KRON'S METHOD OF SUBSPACES*

BY

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Introduction. Gabriel Kron has introduced new and powerful methods of applying tensor analysis to complicated engineering problems, presenting his major contributions in the field of electrical engineering. The manner of presentation, and the rarity of a simultaneous knowledge of the hitherto almost unrelated subjects of electrical engineering and tensor analysis, have unfortunately served to limit his audience. Recent experimental confirmation of some of his investigations dealing with equivalent circuits, however, has attracted the serious attention of a wider engineering following.

In view of the growing importance of the whole subject, and of the controversy which has surrounded it, it has seemed desirable to present some particular aspect of Kron's work in a form which may appeal to a less highly specialized audience. To avoid complications as far as possible, the present paper must ignore such important topics as electrical networks, electrical machines, and equivalent circuits. It confines itself to purely dynamical problems, and to that particular idea of Kron's which may be called *the method of subspaces.*†

Much discussion has arisen over Kron's claim that he uses tensor analysis. It is the considered opinion of the present writer that Kron does indeed make a full and proper use of tensor analysis. Possibly the belief that Kron employs only matrices may have arisen from the fact that, in order to present his actual mathematical *procedure* in a form that may be understood and used by those not familiar with the intricacies of the tensor calculus, he often presents this procedure in matrix form. However, he is always careful to point out that the underlying concepts are wholly tensorial in character. In the present paper the method of subspaces will first be presented in terms of a simple example, and in purely matrix form, merely as a set of rules of procedure, the essentially tensorial significance of the procedure being discussed only after the actual procedure has been brought before the reader. The theoretical discussion will then be followed by three simple, related examples illustrative of various aspects of the method.

Scope of the method of subspaces. Let us consider a system, which may be dynamical, electrodynamical, or otherwise, containing several standard parts, such as a fly-wheel, a governor, a pair of synchronous machines, a system of levers, etc. The equations of performance of the individual parts are usually well known, but the equations of performance of the complex whole will depend on the manner in which they are interconnected. Usually it is extremely difficult to trace out the full influence of each interconnection in setting up the equations of performance. The method

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† Purely dynamical examples of the method of subspaces have been given by Kron in an unpublished manuscript.

of subspaces suggested by Kron yields the equations of performance by a routine and quite straightforward manipulation of the known equations of performance of the constituent parts of the system, a brief inspection sufficing to yield all needed information as to the manner of interconnection.

Simple dynamical example. To illustrate the actual mathematical procedure in its simplest form, we shall first consider a quite trivial dynamical problem. Naturally it will not reveal the power and economy of the method any more than it would the peculiar virtues of, say, the Hamilton-Jacobi equation, were that applied to it. But it will serve to bring the routine mathematical procedure before us without unnecessary distraction from complexities which are merely incidental.

Let us consider the dynamical system S consisting of three particles free to move on a line, the masses being m_1, m_2, m_3, the coordinates[1] x^1, x^2, x^3, and the forces f_1, f_2, f_3. The equations of motion are

$$f_1 = m_1\ddot{x}^1, \qquad f_2 = m_2\ddot{x}^2, \qquad f_3 = m_3\ddot{x}^3. \tag{1}$$

Let us consider now the new dynamical system $\bar{S}$ which arises when the particles 2 and 3 are made to coalesce. Its equations of motion may be written down at once:

$$f_1 = m_1\ddot{x}^1, \qquad f_2 + f_3 = (m_2 + m_3)\ddot{x}^2. \tag{2}$$

Let us suppose, though, that it had been a highly complex system of interconnected simple parts. We would then welcome a routine method of obtaining (2) from (1) which required no detailed thought and avoided constant preoccupation with the effects of the interconnections. The method of subspaces would be applied to the present problem in the following routine manner:

The first step is to write equations (1) of system S in matrix form,

$$F = M\ddot{X}, \tag{3}$$

i.e.,

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} \ddot{x}^1 \\ \ddot{x}^2 \\ \ddot{x}^3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \tag{4}$$

where, it will be noted, the masses form a square array rather than a single row or column such as one might at first expect.

Next, a relationship is set up between the coordinates x^1, x^2, x^3 of S and the coordinates $\bar{x}^1, \bar{x}^2$ of $\bar{S}$. This relationship can be taken to be

$$x^1 = \bar{x}^1, \qquad x^2 = \bar{x}^2, \qquad x^3 = \bar{x}^2 \text{ (not } \bar{x}^3). \tag{5}$$

From this is obtained the matrix C defined by

$$X = C\bar{X}. \tag{6}$$

It is

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{7}$$

[1] We use the tensor practice of placing the indices of contravariant quantities above the symbol. x^1, x^2, x^3 do not stand for x, x-squared, x-cubed.

