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Krasnoselskii-type fixed-point theorems for weakly sequentially continuous mappings

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Abstract

In this article, we establish some fixed-point results of Krasnoselskii type for the sum of two weakly sequentially continuous mappings that extend previous ones. In the last section, we apply such results to study the existence of solutions to a nonlinear integral equation modelled in a Banach space.

1. *Introduction*

From a mathematical point of view, many problems arising from diverse areas of natural science involve the existence of solutions of nonlinear equations of the form

$$Au + Bu = u, \quad u \in M, \tag{1.1}$$

where M is a closed and convex subset of a Banach space X (see, for example, [**4**, **19**, **21**]). In particular, many nonlinear integral equations can be formulated in this form (see [**6**, **11**, **14**, **20**, **21**, **24**, **25**]). Krasnoselskii's fixed-point theorem [**18**] was one of the first results for solving equations of the form (1.1). To solve this equation, Krasnoselskii had to impose that (i) M is a bounded set, (ii) A is a continuous and compact mapping, (iii) B is a strict contraction and (iv) $A(M) + B(M) \subseteq M$. Nevertheless, in several applications, the verification of conditions (i)–(iv) is, in general, either quite hard to be done or even some of them fail. As a tentative approach to grappling with those difficulties, many interesting articles have appeared relaxing some of the assumptions (i)–(iv). For instance, in [**9**], Burton and Kirk used an alternative Leray–Schauder type to avoid the boundedness of M; in [**8**], Burton replaced assumption (iv) by (if $u = Bu + Ay$ with $y \in M$, then $u \in M$). Condition (ii) involves continuity and compactness; since the infinite-dimensional Banach spaces are not locally compact, this condition seems quite strong, hence O'Regan [**22**] and Barroso and Teixeira [**6**] gave new versions of Krasnoselskii's theorem for weakly sequentially continuous mappings, since one of the advantages of the weak topology is the fact that if the set M is weakly compact, then every weakly sequentially continuous map $T : M \to X$ is weakly continuous, and therefore $T(M)$ is also weakly compact. As a consequence, a lot of applications to problems with lack of compactness are solved. Since then, some authors have worked using the weak sequential continuity of the maps (see, for example, [**4**, **23**, **26**] and the references therein). Condition (iii) is also quite restrictive, so some authors have replaced it by a more general condition (many times B is assumed to be a ϕ-contraction [**23**]).

The main goal of the present paper is to establish new variants of Krasnoselskii's theorem in the spirit of the articles [**6**, **22**, **23**, **26**] which allow us to relax the assumptions (i)–(iv) and also to extend several results included in such works. Two examples will be discussed in the last section. The first one will provide the existence of solutions for nonlinear integral equations

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of the form

$$u(t) = f(t, u(t)) + \int_0^t g(s, u(s))\, ds, \quad u \in C(0, T; E),$$

where E is a reflexive Banach space and $f : [0, T] \times E \to E$ is not necessary a lipschitzian map. So, the case where f is a ϕ-contraction is a particular one of our framework. To justify our results, we will discuss a concrete example where the function f is not lipschitzian. Thus, the previous known results cannot be applied in this case.

2. *Preliminaries*

Throughout this paper, we suppose that $(X, \|\cdot\|)$ is a real Banach space. As usual, for any $r > 0$, B_r denotes the closed ball in X centred at 0_X with radius r. Here $\rightharpoonup$ denotes weak convergence and $\longrightarrow$ denotes strong convergence in X, respectively. Let $\mathcal{B}(X)$ denote the collection of all nonempty bounded subsets of X. Then $\mathcal{W}(X)$ is the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X.

Recall that the notion of the measure of weak noncompactness was introduced by De Blasi [**10**]; it is the map $\omega : \mathcal{B}(X) \to [0, \infty)$ defined in the following way:

$$\omega(M) := \inf\{r > 0 : \text{ there exists } W \in \mathcal{W}(X) \text{ with } M \subseteq W + B_r\},$$

for every $M \in \mathcal{B}(X)$. For convenience we recall some basic properties of $\omega(\cdot)$ needed below (see [**1**] or [**10**]).

Let M_1, M_2 be two elements of $\mathcal{B}(X)$. Then the following conditions are satisfied:

(1) if $M_1 \subseteq M_2$, then $\omega(M_1) \leqslant \omega(M_2)$;
(2) $\omega(M_1) = 0$ if and only if $\overline{M_1}^{\omega} \in \mathcal{W}(X)$ ($\overline{M_1}^{\omega}$ means the weak closure of M_1);
(3) $\omega(\overline{M_1}^{\omega}) = \omega(M_1)$;
(4) $\omega(M_1 \cup M_2) = \max\{\omega(M_1), \omega(M_2)\}$;
(5) $\omega(\lambda M_1) = |\lambda|\omega(M_1)$ for all $\lambda \in \mathcal{R}$;
(6) $\omega(\mathrm{co}(M_1)) = \omega(M_1)$;
(7) $\omega(M_1 + M_2) \leqslant \omega(M_1) + \omega(M_2)$.

A mapping $T : M \subseteq X \to X$ is said to be ω-k-contraction, for some $k \in [0, 1)$, if T satisfies $\omega(T(A)) \leqslant k\omega(A)$ for every bounded set $A \subseteq M$ with $\omega(A) > 0$. The map T is said to be ω-condensing if $\omega(T(A)) < \omega(A)$. On the other hand, T is said to be a ϕ-contraction if there exists a continuous nondecreasing function $\phi : [0, \infty) \to [0, +\infty)$ such that $\phi(0) = 0$, $\phi(r) < r$ for any $r > 0$ and $||T_x - T_y|| \leqslant \phi(||x - y||)$ for ever $x, y \in D(T)$. Finally, T is said to be nonexpansive if $||T_x - T_y|| \leqslant ||x - y||$ for every $x, y \in D(T)$. Of course, every ϕ-contraction map is in fact a nonexpansive mapping.

Let X be a Banach space and $T : D(T) \subseteq X \to X$ be a mapping. The map T is said to be pseudocontractive if, for every $x, y \in D(T)$ and for all $r > 0$, the inequality

$$\|x - y\| \leqslant \|(1 + r)(x - y) + r(Ty - Tx)\|$$

holds. Pseudocontractive mappings are easily seen to be more general than nonexpansive mappings. The interest in these mappings also stems from the fact that they are firmly connected to the well-known class of accretive mappings. Specifically, T is pseudocontractive if and only if $I - T$ is accretive, where I is the identity mapping.

