Declarations:

Funding: The research of Mr. P. Howlader is supported by the *Council of Scientific and Industrial Research* (CSIR) India - Research Grant No. 09/092(0950)/2016-EMR-I.

Conflicts of interest/Competing interests: Not applicable.

Availability of data and material: Not applicable

Code availability: Not applicable.

Kripke Contexts, Double Boolean Algebras with Operators and Corresponding Modal Systems

Prosenjit Howlader[1] and Mohua Banerjee[1*]

[1]Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, India.

*Corresponding author(s). E-mail(s): mohua@iitk.ac.in;
Contributing authors: prosen@iitk.ac.in;

Abstract

The notion of a context in formal concept analysis and that of an approximation space in rough set theory are unified in this study to define a Kripke context. For any context (G,M,I), a relation on the set G of objects and a relation on the set M of properties are included, giving a structure of the form ((G,R), (M,S), I). A Kripke context gives rise to complex algebras based on the collections of protoconcepts and semiconcepts of the underlying context. On abstraction, double Boolean algebras (dBas) with operators and topological dBas are defined. Representation results for these algebras are established in terms of the complex algebras of an appropriate Kripke context. As a natural next step, logics corresponding to classes of these algebras are formulated. A sequent calculus is proposed for contextual dBas, modal extensions of which give logics for contextual dBas with operators and topological contextual dBas. The representation theorems for the algebras result in a protoconcept-based semantics for these logics.

Keywords: Formal concept analysis, Rough set theory, Boolean algebra with operators, Double Boolean algebra, Modal logic

MSC Classification: 06E25 , 03Gxx , 03G05 , 03B60 , 03B45

1 Introduction

Formal concept analysis (FCA) [1] and rough set theory [2] are both well-established areas of study with applications in several domains including knowledge representation and data analysis. There has also been a lot of study connecting and comparing the two areas, e.g. in [3–11], and the work presented here is motivated by such studies from the perspective of algebra and logic.

The central objects of FCA are contexts and concepts of a context [12]. A *context* is a triple $\mathbb{K} := (G, M, I)$, where G is the set of *objects*, and M is the set of *attributes* and $I \subseteq G \times M$. For any $A \subseteq G, B \subseteq M$, the following sets are defined: $A' := \{m \in M : \text{for all } g \in G(g \in A \implies gRm)\}$, and $B' := \{g \in G : \text{for all } m \in M(m \in M \implies gRm)\}$. A pair (A, B) is called a *concept* of $\mathbb{K}$, if $A' = B$ and $B' = A$. For a concept (A, B), A is its *extent* and B its *intent*. $\mathcal{B}(\mathbb{K})$ denotes the set of all concepts of $\mathbb{K}$. An order relation $\leq$ is obtained on $\mathcal{B}(\mathbb{K})$ as follows: for $(A_1, B_1), (A_2, B_2) \in \mathcal{B}(\mathbb{K}), (A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subseteq A_2$ (equivalent to $B_2 \subseteq B_1$).

The notion of a concept was generalized to that of semiconcepts and protoconcepts in [13, 14]. A pair (A, B) is called a *semiconcept* of $\mathbb{K}$, if $A' = B$ or $B' = A$. (A, B) is called a *protoconcept* of $\mathbb{K}$, if $A'' = B'$ (equivalently $A' = B''$). $\mathfrak{H}(\mathbb{K})$ and $\mathfrak{P}(\mathbb{K})$ denote the sets of all semiconcepts and protoconcepts of $\mathbb{K}$ respectively. It is observed that $\mathcal{B}(\mathbb{K}) \subseteq \mathfrak{H}(\mathbb{K}) \subseteq \mathfrak{P}(\mathbb{K})$. The partial order $\leq$ on $\mathcal{B}(\mathbb{K})$ is extended to the set $\mathfrak{P}(\mathbb{K})$ as: for any $(A, B), (C, D) \in \mathfrak{P}(\mathbb{K})$, $(A, B) \sqsubseteq (C, D)$ if and only if $A \subseteq C$ and $D \subseteq B$.
The following operations are defined on $\mathfrak{P}(\mathbb{K})$. For $(A_1, B_1), (A_2, B_2)$ in $\mathfrak{P}(\mathbb{K})$,

$$
\begin{aligned}
(A_1, B_1) \sqcap (A_2, B_2) &:= (A_1 \cap A_2, (A_1 \cap A_2)^{'}),\\
(A_1, B_1) \sqcup (A_2, B_2) &:= ((B_1 \cap B_2)^{'}, B_1 \cap B_2),\\
\neg(A, B) &:= (G \setminus A, (G \setminus A)^{'}),\\
\lrcorner(A, B) &:= ((M \setminus B)^{'}, M \setminus B),\\
\top &:= (G, \emptyset),\\
\bot &:= (\emptyset, M).
\end{aligned}
$$

With these operations, the protoconcepts of any context form an algebraic structure called *double Boolean algebra* (dBa) [14]. The structure of a dBa is such that there are two negation operators in it, which result in two Boolean algebras being derived from it – justifying the name. The set of semiconcepts, with the same operations as above, forms a subalgebra of the algebra of protoconcepts. In this work, our interest lies in *contextual* and *pure* dBas [14, 15], the structures formed by protoconcepts and semiconcepts respectively.

There may be circumstances in which the objects and properties defining a context are indistinguishable with respect to certain attributes. For example, two diseases may be indistinguishable by the symptoms available. Indistinguishability of objects and properties have motivated authors [5, 8, 11, 16] to study "indiscernibility" relations on the set of objects and the set of properties. Rough set-theoretic notions of approximation spaces and approximation operators [2, 17] are then introduced in FCA.

A *Pawlakian approximation space* is a pair (W, E), where W is a set and E is an equivalence relation on W. This is generalised to a pair (W, E) with E any binary relation on W, and called a *generalised approximation space* [18]. For $x \in W$, $E(x) := \{y \in W \ : \ xRy\}$. The *lower* and *upper approximations* of any $A(\subseteq W)$ are defined respectively as $\underline{A}_E := \{x \in W \ : \ E(x) \subseteq A\}$, and $\overline{A}^E := \{x \in W \ : \ E(x) \cap A \neq \emptyset\}$. Kent introduced the notion of approximation space into FCA [8, 16], and defined lower and upper approximations of contexts and concepts. The work of Saquer and Deogun [11] differs from that of Kent in choosing the "indiscernibility" relations. Kent considers an indiscernibility relation on the set G of objects which is externally given by some agent, whereas Saquer and Deogun consider a relation that is determined by the given context. For a given context $\mathbb{K} := (G, M, I)$, relations E_1, E_2 are defined on the set G of objects and the set M of properties respectively, as follows.

(a) For $g_1, g_2 \in G$, $g_1 E_1 g_2$ if and only if $I(g_1) = I(g_2)$.

(b) For $m_1, m_2 \in M$, $m_1 E_2 m_2$ if and only if $I^{-1}(m_1) = I^{-1}(m_2)$.

Furthermore, for $A \subseteq G, B \subseteq M$, lower and upper approximations are defined in terms of concepts of $\mathbb{K}$, and using these, approximations of any pair (A, B) that is not a concept, are given. Apart from Saquer and Deogun, Hu et.al. [5] introduce approximation spaces on the sets of objects and properties. In [5], for a given context $\mathbb{K} := (G, M, I)$, relations J_1, J_2 are defined on G and M respectively, as follows.

(a) For $g_1, g_2 \in G$, $g_1 J_1 g_2$ if and only if $I(g_1) \subseteq I(g_2)$.

(b) For $m_1, m_2 \in M$, $m_1 J_2 m_2$ if and only if $I^{-1}(m_1) \subseteq I^{-1}(m_2)$.

The relations E_1, E_2 are equivalence relations [11], while the relations J_1, J_2 are partial order relations [5]. These observations have motivated us to define the Kripke context, which unifies within a single framework, the notions of a context of FCA and approximation space of rough set theory.

Definition 1 A *Kripke context* based on a context $\mathbb{K} := (G, M, I)$ is a triple $\mathbb{KC} := ((G, R), (M, S), I)$, where R, S are relations on G and M respectively.

So a Kripke context consists of a context of FCA and two Kripke frames, which in the terminology of rough set theory, are generalised approximation spaces. Note that for a context $\mathbb{K} := (G, M, I)$, we get a Kripke context $\mathbb{KC}_{DS} := ((G, E_1), (M, E_2), I)$. Moreover, $\mathbb{KC}_{DS}$ is an example such that the relations E_1 and E_2 are reflexive, symmetric and transitive. This observation has led us to define reflexive, symmetric or transitive Kripke contexts, where the relations R and S are reflexive, symmetric or transitive.

It is shown that, using the lower and upper approximation operators induced by the approximation space (G, R), (M, S) in a Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$, one can define unary operators f_R and f_S on the set

$\mathfrak{P}(\mathbb{K})$ of protoconcepts of the underlying context $\mathbb{K} := (G, M, I)$ such that f_R is an interior-type operator, while f_S is a closure-type operator. The Kripke context thus leads to complex algebras. The algebra of protoconcepts with the operators f_R and f_S, is called the *full complex algebra of* $\mathbb{KC}$. Any subalgebra of the full complex algebra of $\mathbb{KC}$ is called a complex algebra. For a Kripke context $\mathbb{KC}$, the algebra of semiconcepts $\mathfrak{H}(\mathbb{K})$ with operators $f_R{\restriction}\mathfrak{H}(\mathbb{K})$ and $f_S{\restriction}\mathfrak{H}(\mathbb{K})$ is an instance of a complex algebra of $\mathbb{KC}$. We show how, in terms of approximation spaces and operators f_{E_1} and f_{E_2}, the full complex algebra of the Kripke context $\mathbb{KC}_{DS}$ can be utilised to compute all the approximation operators defined in the work of Saquer and Deogun [11].

To understand the equational theory of the full complex algebra of protoconcepts and the complex algebra of semiconcepts, abstractions of these structures are defined: these are the *double Boolean algebras with operators* (dBao) and *topological dBas* respectively. An immediate example of a dBao is a Boolean algebra with operators [19]; a topological Boolean algebra [20] gives an instance of a topological dBa. It is shown that the full complex algebra of $\mathbb{KC}$ forms a contextual dBao, while the complex algebra of semiconcepts forms a pure dBao. For a reflexive and transitive Kripke context, the full complex algebra forms a topological contextual dBa and the complex algebra of semiconcepts forms a topological pure dBa. Representation theorems for these classes of algebras are then proved, in terms of the complex algebras of protoconcepts and semiconcepts of an appropriate Kripke context. The results are based on the representations obtained for dBas by Wille [14] and Balbiani [21].

As a natural next step, logics corresponding to dBaos are formulated. A sequent calculus, denoted **CDBL**, is proposed for contextual dBas. **CDBL** is extended to **MCDBL** and **MCDBL4** for the contextual dBaos and topological contextual dBas respectively. Due to the representation theorems for the algebras, one is able to get another semantics for these logics, based on protoconcepts of contexts.

Section 2 gives the preliminaries required for this work. Kripke contexts, their examples and the related complex algebras are studied in Section 3. In particular, we indicate in Section 3.1 how the various approximations defined in [11] can be expressed using terms of the full complex algebra of $\mathbb{KC}_{DS}$. The dBaos and the topological dBa along with the representation results are presented in Section 4. In Section 5, the logics corresponding to the algebras are studied. **CDBL** for the class of contextual dBas is discussed in Section 5.1; in Section 5.2, **CDBL** is extended to **MCDBL** and **MCDBL4**. In Section 5.3, the protoconcept-based semantics for the logics is given. Section 6 concludes the article.

In our presentation, the symbols $\Rightarrow$, $\Leftrightarrow$, *and*, *or* and *not* will be used with the usual meanings in the metalanguage. Throughout, for a map f on X, $f{\restriction}A$ denotes the restriction of the map f on $A \subseteq X$, $\mathcal{P}(X)$ denotes the power set of any set X, and the complement of $A \subseteq X$ in a set X is denoted A^c. For basic notions on universal algebra and lattices, we refer to [22, 23].

2 Preliminaries

In the following subsections, we present basic notions and results related to dBas, Boolean algebras with operators and approximation operators. Our primary references are [11, 12, 14, 19–21].

2.1 Double Boolean algebra

A double Boolean algebra is defined as follows.

Definition 2 [14] An algebra $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$, satisfying the following properties is called a *double Boolean algebra* (dBa). For any $x, y, z \in D$,

$(1a)$ $(x \sqcap x) \sqcap y = x \sqcap y$	$(1b)$ $(x \sqcup x) \sqcup y = x \sqcup y$
$(2a)$ $x \sqcap y = y \sqcap x$	$(2b)$ $x \sqcup y = y \sqcup x$
$(3a)$ $x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$	$(3b)$ $x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$
$(4a)$ $\neg(x \sqcap x) = \neg x$	$(4b)$ $\lrcorner(x \sqcup x) = \lrcorner x$
$(5a)$ $x \sqcap (x \sqcup y) = x \sqcap x$	$(5b)$ $x \sqcup (x \sqcap y) = x \sqcup x$
$(6a)$ $x \sqcap (y \vee z) = (x \sqcap y) \vee (x \sqcap z)$	$(6b)$ $x \sqcup (y \wedge z) = (x \sqcup y) \wedge (x \sqcup z)$
$(7a)$ $x \sqcap (x \vee y) = x \sqcap x$	$(7b)$ $x \sqcup (x \wedge y) = x \sqcup x$
$(8a)$ $\neg\neg(x \sqcap y) = x \sqcap y$	$(8b)$ $\lrcorner\lrcorner(x \sqcup y) = x \sqcup y$
$(9a)$ $x \sqcap \neg x = \bot$	$(9b)$ $x \sqcup \lrcorner x = \top$
$(10a)$ $\neg\bot = \top \sqcap \top$	$(10b)$ $\lrcorner\top = \bot \sqcup \bot$
$(11a)$ $\neg\top = \bot$	$(11b)$ $\lrcorner\bot = \top$

(12) $(x \sqcap x) \sqcup (x \sqcap x) = (x \sqcup x) \sqcap (x \sqcup x)$,

where $x \vee y := \neg(\neg x \sqcap \neg y)$, and $x \wedge y := \lrcorner(\lrcorner x \sqcup \lrcorner y)$. A quasi-order relation $\sqsubseteq$ on $\mathbf{D}$ is defined as follows: $x \sqsubseteq y$ if and only if $x \sqcap y = x \sqcap x$ and $x \sqcup y = y \sqcup y$, for any $x, y \in D$.

A dBa $\mathbf{D}$ is called *contextual* if $\sqsubseteq$ is a partial order. A contextual dBa is also known as a regular dBa [24]. $\mathbf{D}$ is *pure* if for all $x \in D$, either $x \sqcap x = x$ or $x \sqcup x = x$. In the following, let $\boldsymbol{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be a dBa. Let us give some notations that shall be used:
$D_\sqcap := \{x \in D \ : \ x \sqcap x = x\}$, $D_\sqcup := \{x \in D \ : \ x \sqcup x = x\}$, $D_p := D_\sqcap \cup D_\sqcup$.
For $x \in D$, $x_\sqcap := x \sqcap x$ and $x_\sqcup := x \sqcup x$.

Proposition 1 [14] $\mathbf{D}_p := (D_p, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is the largest pure subalgebra of $\mathbf{D}$. Moreover, if $\mathbf{D}$ is pure, $\mathbf{D}_p = \mathbf{D}$.

Proposition 2 [21] Every pure dBa $\mathbf{D}$ is contextual.

Proposition 3 [25]

1. $\mathbf{D}_\sqcap := (D_\sqcap, \sqcap, \vee, \neg, \bot, \neg\bot)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcap$ and is denoted by $\sqsubseteq_\sqcap$.
2. $\mathbf{D}_\sqcup := (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$ is a Boolean algebra whose order relation is the restriction of $\sqsubseteq$ to $D_\sqcup$ and it is denoted by $\sqsubseteq_\sqcup$.

3. $x \sqsubseteq y$ if and only if $x \sqcap x \sqsubseteq y \sqcap y$ and $x \sqcup x \sqsubseteq y \sqcup y$ for $x, y \in D$, that is, $x_\sqcap \sqsubseteq_\sqcap y_\sqcap$ and $x_\sqcup \sqsubseteq y_\sqcup$.

Proposition 4 [26] Let $x, y, a \in D$. Then the following hold.