If we denote the transposed matrix by C^{tr}, then the new forces $\bar{f}_1$, $\bar{f}_2$ are given by the routine matrix multiplication

$$\bar{F} = C^{tr}F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_3 \end{bmatrix}. \tag{8}$$

The new masses are given by routine matrix multiplications as follows:

$$\bar{M} = C^{tr}MC = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 + m_3 \end{bmatrix}. \tag{9}$$

Finally the new equations of motion are

$$\bar{F} = \bar{M}\ddot{\bar{X}}, \tag{10}$$

i.e.,

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 + m_3 \end{bmatrix}\begin{bmatrix} \ddot{\bar{x}}^1 \\ \ddot{\bar{x}}^2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 + f_3 \end{bmatrix}.$$

This yields the two equations

$$m_1\ddot{\bar{x}}^1 = f_1, \qquad (m_2 + m_3)\ddot{\bar{x}}^2 = f_2 + f_3,$$

which are equivalent to (2) above.

The tensor form of the problem. Using Latin indices for the range 1, 2, 3, and Greek for the range 1, 2, we may write the various expressions and equations above in the familiar index notation of the tensor calculus.

The equations of motion of S may be written[2]

$$f_a = m_{ab}\ddot{x}^b, \tag{1'}$$

the matrix C may be written

$$C^a_\alpha = \frac{\partial x^a}{\partial \bar{x}^\alpha}, \tag{7'}$$

and the relations between the forces, etc., in S and $\bar{S}$ may be put in the form

$$\bar{f}_\alpha = \frac{\partial x^a}{\partial \bar{x}^\alpha} f_a, \quad (8'); \qquad \bar{m}_{\alpha\beta} = \frac{\partial x^a}{\partial \bar{x}^\alpha}\frac{\partial x^b}{\partial \bar{x}^\beta} m_{ab}, \quad (9'); \qquad x^a = \frac{\partial x^a}{\partial \bar{x}^\alpha}\bar{x}^\alpha. \quad (6').$$

Also the equations of motion of $\bar{S}$ are

$$\bar{f}_\alpha = \bar{m}_{\alpha\beta}\ddot{\bar{x}}^\beta. \tag{10'}$$

Provided we interpret the time derivatives as absolute derivatives, or alterna-

[2] According to the summation convention of the index notation, a repeated index in a single expression indicates summation over its whole range of values.

tively, as is permissible in the present simple case, provided we avoid non-linear coordinate transformations, the above have the form of tensor equations and tensor transformations. The mere fact that the work can be expressed in the tensor notation does not, in itself, imply tensorial character. The essential criterion of tensor character is the tensor law of transformation. In view of (7′), equations (8′), (9′), and (6′) are tensor transformations, and equation (10′) is the result of transforming the tensor equation (1′). The fact that C is singular does not destroy the tensor character of the transformations; its significance will appear shortly.

The tensor theory. Since tensor equations have objective significance we may look for a geometrical picture of the process described above. Naturally this will be sought in configuration space. The basic tensorial and geometrical significance of Lagrangean dynamics being quite familiar, the corresponding significance of Kron's method may be explained here quite briefly.

As is well known, in Lagrangean dynamics the motion of the dynamical system S is represented by the motion of a point in a three dimensional configuration space, K, having x^a as coordinates and m_{ab} as metrical tensor. When particles 2 and 3 coalesce, the system S loses one degree of freedom and becomes the system $\bar{S}$. Thus the trajectory of $\bar{S}$ belongs to a two dimensional configuration space, $\bar{K}$, which is in fact a subspace of K, for it is defined by a relation (in more general cases, by a set of relations) between the coordinates of K. Specifically, the subspace here is defined by the relation

$$x^2 = x^3.$$

In parametric form this subspace is given by the relations (5) above, which represent the three coordinates x^a as functions of the two variables $\bar{x}^\alpha$. (Compare with the relations $x=\cos\theta\cos\varphi$, $y=\cos\theta\sin\varphi$, $z=\sin\theta$ which express the Cartesian coordinates x, y, z as functions of the two parameters θ, φ. This defines the two dimensional subspace of ordinary three dimensional space constituting the surface of a unit sphere.)