Recall that a mapping $A : D(A) \to X$ is said to be *accretive* if the inequality $\|x - y + \lambda(Ax - Ay)\| \geqslant \|x - y\|$ holds for all $\lambda \geqslant 0$, $x, y \in D(A)$. An alternative way to define accretive mapping is the use of the duality map; that is, A is accretive if the inequality

$$\langle Ax - Ay, x - y\rangle_s \geqslant 0$$

holds for every $x, y \in D(A)$. Here the function $\langle \cdot, \cdot \rangle_s : X \times X \to \mathcal{R}$ is defined by $\langle y, x \rangle_s = \max\{x^*(y) : x^* \in J(x)\}$, where $J : X \to 2^{X^*}$ is the duality mapping on X, that is, $J(x) = \{x^* \in X^* : x^*(x) = \|x\|^2, \|x^*\| = \|x\|\}$. If, in addition, $\mathcal{R}(I + \lambda A)$ is for one, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive. Accretive operators were introduced by Browder [7] and Kato [17] independently.

A mapping $T : D(T) \subseteq X \to X$ is said to be ψ-expansive if there exists a function $\psi : [0, +\infty) \to [0, +\infty)$ satisfying the following conditions:

(1) $\psi(0) = 0$;
(2) $\psi(r) > 0$ for $r > 0$;
(3) either it is continuous or nondecreasing;

such that, for every $x, y \in D(T)$, the inequality $\|T(x) - T(y)\| \geqslant \psi(\|x - y\|)$ holds (see, for example, [14, 16]).

3. *Fixed-point theorems*

Let us now recall the following result by Emmanuele [13].

Proposition 3.1 [13]. *Let M be a nonempty closed convex bounded subset of a Banach space X and let $A : M \to M$ be a weakly continuous mapping. If A is ω-condensing, then it has a fixed point in M.*

Although it is not always possible to show that a given mapping between functional Banach spaces is weakly continuous, quite often its weak sequential continuity can be checked easily. This follows, among other things, from the fact that Lebesgue's dominated convergence theorem is valid for sequences but it fails for nets. Thus, our first step in this section is to improve the above proposition in the following sense.

Lemma 3.2. *Let M be a nonempty closed convex bounded subset of a Banach space X and let $A : M \to M$ be a weakly sequentially continuous mapping. If A is ω-condensing, then it has a fixed point in M.*

Evidently the result of this lemma remains valid if we assume that A is a ω-k-contraction for some $k \in [0, 1)$.

Proof. Given an element $x_0 \in M$, define the set

$$\Sigma := \{D \subseteq M : \ D \text{ is closed convex, } x_0 \in D \text{ and } A : D \to D\},$$

and put

$$B := \bigcap_{D \in \Sigma} D.$$

Let K be the set

$$K = \overline{\text{co}}\{A(B) \cup \{x_0\}\}.$$

Since $x_0 \in B$ and $A : B \to B$, it follows that $K \subseteq B$. This implies that $A(K) \subseteq A(B) \subseteq K$. Moreover, since $x_0 \in K$, we obtain that $K \in \Sigma$. Therefore, we may conclude that $K = B$. On the other hand, using the properties of $\omega(\cdot)$, we get

$$\omega(K) = \omega(\overline{\text{co}}\{A(B) \cup \{x_0\}\}) = \omega(A(B)) = \omega(A(K)) < \omega(K),$$

which implies that $\omega(K) = 0$. So K is weakly compact.

Since $A : B \to B$ is weakly sequentially continuous and B is weakly compact, by the Eberlein–Šmulian theorem [**12**] we infer that $A : B \to B$ is in fact weakly continuous. So, the use of Proposition 3.1 achieves the proof. $\square$

Remark 1. In the framework of Banach spaces, the above lemma can be thought of as an extension of Arino, Gautier and Penot [**2**, Theorem 1].

Now our purpose is to establish a sharpening of Taoudi [**26**, Theorem 2.1].

Theorem 3.3. *Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A : M \to X$ and $B : X \to X$ be two weakly sequentially continuous mappings. If A, B satisfy the following conditions*:

(i) *B is pseudocontractive, continuous and ω-k-contraction for some $k \in [0, 1[$*;
(ii) *$I - B$ is ψ-expansive*;
(iii) *A is a ω-s-contraction for some $s \in [0, 1 - k[$*;
(iv) *$(x = Bx + Ay,\ y \in M) \Rightarrow x \in M$*;

then the equation $x = A(x) + B(x)$ has a solution.

Proof. It is easily checked that $x \in M$ is a solution for the equation $x = B(x) + A(x)$ if and only if x is a fixed point for the operator $(I - B)^{-1} \circ A$, whenever it is well defined. In order to prove the latter, we have to check the following properties.

(a) The operator $(I - B)$ has an inverse over $R(I - B) := (I - B)(M)$.
This is equivalent to seeing that $I - B$ is injective. Consider $x, y \in M$, $x \neq y$. Since $I - B$ is ψ-expansive, we have

$$\|(I - B)(x) - (I - B)(y)\| \geqslant \psi(\|x - y\|) > 0.$$

(b) The domain of $(I - B)^{-1}$ contains the range of A.
Take $y \in M$ and consider $A(y)$. We have to check if there exists some $x \in M$ such that $(I - B)(x) = A(y)$ (which is equivalent to seeing if $x = Bx + Ay$). Define the map $T : X \to X$ by $Tx = Bx + Ay$, $x \in X$. Since B is continuous and pseudocontractive, T is also a continuous pseudocontractive map. Since $I - B$ is ψ-expansive, we have

$$\begin{aligned}\|(I - T)(x) - (I - T)(z)\| &= \|x - z + Tz - Tx\| \\ &= \|x - z + Bz - Bx\| \\ &= \|(I - B)(x) - (I - B)(z)\| \\ &\geqslant \psi(\|x - z\|).\end{aligned}$$

Hence $I - T : X \to X$ is accretive, continuous and ψ-expansive. By Barbu [**3**, Corollary 3.2], $I - T$ is m-accretive and hence, by García-Falset and Morales [**16**, Theorem 8] along with [**14**, Remark 3.8], we conclude that $I - T$ is surjective. Hence, $R(I - T) = X$ and therefore there exists $x \in X$ such that $0 = x - Tx$, that is, $x = Bx + Ay$ and, by assumption (iv), $x \in M$. Hence $Ay = (I - B)(x)$ and consequently the map $(I - B)^{-1} \circ A : M \to M$ is well defined.

