE ISLAND 02903
USA
AND
INSTITUT HENRI POINCARÉ,
11 RUE PIERRE ET MARIE CURIE
75005 PARIS
FRANCE
E-MAIL: ivan.corwin@gmail.com

STANFORD UNIVERSITY
BUILDING 380
SLOAN MATHEMATICAL CENTER,
STANFORD, CALIFORNIA 94305
USA
E-MAIL: lctsai@stanford.edu