By the well known theory of subspaces, the projections of the *covariant* tensors f_a, m_{ab}, are given by the singular transformations (8′), (9′). Thus the equations of motion of $\bar{S}$ are the projection on $\bar{K}$ of the equations of motion of S. And, since the initial conditions of S and $\bar{S}$ coincide in $\bar{K}$, the trajectory of $\bar{S}$ is the projection on $\bar{K}$ of the trajectory of S.

There are two ways of viewing the relationship between the systems S and $\bar{S}$:

(α). We may regard S as the same physical system as $\bar{S}$, the forces between particles 2 and 3 which keep them together being included explicitly in the force vector f_a. The trajectory of S in K is then identical with that of $\bar{S}$ in $\bar{K}$.

(β). We may regard S as a different physical system from $\bar{S}$ inasmuch as particles 2 and 3 are not united in S. The forces in $\bar{S}$ are the same as those in S except for those forces in S which tend to separate particles 2 and 3. These latter forces have components in K which are normal to the subspace $\bar{K}$, and thus have zero projection on $\bar{K}$.

Both viewpoints are of significance, and more will be said about them later.

A formal proof. A formal proof will now be given that the transition to a subspace and the tensor transformations that go with it, are justified in the general case. This proof will thus also justify the general procedure given by Kron, of which the above example was a particular illustration.

The proof will be made brief by basing it on certain standard results in dynamics and tensor analysis.

Let us consider a rigorous proof of the validity of the Lagrangean equations, such as is given, for instance, in Whittaker's *Analytical Dynamics*, third edition, starting on page 34. We are concerned here with the case in which t does not enter explicitly, and we shall use the tensor notation. The dynamical system under consideration in Whittaker has n degrees of freedom, and n generalized coordinates. Let us denote the latter by $\bar{x}^\alpha$, using the Greek indices α, β, γ for the range 1 to n. We denote the number of individual particles in the system by $N/3$, so that their combined coordinates number N and we denote these N coordinates by $\tilde{x}^\lambda$, using λ, μ, ν for the range 1 to N. In this notation, the proof of the Lagrangian equations has the following outline:

The equations of motion of the $N/3$ individual particles, in a self-explanatory notation, have the form

$$\tilde{m}_{\lambda\mu}\ddot{\tilde{x}}^\mu = \tilde{f}_\lambda. \tag{11}$$

These N equations are not independent, since the N coordinates $\tilde{x}^\lambda$ are related, as are the N forces $\tilde{f}_\lambda$. The relations between the coordinates $\tilde{x}^\lambda$ are defined by equations

$$\tilde{x}^\lambda = \tilde{x}^\lambda(\bar{x}^\alpha) \tag{12}$$

which express them in terms of the n generalized coordinates of the system. (These correspond to Whittaker's equations $x_i = f_i(q_1, q_2, \cdots, q_n, t)$, etc., with t omitted.) From (12) may be computed the quantities $\partial\tilde{x}^\lambda/\partial\bar{x}^\alpha$. The equations of motion (11) of the individual particles are multiplied individually by such quantities and then added in groups, the process being precisely that described by the equation

$$\frac{\partial\tilde{x}^\lambda}{\partial\bar{x}^\alpha}\tilde{m}_{\lambda\mu}\ddot{\tilde{x}}^\mu = \frac{\partial\tilde{x}^\lambda}{\partial\bar{x}^\alpha}\tilde{f}_\lambda, \tag{13}$$

where the summation convention is employed, as usual. After some manipulation, the left-hand side is then reduced to the standard Lagrangean form, and the right-hand side is interpreted as a set of generalized forces in the familiar manner.

Later, on page 39, Whittaker gives an explicit form of the Lagrangean equations for the case in which t does not enter explicitly. This reveals that the left-hand side has the form of a covariant derivative with respect to $\bar{m}_{\alpha\beta}$ as metrical tensor. The equations, in fact, may be written

$$\bar{m}_{\alpha\beta}\dot{\bar{x}}^\beta{}_{,\gamma}\dot{\bar{x}}^\gamma = \bar{f}_\alpha. \tag{14}$$

Since, in the original form (11), the coordinates were Cartesian for each individual particle, the ordinary derivatives there coincided with the covariant derivatives; thus (11) may be written as

$$\tilde{m}_{\lambda\mu}\dot{\tilde{x}}^\mu{}_{,\nu}\dot{\tilde{x}}^\nu = \tilde{f}_\lambda, \tag{15}$$

the subscript preceded by a comma here denoting the covariant derivative with respect to $\tilde{m}_{\lambda\mu}$ as metrical tensor.