In order to achieve the proof, we will apply Lemma 3.2, so we have only to prove that the map $(I - B)^{-1} \circ A : M \to M$ is weakly sequentially continuous and ω-condensing. Indeed, consider (x_n) a sequence in M which is weakly convergent to x. In this case, the set $\{x_n : n \in \mathcal{N}\}$ is relatively weakly compact and since A is weakly sequentially continuous, $\{A(x_n) : n \in \mathcal{N}\}$ is also relatively weakly compact. Using the equality

$$(I-B)^{-1} \circ A = A + B \circ (I-B)^{-1} \circ A \tag{3.1}$$

and the fact that B is ω-condensing, we obtain

$$\begin{aligned}\omega(\{(I-B)^{-1} \circ A(x_n) : \ n \in \mathcal{N}\}) &= \omega(\{A(x_n) : \ n \in \mathcal{N}\} + \{B(I-B)^{-1} \circ A(x_n) : \ n \in \mathcal{N}\}) \\ &\leqslant \omega(\{B(I-B)^{-1} \circ A(x_n) : \ n \in \mathcal{N}\}) \\ &< \omega(\{(I-B)^{-1} \circ A(x_n) : \ n \in \mathcal{N}\}).\end{aligned}$$

Hence, $\{(I-B)^{-1} \circ A(x_n) : \ n \in \mathcal{N}\})$ is relatively weakly compact. Consequently, there exists a subsequence (x_{n_k}) such that

$$(I-B)^{-1} \circ A(x_{n_k}) \rightharpoonup y.$$

Going back to (3.1), the weak sequential continuity of A and B yields $y = B(y) + A(x)$ and thus $y = (I-B)^{-1} \circ A(x)$. Accordingly we have

$$(I-B)^{-1} \circ A(x_{n_k}) \rightharpoonup (I-B)^{-1} \circ A(x).$$

Now a standard argument shows that $(I-B)^{-1} \circ A : M \to M$ is weakly sequentially continuous. Next, we will prove that $(I-B)^{-1} \circ A$ is ω-condensing. To do so, let S be a subset M. If we set $J := (I-B)^{-1} \circ A$, using equality (3.1), we infer

$$\omega(J(S)) = \omega(A(S) + B(J(S))).$$

The properties of $\omega(\cdot)$ and the assumptions on A and B imply that

$$\omega(J(S)) \leqslant \omega(A(S)) + \omega(B(J(S))) \leqslant s\omega(S) + k\omega(J(S)),$$

and therefore

$$\omega(J(S)) \leqslant \frac{s}{1-k}\omega(S).$$

This inequality means, in particular, that J is ω-condensing. □

REMARK 2. (a) If in the previous result we assume that $A(M)$ is weakly relatively compact, then we can replace the assumption 'B is a ω-k-contraction for some $k \in [0,1[$' by 'B is ω-condensing'. In these conditions assumption (iii) in Theorem 3.3 is redundant.

(b) Checking the proof of the previous theorem, it is not difficult to show that the same conclusion holds if we replace the condition '$B : X \to X$ is weakly sequentially continuous and ω-k-contraction' by '$B|_M := B : M \to X$ is weakly sequentially continuous and ω-k-contraction'.

THEOREM 3.4. *Let X be a Banach space. Let M be a nonempty closed convex and bounded subset of X and let $A : M \to X$ and $B : X \to X$ be two mappings with A weakly sequentially continuous. If A, B satisfy the following conditions*:

(i) *$A(M)$ is relatively weakly compact*;
(ii) *B is continuous, pseudocontractive and $B|_M$ is weakly sequentially continuous and ω-condensing*;
(iii) *if (x_n) is a sequence of elements of M such that $(I-B)(x_n) \rightharpoonup y$, then (x_n) has a weakly convergent subsequence*;
(iv) *if $\lambda \in]0,1[$ and $x = \lambda Bx + Ay$ for some $y \in M$, then $x \in M$*;

then the equation $x = A(x) + B(x)$ has a solution.

Proof. First, let us note that since B is continuous and ω-condensing, λB also is continuous and ω-condensing whenever $\lambda \in]0,1[$. Next, we are going to check that, for every $\lambda \in]0,1[$, the mapping λB is pseudocontractive. Indeed, since B is pseudocontractive, there exists $j \in J(x-y)$ such that

$$\langle (x-Bx)-(y-By), j\rangle \geqslant 0;$$

then

$$\begin{aligned}\langle x-\lambda Bx-(y-\lambda By), x-y\rangle_s &\geqslant \langle x-\lambda Bx-(y-\lambda By), j\rangle \\ &= \langle \lambda x-\lambda Bx-(\lambda y-\lambda By), j\rangle + \langle (1-\lambda)(x-y), j\rangle \\ &= \lambda\langle (x-Bx)-(y-B(y), j\rangle + (1-\lambda)\|x-y\|^2 \\ &\geqslant (1-\lambda)\|x-y\|^2 \geqslant 0.\end{aligned}$$

Hence, λB is pseudocontractive. On the other hand, the same inequality yields

$$\|x-\lambda Bx-(y-\lambda By)\|\|x-y\| \geqslant \langle x-\lambda Bx-(y-\lambda By), x-y\rangle_+ \geqslant (1-\lambda)\|x-y\|^2.$$

Consequently,

$$\|x-\lambda Bx-(y-\lambda By)\| \geqslant \psi(\|x-y\|),$$

where $\psi(t)=(1-\lambda)t$ and therefore $I-\lambda B$ is ψ-expansive. The above argument shows that A and λB fulfil the condition of Theorem 3.3. Consequently, for each $\lambda \in]0,1[$, there exists $x_\lambda \in M$ such that

$$x_\lambda = A(x_\lambda)+\lambda B(x_\lambda).$$

The rest of the proof works in the same way as the proof of Taoudi [**26**, Theorem 2.4]. □

Next, we derive the following result, which extends in [**6**, Corollary 2.11].

Corollary 3.5. *Let X be a Banach space. Let M be a nonempty weakly compact convex subset of X and let $A : M \to X$ and $B : X \to X$ be two mappings such that A and $B|_M$ are weakly sequentially continuous. If A, B satisfy the following conditions*:
(i) *B is continuous and pseudocontractive*;
(ii) *if $\lambda \in]0,1[$ and $x = \lambda Bx + Ay$ for some $y \in M$, then $x \in M$*;
then the equation $x = A(x) + B(x)$ has a solution.