1. $x \sqcap \bot = \bot$ and $x \sqcup \bot = x \sqcup x$ that is $\bot \sqsubseteq x$.
2. $x \sqcup \top = \top$ and $x \sqcap \top = x \sqcap x$ that is $x \sqsubseteq \top$.
3. $x = y$ implies that $x \sqsubseteq y$ and $y \sqsubseteq x$.
4. $x \sqsubseteq y$ and $y \sqsubseteq x$ if and only if $x \sqcap x = y \sqcap y$ and $x \sqcup x = y \sqcup y$.
5. $x \sqcap y \sqsubseteq x, y \sqsubseteq x \sqcup y, x \sqcap y \sqsubseteq y, x \sqsubseteq x \sqcup y$.
6. $x \sqsubseteq y$ implies $x \sqcap a \sqsubseteq y \sqcap a$ and $x \sqcup a \sqsubseteq y \sqcup a$.

Proposition 5 [27] For any $x, y \in D$, the following hold.

1. $\neg x \sqcap \neg x = \neg x$ and $\lrcorner x \sqcup \lrcorner x = \lrcorner x$, that is, $\neg x = (\neg x)_\sqcap \in D_\sqcap$, $\lrcorner x = (\lrcorner x)_\sqcup \in D_\sqcup$.
2. $x \sqsubseteq y$ if and only if $\neg y \sqsubseteq \neg x$ and $\lrcorner y \sqsubseteq \lrcorner x$.
3. $\neg\neg x = x \sqcap x$ and $\lrcorner\lrcorner x = x \sqcup x$.
4. $x \vee y \in D_\sqcap, x \wedge y \in D_\sqcup$.
5. $\neg\neg\bot = \bot$, and $\lrcorner\lrcorner\top = \top$.
6. $\neg(x \sqcap y) = \neg x \vee \neg y$ and $\lrcorner(x \sqcup y) = \lrcorner x \wedge \lrcorner y$.

Definition 3 A subset F of D is a *filter* in $\mathbf{D}$ if and only if $x \sqcap y \in F$ for all $x, y \in F$, and for all $z \in D$ and $x \in F, x \sqsubseteq z$ implies that $z \in F$. An *ideal* in a dBa is defined dually.
A filter F (ideal I) is *proper* if and only if $F \neq D$ ($I \neq D$). A proper filter F (ideal I) is called *primary* if and only if $x \in F$ or $\neg x \in F$ ($x \in I$ or $\lrcorner x \in I$), for all $x \in D$. The set of primary filters is denoted by $\mathcal{F}_{pr}(\mathbf{D})$; the set of all primary ideals is denoted by $\mathcal{I}_{pr}(\mathbf{D})$.
A *base* F_0 for a filter F is a subset of D such that $F = \{x \in D \ : \ z \sqsubseteq x$ for some $z \in F_0\}$. A *base* for an ideal is defined similarly.
For a subset X of D, $F(X)$ and $I(X)$ denote the filter and ideal generated by X respectively.

Lemma 1 [26] Let F be a filter and I an ideal of $\mathbf{D}$. Then for any element $x \in D$,

1. $F(F \cup \{x\}) = \{a \in D \ : \ x \sqcap w \sqsubseteq a$ for some $w \in F\}$.
2. $I(I \cup \{x\}) = \{a \in D \ : \ a \sqsubseteq x \sqcup w$ for some $w \in I\}$.

The following are introduced in [14] to prove representation theorems for dBas.
$\mathcal{F}_p(\mathbf{D}) := \{F \subseteq D : F$ is a filter of $\mathbf{D}$ and $F \cap D_\sqcap$ is a prime filter in $\mathbf{D}_\sqcap\}$.
$\mathcal{I}_p(\mathbf{D}) := \{I \subseteq D : I$ is an ideal of $\mathbf{D}$ and $I \cap D_\sqcup$ is a prime ideal in $\mathbf{D}_\sqcup\}$.

Proposition 6 [27] $\mathcal{F}_p(\mathbf{D}) = \mathcal{F}_{pr}(\mathbf{D})$ and $\mathcal{I}_p(\mathbf{D}) = \mathcal{I}_{pr}(\mathbf{D})$.

Lemma 2 [14]

1. For any filter F of $\mathbf{D}$, $F \cap D_{\sqcap}$ and $F \cap D_{\sqcup}$ are filters of the Boolean algebras $\mathbf{D}_{\sqcap}$, $\mathbf{D}_{\sqcup}$ respectively.
2. Each filter F_0 of the Boolean algebra $\mathbf{D}_{\sqcap}$ is the base of some filter F of $\mathbf{D}$ such that $F_0 = F \cap D_{\sqcap}$. Moreover if F_0 is prime, $F \in \mathcal{F}_p(\mathbf{D})$.

It is straightforward to show that similar results hold for ideals of dBas.

For a context $\mathbb{K} := (G, M, I)$ and sets $A \subseteq G, B, \subseteq M$, recall the sets A', B' and the operations on protoconcepts of $\mathbb{K}$ defined in Section 1.

Lemma 3 [23]

1. $A \subseteq A''$ and $B \subseteq B''$.
2. $A \subseteq X$ implies that $X' \subseteq A'$, $B \subseteq Y$ implies that $Y' \subseteq B'$, for any $X \subseteq G$ and $Y \subseteq M$.

Theorem 7 [14]

1. $\underline{\mathfrak{P}}(\mathbb{K}) := (\mathfrak{P}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is a contextual dBa.
2. $\underline{\mathfrak{H}}(\mathbb{K}) := (\mathfrak{H}(\mathbb{K}), \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is a pure dBa. Moreover, $\underline{\mathfrak{H}}(\mathbb{K}) = \underline{\mathfrak{P}}(\mathbb{K})_p$.

Theorem 8 [14]

1. The power set Boolean algebra $(\mathcal{P}(G), \cap, \cup, ^c, G, \emptyset)$ is isomorphic to the Boolean algebra $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcap} := (\mathfrak{P}(\mathbb{K})_{\sqcap}, \sqcap, \vee, \neg, \bot, \neg\bot)$, where any $A (\subseteq G)$ is mapped to $(A, A') \in \mathfrak{P}(\mathbb{K})_{\sqcap}$.
2. The power set Boolean algebra $(\mathcal{P}(M), \cup, \cap, ^c, M, \emptyset)$ is anti-isomorphic to the Boolean algebra $\underline{\mathfrak{P}}(\mathbb{K})_{\sqcup} := (\mathfrak{P}(\mathbb{K})_{\sqcup}, \sqcup, \wedge, \lrcorner, \top, \lrcorner\top)$, where any $B (\subseteq M)$ is mapped to $(B', B) \in \mathfrak{P}(\mathbb{K})_{\sqcup}$.

Let us now move to representation theorems for dBas. The following notations and results are needed. Let $\mathbf{D}$ be a dBa. For any $x \in D$, $F_x := \{F \in \mathcal{F}_p(\mathbf{D}) \ : \ x \in F\}$ and $I_x := \{I \in \mathcal{I}_p(\mathbf{D}) \ : \ x \in I\}$.

Lemma 4 [10, 14] Let $x \in D$. Then the following hold.

1. $(F_x)^c = F_{\neg x}$ and $(I_x)^c = I_{\lrcorner x}$.
2. $F_{x \sqcap y} = F_x \cap F_y$ and $I_{x \sqcup y} = I_x \cap I_y$.

To prove the representation theorem, Wille uses the *standard context* corresponding to the dBa $\mathbf{D}$, defined as $\mathbb{K}(\mathbf{D}) := (\mathcal{F}_p(\mathbf{D}), \mathcal{I}_p(\mathbf{D}), \Delta)$, where for all $F \in \mathcal{F}_p(\mathbf{D})$ and $I \in \mathcal{I}_p(\mathbf{D})$, $F \Delta I$ if and only if $F \cap I = \emptyset$. Then we have

Lemma 5 [14] For all $x \in \mathbf{D}$, $F_x' = I_{x_{\sqcap\sqcup}}$ and $I_x' = F_{x_{\sqcup\sqcap}}$.

Theorem 9 [14] The map $h : \mathbf{D} \to \underline{\mathfrak{P}}(\mathbb{K}(\mathbf{D}))$ defined by $h(x) := (F_x, I_x)$ for all $x \in \mathbf{D}$ is a quasi-embedding.

As a consequence of the above theorem, we have

Corollary 1 For a contextual dBa $\mathbf{D}$, the map $h : \mathbf{D} \to \underline{\mathfrak{P}}(\mathbb{K}(\mathbf{D}))$ defined by $h(x) := (F_x, I_x)$ for all $x \in \mathbf{D}$ is an embedding.

Theorem 10 [21] Let $\mathbf{D}$ be a pure dBa. The map $h : \mathbf{D} \to \underline{\mathfrak{H}}(\mathbb{K}(\mathbf{D}))$ defined by $h(x) := (F_x, I_x)$ for all $x \in \mathbf{D}$ is an embedding.

2.2 Boolean algebras with operators

In the literature, there are several definitions of Boolean algebras with additional operators. In this section, we mention the ones to be used in this work.

Definition 4 [19] *A Boolean algebra with operators* (Bao) is an algebra $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, f)$ such that $(B, \vee, \wedge, \neg, 0)$ is a Boolean algebra and $f : B \to B$ satisfies the following.

Normality: $f(0) = 0$, *Additivity*: $f(x \vee y) = f(x) \vee f(y)$.

Note that [19] gives a general definition of Baos with more than one operator. In [20], a Boolean algebra $(B, \vee, \wedge, \neg, 0)$ with only an additive operator f is taken as the definition of Bao.

Definition 5 [20] An algebra $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, f)$ is called a *closure algebra* if $(B, \vee, \wedge, \neg, 0)$ is a Boolean algebra and for all $x, y \in B$, $f : B \to B$ satisfies the following conditions.

1. $f(0) = 0$. 2. $f(x \vee y) = f(x) \vee f(y)$.
3. $ff(x) = f(x)$. 4. $x \leq f(x)$.

Note that for a closure algebra $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, f)$, one can define an operator g on B as: $g(x) := \neg f(\neg x)$, for all $x \in B$. Then for all $x, y \in B$,

$1'$. $g(1) = 1$. $2'$. $g(x \wedge y) = g(x) \wedge g(y)$.
$3'$. $gg(x) = g(x)$. $4'$. $g(x) \leq x$.

An algebra $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, g)$, where $(B, \vee, \wedge, \neg, 0)$ is a Boolean algebra and g satisfies $1', 2', 3', 4'$ is called a *topological Boolean algebra* in [28]. Moreover, for a topological Boolean algebra $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, g)$, one can define an operator $g^{\delta}(x) := \neg g(\neg x)$, for all $x \in D$ such that $\mathfrak{A} := (B, \vee, \wedge, \neg, 0, g^{\delta})$ is a closure algebra. In other words, a closure algebra and a topological Boolean algebra of [28] are dual to each other and one can be obtained from the other. In this work, by a topological Boolean algebra, we shall mean a closure algebra.

2.3 Approximation operators

Recall the definitions of lower and upper approximation operators in an approximation space given in Section 1. If the relation is clear from the context, we shall omit the subscript and denote $\underline{A}_E$ by $\underline{A}$, $\overline{A}^E$ by $\overline{A}$.

Proposition 11 [18]

I. For an approximation space (W, E), the following hold.

(i) $\overline{A} = (\underline{(A^c)})^c, \underline{A} = (\overline{(A^c)})^c$.

(ii) $\underline{W} = W$.

(iii) $\underline{A \cap B} = \underline{A} \cap \underline{B}, \overline{A \cup B} = \overline{A} \cup \overline{B}$.

(iv) $A \subseteq B$ implies that $\underline{A} \subseteq \underline{B}, \overline{A} \subseteq \overline{B}$.

II. Moreover if E is a reflexive and transitive relation then the following hold.

(v) $\underline{A} \subseteq A$ and $A \subseteq \overline{A}$.

(vi) $\underline{(\underline{A})} = \underline{A}$ and $\overline{(\overline{A})} = \overline{A}$.

Let $\mathbb{K} := (G, M, I)$ be a context and recall the approximation spaces (G, E_1) and (M, E_2) mentioned in Section 1. In [11], $A \subseteq G$ and $B \subseteq M$ are called *feasible* if $A'' = A$ and $B'' = B$. Then the concept approximation(s) of A are defined as follows.

- If A is feasible, the concept approximation of A is (A, A').
- If A is not feasible, A is considered as s rough set of the approximation space (G, E_1), and its concept approximations are defined with the help of its lower approximation $\underline{A}_{E_1}$ and upper approximation $\overline{A}^{E_1}$. The *lower concept approximation* of A is the pair $((\underline{A}_{E_1})'', (\underline{A}_{E_1})')$, while its *upper concept approximation* is $((\overline{A}^{E_1})'', (\overline{A}^{E_1})')$.

For $B \subseteq M$:

- if B is feasible, the concept approximation of B is (B', B);
- if B is non-feasible, the lower and upper concept approximations of B are defined by $((\overline{B}^{E_2})', (\overline{B}^{E_2})'')$ and $((\underline{B}_{E_2})', (\underline{B}_{E_2})'')$ respectively.

A pair (A, B) is called a *non-definable* concept, if it is not a concept of the context $\mathbb{K}$. A concept is said to approximate such a pair (A, B), if its extent approximates A and intent approximates B. The four possible cases for A, B are considered: (i) both A and B are feasible, (ii) A is feasible and B is not, (iii) B is feasible and A is not, and (iv) both A and B are not feasible. In case both A and B are feasible and $A' = B$ then the pair (A, B) itself constitutes a concept and no approximations are needed. For the other cases, the lower approximation of (A, B) is obtained in terms of the meet of the lower concept approximations of its individual components, while the upper approximation of (A, B) is obtained in terms of the join of the upper concept approximations of its individual components. For example, consider case (iv), when both A and B are not feasible.

The *lower approximation* of (A, B) is defined by $\underline{(A, B)} :=$ $((\underline{A}_{E_1})'', (\underline{A}_{E_1})') \sqcap ((\overline{B}^{E_2})', (\overline{B}^{E_2})'') = ((\underline{A}_{E_1})'' \cap (\overline{B}^{E_2})', ((\underline{A}_{E_1})'' \cap (\overline{B}^{E_2})')')$.
The *upper approximation* of (A, B) is defined by $\overline{(A, B)} :=$ $((\overline{A}^{E_1})'', (\overline{A}^{E_1})') \sqcup ((\underline{B}_{E_2})', (\underline{B}_{E_2})'') = (((\overline{A}^{E_1})' \cap (\underline{B}_{E_2})'')', (\overline{A}^{E_1})' \cap (\underline{B}_{E_2})'')$.
Let us illustrate these notions by an example. The following context (G, M, I) is a subcontext of a context given by Wille [12] with some modifications. $G := \{Leech, Bream, Frog, Dog, Cat\}$ and $M := \{a, b, c, g\}$, where a:= needs water to live, b:= lives in water, c:= lives on land, g:=can move around. I is given by Table 1, where * as an entry corresponding to object x and property y means xIy holds.

Table 1: Context $\mathbb{K}$

	a	b	c	g
Leech	*	*		*
Bream	*	*		*
Frog	*	*	*	*
Dog	*		*	*
Cat	*		*	*

Observe that the properties a and g are indiscernible by objects, while Leech and Bream as well as Dog and Cat are indiscernible by properties. The induced approximation spaces are $(G, \{\{Leech, Bream\}, \{Frog\}, \{Dog, Cat\}\})$ and $(M, \{\{a, \mathrm{g}\}, \{b\}, \{c\}\})$.

Let $A := \{Leech, Bream, Dog\}$ and $B := \{a, c\}$. A is not feasible, as $A'' \neq A$. B is also non-feasible. The upper and lower concept approximations of A are $(G, \{a, g\})$ and $(\{Leech, Bream, Frog\}, \{a, b, \mathrm{g}\})$, respectively. The upper and lower concept approximations of B are both given by $(\{Frog, Dog, Cat\}, \{a, \mathrm{g}, c\})$. Moreover, (A, B) is a non-definable concept. The lower approximation of (A, B) is $(\{Frog\}, M)$ and the upper approximation is $(G, \{a, g\})$.

3 Kripke context

As given by Definition 1 in Section 1, a Kripke context based on a context $\mathbb{K} := (G, M, I)$ is a triple $\mathbb{KC} := ((G, R), (M, S), I)$, where R, S are binary relations on G and M respectively. Let us give a couple of examples of Kripke contexts. The first example is based on Pawlakian approximation spaces.

Example 1 $\mathbb{KC} := ((G, R), (M, S), I)$, where $G := \{D_1, D_2, D_3, D_4\}$ represents a collection of diseases and $M := \{S_1, S_2, S_3, S_4, S_5\}$ a collection of symptoms. $D_i I S_j$ holds if disease D_i has symptom S_j, and I is given by Table 2. Equivalence relations R on G and S on M are then induced as follows, relating respectively, the diseases that have the same set of symptoms, and the symptoms that apply to the same set of diseases:

D_iRD_j, if and only if $I(D_i) = I(D_j)$, $i, j \in \{1, 2, 3, 4\}$ and S_iRS_j, if and only if $I^{-1}(S_i) = I^{-1}(S_j)$, $i, j \in \{1, 2, 3, 4, 5\}$.
One thus gets the approximation spaces (G, R) and (M, S).