It will be observed that the initial step, represented by the equation (13), is a transition to a subspace, complete with a singular transformation matrix $\partial\tilde{x}^\lambda/\partial\bar{x}^\alpha$

of the type which, for some reason, excites the critics of Kron when it is used by him. The relations between $\tilde{m}_{\lambda\mu}$ and $\bar{m}_{\alpha\beta}$, and between $\tilde{f}_\lambda$ and $\bar{f}_\alpha$, follow from the analysis and are the usual tensor transformations, with singular transformation matrix.

Now Lagrange goes from an initial configuration space of N dimensions to a final subspace of n dimensions in one step. Kron's idea is, in theory, simply to make this transition in more than one step, using subsidiary subspaces as resting places when the mathematics tends to become too complicated. For example, he would, in theory, go from the first space having coordinates $\tilde{x}^\lambda$ to an intermediary subspace having coordinates x^a and then to the final subspace having coordinates $\bar{x}^\alpha$ which is also a subspace of the intermediary subspace. In practice, of course, as is also the case in the Lagrangean method, the initial space having coordinates $\tilde{x}^\lambda$ is entirely neglected, having served its sole purpose in providing the theoretical basis for the equations and procedures actually used.

Since[3]

$$\frac{\partial \tilde{x}^\lambda}{\partial x^a}\frac{\partial x^a}{\partial \bar{x}^\alpha} = \frac{\partial \tilde{x}^\lambda}{\partial \bar{x}^\alpha},$$

the transition in several steps will yield the same result as the transition in one step, the various tensors involved being transformed according to the standard tensor law.

Thus the proof of Kron's theory and procedure is a direct corollary of the proof of Lagrange's equations and the tensor theory of subspaces.

The proof given in Whittaker is based on viewpoint (α), since the forces $\tilde{f}_\lambda$ between the individual particles are regarded as the forces actually existing between them in the ultimate system. The trajectory in the N dimensional space is actually confined to the n dimensional subspace.

It is important to note, though, that the proof is equally valid for viewpoint (β). For those forces which do no work do not contribute to the values of the generalized forces of the ultimate system. Since they do not affect the ultimate system, it is clear that, for the purpose of setting up the equations of that system, they may be omitted from the N basic equations of the individual particles. When these forces are ignored, however, the forces between the individual particles are very much changed, especially in the case of inelastic bodies. The system of individual particles is then no longer physically equivalent to the ultimate system. It has a quite different motion (for instance, the particles of a rigid body here move in divergent directions) and its trajectory, in general, spans the whole N dimensional space. Nevertheless, according to the above reasoning, the projection of its trajectory on the n dimensional subspace coincides with the trajectory of the ultimate system.

Non-linear transformations. Since Kron has made the widest application of his method of subspaces to electrical networks and other electrodynamical problems in which the interconnection transformation is very often linear, the impression has sometimes arisen that the method is applicable only to situations in which this linearity is present. The following examples of the method involve non-linear transformations. They are simple enough so that the more usual methods of solution are

[3] When the transformations are non-singular, this is the basis of the important "group property" of tensor transformations. In the absence of inverses the term "group" is inappropriate here, but provided the succession is always to a subspace of the preceding space or subspace, as it always is here, the usual combination properties of tensor transformations are preserved.

hardly more complicated than those by Kron's method, but this is inevitable in simple problems. The power of Kron's method begins to be felt when the systems in question involve a larger number of interconnections, and of more complicated mechanisms than the simple rods of the examples below.

Illustrative example I. We begin with a system consisting of two rods hinged together without friction, one rod being suspended by its free end from a fixed point O (Fig. 1). We denote the masses of the rods by m_1, m_2, their lengths by $2a_1$, $2a_2$, and their moments of inertia about their centers of gravity, which we shall assume to coincide with their mid-points, by I_1, I_2. In addition to gravity, let us consider a force F, not necessarily conservative, which acts horizontally and to the right at the mid-point of the lower rod. The system has two degrees of freedom, and we may take as the generalized coordinates the angles θ, φ which the rods make with the vertical.

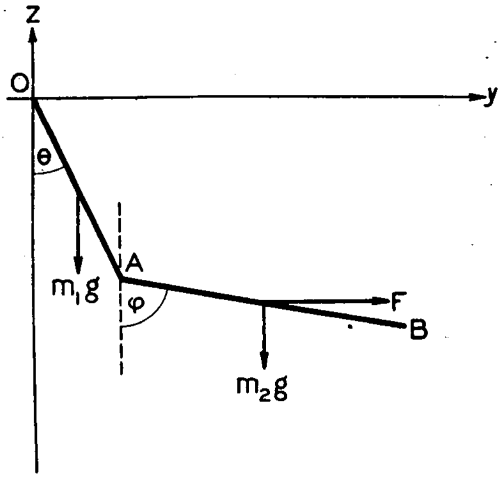

FIG. 1. System I.