Proof. Since M is a weakly compact set and A, $B|_M$ are weakly sequentially continuous mappings, the following conditions are satisfied:
(a) $A(M)$ is a weakly compact set;
(b) $B|_M$ is weakly sequentially continuous and ω-condensing;
(c) every sequence (x_n) in M admits a weakly convergent subsequence.
In order to achieve the proof, it is enough to note that conditions (a)–(c) along with (i) and (ii) allow us to apply Theorem 3.4. □

Next, we state a result which is a nonlinear alternative of Leray–Schauder type as well as an improvement of O'Regan [**22**, Theorem 2.3].

Theorem 3.6. *Let Q, C be a nonempty closed convex and bounded subsets of a Banach space X with $Q \subseteq C$. Let U be a weakly open subset of Q (which means that there exists a weakly open set V in X such that $U = V \cap Q$) with $0 \in U$, let $\bar{U}^\omega$, the weak closure of U, be a subset of Q, and let $A : \bar{U}^\omega \to C$ be a weakly sequentially continuous and w-condensing map. Then*

(i) *either the equation* $x = A(x)$ *has a solution or*
(ii) *there exist* $u \in \partial_Q U$ *and* $\lambda \in]0,1[$ *with* $u = \lambda Au$.

Proof. Assume that the second item is false and A does not have a fixed point in $\partial_Q U$ (otherwise we have nothing to do). Define the set

$$H := \{x \in \bar{U}^\omega \,:\, x = \lambda A(x) \text{ for some } \lambda \in [0,1]\}.$$

It is clear that $H \neq \emptyset$ $(0 \in H)$. Since H is contained in $\bar{U}^\omega$, it is bounded. Now using the inclusion $H \subseteq \mathrm{co}(A(H) \cup \{0\})$ together with the fact that A is w-condensing, we get

$$w(H) \leqslant w(A(H)) < w(H).$$

So $w(H) = 0$ and then H is relatively weakly compact. Nevertheless, since A is weakly sequentially continuous, we infer that H is weakly compact. Since (ii) does not hold and A is a fixed point free on $\partial_Q U$, we have $H \cap \partial_Q U = \emptyset$. Applying the Tietze–Urysohn theorem, we conclude that there exists a weakly continuous map $\zeta : \bar{U}^\omega \to [0,1]$ such that $\zeta(H) = 1$ and $\zeta(\partial_Q U) = 0$. Let $\eta : C \to C$ be the function defined by $\eta(x) = \zeta(x)A(x)$ if $x \in \bar{U}^\omega$ and $\eta(x) = 0$ if $x \in C \backslash \bar{U}^\omega$. Note that η is weakly sequentially continuous because A is weakly sequentially continuous. Consider now a subset V of C. Since $\eta(V) \subseteq \mathrm{co}(A(V \cap U) \cup \{0\})$, we have

$$w(\eta(V)) \leqslant w(A(V \cap U)) \leqslant w(A(V)) < w(V).$$

So η is w-condensing. According to Lemma 3.2, there exists $x \in C$ such that $x = \eta(x)$. Now $x \in U$ because $0 \in U$. Consequently, $x = \zeta(x)A(x) = \lambda A(x)$ with $0 \leqslant \lambda = \zeta(x) \leqslant 1$. This shows that $x \in H$ and therefore $\zeta(x) = 1$. Thus, $x = A(x)$. □

THEOREM 3.7. *Let* X *be a Banach space. Let* Q, C *be a nonempty closed convex and bounded subsets of* X *with* $Q \subseteq C$. *In addition, let* U *be a weakly open subset of* Q *with* $0 \in U$, *let* $\bar{U}^\omega$ *be a subset of* Q, *and let* $A : \bar{U}^\omega \to X$ *and* $B : X \to X$ *be two weakly sequentially continuous mappings satisfying the following conditions*:

(i) *B is pseudocontractive, continuous and ω-k-contraction for some* $k \in [0,1[$;
(ii) $I - B$ *is* ψ*-expansive*;
(iii) *A is a ω-s-contraction for some* $s \in [0, 1-k[$;
(iv) $(x = Bx + Ay,\ y \in \bar{U}^\omega) \Rightarrow x \in C$.

Then either the equation $x = A(x) + B(x)$ *has a solution or there exist* $u \in \partial_Q U$ *and* $\lambda \in]0,1[$ *with* $u = \lambda Au + \lambda B(u/\lambda)$.

Proof. The same argument developed in the proof of Theorem 3.3 shows that $(I-B)^{-1} : X \to X$ exists. It is also continuous [**14**]. Moreover, by condition (iii) it is not difficult to see that $(I-B)^{-1} \circ A(\bar{U}^\omega) \subseteq C$. So we have only to prove that the mapping $T := (I-B)^{-1} \circ A : \bar{U}^\omega \to C$ is weakly sequentially continuous and ω-condensing. These two facts can be proved as in the proof of Theorem 3.3. Now applying Theorem 3.6, we get the desired result. □

REMARK 3. (a) As said in Remark 2, if in the previous result we assume that $A(\bar{U}^\omega)$ is weakly relatively compact, then we can replace the assumption 'B is ω-k-contraction for some $k \in [0,1[$' by 'B is ω-condensing' and, moreover, assumption (iii) in such a result is now redundant.

(b) Note that if B is a ϕ-contraction and weakly continuous mapping, then B is pseudocontractive, continuous and ω-condensing (see [**15**, Lemma 3.1]). Consequently, according to Remark 2 (a), Theorem 3.3 is a generalization of Taoudi [**26**, Theorem 2.1] and Theorem

3.4 is an extension of Taoudi [**26**, Theorem 2.4]. Note also that Theorem 3.7 improves [**23**, Theorem 2.1].

In order to show that our results are strictly more general than those quoted in Remark 3, let us give the following example.