Table 2: Context $\mathbb{K}$

	S_1	S_2	S_3	S_4	S_5
D_1	*	*		*	
D_2			*		*
D_3			*	*	*
D_4	*	*		*	

Our next example is motivated by the notion of bisimulation between Kripke frames [19]. It gives a Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$ such that the relation I is in fact, a bisimulation between the Kripke frames (G, R) and (M, S), that is, it satisfies the back and forth conditions: for all $g \in G$ and $m \in M$,
for all $g_1 \in G$ (gRg_1 and $gIm \implies$ there exists $m_1 \in M(mSm_1$ and $g_1Im_1)$);
for all $m_1 \in M$ (mSm_1 and $gIm \implies$ there exists $g_1 \in M(gRg_1$ and $g_1Im_1)$).

Example 2 $\mathbb{KC} := ((G, R), (M, S), I)$, where $G := \{c, d, e\}$, $M := \{a, b\}$, $R := \{(d, e), (c, d)\}$ and $S := \{(a, b), (b, a)\}$. I is given by Table 3. Figure 1 depicts the objects, properties and the three relations R, S, I. Each circular node represents an object and each rectangular node a property. Two circular nodes are connected by an arrow if they are related by R. Similarly for the rectangular nodes. The dotted arrow represents the relation I. From the figure it is clear that I satisfies the back and forth conditions.

Table 3: Context $\mathbb{K}$

	a	b
c	*	
d		*
e	*	

In a Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$, if (G, R) is a Pawlakian approximation space, one gets an interior operator $-_R : \mathcal{P}(G) \to \mathcal{P}(G)$ defined as $-_R(A) := \underline{A}_R$ for all $A \in \mathcal{P}(G)$ (Proposition 11). Similarly, one has the interior operator $-_S : \mathcal{P}(M) \to \mathcal{P}(M)$ defined by $-_M(B) := \underline{B}_S$ for all $B \in \mathcal{P}(M)$, if (M, S) is a Pawlakian approximation space. Now from Theorem 8, we get the isomorphism $f : \mathcal{P}(G) \to \mathfrak{P}(\mathbb{K})_\sqcap$ given by $f(A) := (A, A')$ for all $A \in \mathcal{P}(G)$ and the anti-isomorphism $g : \mathcal{P}(M) \to \mathfrak{P}(\mathbb{K})_\sqcup$ given by $g(B) := (B', B)$ for all $B \in \mathcal{P}(M)$. Taking a cue from the compositions of

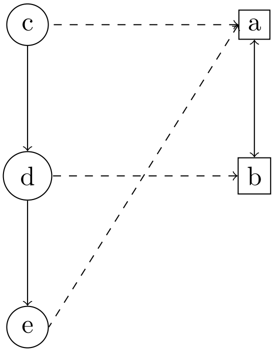

Fig. 1: Kripke Context $\mathbb{KC}$

$f, -_R$ and $g, -_S$, we can define two unary operators f_R and f_S on $\mathfrak{P}(\mathbb{K})$ as given below. It will be seen in Theorem 12 that f_R is an interior-type operator on $\mathfrak{P}(\mathbb{K})$, while g_S is a closure-type operator on $\mathfrak{P}(\mathbb{K})$. For any $(A, B) \in \mathfrak{P}(\mathbb{K})$,

- $f_R((A, B)) := (\underline{A}_R, (\underline{A}_R)')$,
- $f_S((A, B)) := ((\underline{B}_S)', \underline{B}_S)$.

f_R, f_S are well-defined, as $(\underline{A}_R, (\underline{A}_R)')$ and $((\underline{B}_S)', \underline{B}_S)$ are both semiconcepts and hence protoconcepts of $\mathbb{K}$. This implies that the set $\mathfrak{P}(\mathbb{K})$ of protoconcepts is closed under the operators f_R, f_S. We have

Definition 6 Let $\mathbb{KC} := ((G, R), (M, S), I)$ be a Kripke context. The *full complex algebra* of $\mathbb{KC}$, $\underline{\mathfrak{P}}^+(\mathbb{KC}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_R, f_S)$, is the expansion of the algebra $\underline{\mathfrak{P}}(\mathbb{K})$ of protoconcepts with the operators f_R and f_S.
Any subalgebra of $\underline{\mathfrak{P}}^+(\mathbb{KC})$ is called a *complex algebra* of $\mathbb{KC}$.

Let f_R^δ, f_S^δ denote the operators on $\mathfrak{P}(\mathbb{K})$ that are *dual* to f_R, f_S respectively. In other words, for each $x := (A, B) \in \mathfrak{P}(\mathbb{K})$,
$f_R^\delta(x) := \neg f_R(\neg x) = \neg f_R((A^c, A^{c\prime})) = \neg(\underline{A^c}_R, (\underline{A^c}_R)') = ((\underline{A^c}_R)^c, (\underline{A^c}_R)^{c\prime}) = (\overline{A}^R, (\overline{A}^R)')$, by Proposition 11(i).
Similarly $f_S^\delta(x) := \lrcorner f_S(\lrcorner x) = ((\overline{B}^S)', \overline{B}^S)$.
Again, note that $f_R^\delta(x) = (\overline{A}^R, (\overline{A}^R)')$ and $f_S^\delta(x) = ((\overline{B}^S)', \overline{B}^S)$ are semiconcepts of $\mathbb{K}$. Let us now list some properties of f_R and f_S.

Theorem 12 For all $x, y \in \mathfrak{P}(\mathbb{K})$, the following hold.

1. $f_R(x \sqcap y) = f_R(x) \sqcap f_R(y)$ and $f_S(x \sqcup y) = f_S(x) \sqcup f_S(y)$.
2. $f_R(x \sqcap x) = f_R(x)$ and $f_S(x \sqcup x) = f_S(x)$.
3. $f_R(\neg\bot) = \neg\bot$ and $f_S(\lrcorner\top) = \lrcorner\top$.
4. $f_R(\neg x) = \neg f_R^\delta(x)$ and $f_S(\lrcorner x) = \lrcorner f_S^\delta(x)$.

Proof Let $x := (A, B)$ and $y := (C, D)$.
1. We use Proposition 11(iii) in the following equations. $f_R((A, B)\sqcap(C, D)) = f_R(A\cap C, (A \cap C)') = (\underline{A \cap C}_R, (\underline{A \cap C}_R)') = (\underline{A}_R \cap \underline{C}_R, (\underline{A}_R \cap \underline{C}_R)') = (\underline{A}_R, (\underline{A}_R)') \sqcap (\underline{C}_R, (\underline{C}_R)') = f_R((A, B)) \sqcap f_R((C, D))$.
$f_S((A, B) \sqcup (C, D)) = f_S((B \cap D)', B \cap D) = ((\underline{B \cap D}_S)', \underline{B \cap D}_S) = f_S((A, B)) \sqcup f_S((C, D))$.
2. $f_R((A, B) \sqcap (A, B)) = f_R((A, A')) = (\underline{A}_R, (\underline{A}_R)') = f_R((A, B))$. Similarly, one can show that $f_S((A, B) \sqcup (A, B)) = ((\underline{B}_S)', \underline{B}_S)$.
3. $f_R(\neg\bot) = f_R((G, G')) = (\underline{G}_R, (\underline{G}_R)') = (G, G') = \neg\bot$, by Proposition 11(ii). Similarly, one gets $f_S(\lrcorner\top) = \lrcorner\top$.
4. $f_R(\neg(A, B)) = f_R(A^c, A^{c\prime}) = (\underline{A^c}_R, (\underline{A^c}_R)') = ((\overline{A}^R)^c, (\overline{A}^R)^{c\prime})$ by Proposition 11(i). So $f_R(\neg(A, B)) = \neg(\overline{A}^R, (\overline{A}^R)') = \neg f^\delta_R((A, B))$. Similarly, one can show that $f_S(\lrcorner(A, B)) = \lrcorner f^\delta_S((A, B))$. □

Using Theorem 12(1,3,4), one obtains

Corollary 2 For all $x, y \in \mathfrak{P}(\mathbb{K})$,

1. $f^\delta_R(x \vee y) = f^\delta_R(x) \vee f^\delta_R(y)$ and $f^\delta_S(x \wedge y) = f^\delta_S(x) \wedge f^\delta_S(y)$.
2. $f^\delta_R(\bot) = \bot$ and $f^\delta_S(\top) = \top$.

Consider the restriction maps $f_R\restriction\mathfrak{P}(\mathbb{K})_\sqcap$ and $f_S\restriction\mathfrak{P}(\mathbb{K})_\sqcup$. From Theorem 12(2), it follows that $\mathfrak{P}(\mathbb{K})_\sqcap$ and $\mathfrak{P}(\mathbb{K})_\sqcup$ are closed under $f_R\restriction\mathfrak{P}(\mathbb{K})_\sqcap$ and $f_S\restriction\mathfrak{P}(\mathbb{K})_\sqcup$ respectively. Using Theorem 12(1,3) and Corollary 2, we get

Corollary 3 $\underline{\mathfrak{P}}(\mathbb{KC})^+_\sqcap := (\mathfrak{P}(\mathbb{K})_\sqcap, \sqcap, \vee, \neg, \bot, f^\delta_R\restriction\mathfrak{P}(\mathbb{K})_\sqcap)$ and $\underline{\mathfrak{P}}(\mathbb{KC})^+_\sqcup := (\mathfrak{P}(\mathbb{K})_\sqcup, \sqcup, \wedge, \lrcorner, \top, f_S\restriction\mathfrak{P}(\mathbb{K})_\sqcup)$ are Baos.

We next consider a Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$ where the relations R, S satisfy certain properties that are of particular relevance here.

Definition 7

1. $\mathbb{KC}$ is *reflexive from the left*, if R is reflexive.
2. $\mathbb{KC}$ is *reflexive from the right*, if S is reflexive.
3. $\mathbb{KC}$ is *reflexive*, if it is reflexive from both left and right.

The cases for *symmetry* and *transitivity* of $\mathbb{KC}$ are similarly defined.

Observe that the Kripke context in Example 2 is symmetric from the right.

Theorem 13 Let $\mathbb{KC} := ((G, R), (M, S), I)$ be a reflexive and transitive Kripke context. Then for all $x \in \mathfrak{P}(\mathbb{K})$, the following hold.

1. $f_R(x) \sqsubseteq x$ and $x \sqsubseteq f_S(x)$.
2. $f_Rf_R(x) = f_R(x)$ and $f_Sf_S(x) = f_S(x)$.

Proof 1. Let $(A,B) \in \mathfrak{P}(\mathbb{K})$. By Proposition 11(v) $\underline{A}_R \subseteq A$ and $\underline{B}_S \subseteq B$, which implies that $A' \subseteq (\underline{A}_R)'$ and $B' \subseteq (\underline{B}_S)'$. Now $B'' = A'$ and $A'' = B'$, as $(A,B) \in \mathfrak{P}(\mathbb{K})$. By Lemma 3, $A \subseteq A''$ and $B \subseteq B''$. So $A \subseteq B'$ and $B \subseteq A'$, which implies that $B \subseteq (\underline{A}_R)'$ and $A \subseteq (\underline{B}_S)'$. Therefore $f_R((A,B)) = (\underline{A}_R, (\underline{A}_R)') \sqsubseteq (A,B)$ and $(A,B) \sqsubseteq f_S((A,B)) = ((\underline{B}_S)', \underline{B}_S)$.
2. $f_R f_R((A,B)) = f_R((\underline{A}_R, (\underline{A}_R)')) = (\underline{(\underline{A}_R)}_R, (\underline{(\underline{A}_R)}_R)') = (\underline{A}_R, (\underline{A}_R)') = f_R((A,B))$, by Proposition 11(vi). Similarly, one can show that $f_S f_S((A,B)) = f_S((A,B))$. □

Theorems 13 and 12(4) give

Corollary 4 For all $x \in \mathfrak{P}(\mathbb{K})$, $x \sqsubseteq f_R^\delta(x)$ and $f_R^\delta f_R^\delta(x) = f_R^\delta(x)$.

Further, using Theorems 12, 13 and Corollaries 2, 4, we get

Corollary 5 $\underline{\mathfrak{P}}(\mathbb{KC})^+_\sqcap := (\mathfrak{P}(\mathbb{K})_\sqcap, \sqcap, \vee, \neg, \bot, f_R^\delta \restriction \mathfrak{P}(\mathbb{K})_\sqcap)$ and $\underline{\mathfrak{P}}(\mathbb{KC})^+_\sqcup := (\mathfrak{P}(\mathbb{K})_\sqcup, \sqcup, \wedge, \lrcorner, \top, f_S \restriction \mathfrak{P}(\mathbb{K})_\sqcup)$ are topological Boolean algebras.

3.1 Complex algebra to concept approximation

Recall the Kripke context $\mathbb{KC}_{DS} := ((G,E_1),(M,E_2),I)$ defined in Section 1, where $(G,E_1),(M,E_2)$ are Pawlakian approximation spaces. We observe that terms of the full complex algebra $\underline{\mathfrak{P}}^+(\mathbb{KC}_{DS})$ are able to express the various notions of concept approximations mentioned in Section 2.3. Indeed, for $\mathbb{KC}_{DS}$, we get the operators $f_{E_1}, f_{E_2} : \mathfrak{P}(\mathbb{K}) \to \mathfrak{P}(\mathbb{K})$ as above, that is, $f_{E_1}((A,B)) := (\underline{A}_{E_1}, (\underline{A}_{E_1})')$, and $f_{E_2}((A,B)) := ((\underline{B}_{E_2})', \underline{B}_{E_2})$ for any $(A,B) \in \mathfrak{P}(\mathbb{K})$. Moreover, $f^\delta_{E_1}((A,B)) = (\overline{A}^{E_1}, (\overline{A}^{E_1})')$ and $f^\delta_{E_2}((A,B)) = ((\overline{B}^{E_2})', \overline{B}^{E_2})$. Let $A \subseteq G$ and $B \subseteq M$.
If A and B are feasible then the concept approximations of A and B are (A,A') and (B',B) respectively and these are elements of $\mathfrak{P}(\mathbb{K})$.
Suppose A and B are both non-feasible sets. Let $x, y \in \mathfrak{P}(\mathbb{K})$ be such that the extent of x is A and intent of y is B. Then we have the following.
The lower concept approximation of A, $((\underline{A}_{E_1})'', (\underline{A}_{E_1})') = (\underline{A}_{E_1}, (\underline{A}_{E_1})') \sqcup (\underline{A}_{E_1}, (\underline{A}_{E_1})') = f_{E_1}(x) \sqcup f_{E_1}(x)$.
The upper concept approximation of A, $((\overline{A}^{E_1})'', (\overline{A}^{E_1})') = (\overline{A}^{E_1}, (\overline{A}^{E_1})') \sqcup (\overline{A}^{E_1}, (\overline{A}^{E_1})') = f^\delta_{E_1}(x) \sqcup f^\delta_{E_1}(x)$.
The lower concept approximation of B, $((\overline{B}^{E_2})', (\overline{B}^{E_2})'') = ((\overline{B}^{E_2})', \overline{B}^{E_2}) \sqcap ((\overline{B}^{E_2})', \overline{B}^{E_2}) = f^\delta_{E_2}(y) \sqcap f^\delta_{E_2}(y)$.
The upper concept approximation of B, $((\underline{B}_{E_2})', (\underline{B}_{E_2})'') = ((\underline{B}_{E_2})', \underline{B}_{E_2}) \sqcap ((\underline{B}_{E_2})', \underline{B}_{E_2}) = f_{E_2}(y) \sqcap f_{E_2}(y)$.
Now by definition, approximations of any pair (A,B) are obtained using the concept approximations of A and B. As shown above, the latter are all expressible by the terms of the full complex algebra, and hence we have the

observation. For instance, suppose, (A, B) is a non-definable concept of $\mathbb{K}$ with A and B non-feasible.
The lower approximation of (A, B), $((\underline{A}_{E_1})'' \cap (\overline{B}^{E_2})', ((\underline{A}_{E_1})'' \cap (\overline{B}^{E_2})')')$ $= (f_{E_1}(x) \sqcup f_{E_1}(x)) \sqcap (f^{\delta}_{E_2}(y) \sqcap f^{\delta}_{E_2}(y)) = (f_{E_1}(x) \sqcup f_{E_1}(x)) \sqcap f^{\delta}_{E_2}(y)$.
The upper approximation of (A, B), $(((\overline{A}^{E_1})' \cap (\underline{B}_{E_2})'')', (\overline{A}^{E_1})' \cap (\underline{B}_{E_2})'') =$ $(f^{\delta}_{E_1}(x) \sqcup f^{\delta}_{E_1}(x)) \sqcup (f_{E_2}(y) \sqcap f_{E_2}(y)) = (f_{E_2}(y) \sqcap f_{E_2}(y)) \sqcup f^{\delta}_{E_1}(x)$.