The problem is to set up the equations of motion. From previous experience with Lagrangean dynamics, we may regard a single rod as a *known system*, in the sense that we can instantly write down its equations of motion, or have already tabulated them for quick reference. The present system consists of two of these known systems interconnected. We therefore begin by considering the system consisting of the two rods not interconnected, the forces being the same as those acting externally on the original system.[4] Kron calls this system the *primitive system*. It is shown in Fig. 2, and has here four degrees of freedom. We may take the four generalized coordinates to be the angles θ, φ above together with the coordinates y, z of the center of gravity of the lower rod. We let Latin indices refer to the primitive system, and Greek to the actual system under discussion. For the primitive system the metrical tensor and the force vector can be written down at once. They are

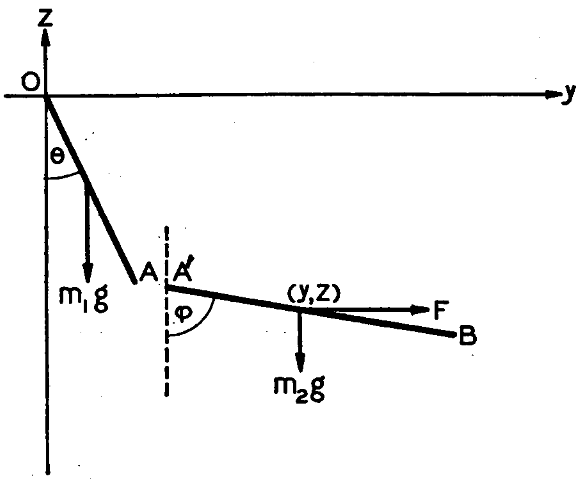

FIG. 2. Primitive system of system I.

[4] In general one must include all forces that do work, including dissipative forces.

m_{ab} = (16)

	θ	φ	y	z
θ	$I_1 + m_1 a_1^2$	0	0	0
φ	0	I_2	0	0
y	0	0	m_2	0
z	0	0	0	m_2

and

f_a = . (17)

θ	$-m_1 g a_1 \sin\theta$
φ	0
y	F
z	$-m_2 g$

The restraint arising from the interconnection of the two rods imposes the following two conditions on the four coordinates x^a of the primitive system:

$$y = 2a_1 \sin\theta + a_2 \sin\varphi, \qquad z = -2a_1 \cos\theta - a_2 \cos\varphi. \tag{18}$$

These two equations define the subspace of the configuration space of the primitive system to which the given system is confined. We may express them in the form of a transformation, that is to say, in parametric form, by writing

$$\theta = \bar\theta, \quad \varphi = \bar\varphi, \quad y = 2a_1 \sin\bar\theta + a_2 \sin\bar\varphi, \quad z = -2a_1\cos\bar\theta - a_2\cos\bar\varphi, \tag{19}$$

which is of the form

$$x^a = x^a(\bar x^\alpha). \tag{20}$$

The transformation matrix C^a_α, or $\partial x^a/\partial \bar x^\alpha$, is (in the form C^{tr})

$\dfrac{\partial x^a}{\partial \bar x^\alpha}$ = . (21)

$\bar x^\alpha$ \ x^a	θ	φ	y	z
$\bar\theta$	1	0	$2a_1\cos\bar\theta$	$2a_1\sin\bar\theta$
$\bar\varphi$	0	1	$a_2\cos\bar\varphi$	$a_2\sin\bar\varphi$

Thus the metrical tensor for the given system, namely the projection $\bar m_{\alpha\beta}$ of m_{ab}, is given by

$$\bar m_{\alpha\beta} = \frac{\partial x^a}{\partial \bar x^\alpha}\frac{\partial x^b}{\partial \bar x^\beta} m_{ab},$$

i.e.,

$$\bar m_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 2a_1\cos\bar\theta & 2a_1\sin\bar\theta \\ 0 & 1 & a_2\cos\bar\varphi & a_2\sin\bar\varphi \end{bmatrix} \begin{bmatrix} I_1 + m_1 a_1^2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2a_1\cos\bar\theta & a_2\cos\bar\varphi \\ 2a_1\sin\bar\theta & a_2\sin\bar\varphi \end{bmatrix}$$