EXAMPLE 1. Consider the Banach space $(l_2, \|\cdot\|_2)$ and let $(e_n = (\delta_{i,n}))$ be the usual Schauder basis of such a space. Define the operator

$$T\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{i=1}^{\infty}(x_i - z_i(x))e_i,$$

where

$$z_i(x) = \begin{cases} x_{2k} & i = 2k-1, \\ -x_{2k-1} & i = 2k. \end{cases}$$

Let us see that T is pseudocontractive. For $x, y \in l_2$, we can write

$$\begin{aligned}
&\langle x - T(x) - (y - T(y)), x - y\rangle \\
&= \left\langle \sum_{k=1}^{\infty}(x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{\infty}(y_{2k-1} - x_{2k-1})e_{2k}, \sum_{k=1}^{\infty}(x_k - y_k)e_k \right\rangle \\
&= \sum_{k=1}^{\infty}(x_{2k} - y_{2k})(x_{2k-1} - y_{2k-1}) + \sum_{k=1}^{\infty}(y_{2k-1} - x_{2k-1})(x_{2k} - y_{2k}) \\
&= 0.
\end{aligned}$$

This equality shows that T is a pseudocontractive mapping. Moreover, T is linear and satisfies

$$\|Tx - Ty\| = \sqrt{2}\|x - y\|.$$

So T is not nonexpansive. On the other hand, if we take $\psi(t) = t$ and $x, y \in l_2$, then

$$\begin{aligned}
\|x - T(x) - (y - T(y))\| &= \left\| \sum_{k=1}^{\infty}(x_{2k} - y_{2k})e_{2k-1} + \sum_{k=1}^{\infty}(y_{2k-1} - x_{2k-1})e_{2k} \right\| \\
&= \|x - y\| \\
&= \psi(\|x - y\|),
\end{aligned}$$

which means that $I - T$ is ψ-expansive.

COROLLARY 3.8. *Let X be a Banach space. Let Q, C be a nonempty closed convex and bounded subset of X with $Q \subseteq C$. In addition, let U be a weakly open subset of Q with $0 \in U$, let $\bar{U}^{\omega}$ be a subset of Q, $A : \bar{U}^{\omega} \to X$ weakly sequentially continuous with $A(\bar{U}^{\omega})$ relatively weakly compact and $B : X \to X$ satisfying the following conditions*:

(i) *B is linear and bounded*;
(ii) *there exists $p \in \mathcal{N}$ such that B^p is pseudocontractive and $I - B^p$ is ψ-expansive*;
(iii) *$(x = Bx + Ay,\ y \in \bar{U}^{\omega}) \Rightarrow x \in C$.*

Then either the equation $x = A(x) + B(x)$ has a solution or there exist $u \in \partial_Q U$ and $\lambda \in]0, 1[$ with $u = \lambda Au + \lambda B(u/\lambda)$.

Proof. Following the arguments of the proof of Theorem 3.3, we infer that there exists $(I - B^p)^{-1} : X \to X$ and it is continuous and so $(I - B)^{-1} = (I - B^p)^{-1} \sum_{k=0}^{p-1} B^k$. This fact means that $(I-B)^{-1}$ is a bounded linear operator and thus it is both continuous and weakly continuous. Now, since A is weakly sequentially continuous, $A(\bar{U}^{\omega})$ is relatively weakly compact and (iii) is satisfied. So, the mapping $T:=(I-B)^{-1}\circ A:\bar{U}^{\omega}\to C$ is under the conditions of Theorem 3.6, which allows us to achieve the proof. □

Example 1 shows that even in the case where B is a linear operator, our results can be applied in a more general framework.

THEOREM 3.9. *Let X be a Banach space, let M be a nonempty convex weakly compact subset of X, and let $A:M\to X$ and $B:X\to X$ be two continuous mappings. If A,B satisfy the following conditions*:

(i) *A is weakly sequentially continuous*;
(ii) *B is pseudocontractive and $I-B$ is ψ-expansive*;
(iii) $(x=Bx+Ay,\ y\in M)\Rightarrow x\in M$;

then there exists a sequence $(x_n)_{n\in\mathcal{N}}$ in M such that $x_n-(Ax_n+Bx_n)\rightharpoonup 0$.

Proof. Arguing as in the proof of Theorem 3.3, one sees that $(I-B)^{-1}A(M)\subset M$. Since $(I-B)^{-1}A$ is continuous [**15**], according to [**5**, Theorem 3.3], it has a weak-approximate fixed-point sequence, that is, there is a sequence $(y_n)_{n\in\mathcal{N}}$ in M such that $y_n-(I-B)^{-1}Ay_n\rightharpoonup 0$. The weak compactness of M implies that $y_n\rightharpoonup y$ with $y\in M$. Hence, we infer that $(I-B)^{-1}Ay_n\rightharpoonup y$. On the other hand, the weak sequential continuity of A implies that

$$Ay_n\rightharpoonup Ay \tag{3.2}$$

and

$$A(I-B)^{-1}Ay_n\rightharpoonup Ay. \tag{3.3}$$

Next, let $(x_n)_{n\in\mathcal{N}}$ be a sequence of M defined by $x_n=(I-B)^{-1}Ay_n$. Hence, we have

$$x_n-(Ax_n+Bx_n)=(I-B)x_n-Ax_n=Ay_n-Ax_n=Ay_n-A(I-B)^{-1}Ay_n.$$

Now, using (3.2) and (3.3), we get $x_n-(Ax_n+Bx_n)\rightharpoonup 0$ as $n\to\infty$, which ends the proof. □

REMARK 4. If in the result above the space is reflexive, then we can drop the weak compactness assumption of M and replace it by its boundedness and closedness because any closed bounded convex subset of a reflexive Banach space is weakly compact.

Let M be a nonempty closed convex subset of a Banach space X, let $A:M\to X$ be a weakly sequentially continuous mapping and let $B:X\to X$ be a mapping. If A,B satisfy the following conditions:

(i) B is pseudocontractive continuous and $I-B$ is ψ-expansive;
(ii) $(x=Bx+Ay,\ \text{for some } y\in M)\Rightarrow x\in M$;
(iii) $\mathcal{F}:=\{x\in M:\ x=Bx+Ay,\ y\in M)\}$ is bounded;
(vi) $B:\mathcal{F}\to X$ is weakly sequentially continuous;

and if $C:=\overline{co}(\mathcal{F})$ then, as seen in the proof of Theorem 3.3, the operator $(I-B)^{-1}A$ is well defined on C and we have $(I-B)^{-1}A(C)\subset C$. Assume further that

(v) the map $T:=(I-B)^{-1}A:C\to C$ is ω-condensing.