4 The algebras

In this section, we study abstractions of the algebraic structure $\underline{\mathfrak{P}}^+(\mathbb{KC})$ obtained in Section 3. These are dBas with operators (Definition 8), and topological dBas (Definition 9).

4.1 Double Boolean algebras with operators

Definition 8 A structure $\mathfrak{O} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ is a *dBa with operators* (dBao) provided
1. $(D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a dBa and
2. $\mathbf{I}, \mathbf{C}$ are monotonic operators on D satisfying the following for any $x, y \in D$.

$1a$ $\mathbf{I}(x \sqcap y) = \mathbf{I}(x) \sqcap \mathbf{I}(y)$	$1b$ $\mathbf{C}(x \sqcup y) = \mathbf{C}(x) \sqcup \mathbf{C}(y)$
$2a$ $\mathbf{I}(\neg\bot) = \neg\bot$	$2b$ $\mathbf{C}(\lrcorner\top) = \lrcorner\top$
$3a$ $\mathbf{I}(x \sqcap x) = \mathbf{I}(x)$	$3b$ $\mathbf{C}(x \sqcup x) = \mathbf{C}(x)$

A *contextual dBao* is a dBao in which the underlying dBa is contextual. If the underlying dBa is pure, the dBao is called a *pure dBao*.
The duals of $\mathbf{I}$ and $\mathbf{C}$ with respect to $\neg, \lrcorner$ are defined as $\mathbf{I}^{\delta}(a) := \neg\mathbf{I}(\neg a)$ and $\mathbf{C}^{\delta}(a) := \lrcorner\mathbf{C}(\lrcorner a)$ for all $a \in D$.

Any Bao provides a trivial example of a contextual and pure dBao. Indeed, in a Bao $(B, \sqcap, \sqcup, \neg, \top, \bot, f)$, setting $\lrcorner = \neg$, $\mathbf{C} := f$ and $\mathbf{I} := f^{\delta}$, one obtains the dBao $(B, \sqcap, \sqcup, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$. Due to the idempotence of the operators $\sqcap, \sqcup$ in the Boolean algebra $(B, \sqcap, \sqcup, \neg, \top, \bot)$, the dBa $(B, \sqcap, \sqcup, \neg, \lrcorner, \top, \bot)$ is pure; as $B_{\sqcap} = B_{\sqcup} = B$, the dBa is contextual as well.

An immediate consequence is the following.

Theorem 14 Let $\mathfrak{O} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao. Then

1. $\mathfrak{O}_p := (D_p, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}\lceil D_p, \mathbf{C}\lceil D_p)$ is the largest pure subalgebra of $\mathfrak{O}$.
2. If $\mathfrak{O}$ is pure, it is contextual and moreover, $\mathfrak{O} = \mathfrak{O}_p$.

Proof 1. From Proposition 1 it follows that $(D_p, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is the largest pure subalgebra of $\mathbf{D}$. To complete the proof it is sufficient to show that D_p is closed under $\mathbf{I}$ and $\mathbf{C}$, which follows from Definition 8(1a, 3a, 1b, 3b).
2. Proposition 2 gives the first part. For any pure dBa, $D = D_p$. □

As intended, the sets of protoconcepts and semiconcepts of a context provide examples of dBaos:

Theorem 15 Let $\mathbb{KC} := ((G, R), (M, S), I)$ be a Kripke context based on the context $\mathbb{K} := (G, M, I)$. Then the following hold.

1. $\underline{\mathfrak{P}}^+(\mathbb{KC}) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_R, f_S)$ is a contextual dBao.
2. $\underline{\mathfrak{H}}^+(\mathbb{KC}) := (\mathfrak{H}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_R{\restriction}\mathfrak{H}(\mathbb{K}), f_S{\restriction}\mathfrak{H}(\mathbb{K}))$ is a pure dBao. It is the largest pure subalgebra of $\underline{\mathfrak{P}}^+(\mathbb{KC})$, that is, $\underline{\mathfrak{P}}^+(\mathbb{KC})_p = \underline{\mathfrak{H}}^+(\mathbb{KC})$.

Proof 1. From Theorem 7 it follows that $(\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ is a dBa. To show monotonicity of f_R, f_S, let $(A, B), (C, D) \in \mathfrak{P}(\mathbb{K})$ and $(A, B) \sqsubseteq (C, D)$. Then, by definition of $\sqsubseteq$, $A \subseteq C$ and $D \subseteq B$, and by using Proposition 11(iv), $\underline{A}_R \subseteq \underline{C}_R$, which implies $(\underline{C}_R)' \subseteq (\underline{A}_R)'$. Hence $f_R((A, B)) \sqsubseteq f_R((C, D))$. Similar to the above, we can show the monotonicity of f_S. Rest of the proof follows from Theorem 12.
2. From Theorem 7, it follows that $\mathfrak{P}(\mathbb{K})_p = \mathfrak{H}(\mathbb{K})$. By Theorem 14(2), $\underline{\mathfrak{P}}^+(\mathbb{KC})_p = \underline{\mathfrak{H}}^+(\mathbb{KC})$ is the largest pure subalgebra of $\underline{\mathfrak{P}}^+(\mathbb{KC})$. □

The following lists some basic properties of the operators $\mathbf{I}, \mathbf{C}$ and their duals in a dBao.

Lemma 6 Let $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao. Then the following hold for any $a, x, y \in D$.
1. $\neg\mathbf{I}^\delta(\neg a) = \mathbf{I}a$ and $\lrcorner\mathbf{C}^\delta(\lrcorner a) = \mathbf{C}(a)$.
2. $\mathbf{I}(\neg a) = \neg\mathbf{I}^\delta(a)$ and $\mathbf{I}^\delta(\neg a) = \neg\mathbf{I}(a)$.
3. $\mathbf{C}(\lrcorner a) = \lrcorner\mathbf{C}^\delta(a)$ and $\mathbf{C}^\delta(\lrcorner a) = \lrcorner\mathbf{C}(a)$.
4. $\mathbf{I}^\delta$ and $\mathbf{C}^\delta$ both are monotonic.
5. $\mathbf{I}^\delta(a \sqcap a) = \mathbf{I}^\delta(a)$ and $\mathbf{C}^\delta(a \sqcup a) = \mathbf{C}^\delta(a)$.
6. $\mathbf{I}^\delta(x \vee y) = \mathbf{I}^\delta(x) \vee \mathbf{I}^\delta(y)$ and $\mathbf{C}^\delta(x \wedge y) = \mathbf{C}^\delta(x) \wedge \mathbf{C}^\delta(y)$.
7. $\mathbf{I}^\delta(\bot) = \bot$ and $\mathbf{C}^\delta(\top) = \top$.
8. $\mathbf{I}^\delta(x) \sqcap \mathbf{I}^\delta(x) = \mathbf{I}^\delta(x)$ and $\mathbf{C}^\delta(x) \sqcup \mathbf{C}^\delta(x) = \mathbf{C}^\delta(x)$.

Proof The proof is obtained in a straightforward manner. We use 1, 2, 3 and 5 of Proposition 5, (8a), (8b) of Definition 2 and $3a, 3b$ of Definition 8. □

We noted earlier that a Bao provides an example of a dBao. The converse question is addressed in Theorems 16 and 17 below.

Theorem 16 Let $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao such that for all $a \in D$ $\neg a = \lrcorner a$, $\neg\neg a = a$. Then $(D, \sqcup, \sqcap, \neg, \top, \bot, \mathbf{C})$ and $(D, \sqcup, \sqcap, \neg, \top, \bot, \mathbf{I}^\delta)$ are Baos.

Proof That $(D, \sqcup, \sqcap, \neg, \top, \bot)$ forms a Boolean algebra is not difficult to prove, and the proof is given in the Appendix. In particular, one can show that $y \sqcup z = y \vee z$ and $y \sqcap z = y \wedge z$ for any $y, z \in D$. It is then easy to verify that $\mathbf{C}$ and $\mathbf{I}^\delta$ are additive and normal. Indeed, Definition 8(1b) implies that $\mathbf{C}$ is additive. As $\lrcorner\top = \bot$, by Definition 8(3b), it is normal. On the other hand, as $y \sqcup z = y \vee z$ for all $y, z \in D$, from Lemma 6(6) it follows that $\mathbf{I}^\delta(x \sqcup y) = \mathbf{I}^\delta(x \vee y) = \mathbf{I}^\delta(x) \vee \mathbf{I}^\delta(y) = \mathbf{I}^\delta(x) \sqcup \mathbf{I}^\delta(y)$. $\mathbf{I}^\delta(\bot) = \bot$ by Lemma 6(7). □

Theorem 17 Let $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao. Then $\mathfrak{D}_\sqcap := (D_\sqcap, \sqcap, \vee, \neg, \bot, \mathbf{I}^\delta\lceil D_\sqcap)$ and $\mathfrak{D}_\sqcup := (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \mathbf{C}\lceil D_\sqcup)$ are Baos.

Proof By Proposition 3, $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are Boolean algebras. Let $x \in D_\sqcap$. Then $\mathbf{I}^\delta\lceil D_\sqcap(x) \sqcap \mathbf{I}^\delta\lceil D_\sqcap(x) = \mathbf{I}^\delta(x) \sqcap \mathbf{I}^\delta(x) = \mathbf{I}^\delta(x) = \mathbf{I}^\delta\lceil D_\sqcap(x)$, by Lemma 6(8). So $D_\sqcap$ is closed under $\mathbf{I}^\delta\lceil D_\sqcap$. Similarly, $D_\sqcup$ is closed under $\mathbf{C}\lceil D_\sqcup$. That both $\mathbf{I}^\delta\lceil D_\sqcap$ and $\mathbf{C}\lceil D_\sqcup$ are additive and normal follows from Lemma 6(6,7) and Definition 8. $\square$

The following result addresses the converse of Theorem 17.

Theorem 18 Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot,)$ be a dBa such that $\mathfrak{D}_\sqcap := (D_\sqcap, \sqcap, \vee, \neg, \bot, \overline{\mathbf{I}})$ and $\mathfrak{D}_\sqcup := (D_\sqcup, \sqcup, \wedge, \lrcorner, \top, \overline{\mathbf{C}})$ are Baos. Then $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ is a dBao, where $\mathbf{I}(x) := \neg\overline{\mathbf{I}}(\neg x)$ and $\mathbf{C}(x) := \overline{\mathbf{C}}(x \sqcup x)$ for all $x \in D$.

Proof Let $x, y \in D$. Using Proposition 5(6), $\mathbf{I}(x \sqcap y) = \neg\overline{\mathbf{I}}(\neg(x \sqcap y)) = \neg\overline{\mathbf{I}}(\neg x \vee \neg y) = \neg(\overline{\mathbf{I}}(\neg x) \vee \overline{\mathbf{I}}(\neg y))$, as $\neg x, \neg y \in D_\sqcap$ by Proposition 5(1). As $\overline{\mathbf{I}}(\neg x), \overline{\mathbf{I}}(\neg y) \in D_\sqcap$, using definition of $\vee$ we have $\mathbf{I}(x \sqcap y) = \neg\overline{\mathbf{I}}(\neg x) \sqcap \neg\overline{\mathbf{I}}(\neg y) = \mathbf{I}(x) \sqcap \mathbf{I}(y)$. Using Proposition 5(5), $\mathbf{I}(\neg\bot) = \neg\overline{\mathbf{I}}(\neg\neg\bot) = \neg\overline{\mathbf{I}}(\bot) = \neg\bot$. By Definition 2(4a), $\mathbf{I}(x \sqcap x) = \neg\overline{\mathbf{I}}(\neg(x \sqcap x)) = \neg\overline{\mathbf{I}}(\neg x) = \mathbf{I}(x)$.
$\mathbf{C}(\lrcorner\top) = \overline{\mathbf{C}}(\lrcorner\top \sqcup \lrcorner\top) = \overline{\mathbf{C}}(\lrcorner\top) = \lrcorner\top$, as $\top \in D_\sqcup$. That $\mathbf{C}(x \sqcup x) = \mathbf{C}(x)$ is immediate from Definition 2. Finally, one shows that $\mathbf{C}(x \sqcup y) = \mathbf{C}(x) \sqcup \mathbf{C}(y)$ for all $x, y \in D$. Let $x, y \in D$. Using commutativity and associativity of $\sqcup$ and Definition 2(1b), additivity of $\overline{\mathbf{C}}$ and the fact that $x \sqcup x, y \sqcup y \in D_\sqcup$, we have the following equalities. $\mathbf{C}(x \sqcup y) = \overline{\mathbf{C}}((x \sqcup y) \sqcup (x \sqcup y)) = \overline{\mathbf{C}}((x \sqcup x) \sqcup (y \sqcup y)) = \overline{\mathbf{C}}(x \sqcup x) \sqcup \overline{\mathbf{C}}(y \sqcup y) = \mathbf{C}(x) \sqcup \mathbf{C}(y)$. So $\mathfrak{D}$ is a dBao. $\square$

We end this part by noting a close connection between the full complex algebra of a Kripke frame and that of a corresponding Kripke context. Let (W, R) be a Kripke frame and $\mathfrak{F}^+ := (\mathcal{P}(W), \cap, \cup, ^c, W, \emptyset, m_R)$ be the full complex algebra [19], where for all $A \in \mathcal{P}(W)$, $m_R(A) := \{w \in W : R(w) \cap A \neq \emptyset\} = \overline{A}^R$. This is a Bao, and as observed earlier, yields the dBao $(\mathcal{P}(W), \cap, \cup, ^c, W, \emptyset, m_R^\delta, m_R)$. For the Kripke frame (W, R), let us define the Kripke context $\mathbb{KC}_0 := ((W, R), (W, R), \neq)$. By Definition 6, we have the full complex algebra of $\mathbb{KC}_0$ as $\underline{\mathfrak{P}}^+(\mathbb{KC}_0) := (\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, f_1, f_2)$, where $f_1((A, B)) := (\underline{A}_R, (\underline{A}_R)')$, $f_2((A, B)) := ((\underline{B}_R)', \underline{B}_R)$ for all $(A, B) \in \mathfrak{P}(\mathbb{K})$. Then we get

Theorem 19 For the full complex algebra $\underline{\mathfrak{P}}^+(\mathbb{KC}_0)$, the following hold.

1. $\neg x = \lrcorner x$, $\neg\neg x = x$ and $f_1(x) = \neg f_2(\neg x)$ for all $x \in \mathfrak{P}(\mathbb{K})$.
2. $(\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \top, \bot, f_2)$ is a Bao, which is isomorphic to $\mathfrak{F}^+$.

Proof 1. Let $A \subseteq W$ and $x \in A^c$. Then for all $a \in A$, $x \neq a$, which implies that $x \in A'$. Now let $x \in A'$. Then $x \neq a$, for all $a \in A$, which implies that $x \in A^c$. So

$A' = A^c$, and $A'' = A^{c\prime} = A^{cc} = A$. Therefore $(A, B) \in \mathfrak{P}(\mathbb{K})$ if and only if $A = B^c$, which is equivalent to $A^c = B$.
Let $(A, A^c) \in \mathfrak{P}(\mathbb{K})$. Then $\neg(A, A^c) = (A^c, A) = \lrcorner(A, A^c)$ and $\neg\neg(A, A^c) = (A, A^c)$.
$f_2((A, A^c)) := (\underline{(A^c)'}_R, \underline{(A^c)}_R)$, giving
$\neg f_2(\neg(A, A^c)) = \neg f_2((A^c, A)) = \neg((\underline{A}_R)', \underline{A}_R) = ((\underline{A}_R)'^c, (\underline{A}_R)'^{c\prime}) = (\underline{A}_R, (\underline{A}_R)')$.
So $f_1((A, A^c)) = (\underline{A}_R, (\underline{A}_R)') = \neg f_2(\neg(A, A^c))$.
2. By Theorem 16 it follows that $(\mathfrak{P}(\mathbb{K}), \sqcup, \sqcap, \neg, \top, \bot, f_2)$ is a Bao.
Let us define a map f from $\mathcal{P}(W)$ to $\mathfrak{P}(\mathbb{K})$ by $f(A) := (A, A^c)$ for all $A \subseteq W$. It is clear that f is well-defined. To show f is a homomorphism, let $A, B \subseteq W$. $f(A \cap B) = (A \cap B, (A \cap B)^c) = (A, A^c) \sqcap (B, B^c) = f(A) \sqcap f(B)$ and $f(A \cup B) = (A \cup B, (A \cup B)^c) = (A, A^c) \sqcup (B, B^c) = f(A) \sqcup f(B)$. $f(A^c) = (A^c, A) = \neg f(A) = \lrcorner f(A)$ and $f(W) = (W, \emptyset) = \top$ $f(\emptyset) = (\emptyset, W) = \bot$. $f(m_R(A)) = (\overline{A}^R, (\overline{A}^R)^c) = ((\underline{A^c}_R)^c, \underline{A^c}_R) = f_2((A, A^c)) = f_2(f(A))$.
Injectivity and surjectivity of f follow trivially. □

From Theorem 19, we may conclude that the dBao $\underline{\mathfrak{P}}^+(\mathbb{K}\mathbb{C}_0)$ is identifiable with the Bao $\mathfrak{F}^+$.

4.1.1 Representation theorems for dBaos

For every dBao $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$, we construct a Kripke context based on the standard context $\mathbb{K}(\mathbf{D}) := (\mathcal{F}_p(\mathbf{D}), \mathcal{I}_p(\mathbf{D}), \Delta)$ corresponding to the underlying dBa $\mathbf{D}$. For that, relations R and S are defined on $\mathcal{F}_p(\mathbf{D})$ and $\mathcal{I}_p(\mathbf{D})$ respectively as follows.
For all $u, u_1 \in \mathcal{F}_p(\mathbf{D})$, uRu_1 if and only if $\mathbf{I}^\delta(a) \in u$ for all $a \in u_1$.
For all $v, v_1 \in \mathcal{I}_p(\mathbf{D})$, vSv_1 if and only if $\mathbf{C}^\delta(a) \in v$ for all $a \in v_1$.
The following results are required to get (Representation) Theorem 20.

Lemma 7 If F is a primary filter (ideal) of a dBa $\mathbf{D}$, then for any $x \in D$, exactly one of the elements x and $\neg x$ belongs to F.

Proof Proof follows from the definition of a primary filter (ideal). □

Lemma 8 Let $\mathfrak{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao. The following hold.

1. For all $u, u_1 \in \mathcal{F}_p(\mathbf{D})$, uRu_1 if and only if for all $a \in D$, $\mathbf{I}a \in u$ implies that $a \in u_1$.
2. For all $v, v_1 \in \mathcal{I}_p(\mathbf{D})$, vSv_1 if and only if for all $a \in D$, $\mathbf{C}a \in v$ implies that $a \in v_1$.

Proof 1. For all $a \in D$, suppose $\mathbf{I}a \in u$ implies that $a \in u_1$. If possible, assume $u\not R u_1$. Then there exists $a_1 \in u_1$ such that $\mathbf{I}^\delta(a_1) \notin u$. So $\neg\mathbf{I}^\delta(a_1) \in u$, which implies that $\mathbf{I}(\neg a_1) \in u$ by Lemma 6(2). As $a_1 \in u_1$, $\neg a_1 \notin u_1$, which contradicts that $\mathbf{I}(\neg a_1) \in u$. Hence uRu_1.

Now, we assume that uRu_1 and let $a_1 \in D$ such that $\mathbf{I}a_1 \in u$. If possible, suppose $a_1 \notin u_1$. Then $\neg a_1 \in u_1$. So $\mathbf{I}^\delta(\neg a_1) \in u$ as uRu_1. Therefore by Lemma 6, $\mathbf{I}^\delta(\neg a_1) = \neg\mathbf{I}(a_1) \in u$, which is a contradiction by Lemma 7. Hence $a_1 \in u_1$.
Proof of 2 is similar to the above. □

Lemma 9 Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be a dBa. For all $a, b \in D$, the following hold.

1. If $a \sqcap b = \bot$ then $a \sqcap a \sqsubseteq \neg b$.
2. If $a \sqcap a \sqsubseteq \neg b$ then $a \sqcap b \sqsubseteq \bot$.
3. If $a \sqcup b = \top$ then $\lrcorner b \sqsubseteq a \sqcup a$.
4. If $\lrcorner b \sqsubseteq a \sqcup a$ then $\top \sqsubseteq a \sqcup b$.

In particular, if $\mathbf{D}$ is a contextual dBa then $a \sqcap b = \bot$ if and only if $a \sqcap a \sqsubseteq \neg b$, and $a \sqcup b = \top$ if and only if $\lrcorner b \sqsubseteq a \sqcup a$.

Proof 1. Let $a, b \in D$ and $a \sqcap b = \bot$. Then by Definition 2(1a) and the associative law, $\bot = (a \sqcap a) \sqcap (b \sqcap b)$. So $\bot \vee \neg(b \sqcap b) = ((a \sqcap a) \sqcap (b \sqcap b)) \vee \neg(b \sqcap b)$. By Definition 2(6a), $\bot \vee \neg(b \sqcap b) = ((a \sqcap a) \vee \neg(b \sqcap b)) \sqcap ((b \sqcap b) \vee \neg(b \sqcap b))$. Now $(a \sqcap a) \vee \neg(b \sqcap b) = \neg(\neg(a \sqcap a) \sqcap \neg\neg(b \sqcap b)) = \neg(\neg a \sqcap (b \sqcap b))$ by Definition 2(4a) and Proposition 5(3). So $(a \sqcap a) \vee \neg(b \sqcap b) = \neg(\neg a \sqcap b)$ by Definition 2(1a). Similarly, we can show that $\bot \vee \neg(b \sqcap b) = \neg(b \sqcap \neg\bot)$. Therefore $\bot \vee \neg(b \sqcap b) = \neg(b \sqcap \neg\bot) = \neg(b \sqcap (\top \sqcap \top))$ by Definition 2(10a). Using Definition 2(1a) and Proposition 4(2), $\bot \vee \neg(b \sqcap b) = \neg(b \sqcap \top) = \neg(b \sqcap b) = \neg b$, where the last equality follows from Definition 2(4a). This implies that $\neg b = \neg(\neg a \sqcap b) \sqcap \neg\bot = \neg(\neg a \sqcap b)$, as $\neg(\neg a \sqcap b), b \sqcap b, \neg\bot \in D_\sqcap$. $\neg\neg a \sqsubseteq \neg(\neg a \sqcap b)$, as $\neg a \sqcap b \sqsubseteq \neg a$. So $a \sqcap a \sqsubseteq \neg(\neg a \sqcap b) = \neg b$.
2. Let $a \sqcap a \sqsubseteq \neg b$. Then $a \sqcap a \sqcap b \sqsubseteq \neg b \sqcap b$ by Proposition 4(6) and by Definition 2(1a), $a \sqcap b \sqsubseteq \bot$.
Now if $\mathbf{D}$ is a contextual dBa then $\sqsubseteq$ becomes a partial order. Therefore from the above it follows that $a \sqcap b = \bot$ if and only if $a \sqcap a \sqsubseteq \neg b$.
The other parts can be proved dually. □

Lemma 10 Let $\mathfrak{O}$ be a dBao and $\mathbb{KC}(\mathfrak{O}) := ((\mathcal{F}_p(\mathbf{D}), R), (\mathcal{I}_p(\mathbf{D}), S), \Delta)$. Then for all $a \in D$ the following hold.

1. $\overline{F_a}^R = F_{\mathbf{I}^\delta(a)}$ and $\underline{F_a}_R = F_{\mathbf{I}(a)}$.
2. $\overline{I_a}^S = I_{\mathbf{C}^\delta(a)}$ and $\underline{I_a}_S = I_{\mathbf{C}(a)}$.

Proof 1. Let $F \in \overline{F_a}^R$. Then there exists $F_1 \in F_a$ such that FRF_1, which implies that $\mathbf{I}^\delta(a) \in F$, as $a \in F_1$. So $\overline{F_a}^R \subseteq F_{\mathbf{I}^\delta(a)}$.
Let $F \in F_{\mathbf{I}^\delta(a)}$ and we show that $F \in \overline{F_a}^R$. We must then find a primary filter $F_1 \in F_a$ such that FRF_1. Let $F_0 := \{x \in D : \mathbf{I}x \in F\}$ and $F_{01} := \{x \sqcap a : x \in F_0\}$. Then F_{01} is closed under $\sqcap$ and $F_{01} \subseteq D_\sqcap$. Next we show that $\bot \notin F_{01}$. If possible, suppose $\bot \in F_{01}$. Then there exists $x_1 \in F_0$ such that $x_1 \sqcap a = \bot$, which implies that $a \sqcap a \sqsubseteq \neg x_1$ by Lemma 9(1). So $\mathbf{I}^\delta(a \sqcap a) \sqsubseteq \mathbf{I}^\delta(\neg x_1)$, whence $\mathbf{I}^\delta(a) \sqsubseteq \mathbf{I}^\delta(\neg x_1)$ by Lemma 6(4,5). $\mathbf{I}^\delta(\neg x_1) \in F$, as $\mathbf{I}^\delta(a) \in F$ and F is a filter, which implies that

$\neg\mathbf{I}(x_1) \in F$. So $\mathbf{I}(x_1) \notin F$ which contradicts that $x_1 \in F_0$. Therefore $\bot \notin F_{01}$. Since $\mathbf{D}_\sqcap$ is a Boolean algebra and $F_{01} \subseteq D_\sqcap$, there exists a prime filter F_2 containing F_{01}. So $F_3 := \{x \in D : y \sqsubseteq x \text{ for some } y \in F_2\}$ is a primary filter containing F_2 by Lemma 2 and Proposition 6. For all $x \in F_0$, $x \sqcap a \in F_{01} \subseteq F_2$ and $x \sqcap a \sqsubseteq x, x \sqcap a \sqsubseteq a$, which implies that $F_0 \subseteq F_3$ and $a \in F_3$. By Lemma 8(1) it follows that FRF_3. Therefore $F \in \overline{F_a}^R$.
Using Proposition 11(i), Lemmas 4 and 6(1), we get
$\underline{F_a}_R = ((\overline{F_a^c})^R)^c = ((\overline{F_{\neg a}})^R)^c = F^c_{\mathbf{I}^\delta(\neg a)} = F_{\neg\mathbf{I}^\delta(\neg a)} = F_{\mathbf{I}(a)}$.
2 can be proved dually. $\square$

The Kripke context $\mathbb{KC}(\mathfrak{O})$ of Lemma 10 is used to obtain the representation theorem.

Theorem 20 (Representation theorem) Let $\mathfrak{O} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ be a dBao. The following hold.

1. $\mathfrak{O}$ is quasi-embeddable into the full complex algebra $\underline{\mathfrak{P}}^+(\mathbb{KC}(\mathfrak{O}))$ of the Kripke context $\mathbb{KC}(\mathfrak{O})$. $h : D \to \mathfrak{P}(\mathbb{K}(\mathbf{D}))$ defined by $h(x) := (F_x, I_x)$ for all $x \in D$, is the required quasi-embedding.
2. If $\mathfrak{O}$ is a contextual dBao then the quasi-embedding h is an embedding.
3. $\mathfrak{O}_p$ is embeddable into the largest pure subalgebra $\underline{\mathfrak{H}}^+(\mathbb{KC}(\mathfrak{O}))$ of $\underline{\mathfrak{P}}^+(\mathbb{KC}(\mathfrak{O}))$.

Proof 1. Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top, \bot)$ be the underlying dBa. By Theorem 9, we know that the map $h : D \to \mathfrak{P}(\mathbb{K}(\mathbf{D}))$ defined by $h(x) := (F_x, I_x)$ for all $x \in D$ is a quasi-embedding. To show h is a dBao homomorphism, we prove that for any $x \in D$, $h(\mathbf{I}x) = f_R(h(x))$ and $h(\mathbf{C}x) = f_S(h(x))$, that is, $(F_{\mathbf{I}x}, I_{\mathbf{I}x}) = (\underline{F_x}_R, (\underline{F_x}_R)')$ and $(F_{\mathbf{C}x}, I_{\mathbf{C}x}) = ((\underline{I_x}_S)', \underline{I_x}_S)$. By Lemma 10(1), $\underline{F_x}_R = F_{\mathbf{I}x}$. By Lemma 5, $F'_{\mathbf{I}x} = I_{\mathbf{I}x_{\sqcap\sqcup}} = I_{(\mathbf{I}x\sqcap\mathbf{I}x)\sqcup(\mathbf{I}x\sqcap\mathbf{I}x)} = I_{\mathbf{I}x\sqcup\mathbf{I}x} = I_{\mathbf{I}x}$, the last two equalities hold, as $\mathbf{I}x \sqcap \mathbf{I}x = \mathbf{I}(x \sqcap x) = \mathbf{I}x$ and by Lemma 4(1). So $(\underline{F_x}_R)' = I_{\mathbf{I}x}$.
Similar to the above, using Lemma 10(2) and Lemma 5, we can show that $(F_{\mathbf{C}x}, I_{\mathbf{C}x}) = ((\underline{I_x}_S)', \underline{I_x}_S)$. Hence h is the required quasi-embedding from the dBao $\mathfrak{O}$ into $\underline{\mathfrak{P}}^+(\mathbb{KC}(\mathfrak{O}))$.
2. Since $\mathfrak{O}$ is contextual, the quasi-order is a partial order. As a result, h becomes injective.
3. Let $x \in D_p$. Then either $x \sqcap x = x$ or $x \sqcup x = x$. If $x \sqcap x = x$, $h(x) = (F_x, I_x) = (F_x, F'_x)$, by Lemmas 4 and 5. Now if $x \sqcup x = x$, $h(x) = (F_x, I_x) = (I'_x, I_x)$, by Lemmas 4 and 5. So $h\restriction D_p$ is an injective dBao homomorphism from $\mathfrak{O}_p$ to $\underline{\mathfrak{H}}^+(\mathbb{KC}(\mathfrak{O}))$, as $\mathfrak{O}_p$ is pure and by Proposition 2. $\square$

Corollary 6 Let $\mathfrak{O}$ be a pure dBao. Then $\mathfrak{O}$ is embeddable into the complex algebra $\underline{\mathfrak{H}}^+(\mathbb{KC}(\mathfrak{O}))$ of the Kripke context $\mathbb{KC}(\mathfrak{O})$.

Proof Proof follows from Theorems 14(2) and 20(3). $\square$

4.2 Topological double Boolean algebras

Definition 9 A dBao $\mathfrak{D} := (D, \sqcap, \sqcup, \neg, \lrcorner, \top, \bot, \mathbf{I}, \mathbf{C})$ is called a *topological dBa* if the following hold.

4*a* $\mathbf{I}(x) \sqsubseteq x$ 4*b* $x \sqsubseteq \mathbf{C}(x)$
5*a* $\mathbf{II}(x) = \mathbf{I}(x)$ 5*b* $\mathbf{CC}(x) = \mathbf{C}(x)$

A *topological contextual dBa* is a topological dBa in which the underlying dBa is contextual. If the underlying dBa is pure, the topological dBa is called a *topological pure dBa.*

Again, as intended, we obtain a class of examples of topological dBas from the sets of protoconcepts and semiconcepts of contexts.

Theorem 21 Let $\mathbb{KC} := ((G, R), (M, S), I)$ be a reflexive and transitive Kripke context. Then the following hold.

1. $\underline{\mathfrak{P}}^{+}(\mathbb{KC})$ is a topological contextual dBa.
2. $\underline{\mathfrak{P}}^{+}(\mathbb{KC})_p = \underline{\mathfrak{H}}^{+}(\mathbb{KC})$ is a topological pure dBa.

Proof 1. Proof follows from Theorems 15 and 13.
2. Proof is similar to the proof of Theorem 15(2). □

Now, we will show that for a topological dBa $\mathfrak{D}$, $\mathbb{KC}(\mathfrak{D})$ is a reflexive and transitive Kripke context. For that, we first prove the following lemma.

Lemma 11 Let $\mathfrak{D}$ be a topological dBa. Then for all $a \in D$, $\mathbf{I}^{\delta}\mathbf{I}^{\delta}(a) = \mathbf{I}^{\delta}(a)$ and $\mathbf{C}^{\delta}\mathbf{C}^{\delta}(a) = \mathbf{C}^{\delta}(a)$.