Let us now prove that T is weakly sequentially continuous on C. Let (x_n) be a sequence in C converging weakly to x. So the set $\{x_n,\ n\in\mathcal{N}\}$ is relatively weakly compact. Since T is ω-condensing, we get the relative weak compactness of the set $\{Tx_n,\ n\in\mathcal{N}\}$. Accordingly there exists a subsequence (x_{n_k}) of (x_n) such that $Tx_{n_k} \rightharpoonup y$. Now using the equality $T = A + BT$, the weak sequential continuity of A, the fact that $Tx_n \in \mathcal{F}$ for all $n \in \mathcal{N}$ and the fact that B is weakly sequentially continuous on $\mathcal{F}$, we get $y = Ax + By$ and therefore $y = Tx$. Thus, $Tx_{n_k} \rightharpoonup Tx$. By a standard argument (see, for example, [**22**]), we get the weak sequential continuity of T.

Summarizing the discussion above and using Lemma 3.2, we obtain the following theorem.

THEOREM 3.10. *Let M be a nonempty closed convex subset of a Banach space X, let $A : M \to X$ be a weakly sequentially continuous map and let $B : X \to X$ be a mapping. Further, if the conditions* (i)–(v) *hold true, then the operator $A + B$ has a fixed point in M.*

As an easy consequence of this theorem, the next corollary is an improvement of Barroso and Teixeira [**6**, Theorem 2.9].

COROLLARY 3.11. *Let M be a nonempty closed convex subset of a Banach space X. Assume that $A : M \to X$ and $B : X \to X$ are two mappings satisfying the following conditions*:
(i) *A is weakly sequentially continuous*;
(ii) *B is continuous, pseudocontractive and $I - B$ is ψ-expansive*;
(iii) *if $u = Bu + Ay$ for some $y \in M$, then $u \in M$*;
(iv) *the set $\mathcal{F} := \{u \in M : u = B(u) + A(v)$ for some $v \in M\}$ is relatively weakly compact*;
(v) *$B : \mathcal{F} \to X$ is weakly sequentially continuous.*
Then the operator $A + B$ has a fixed point in M.

Proof. Let $C := \overline{\mathrm{co}}(\mathcal{F})$. The use of the Krein–Šmulian theorem implies that C is weakly compact. Therefore, the map $(I - B)^{-1}A : C \to C$ is w-condensing. Hence, the result follows from the above theorem. $\square$

REMARK 5. Let us note that all the results of this section remain valid if the De Blasi measure of weak noncompactness is replaced by any other abstract measure of weak noncompactness satisfying the conditions (1)–(7) given in the preliminaries.

4. *A nonlinear integral equation: existence theory*

In this section, we deal with the following nonlinear integral equation:

$$u(t) = f(t, u(t)) + \int_0^t g(s, u(s))\, ds, \quad u \in C(0, T; E), \tag{4.1}$$

where E is a reflexive Banach space. Assume that f and g satisfy the following conditions:
(H1) $f : [0, T] \times E \to E$ is weakly sequentially continuous, and uniformly continuous on bounded sets;
(H2) for every $t \in [0, T]$, $f(t, \cdot) : E \to E$ is pseudocontractive and $I - f(t, \cdot)$ is ψ-expansive with the same ψ for every $t \in [0, T]$;
(H3) $\|u\| \leqslant \|u - f(t, u)\|$ for all $(t, u) \in [0, T] \times E$;
(H4) for each $t \in [0, T]$ the map $g(t, \cdot) : E \to E$ is weakly sequentially continuous;
(H5) for each $u \in C(0, T, E)$ the map $t \longmapsto g(t, u(t))$ is Pettis integrable on $[0, T]$;
(H6) there exists $\alpha \in L^1(0, T)$ and a nondecreasing continuous function $\varphi : [0, \infty[\to (0, \infty[$ such that $\|g(s, x)\| \leqslant \alpha(s)\varphi(\|x\|)$ for almost everywhere $s \in [0, T]$, and all $x \in E$. Moreover, $\int_0^T \alpha(t)\, dt < \int_0^\infty dt/\varphi(t)$.

THEOREM 4.1. *Equation* (4.1) *has a solution in* $C(0,T;E)$ *whenever assumptions* (H1)–(H6) *are satisfied.*

Proof. As in the proof of Barroso and Teixeira [**6**, Theorem 5.1], we introduce the following functions:

$$J(z) := \int_0^z \frac{ds}{\varphi(s)} \quad \text{and} \quad b(t) := J^{-1}\left(\int_0^t \alpha(s)\,ds\right).$$

Define the set

$$M := \{u \in C(0,T;E) : \ \|u(t)\| \leqslant b(t), \text{ for all } t \in [0,T]\}.$$

Consider now the mappings $A, B : C(0,T;E) \to C(0,T;E)$ defined as follows:

$$A(u)(t) := \int_0^t g(s,u(s))\,ds \quad \text{and} \quad B(u)(t) = f(t,u(t)).$$

Our strategy consists in showing that $A+B$ has a fixed point, since such a fixed point is a solution of equation (4.1).

Steps (1)–(3) in the proof of Barroso and Teixeira [**6**, Theorem 5.1] yield the following:

(1) M is a closed, bounded and convex subset of $C(0,T;E)$;
(2) $A(M) \subseteq M$;
(3) given $u \in M$, then $\|Au(t) - Au(s)\| \leqslant |b(t) - b(s)|$;
(4) $A(M)$ is weakly equicontinuous and relatively weakly compact;
(5) $A : M \to C(0,T,E)$ is weakly sequentially continuous.

The properties of the mapping B will be given in several steps.

Step 1: The operator B is continuous.

Let (u_n) be a sequence in $C(0,T,E)$ that converges to u, set $k := \sup\{\|u_n\|_\infty : \ n \in \mathcal{N}\}$ and consider $B_k := \{x \in E : \ \|x\| \leqslant k\}$. It is clear that $u_n(t)$ and $u(t)$ belong to B_k for every $t \in [0,T]$ and for all $n \in \mathcal{N}$. Since, by the condition (H1), f is uniformly continuous on bounded sets, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in B_k$ and $\|x - y\| \leqslant \delta$, then $\|f(t,x) - f(t,y)\| \leqslant \epsilon$. Moreover, since $u_n \to u$, there exists $n_0 \in \mathcal{N}$ such that $\|u_n(t) - u(t)\| \leqslant \delta$ for all $t \in [0,T]$ and all $n \geqslant n_0$. Therefore, $\|f(t,u_n(t)) - f(t,u(t))\| \leqslant \epsilon$ for all $n \geqslant n_0$, which proves the continuity of B.

Step 2: The operator B is pseudocontractive.