Proof Let $a \in D$. By Definition 9(5a), $\mathbf{II}(\neg a) = \mathbf{I}(\neg a)$, which implies that $\neg\mathbf{II}(\neg a) = \neg\mathbf{I}(\neg a)$. By Lemma 6(2), $\mathbf{I}^{\delta}(\neg\mathbf{I}\neg a) = \mathbf{I}^{\delta}(a)$, whence $\mathbf{I}^{\delta}\mathbf{I}^{\delta}(a) = \mathbf{I}^{\delta}(a)$. Similarly, we can show that $\mathbf{C}^{\delta}\mathbf{C}^{\delta}(a) = \mathbf{C}^{\delta}(a)$. □

We now have

Theorem 22 $\mathbb{KC}(\mathfrak{D}) := ((\mathcal{F}_p(\mathbf{D}), R), (\mathcal{I}_p(\mathbf{D}), S), \Delta)$ is a reflexive and transitive Kripke context.

Proof To show R is reflexive, let $F \in \mathcal{F}_p(\mathbf{D})$ and $\mathbf{I}a \in F$ for some $a \in D$. By Definition 9(4a), $\mathbf{I}a \sqsubseteq a$, which implies that $a \in F$, as F is a filter. So FRF by Lemma 8.
To show R is transitive, let $F, F_1, F_2 \in \mathcal{F}_p(\mathfrak{D})$ such that FRF_1 and F_1RF_2. We show that FRF_2. Let $a \in F_2$. Then $\mathbf{I}^{\delta}(a) \in F_1$, as F_1RF_2, which implies that $\mathbf{I}^{\delta}\mathbf{I}^{\delta}(a) \in F$, as FRF_1. So $\mathbf{I}^{\delta}(a) = \mathbf{I}^{\delta}\mathbf{I}^{\delta}(a) \in F$, using Lemma 11. Thus FRF_2.
Similarly, one can show that S is reflexive and transitive. □

Combining Theorem 20, Corollary 6 and Theorem 22, we get the representation results for topological dBas in terms of reflexive and transitive Kripke contexts.

Theorem 23 A topological dBa $\mathfrak{O}$ is quasi-embeddable into the full complex algebra $\underline{\mathfrak{P}}^{+}(\mathbb{KC}(\mathfrak{O}))$ of the reflexive and transitive Kripke context $\mathbb{KC}(\mathfrak{O})$.
$\mathfrak{O}_p$ is embeddable into the complex algebra $\underline{\mathfrak{H}}^{+}(\mathbb{KC}(\mathfrak{O}))$ of $\mathbb{KC}(\mathfrak{O})$. Moreover,

1. If $\mathfrak{O}$ is a topological contextual dBa then $\mathfrak{O}$ is embeddable into $\underline{\mathfrak{P}}^{+}(\mathbb{KC}(\mathfrak{O}))$.
2. If $\mathfrak{O}$ is a topological pure dBa then $\mathfrak{O}$ is embeddable into the complex algebra $\underline{\mathfrak{H}}^{+}(\mathbb{KC}(\mathfrak{O}))$ of $\mathbb{KC}(\mathfrak{O})$.

5 Logics corresponding to the algebras

We next formulate the logic **CDBL** for contextual dBas. The logic **MCDBL** for the class of contextual dBaos, and its extension **MCDBL4** for topological contextual dBas are both defined with **CDBL** as their base. In Section 5.3, it is shown that, apart from the algebraic semantics, the logics can be imparted a protoconcept-based semantics, due to the representation theorems for the respective classes of algebras obtained in Sections 4.

5.1 CDBL

The language $\mathfrak{L}$ of **CDBL** consists of a countably infinite set **PV** of propositional variables, propositional constants $\bot, \top$, and logical connectives $\sqcup, \sqcap, \neg, \lrcorner$. The set $\mathfrak{F}$ of formulae is given by the following scheme:

$$\top \mid \bot \mid p \mid \alpha \sqcup \beta \mid \alpha \sqcap \beta \mid \neg\alpha \mid \lrcorner\alpha,$$

where $p \in$ **PV**. $\vee$ and $\wedge$ are definable connectives: $\alpha \vee \beta := \neg(\neg\alpha \sqcap \neg\beta)$ and $\alpha \wedge \beta := \lrcorner(\lrcorner\alpha \sqcup \lrcorner\beta)$ for all $\alpha, \beta \in \mathfrak{F}$. A *sequent* in **CDBL** is a pair of formulae denoted by $\alpha \vdash \beta$ for $\alpha, \beta \in \mathfrak{F}$. If $\alpha \vdash \beta$ and $\beta \vdash \alpha$, we use the abbreviation $\alpha \dashv\vdash \beta$.

The *axioms* of **CDBL** are given by the following schema.

1 $\alpha \vdash \alpha$.

Axioms for $\sqcap$ *and* $\sqcup$:

2a $\alpha \sqcap \beta \vdash \alpha$	2b $\alpha \vdash \alpha \sqcup \beta$
3a $\alpha \sqcap \beta \vdash \beta$	3b $\beta \vdash \alpha \sqcup \beta$
4a $\alpha \sqcap \beta \vdash (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta)$	4b $(\alpha \sqcup \beta) \sqcup (\alpha \sqcup \beta) \vdash \alpha \sqcup \beta$

Axioms for $\neg$ *and* $\lrcorner$:

5a $\neg(\alpha \sqcap \alpha) \vdash \neg\alpha$	5b $\lrcorner\alpha \vdash \lrcorner(\alpha \sqcup \alpha)$
6a $\alpha \sqcap \neg\alpha \vdash \bot$	6b $\top \vdash \alpha \sqcup \lrcorner\alpha$
7a $\neg\neg(\alpha \sqcap \beta) \dashv\vdash (\alpha \sqcap \beta)$	7b $\lrcorner\lrcorner(\alpha \sqcup \beta) \dashv\vdash (\alpha \sqcup \beta)$

Generalization of the law of absorption:

$8a$ $\alpha \sqcap \alpha \vdash \alpha \sqcap (\alpha \sqcup \beta)$ $8b$ $\alpha \sqcup (\alpha \sqcap \beta) \vdash \alpha \sqcup \alpha$
$9a$ $\alpha \sqcap \alpha \vdash \alpha \sqcap (\alpha \vee \beta)$ $9b$ $\alpha \sqcup (\alpha \wedge \beta) \vdash \alpha \sqcup \alpha$

Laws of distribution:
$10a$ $\alpha \sqcap (\beta \vee \gamma) \dashv\vdash (\alpha \sqcap \beta) \vee (\alpha \sqcap \gamma)$ $10b$ $\alpha \sqcup (\beta \wedge \gamma) \dashv\vdash (\alpha \sqcup \beta) \wedge (\alpha \sqcup \gamma)$

Axioms for $\bot, \top$:
$11a$ $\bot \vdash \alpha$ $\quad$ $11b$ $\alpha \vdash \top$
$12a$ $\neg\top \vdash \bot$ $\quad$ $12b$ $\top \vdash \lrcorner\bot$
$13a$ $\neg\bot \dashv\vdash \top \sqcap \top$ $\quad$ $13b$ $\lrcorner\top \dashv\vdash \bot \sqcup \bot$

The compatibility axiom:
14 $(\alpha \sqcup \alpha) \sqcap (\alpha \sqcup \alpha) \dashv\vdash (\alpha \sqcap \alpha) \sqcup (\alpha \sqcap \alpha)$

Rules of inference of **CDBL** are as follows.
For $\sqcap$ *and* $\sqcup$:

$$\frac{\alpha \vdash \beta}{\alpha \sqcap \gamma \vdash \beta \sqcap \gamma}\ (R1) \quad \frac{\alpha \vdash \beta}{\gamma \sqcap \alpha \vdash \gamma \sqcap \beta}\ (R1)'$$
$$\frac{\alpha \vdash \beta}{\alpha \sqcup \gamma \vdash \beta \sqcup \gamma}\ (R2) \quad \frac{\alpha \vdash \beta}{\gamma \sqcup \alpha \vdash \gamma \sqcup \beta}\ (R2)'$$

For $\neg, \lrcorner$:

$$\frac{\alpha \vdash \beta}{\neg\beta \vdash \neg\alpha}\ (R3) \quad \frac{\alpha \vdash \beta}{\lrcorner\beta \vdash \lrcorner\alpha}\ (R3)'$$

Transitivity:

$$\frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}\ (R4)$$

Order:

$$\frac{\alpha \sqcap \beta \vdash \alpha \sqcap \alpha \quad \alpha \sqcap \alpha \vdash \alpha \sqcap \beta \quad \alpha \sqcup \beta \vdash \beta \sqcup \beta \quad \beta \sqcup \beta \vdash \alpha \sqcup \beta}{\alpha \vdash \beta}(R5)$$

$(R5)$ captures the order relation of the contextual dBas.

Derivability is defined in the standard manner: a sequent S is *derivable* (or *provable*) in **CDBL**, if there exists a finite sequence of sequents $S_1, \ldots, S_m$ such that S_m is the sequent S and for all $k \in \{1, \ldots, m\}$ either S_k is an axiom or S_k is obtained by applying rules of **CDBL** to elements from $\{S_1, \ldots, S_{k-1}\}$. Let us give a few examples of derived rules and sequents.

Proposition 24 The following rules are derivable in **CDBL**.

$$\frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \sqcap \alpha \vdash \beta \sqcap \gamma}\ (R6) \qquad \frac{\beta \vdash \alpha \quad \gamma \vdash \alpha}{\beta \sqcup \gamma \vdash \alpha \sqcup \alpha}\ (R7)$$

Proof $(R6)$ is derived using $(R1), (R1)'$ and $(R4)$, while for $(R7)$ one uses $(R2), (R2)'$ and $(R4)$. □

Theorem 25 For $\alpha, \beta, \gamma \in \mathfrak{F}$, the following are provable in **CDBL**.

$1a$ $(\alpha \sqcap \beta) \dashv\vdash (\beta \sqcap \alpha)$.	$1b$ $\alpha \sqcup \beta \dashv\vdash \beta \sqcup \alpha$.
$2a$ $\alpha \sqcap (\beta \sqcap \gamma) \dashv\vdash (\alpha \sqcap \beta) \sqcap \gamma$.	$2b$ $\alpha \sqcup (\beta \sqcup \gamma) \dashv\vdash (\alpha \sqcup \beta) \sqcup \gamma$.
$3a$ $(\alpha \sqcap \alpha) \sqcap \beta \dashv\vdash (\alpha \sqcap \beta)$.	$3b$ $(\alpha \sqcup \alpha) \sqcup \beta \dashv\vdash \alpha \sqcup \beta$.
$4a$ $\neg\alpha \vdash \neg(\alpha \sqcap \alpha)$.	$4b$ $\lrcorner(\alpha \sqcup \alpha) \vdash \lrcorner\alpha$.
$5a$ $\alpha \sqcap (\alpha \sqcup \beta) \vdash (\alpha \sqcap \alpha)$.	$5b$ $\alpha \sqcup \alpha \vdash \alpha \sqcup (\alpha \sqcap \beta)$.
$6a$ $\alpha \sqcap (\alpha \vee \beta) \vdash \alpha \sqcap \alpha$.	$6b$ $\alpha \sqcup \alpha \vdash \alpha \sqcup (\alpha \wedge \beta)$.
$7a$ $\bot \vdash \alpha \sqcap \neg\alpha$.	$7b$ $\alpha \sqcup \lrcorner\alpha \vdash \top$.
$8a$ $\bot \vdash \neg\top$.	$8b$ $\lrcorner\bot \vdash \top$.

Proof The proofs are straightforward and one makes use of axioms 2a, 3a, 4a, Proposition 24 and the rule $(R4)$ in most cases. For instance, here is a proof for $1a$:

$$\dfrac{4a\ (\alpha \sqcap \beta) \vdash (\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \qquad \dfrac{3a\ \alpha \sqcap \beta \vdash \beta \quad \alpha \sqcap \beta \vdash \alpha\ 2a}{(\alpha \sqcap \beta) \sqcap (\alpha \sqcap \beta) \vdash \beta \sqcap \alpha}\ (R6)}{(\alpha \sqcap \beta) \vdash (\beta \sqcap \alpha)}\ (R4)$$

Interchanging α and β in the above, we get $(\beta \sqcap \alpha) \vdash (\alpha \sqcap \beta)$.
$(4a)$ follows from axiom 2a and $(R3)$. $(7a), (8a)$ follow from axiom 11a. The remaining proofs are given in the Appendix. Note that the proofs of $(ib), i = 1, 2, 3, 4, 5, 6, 7, 8$, are obtained using the axioms and rules dual to those used to derive (ia). □

Definitions of valuations on the algebras and satisfaction of sequents are as follows.

Definition 10 Let $\mathbf{D} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top_D, \bot_D)$ be a contextual dBa. A *valuation* $v : \mathfrak{F} \to D$ on $\mathbf{D}$ is a map such that for all $\alpha, \beta \in \mathfrak{F}$ the following hold.

1. $v(\alpha \sqcup \beta) := v(\alpha) \sqcup v(\beta)$.	4. $v(\alpha \sqcap \beta) := v(\alpha) \sqcap v(\beta)$.
2. $v(\lrcorner\alpha) := \lrcorner v(\alpha)$.	5. $v(\neg\alpha) := \neg v(\alpha)$.
3. $v(\top) := \top_D$.	6. $v(\bot) := \bot_D$.

Definition 11 *A sequent* $\alpha \vdash \beta$ is said to be *satisfied* by a valuation v on a contextual dBa $\mathbf{D}$ if and only if $v(\alpha) \sqsubseteq v(\beta)$. $\alpha \vdash \beta$ is *true* in $\mathbf{D}$ if and only if for all valuations v on $\mathbf{D}$, v satisfies $\alpha \vdash \beta$. $\alpha \vdash \beta$ is *valid* in the class of all contextual dBas if and only if it is true in every contextual dBa.

Theorem 26 (Soundness) If a sequent $\alpha \vdash \beta$ is provable in **CDBL** then it is valid in the class of all contextual dBas.

Proof The proof that all the axioms of **CDBL** are valid in the class of all contextual dBas is straightforward and can be obtained using Proposition 4 and Definition 2. One then needs to verify that the rules of inference preserve validity. Using Proposition 4, one can show that $(R1), (R2), (R1)'$ and $(R2)'$ preserve validity. The cases for $(R3)$ and $(R3)'$ follow from Proposition 5.

To show $(R5)$ preserves validity, let the sequents $\alpha \sqcap \beta \vdash \alpha \sqcap \alpha, \alpha \sqcap \alpha \vdash \alpha \sqcap \beta, \alpha \sqcup \beta \vdash \beta \sqcup \beta$, and $\beta \sqcup \beta \vdash \alpha \sqcup \beta$ be valid in the class of all contextual dBas. Let $\mathbf{D}$ be a contextual dBa and v a valuation in $\mathbf{D}$. Then v satisfies each sequent, which implies that $v(\alpha\sqcap\beta) \sqsubseteq v(\alpha\sqcap\alpha), v(\alpha\sqcap\alpha) \sqsubseteq v(\beta\sqcap\alpha), v(\alpha\sqcup\beta) \sqsubseteq v(\beta\sqcap\beta)$ and $v(\beta\sqcup\beta) \sqsubseteq v(\alpha\sqcup\beta)$. So $v(\alpha \sqcap \beta) = v(\alpha \sqcap \alpha)$ and $v(\alpha \sqcup \beta) = v(\beta \sqcup \beta)$, as **D** is contextual. This gives $v(\alpha) \sqcap v(\beta) = v(\alpha) \sqcap v(\alpha)$ and $v(\alpha) \sqcup v(\beta) = v(\beta) \sqcup v(\beta)$. Thus $v(\alpha) \sqsubseteq v(\beta)$, whence $\alpha \vdash \beta$ is satisfied by v. $\square$

As usual, the completeness theorem is proved using the Lindenbaum-Tarski algebra of **CDBL**, and the algebra is constructed in the standard way as follows. A relation $\equiv_\vdash$ is defined on $\mathfrak{F}$ by: $\alpha \equiv_\vdash \beta$ if and only if $\alpha \dashv\vdash \beta$, for $\alpha, \beta \in \mathfrak{F}$. $\equiv_\vdash$ is a congruence relation on $\mathfrak{F}$ with respect to $\sqcup$, $\sqcap$, $\neg$, $\lrcorner$. The quotient set $\mathfrak{F}/\equiv_\vdash$ with operations induced by the logical connectives, give the Lindenbaum-Tarski algebra $\mathcal{L}(\mathfrak{F}) := (\mathfrak{F}/\equiv_\vdash, \sqcup, \sqcap, \neg, \lrcorner, [\top], [\bot])$. The axioms in **CDBL** and Theorem 25 ensure that $\mathcal{L}(\mathfrak{F})$ is a dBa. One then has

Proposition 27 For any formula α and β the following are equivalent.

1. $\alpha \vdash \beta$ is provable in **CDBL**.
2. $[\alpha] \sqsubseteq [\beta]$ in $\mathcal{L}(\mathfrak{F})$ of **CDBL**.