Indeed, by assumption (H2), the inequality

$$\|u(t) - v(t)\| \leqslant \|(1+r)(u(t) - v(t)) + r(f(t,v(t)) - f(t,u(t)))\|$$

holds for all $u, v \in C(0,T;E)$ and for all $t \in [0,T]$. Consequently,

$$\|u - v\|_\infty \leqslant \|(1+r)(u - v) + r(B(v) - B(u))\|_\infty.$$

Step 3: The operator $I - B$ is ψ-expansive.

Assumption (H2) says that $I - f(t,\cdot) : E \to E$ is ψ-expansive, hence the inequality

$$\|(u(t) - f(t,u(t)) - (v(t) - f(t,v(t)))\| \geqslant \psi(\|u(t) - v(t)\|)$$

holds for every $u, v \in C(0,T;E)$ and for all $t \in [0,T]$. Given $u, v \in C(0,T;E)$, the above inequality yields

$$\|(u - B(u)) - (v - B(v))\|_\infty \geqslant \psi(\|u(t) - v(t)\|), \quad \text{for all } t \in [0,T].$$

On the other hand, since u, v are continuous functions, there exists $t_0 \in [0,T]$ such that $\|u - v\|_\infty = \|u(t_0) - v(t_0)\|$. Therefore,

$$\|(u - B(u)) - (v - B(v))\|_\infty \geqslant \psi(\|u(t_0) - v(t_0)\|) = \psi(\|u - v\|_\infty).$$

Step 4: Suppose that $u = B(u) + A(y)$ for some $y \in M$. We will show that $u \in M$.

It follows from the hypothesis (H3) that

$$\|u(t)\| \leqslant \|u(t) - B(u)(t)\| = \|A(v)(t)\|.$$

Since $A(M) \subseteq M$, we conclude that $u \in M$.

Step 5: The operator B is weakly sequentially continuous on $\mathcal{F} := \{u \in M : \ u = Bu + Ay$ for some $y \in M\}$.

First, we claim that $\mathcal{F}$ is a weakly equicontinuous set. Let u be an element of $\mathcal{F}$; then there exists $v \in M$ such that $u = Bu + Av$. Since, for any $t \in [0,T]$, $I - f(t,\cdot)$ is ψ-expansive with the same ψ, we infer that

$$\begin{aligned}\psi(\|u(t) - u(s)\|) &\leqslant \|(u(t) - f(t,u(t))) - (u(s) - f(t,u(s)))\| \\ &\leqslant \|(u(t) - Bu(t)) - (u(s) - Bu(s))\| \\ &\quad + \|f(s,u(s)) - f(t,u(s))\| \\ &= \|Av(t) - Av(s)\| + \|f(s,u(s)) - f(t,u(s))\| \\ &\leqslant |b(t) - b(s)| + \|f(s,u(s)) - f(t,u(s))\|,\end{aligned}$$

the properties of f, ψ along with the inequality above imply that $\mathcal{F}$ is a weakly equicontinuous set as we claimed.

Since $B(\mathcal{F}) \subseteq \mathcal{F} - A(\mathcal{F})$ and $\mathcal{F}$, $A(\mathcal{F})$ are weakly equicontinuous so is $B(\mathcal{F})$. Let (u_n) be a sequence in $\mathcal{F}$ weakly convergent to $u \in M$. Then $u_n(t) \rightharpoonup u(t)$ in E for all $t \in [0,T]$. Since f is weakly sequentially continuous, one has that $f(t,u_n(t)) \rightharpoonup f(t,u(t))$ in E, thus $B(u_n)(t) \rightharpoonup B(u)(t)$ in E for all $t \in [0,T]$.

On the other hand, consider the set $\{B(u_n) : \ n \in \mathcal{N}\}$. Since (u_n) is a bounded sequence in $C(0,T,E)$, it is clear that the set

$$\{u_n(t) \in E : n \in \mathcal{N} \text{ and } t \in [0,T]\}$$

is bounded in E. Since E is reflexive and f is weakly sequentially continuous, it follows that

$$\{f(t,u_n(t)) \in E : n \in \mathcal{N} \text{ and } t \in [0,T]\}$$

is a bounded set in E. This means that (Bu_n) is a bounded sequence in $C(0,T,E)$. These facts allow us to apply the Ascoli–Arzela theorem and conclude that there exists a subsequence (u_{n_k}) of (u_n) such that $Bu_{n_k} \rightharpoonup v$ for some $v \in C(0,T,E)$. Hence, $Bu = v$ and by a standard method, we obtain that $Bu_n \rightharpoonup Bu$.

Step 6: $\mathcal{F}$ is relatively weakly compact.

Let (u_n) be an arbitrary sequence in $\mathcal{F}$. Step 5 shows that $\mathcal{F}$ is weakly equicontinuous and hence so is $\{u_n : \ n \in \mathcal{N}\}$. Moreover, for all $t \in [0,T]$, the reflexivity of E implies that the set $\{u_n(t) : \ n \in \mathcal{N}\}$ is relatively weakly compact. Thus, invoking first the Ascoli–Arzela and later the Eberlein–Šmulian theorem, we obtain that $\mathcal{F}$ is relatively weakly compact.

Summarizing the argument above, we see that A and B fulfil the conditions of Corollary 3.11 and therefore $A + B$ has at least one fixed point in M. □

Example 2. Consider the following nonlinear integral equation:

$$u(t) = -u^3(t) + \int_0^t s^2 \sqrt{|u(s)| + 1}\, ds, \quad u \in C(0,T;\mathcal{R}). \tag{4.2}$$

In order to show that such an equation admits a solution in the Banach space $C(0,T;\mathcal{R})$, we are going to check that conditions of Theorem 4.1 are satisfied. In this case, $E := \mathcal{R}$; then E is a reflexive Banach space.

Define $f : \mathcal{R} \to \mathcal{R}$ by $f(x) = -x^3$. It is obvious that f is continuous (then weakly continuous) and uniformly continuous on bounded sets of E.

The function f is pseudocontractive. Indeed, let $(x, y) \in \mathcal{R}^2$ and $r > 0$; we can write

$$\begin{aligned} |(1+r)(x-y) + r(f(y) - f(x))| &= |(1+r)(x-y) + r(x^3 - y^3)| \\ &= |x-y||1 + r(1 + x^2 + xy + y^2)| \\ &\geqslant |x-y||1+r| \\ &\geqslant |x-y|, \end{aligned}$$

which proves our claim.