Proof For 1 $\implies$ 2, we make use of $(R1)'$, $(R4)$, axiom 2a and Theorem 25(2a, 3a).

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \vdash \beta}{\alpha \sqcap \alpha \vdash \alpha \sqcap \beta}}{\alpha \sqcap \beta \vdash \alpha}}{\alpha \sqcap (\alpha \sqcap \beta) \vdash \alpha \sqcap \alpha \quad \alpha \sqcap \beta \vdash \alpha \sqcap (\alpha \sqcap \beta)}}{\alpha \sqcap \beta \vdash \alpha \sqcap \alpha}$$

So $\alpha \sqcap \alpha \dashv\vdash \alpha \sqcap \beta$, which implies that $[\alpha] \sqcap [\alpha] = [\alpha \sqcap \alpha] = [\alpha \sqcap \beta] = [\alpha] \sqcap [\beta]$. Dually we can show that $[\alpha] \sqcup [\beta] = [\beta] \sqcup [\beta]$. Therefore $[\alpha] \sqsubseteq [\beta]$.
For 2 $\implies$ 1, suppose $[\alpha] \sqsubseteq [\beta]$. Then $[\alpha]\sqcap[\beta] = [\alpha]\sqcap[\alpha]$. So $[\alpha\sqcap\beta] = [\alpha\sqcap\alpha]$. Similarly we can show that $[\alpha \sqcup \beta] = [\beta \sqcup \beta]$. Therefore $\alpha \sqcap \beta \dashv\vdash \alpha \sqcap \alpha$ and $\alpha \sqcup \beta \dashv\vdash \beta \sqcup \beta$. Now using (R5), $\alpha \vdash \beta$. $\square$

It is worth noting that the axioms of **CDBL** are obtained by converting the dBa axioms into sequents. Nonetheless, the system is complete with respect to the class of contextual dBas, because the relation $\equiv_\vdash$ provides a partial order on the set $\mathfrak{F}/\equiv_\vdash$, which forces the Lindenbaum algebra $\mathcal{L}(\mathfrak{F})$ to become a contextual dBa.

Theorem 28 $\mathcal{L}(\mathfrak{F})$ is a contextual dBa.

Proof Follows directly by axiom 1, (R4) and Proposition 27. $\square$

The canonical map $v_0 : \mathfrak{F} \to \mathfrak{F}/\equiv_\vdash$ defined by $v_0(\gamma) := [\gamma]$ for all $\gamma \in \mathfrak{F}$, can be shown to be a valuation on $\mathcal{L}(\mathfrak{F})$.

Theorem 29 (Completeness) If a sequent $\alpha \vdash \beta$ is valid in the class of all contextual dBas then it is provable in **CDBL**.

Proof If $\alpha \vdash \beta$ be valid in the class of all contextual dBas, it is true in $\mathcal{L}(\mathfrak{F})$. Consider the canonical valuation v_0. Then $v_0(\alpha) \sqsubseteq v_0(\beta)$ and so $[\alpha] \sqsubseteq [\beta]$. By Proposition 27, it follows that $\alpha \vdash \beta$ is provable in **CDBL**. $\square$

5.2 MCDBL and MCDBL4

The language $\mathfrak{L}_1$ of **MCDBL** adds two unary modal connectives $\Box$ and $\blacksquare$ to the language $\mathfrak{L}$ of **CDBL**. The formulae are given by the following scheme.

$$\top \mid \bot \mid p \mid \alpha \sqcup \beta \mid \alpha \sqcap \beta \mid \neg\alpha \mid \lrcorner\alpha \mid \Box\alpha \mid \blacksquare\alpha,$$

where $p \in \mathbf{PV}$. The set of formulae is denoted by $\mathfrak{F}_1$. The axiom schema for **MCDBL** consists of all the axioms of **CDBL** and the following.

$15a$ $\Box\alpha \sqcap \Box\beta \dashv\vdash \Box(\alpha \sqcap \beta)$	$15b$ $\blacksquare\alpha \sqcup \blacksquare\beta \dashv\vdash \blacksquare(\alpha \sqcup \beta)$
$16a$ $\Box(\neg\bot) \dashv\vdash \neg\bot$	$16b$ $\blacksquare(\lrcorner\top) \dashv\vdash \lrcorner\top$
$17a$ $\Box(\alpha \sqcap \alpha) \dashv\vdash \Box(\alpha)$	$17b$ $\blacksquare(\alpha \sqcup \alpha) \dashv\vdash \blacksquare(\alpha)$

Rules of inference: All the rules of **CDBL** and the following.

$$\frac{\alpha \vdash \beta}{\Box\alpha \vdash \Box\beta}\ (R8) \qquad \frac{\alpha \vdash \beta}{\blacksquare\alpha \vdash \blacksquare\beta}\ (R9)$$

Definable modal operators are $\Diamond, \blacklozenge$, given by $\Diamond\alpha := \neg\Box\neg\alpha$ and $\blacklozenge\alpha := \lrcorner\blacksquare\lrcorner\alpha$. It is immediate that

Theorem 30 If a sequent $\alpha \vdash \beta$ is provable in **CDBL** then it is provable in **MCDBL**.

A valuation v on a contextual dBao $\mathfrak{O} := (D, \sqcup, \sqcap, \neg, \lrcorner, \top_D, \bot_D, \mathbf{I}, \mathbf{C})$, is a map from $\mathfrak{F}_1$ to D that satisfies the conditions in Definition 10 and the following for the modal operators:

Definition 12 $v(\Box\alpha) := \mathbf{I}(v(\alpha))$ and $v(\blacksquare\alpha) := \mathbf{C}(v(\alpha))$.

Definitions of satisfaction, truth and validity of sequents are given in a similar manner as before.

5.2.1 MCDBLΣ

MCDBL4 is obtained as a special case of the logic **MCDBL**Σ that is defined as follows.

Definition 13 Let Σ be any set of sequents in **MCDBL**. **MCDBL**Σ is the logic obtained from **MCDBL** by adding all the sequents in Σ as axioms.

Note that if $\Sigma = \emptyset$ then **MCDBL**Σ is the same as **MCDBL**. The set Σ required to define **MCDBL4** will be given at the end of this section. Let us briefly discuss some features of **MCDBL**Σ for any Σ – these would then apply to both **MCDBL** and **MCDBL4**.

Let V_Σ denote the class of those contextual dBaos in which the sequents of Σ are valid. As a consequence of axioms 15a, 16a, 17a, 15a, 16b, 17b and rules $(R8)$, $(R9)$, one can conclude that if a sequent $\alpha \vdash \beta$ is provable in **MCDBL**Σ then it is valid in the class V_Σ.

As before, one has the Lindenbaum-Tarski algebra $\mathcal{L}_\Sigma(\mathfrak{F}_1)$ for **MCDBL**Σ; it has additional unary operators induced by the modal operators in $\mathfrak{L}_1$. More precisely, $\mathcal{L}_\Sigma(\mathfrak{F}_1) := (\mathfrak{F}_1/{\equiv_\vdash}, \sqcup, \sqcap, \neg, \lrcorner, [\top], [\bot], f_\square, f_\blacksquare)$, where $f_\square, f_\blacksquare$ are defined as: $f_\square([\alpha]) := [\square\alpha]$, $f_\blacksquare([\alpha]) := [\blacksquare\alpha]$.

Proposition 27 extends to this case. Using this proposition and rules $(R8)$, $(R9)$, one shows that the operators $f_\square, f_\blacksquare$ are monotonic:

Lemma 12 For $\alpha, \beta \in \mathfrak{F}_1$, $[\alpha] \sqsubseteq [\beta]$ in $\mathcal{L}_\Sigma(\mathfrak{F}_1)$ implies that $f_\square([\alpha]) \sqsubseteq f_\square([\beta])$ and $f_\blacksquare([\alpha]) \sqsubseteq f_\blacksquare([\beta])$.

$(\mathfrak{F}_1/{\equiv_\vdash}, \sqcup, \sqcap, \neg, \lrcorner, [\top], [\bot])$ is a contextual dBa; Lemma 12 along with axioms 16a, 16b, 17a, 17b and the result corresponding to Proposition 27 give

Theorem 31 $\mathcal{L}_\Sigma(\mathfrak{F}_1) \in V_\Sigma$.

One then gets in the standard manner,

Theorem 32 (Completeness) If a sequent $\alpha \vdash \beta$ is valid in the class V_Σ then it is provable in **MCDBL**Σ.

MCDBL4 is defined as the logic **MCDBL**Σ where Σ contains the following:

$$18a \; \square\alpha \vdash \alpha \qquad 18b \; \alpha \vdash \blacksquare\alpha$$
$$19a \; \square\square\alpha \dashv\vdash \square\alpha \quad 19b \; \blacksquare\blacksquare\alpha \dashv\vdash \blacksquare\alpha$$

We have thus obtained

Theorem 33 (Soundness and Completeness)

1. $\alpha \vdash \beta$ is provable in **MCDBL** if and only if $\alpha \vdash \beta$ is valid in the class of all contextual dBaos.
2. $\alpha \vdash \beta$ is provable in **MCDBL4** if and only if $\alpha \vdash \beta$ is valid in the class of all topological contextual dBas.

5.3 Protoconcept-based semantics for the logics

As a consequence of the representation result for contextual dBas (Corollary 1), we get another semantics for **CDBL** based on the sets of protoconcepts of contexts. The required basic definitions are derivable from those given in Section 5. However, for the sake of completeness, these are mentioned here. We first define valuations, models and satisfaction for a context $\mathbb{K} := (G, M, I)$.
Valuations associate formulae with protoconcepts of $\mathbb{K}$:
A *valuation* is a map $v : \mathfrak{F} \to \mathfrak{P}(\mathbb{K})$ such that

$v(\alpha \sqcup \beta) := v(\alpha) \sqcup v(\beta)$. $v(\alpha \sqcap \beta) := v(\alpha) \sqcap v(\beta)$.
$v(\neg\alpha) := \neg v(\alpha)$. $v(\lrcorner\alpha) := \lrcorner v(\alpha)$.
$v(\top) := (G, \emptyset)$. $v(\bot) := (\emptyset, M)$.

A *model* for **CDBL** based on the context $\mathbb{K}$ is a pair $\mathbb{M} := (\mathfrak{P}(\mathbb{K}), v)$, where v is a valuation.

Let $\mathcal{K}$ denote the collection of all contexts.

Definition 14 A sequent $\alpha \vdash \beta$ is said to be *satisfied* in a model $\mathbb{M}$ based on $\mathbb{K}$ if $v(\alpha) \sqsubseteq v(\beta)$. $\alpha \vdash \beta$ is *true* in $\mathbb{K}$ if it is satisfied in every model based on $\mathbb{K}$. $\alpha \vdash \beta$ is *valid* in the class $\mathcal{K}$ if it is true in every context $\mathbb{K} \in \mathcal{K}$.

As for any context $\mathbb{K}$ the set $\mathfrak{P}(\mathbb{K})$ of protoconcepts of $\mathbb{K}$ forms a contextual dBa (Theorem 7(1)), and for any model $\mathbb{M} := (\mathfrak{P}(\mathbb{K}), v)$, v is a valuation according to Definition 10, Theorem 26 gives us the soundness of **CDBL** with respect to the above semantics. In other words, if a sequent is provable in **CDBL** then it is valid in the class $\mathcal{K}$.

For the completeness result, we make use of the (Representation) Corollary 1 for contextual dBas and the fact that the Lindenbaum-Tarski algebra $\mathcal{L}(\mathfrak{F})$ is a contextual dBa (Theorem 28). From these it follows that $h : \mathfrak{F}/\equiv_{\vdash} \to \mathfrak{P}(\mathbb{K}(\mathcal{L}(\mathfrak{F})))$ defined as $h([\alpha]) := (F_{[\alpha]}, I_{[\alpha]})$ for all $[\alpha] \in \mathfrak{F}/\equiv_{\vdash}$, is an embedding. Recall the canonical map $v_0 : \mathfrak{F} \to \mathfrak{F}/\equiv_{\vdash}$ defined in Section 5. The composition $v_1 := h \circ v_0$ is then a valuation, which implies that $\mathbb{M}(\mathcal{L}(\mathfrak{F})) := (\mathfrak{P}(\mathbb{K}(\mathcal{L}(\mathfrak{F}))), v_1)$ is a model for **CDBL**.

Theorem 34 (Completeness) If a sequent $\alpha \vdash \beta$ is valid in $\mathcal{K}$ then $\alpha \vdash \beta$ is provable in **CDBL**.

Proof If possible, suppose $\alpha \vdash \beta$ is not provable in **CDBL**. By Proposition 27, $[\alpha] \not\sqsubseteq [\beta]$. By Proposition 3(3), either $[\alpha] \sqcap [\alpha] \not\sqsubseteq_{\sqcap} [\beta] \sqcap [\beta]$ or $[\alpha] \sqcup [\alpha] \not\sqsubseteq_{\sqcup} [\beta] \sqcup [\beta]$. Then there exists a prime filter F_0 in $\mathcal{L}(\mathfrak{F})_{\sqcap}$ (a Boolean algebra by Proposition 3) such that $[\alpha] \sqcap [\alpha] \in F_0$ and $[\beta] \sqcap [\beta] \notin F_0$. By Lemma 2, there exists a filter F in $\mathcal{L}(\mathfrak{F})$ such that $F \cap \mathcal{L}(\mathfrak{F})_{\sqcap} = F_0$ and as F_0 is prime, $F \in \mathcal{F}_p(\mathcal{L}(\mathfrak{F}))$. As $[\alpha] \sqcap [\alpha] \in F_0$, $[\alpha] \sqcap [\alpha] \in F$ and $[\beta] \sqcap [\beta] \notin F$, because $[\beta] \sqcap [\beta] \notin F_0$ and $[\beta] \sqcap [\beta] \in \mathcal{L}(\mathfrak{F})_{\sqcap}$. So $[\alpha] \in F$, as $[\alpha] \sqcap [\alpha] \sqsubseteq [\alpha]$, and $[\beta] \notin F$, otherwise $[\beta] \sqcap [\beta] \in F$. This gives $F \in F_{[\alpha]}$ and $F \notin F_{[\beta]}$, whence $F_{[\alpha]} \not\subseteq F_{[\beta]}$.

In case $[\alpha] \sqcup [\alpha] \not\sqsubseteq_{\sqcap} [\beta] \sqcup [\beta]$, we can dually show that there exists $I \in \mathcal{I}_p(\mathcal{L}(\mathfrak{F}))$ such that $[\alpha] \notin I$ and $[\beta] \in I$ giving $I_{[\beta]} \not\subseteq I_{[\alpha]}$.

So $v_1(\alpha) = (F_{[\alpha]}, I_{[\alpha]}) \not\sqsubseteq (F_{[\beta]}, I_{[\beta]}) = v_1(\beta)$, which implies that $\alpha \vdash \beta$ is not true in the model $\mathbb{M}(\mathcal{L}(\mathfrak{F}))$ – a contradiction. $\square$

In case of **MCDBL** and **MCDBL4**, instead of a context $\mathbb{K} := (G, M, I)$, we consider a Kripke context $\mathbb{KC} := ((G, R), (M, S), I)$ based on $\mathbb{K} := (G, M, I)$. A valuation $v : \mathfrak{F}_1 \to \mathfrak{P}(\mathbb{K})$ extends the one for **CDBL** with the following definitions for the modal operators: $v(\Box\alpha) := f_R(v(\alpha))$ and $v(\blacksquare\alpha) := f_S(v(\alpha))$. Let us denote the class of all Kripke contexts by $\mathcal{KC}$ and that of all reflexive and transitive Kripke contexts by $\mathcal{KC}_{RT}$. Models, satisfaction of sequents is as for **CDBL**. Then it is straightforward to show that **MCDBL** and **MCDBL4** are sound with respect to the classes $\mathcal{KC}$ and $\mathcal{KC}_{RT}$ respectively.

Note that by Theorem 31, for **MCDBL** the Lindenbaum-Tarski algebra $\mathcal{L}_\Sigma(\mathfrak{F}_1)$ is a contextual dBao, while for **MCDBL4**, it is a topological contextual dBa. The completeness of **MCDBL** with respect to the class $\mathcal{KC}$ is then proved in a similar manner as Theorem 34, the representation result given by Theorem 20(2) being used. In case of **MCDBL4**, as a consequence of Theorem 22, $\mathbb{KC}(\mathcal{L}_\Sigma(\mathfrak{F}_1))$ is a reflexive and transitive Kripke context. Using the (Representation) Theorem 23(1), one gets completeness of **MCDBL4** with respect to the class $\mathcal{KC}_{RT}$.