The second step is to see that $I - f$ is ψ-expansive. Indeed,

$$\begin{aligned} |(x - f(x) - (y - f(y))| &= |(x-y) + (x^3 - y^3)| \\ &= |(x-y) + (x-y)(x^2 + xy + y^2)| \\ &= |(x-y)(1 + x^2 + xy + y^2)| \\ &\geqslant |x-y|. \end{aligned}$$

Thus, taking $\psi(t) = t$, we conclude that f is ψ-expansive. Moreover, since $f(0) = 0$, we have

$$|x| \leqslant |x - f(x)| \quad \text{for all } x \in \mathcal{R}.$$

So, f satisfies conditions (H1)–(H3).

Next, we introduce the function $g : [0, T] \times \mathcal{R} \to \mathcal{R}$ as follows:

$$g(s, x) := s^2 \sqrt{|x| + 1}.$$

For each $t \in [0, T]$ the function $g(t, \cdot) : E \to E$ is clearly continuous (then weakly continuous).

On the other hand, given $u \in C(0, T; E)$, the function $t \longmapsto g(t, u(t))$ is continuous and hence it is integrable in $[0, T]$.

Finally, we define the functions $\alpha : \mathcal{R} \to \mathcal{R}$ and $\varphi : [0, \infty[\to (0, \infty)$ by

$$\alpha(s) = s^2 \quad \text{and} \quad \varphi(x) = \sqrt{x+1}.$$

Since g, α and φ satisfy conditions (H4)–(H6), we apply Theorem 4.1 to derive the existence of a solution to equation (4.2).

Remark 6. (i) Equation (4.2) is a particular case of the model integral equation

$$u(t) = f(u(t)) + \int_0^t g(s, u(s))\, ds, \quad u \in C(0, T; E),$$

which is widely considered in the literature. Obviously, this type of equation falls into the class studied in equation (4.1).

(ii) If $f : [0, T] \times E \to E$ is a ϕ-contraction with respect to the second variable, then $f(t, \cdot)$ is pseudocontractive and $I - f(t, \cdot)$ is ψ-expansive. Theorem 4.1 represents an extension of several previous ones which assume such a hypothesis on f (see, for instance, [**6**, Theorem 5.1]).

References

1. J. Appell and E. De Pascale, 'Su alcuni parametri connessi con la misura di non compattezza di Hausdorff in spazi di funzioni misurabili', *Boll. Unione Mat. Ital. B* 3 (1984) 497–515.

2. O. Arino, S. Gautier and J. P. Penot, 'A fixed point theorem for sequentially continuous mappings with application to ordinary differential equations', *Funkcial. Ekvac.* 27 (1984) 273–279.

3. V. Barbu, *Nonlinear semigroups and differential equations in Banach spaces* (Noordhoff International Publishing, Leyden, the Netherlands, 1976).

4. C. S. Barroso, 'Krasnosel'skii's fixed point theorem for weakly continuous maps', *Nonlinear Anal.* 55 (2003) 25–31.

5. C. S. Barroso, 'The approximate fixed point property in Hausdorff topological vector spaces and applications', *Discrete Contin. Dyn. Syst.* 25 (2009) 467–479.
6. C. S. Barroso and E. V. Teixeira, 'A topological and geometric approach to fixed point results for sum of operators and applications', *Nonlinear Anal.* 60 (2005) 625–650.
7. F. E. Browder, 'Nonlinear mappings of nonexpansive and accretive type in Banach spaces', *Bull. Amer. Math. Soc.* 73 (1967) 875–882.
8. T. A. Burton, 'A fixed point theorem of Krasnoselskii type', *Appl. Math. Lett.* 11 (1998) 85–88.
9. T. A. Burton and C. Kirk, 'A fixed point theorem of Krasnoselskii–Schaefer type', *Math. Nachr.* 189 (1998) 23–31.
10. F. S. De Blasi, 'On a property of the unit sphere in Banach spaces', *Bull. Math. Soc. Sci. Math. Roumanie* 21 (1977) 259–262.
11. S. Djebali and Z. Sahnoun, 'Nonlinear alternatives of Schauder and Krasnosel'skii types with applications to Hammerstein integral equations in L^1 spaces', *J. Differential Equations* 249 (2010) 2061–2075.
12. N. Dunford and J. T. Schwartz, *Linear operators, Part I: General theory* (Interscience Publishers, New York, 1958).
13. G. Emmanuele, 'Measure of weak noncompactness and fixed point theorems', *Bull. Math. Soc. Sci. Math. Roumanie* 25 (1981) 353–358.
14. J. García-Falset, 'Existence of fixed points for the sum of two operators', *Math. Nachr.* 283 (2010) 1726–1757.
15. J. Garcia-Falset, K. Latrach, E. Moreno-Gálvez and M-A. Taoudi, 'Schaefer–Krasnoselskii fixed point theorems using a usual measure of weak noncompactness', *J. Differential Equations*, to appear.
16. J. García-Falset and C. H. Morales, 'Existence theorems for m-accretive operators in Banach spaces', *J. Math. Anal. Appl.* 309 (2005) 453–461.
17. T. Kato, 'Nonlinear semigroups and evolution equation', *J. Math. Soc. Japan* 19 (1967) 508–520.
18. M. A. Krasnosel'skii, 'Some problems of nonlinear analysis', *Amer. Math. Soc. Transl. Ser.* 2 10 (1958) 345–409.
19. M. A. Krasnosel'skii, *Topological methods in the theory of nonlinear integral equations* (Pergamon Press, New York, 1964).
20. K. Latrach and M. A. Taoudi, 'Existence results for a generalized nonlinear Hammerstein equation on L^1-spaces', *Nonlinear Anal.* 66 (2007) 2325–2333.
21. K. Latrach, M. A. Taoudi and A. Zeghal, 'Some fixed point theorems of the Schauder and Krasnosel'skii type and application to nonlinear transport equations', *J. Differential Equations* 221 (2006) 256–271.
22. D. O'Regan, 'Fixed point theory for weakly sequentially continuous mappings', *Math. Comput. Modelling* 27 (1998) 1–14.
23. D. O'Regan and M. A. Taoudi, 'Fixed point theorems for the sum of two weakly sequentially continuous mappings', *Nonlinear Anal.* 73 (2010) 283–289.
24. S. Park, 'Generalization of the Krasnoselskii fixed point theorem', *Nonlinear Anal.* 67 (2007) 3401–3410.
25. D. R. Smart, *Fixed point theorems* (Cambridge University Press, Cambridge, 1980).
26. M. A. Taoudi, 'Krasnosel'skii type fixed point theorems under weak topology features', *Nonlinear Anal.* 72 (2010) 478–482.

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