6 Conclusions

In a pioneering work unifying FCA and rough set theory, Yao, Düntsch and Gediga [3, 29] proposed object oriented and property oriented concepts of a context. For a context $\mathbb{K} := (G, M, I)$, its complement is the context $\mathbb{K}^c := (G, M, -R)$, where $-R := G \times M \setminus R$. It has been shown that the lattice of concepts of $\mathbb{K}$ is dually isomorphic (isomorphic) to that of object oriented (property oriented) concepts of $\mathbb{K}^c$. In the line of Wille's work, negation was introduced into the study and object oriented semiconcepts and protoconcepts of a context were proposed in [9, 10]. It was observed that (A, B) is a protoconcept of $\mathbb{K}$, if and only if (A^c, B) is an object oriented protoconcept of $\mathbb{K}^c$. The same holds for semiconcepts of a context. For a context $\mathbb{K}$, object oriented protoconcepts form a dBa, while object oriented semiconcepts form a pure dBa. The entire study presented here may also be done in terms of object oriented semiconcepts and protoconcepts. In particular, one may derive representation results for the algebras introduced here, with the help of corresponding algebras of object oriented semiconcepts and protoconcepts.

A complete [15] dBa $\mathbf{D}$ is one for which the Boolean algebras $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are complete. Vormbrock and Wille [15] have shown that any complete fully contextual (pure) dBa $\mathbf{D}$ for which $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are atomic, is isomorphic to the algebra of protoconcepts (semiconcepts) of some context. This result gives rise to the question of such a characterisation in case of a complete fully contextual dBao $\mathbf{D}$ for which $\mathbf{D}_\sqcap$ and $\mathbf{D}_\sqcup$ are atomic. It appears that, using

Vormbrock and Wille's results and the representation results obtained here for dBaos in terms of the full complex algebra of protoconcepts, one should be able to obtain the desired characterisation.

Another direction of investigation one may pursue, is the duality between the class of all Kripke contexts and that of all dBaos. We have shown in this work that a dBao $\mathfrak{O}$ induces a Kripke context $\mathbb{KC}(\mathfrak{O})$, and on the other hand, a Kripke context $\mathbb{KC}$ induces a dBao $\underline{\mathfrak{P}}^+(\mathbb{KC})$. A natural question then would be: is $\mathbb{KC}(\underline{\mathfrak{P}}^+(\mathbb{KC}))$ isomorphic to $\mathbb{KC}$?

Topological representation results for algebras are well-studied in literature. This would serve as yet another immediate point of investigation for the algebras discussed in this work.

Logics corresponding to dBas, pure dBas and their extensions with operators as defined here, remain an open question. The logic **MCDBL4** for topological contextual dBas is obtained as a special case of **MCDBL**Σ, where Σ is any set of sequents in **MCDBL**. This gives a scheme of obtaining several other logics that may express properties of dBaos and corresponding classes of Kripke contexts besides the ones considered here. For topological contextual dBas and correspondingly, reflexive and transitive Kripke contexts, **MCDBL4** with Σ containing the modal axioms for reflexivity and transitivity, serves the purpose. One may well include other axioms (such as symmetry) in Σ, and investigate the resulting modal systems.

References

[1] Wille, R.: Restructuring lattice theory: an approach based on hierarchies of concepts. In: Rival, I. (ed.) Ordered Sets. NATO Advanced Study Institutes Series (Series C — Mathematical and Physical Sciences), pp. 445–470 (1982). Springer, Dordrecht

[2] Pawlak, Z.: Rough sets. International Journal of Computer and Information Sciences **11**(5), 341–356 (1982)

[3] Yao, Y.Y.: A comparative study of formal concept analysis and rough set theory in data analysis. In: Tsumoto, S., et al. (eds.) International Conference on Rough Sets and Current Trends in Computing, pp. 59–68 (2004). Springer, Berlin

[4] Meschke, C.: Approximations in concept lattices. In: Kwuida, L., et al. (eds.) Formal Concept Analysis, pp. 104–123 (2010). Springer, Berlin

[5] Hu, K., Sui, Y., Lu, Y., Wang, J., Shi, C.: Concept approximation in concept lattice. In: Cheung, D., et al. (eds.) Advances in Knowledge Discovery and Data Mining, pp. 167–173 (2001). Springer, Berlin

[6] Ganter, B., Meschke, C.: A formal concept analysis approach to rough data tables. In: Peters, J.F., et al. (eds.) Transactions on Rough Sets XIV,

pp. 37–61 (2011). Springer, Berlin

[7] Yao, Y.Y., Chen, Y.: Rough set approximations in formal concept analysis. In: Peters, J.F., et al. (eds.) Transactions on Rough Sets V, pp. 285–305 (2006). Springer, Berlin

[8] Kent, R.E.: Rough concept analysis. In: Ziarko, W.P. (ed.) Rough Sets, Fuzzy Sets and Knowledge Discovery, pp. 248–255 (1994). Springer, London

[9] Howlader, P., Banerjee, M.: Algebras from semiconcepts in rough set theory. In: Nguyen, H.S., et al. (eds.) International Joint Conference on Rough Sets, pp. 440–454 (2018). Springer, Cham

[10] Howlader, P., Banerjee, M.: Object oriented protoconcepts and logics for double and pure double Boolean algebras. In: Bello, R., et al. (eds.) International Joint Conference on Rough Sets, pp. 308–323 (2020). Springer, Cham

[11] Saquer, J., Deogun, J.S.: Concept approximations based on rough sets and similarity measures. International Journal of Applied Mathematics and Computer Science **11**(3), 655–674 (2001)

[12] Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations, (1999). Springer-verlag, Berlin

[13] Luksch, P., Wille, R.: A mathematical model for conceptual knowledge systems. In: Bock, H.H., et al. (eds.) Classification, Data Analysis, and Knowledge Organization, pp. 156–162 (1991). Springer, Berlin

[14] Wille, R.: Boolean concept logic. In: Ganter, B., et al. (eds.) Conceptual Structures: Logical, Linguistic, and Computational Issues, pp. 317–331 (2000). Springer, Berlin

[15] Vormbrock, B., Wille, R.: Semiconcept and protoconcept algebras: the basic theorems. In: Ganter, B., Stumme, G., Wille, R. (eds.) Formal Concept Analysis: Foundations and Applications, pp. 34–48 (2005). Springer, Berlin

[16] Kent, R.E.: Rough concept analysis: a synthesis of rough sets and formal concept analysis. Fundamenta Informaticae **27**, 169–181 (1996)

[17] Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning About Data, (1991). Kluwer Academic Publishers, Dordrecht

[18] Yao, Y.Y., Lin, T.Y.: Generalization of rough sets using modal logics. Intelligent Automation and Soft Computing **2**(2), 103–119 (1996)

[19] Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic, (2001). Cambridge University Press, Cambridge

[20] Jonsson, B., Tarski, A.: Boolean algebras with operators. Part I. American Journal of Mathematics **73**(4), 891–939 (1951)

[21] Balbiani, P.: Deciding the word problem in pure double Boolean algebras. Journal of Applied Logic **10**(3), 260–273 (2012)

[22] Burris, S., Sankappanavar, H.P.: A Course in Universal Algebra, (1981). Springer-Verlag, New York-Berlin

[23] Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, (2002). Cambridge University Press, New York

[24] Breckner, B.E., Săcărea, C.: A topological representation of double Boolean lattices. Studia. Universitatis Babeş-Bolyai Mathematica **64**(1), 11–23 (2019)

[25] Vormbrock, B.: A solution of the word problem for free double Boolean algebras. In: Kuznetsov, S.O., et al. (eds.) Formal Concept Analysis, pp. 240–270 (2007). Springer, Berlin

[26] Kwuida, L.: Prime ideal theorem for double Boolean algebras. Discussiones Mathematicae-General Algebra and Applications **27**(2), 263–275 (2007)

[27] Howlader, P., Banerjee, M.: Remarks on prime ideal and representation theorems for double Boolean algebras. In: Valverde-Albacete, F.J., et al. (eds.) CLA 2020, pp. 83–94 (2020). CEUR Workshop Proceedings

[28] Rasiowa, H.: An Algebraic Approach To Non-Classical Logics, (1974). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York

[29] Düntsch, I., Gediga, G.: Modal-style operators in qualitative data analysis. In: Vipin, K., et al. (eds.) Proceedings of the 2002 IEEE International Conference on Data Mining, pp. 155–162 (2002). IEEE Computer Society

Appendix A Proofs

Proof in Theorem 16, that $(D, \sqcap, \sqcup, \neg, \top, \bot)$ *is a Boolean algebra:* Let $\mathfrak{O}$ be a dBao such that for all $a \in D$, $\neg a = \lrcorner a$ and $\neg\neg a = a$. Let $x, y \in D$ such that $x \sqsubseteq y$ and $y \sqsubseteq x$. By Proposition 4(4), $x \sqcap x = y \sqcap y$ and $x \sqcup x = y \sqcup y$. Using Proposition 5(3), $\neg\neg x = \neg\neg y$ and so $x = y$. Therefore $(D, \sqsubseteq)$ is a partially ordered set. From Definition 2(2a and 2b) it follows that $\sqcap, \sqcup$ is commutative, while Definition 2(3a and 3b) gives that $\sqcap, \sqcup$ is associative. Using Definition 2(5a) and Proposition 5(3), $x \sqcap (x \sqcup y) = x \sqcap x = \neg\neg x$. So $x \sqcap (x \sqcup y) = x$. Again using Definition 2(5b) and Proposition 5(3), $x \sqcup (x \sqcap y) = x$. Therefore $(D, \sqcap, \sqcup, \neg, \top, \bot)$ is a bounded complemented lattice. To show it is a distributive lattice, we show that for all $x, y, \in D$ $x \sqcap y = x \wedge y$ and $x \vee y = x \sqcup y$. Rest of the proof follows from Definition 2(6a and 6b).

Let $x, y \in D$. Then $x, y \sqsubseteq x \sqcup y$. Proposition 5(2) gives $\neg(x \sqcup y) \sqsubseteq \neg x, \neg y$. Therefore by Proposition 4(6), $\neg(x \sqcup y) \sqcap \neg y \sqsubseteq \neg x \sqcap \neg y$ and $\neg(x \sqcup y) \sqcap \neg(x \sqcup y) \sqsubseteq \neg(x \sqcup y) \sqcap \neg y$. So $\neg(x \sqcup y) \sqcap \neg(x \sqcup y) \sqsubseteq \neg x \sqcap \neg y$. By Proposition 5(1), $\neg(x \sqcup y) \sqsubseteq \neg x \sqcap \neg y$, and by Proposition 5(2), $\neg(\neg x \sqcap \neg y) \sqsubseteq \neg\neg(x \sqcup y) = (x \sqcup y) \sqcap (x \sqcup y) \sqsubseteq x \sqcup y$. Hence $x \vee y \sqsubseteq x \sqcup y$. Using Proposition 4(5) and Proposition 5(2), $\neg x \sqcap \neg y \sqsubseteq \neg x, \neg y$. So $\neg\neg x \sqsubseteq \neg(\neg x \sqcap \neg y)$ and $\neg\neg y \sqsubseteq \neg(\neg x \sqcap \neg y)$. Therefore $x \sqsubseteq \neg(\neg x \sqcap \neg y) = x \vee y$ and $y \sqsubseteq \neg(\neg x \sqcap \neg y) = x \vee y$. Proposition 4(6) gives $x \sqcup y \sqsubseteq x \vee y$, as $(x \vee y) \sqcup (x \vee y) = \lrcorner\lrcorner(x \vee y) = \neg\neg(x \vee y) = x \vee y$. So $x \sqcup y = x \vee y$. Dually we can show that $x \sqcap y = x \wedge y$.

Proof of Theorem 25:

$2a$.

$$\dfrac{4a\ (\alpha\sqcap\beta)\sqcap\gamma\vdash((\alpha\sqcap\beta)\sqcap\gamma)\sqcap((\alpha\sqcap\beta)\sqcap\gamma) \qquad \dfrac{\dfrac{2a\ (\alpha\sqcap\beta)\sqcap\gamma\vdash(\alpha\sqcap\beta) \quad \alpha\sqcap\beta\vdash\beta\ 3a}{(R4)\ (\alpha\sqcap\beta)\sqcap\gamma\vdash\beta} \qquad (\alpha\sqcap\beta)\sqcap\gamma\vdash\gamma\ 3a}{((\alpha\sqcap\beta)\sqcap\gamma)\sqcap((\alpha\sqcap\beta)\sqcap\gamma)\vdash\beta\sqcap\gamma\ (R6)}}{(\alpha\sqcap\beta)\sqcap\gamma\vdash\beta\sqcap\gamma\ (R4) - (\mathrm{I})}$$

Now,

$$\dfrac{4a\ (\alpha\sqcap\beta)\sqcap\gamma\vdash((\alpha\sqcap\beta)\sqcap\gamma)\sqcap((\alpha\sqcap\beta)\sqcap\gamma) \qquad \dfrac{\dfrac{2a\ (\alpha\sqcap\beta)\sqcap\gamma\vdash\alpha\sqcap\beta \quad \alpha\sqcap\beta\vdash\alpha\ 2a}{(\alpha\sqcap\beta)\sqcap\gamma\vdash\alpha\ (R4)} \qquad (\alpha\sqcap\beta)\sqcap\gamma\vdash\beta\sqcap\gamma\ (\text{from (I) above})}{((\alpha\sqcap\beta)\sqcap\gamma)\sqcap((\alpha\sqcap\beta)\sqcap\gamma)\vdash\alpha\sqcap(\beta\sqcap\gamma)\ (R6)}}{(\alpha\sqcap\beta)\sqcap\gamma\vdash\alpha\sqcap(\beta\sqcap\gamma)\ (R4)}$$

Similarly we can show that $\alpha \sqcap (\beta \sqcap \gamma) \vdash (\alpha \sqcap \beta) \sqcap \gamma$.

$3a$.

$$\dfrac{4a\ (\alpha\sqcap\alpha)\sqcap\beta\vdash((\alpha\sqcap\alpha)\sqcap\beta)\sqcap((\alpha\sqcap\alpha)\sqcap\beta) \qquad \dfrac{\dfrac{2a\ (\alpha\sqcap\alpha)\sqcap\beta\vdash\alpha\sqcap\alpha \quad \alpha\sqcap\alpha\vdash\alpha\ 2a}{(R4)\ (\alpha\sqcap\alpha)\sqcap\beta\vdash\alpha} \qquad (\alpha\sqcap\alpha)\sqcap\beta\vdash\beta\ 3a}{((\alpha\sqcap\alpha)\sqcap\beta)\sqcap((\alpha\sqcap\alpha)\sqcap\beta)\vdash\alpha\sqcap\beta\ (R6)}}{(\alpha\sqcap\alpha)\sqcap\beta\vdash\alpha\sqcap\beta\ (R4)}$$

$$\dfrac{4a\ \alpha\sqcap\beta\vdash(\alpha\sqcap\beta)\sqcap(\alpha\sqcap\beta) \qquad \dfrac{\dfrac{4a\ \alpha\sqcap\beta\vdash(\alpha\sqcap\beta)\sqcap(\alpha\sqcap\beta) \quad \dfrac{2a\ (\alpha\sqcap\beta)\vdash\alpha \quad (\alpha\sqcap\beta)\vdash\alpha\ 2a}{(\alpha\sqcap\beta)\sqcap(\alpha\sqcap\beta)\vdash\alpha\sqcap\alpha\ (R6)}}{(R4)\ \alpha\sqcap\beta\vdash\alpha\sqcap\alpha} \qquad \alpha\sqcap\beta\vdash\beta\ 3a}{(\alpha\sqcap\beta)\sqcap(\alpha\sqcap\beta)\vdash(\alpha\sqcap\alpha)\sqcap\beta\ (R6)}}{\alpha\sqcap\beta\vdash(\alpha\sqcap\alpha)\sqcap\beta\ (R4)}$$

$5a$.

$$\dfrac{4a\ \alpha\sqcap(\alpha\sqcup\beta)\vdash(\alpha\sqcap(\alpha\sqcup\beta))\sqcap(\alpha\sqcap(\alpha\sqcup\beta)) \qquad \dfrac{2a\ \alpha\sqcap(\alpha\sqcup\beta)\vdash\alpha \quad \alpha\sqcap(\alpha\sqcup\beta)\vdash\alpha\ 2a}{(\alpha\sqcap(\alpha\sqcup\beta))\sqcap(\alpha\sqcap(\alpha\sqcup\beta))\vdash\alpha\sqcap\alpha\ (R6)}}{\alpha\sqcap(\alpha\sqcup\beta)\vdash\alpha\sqcap\alpha\ (R4)}$$

$6a$. Proof is identical to that of $5a$.