$$= \begin{bmatrix} I_1 + m_1a_1^2 & 0 & 2a_1m_2 \cos \bar{\theta} & 2a_1m_2 \sin \bar{\theta} \\ 0 & I_2 & a_2m_2 \cos \bar{\varphi} & a_2m_2 \sin \bar{\varphi} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2a_1 \cos \bar{\theta} & a_2 \cos \bar{\varphi} \\ 2a_1 \sin \bar{\theta} & a_2 \sin \bar{\varphi} \end{bmatrix}$$

$$= \begin{bmatrix} I_1 + m_1a_1^2 + 4m_2a_1^2 & 2m_2a_1a_2 \cos (\bar{\theta} - \bar{\varphi}) \\ 2m_2a_1a_2 \cos (\bar{\theta} - \bar{\varphi}) & I_2 + m_2a_2^2 \end{bmatrix}. \qquad (22)$$

Likewise, the force vector is given by

$$\bar{f}_\alpha = \frac{\partial x^a}{\partial \bar{x}^\alpha} f_a,$$

or

$$\bar{f}_\alpha = \begin{bmatrix} 1 & 0 & 2a_1 \cos \bar{\theta} & 2a_1 \sin \bar{\theta} \\ 0 & 1 & a_2 \cos \varphi & a_2 \sin \bar{\varphi} \end{bmatrix} \begin{bmatrix} -m_1ga_1 \sin \bar{\theta} \\ 0 \\ F \\ -m_2g \end{bmatrix}$$

$$= \begin{bmatrix} -(m_1 + 2m_2)ga_1 \sin \bar{\theta} + 2a_1F \cos \bar{\theta} \\ a_2F \cos \bar{\varphi} - a_2m_2g \sin \bar{\varphi} \end{bmatrix}. \qquad (23)$$

The kinetic energy function of the given system is

$$T = \tfrac{1}{2}\bar{m}_{\alpha\beta}\dot{\bar{x}}^\alpha\dot{\bar{x}}^\beta$$
$$= \tfrac{1}{2}(I_1 + m_1a_1^2 + 4m_2a_1^2)\dot{\bar{\theta}}^2 + 2m_2a_1a_2 \cos (\bar{\theta} - \bar{\varphi})\dot{\bar{\theta}}\dot{\bar{\varphi}} + \tfrac{1}{2}(I_2 + m_2a_2^2)\dot{\bar{\varphi}}^2. \qquad (24)$$

The Lagrangean equations for the given system may now be written in the usual form.[5] They are, on dropping the bars over θ and φ, but without simplification,

$$\frac{d}{dt}\left\{(I_1 + m_1a_1^2 + 4m_2a_1^2)\dot{\theta} + 2m_2a_1a_2 \cos (\theta - \varphi)\dot{\varphi}\right\} - \left\{-2m_2a_1a_2 \sin (\theta - \varphi)\dot{\theta}\dot{\varphi}\right\}$$
$$= -(m_1 + 2m_2)ga_1 \sin \theta + 2a_1F \cos \theta,$$

$$\frac{d}{dt}\left\{2m_2a_1a_2 \cos (\theta - \varphi)\dot{\theta} + (I_2 + m_2a_2^2)\dot{\varphi}\right\} - \left\{2m_2a_1a_2 \sin (\theta - \varphi)\dot{\theta}\dot{\varphi}\right\}$$
$$= a_2F \cos \varphi - a_2m_2g \sin \varphi.$$

Since no forces were introduced at the points A, A' of the primitive system, the latter was physically different from the given system, for in the given system opposite forces acted at the hinge A. Thus we have been using viewpoint (β). To make the two systems physically equivalent, it would be necessary to impose appropriate initial conditions on the primitive system and to introduce the proper opposite forces at A and A' corresponding to the reactions at the hinge in the given system. This,

[5] Kron has suggested another method having advantages when the system and its interconnections are complicated.

however, would entail a knowledge of the reactions at A, and the advantage of the (β) viewpoint, which makes it the more appropriate viewpoint for the Kron method here, is that it enables one to proceed without bringing in the reactions at all, except indirectly insofar as they imply the equations of constraint. It is possible to use viewpoint (α) by introducing unknown opposite reactions at A and A' in the primitive, denoting them by some symbol, say R and $-R$; they will automatically cancel when the transition is made to the subspace.

To give some indication of the flexibility of the Kron method and its ability to extract cumulative dividends from such calculations as may previously have been performed, we conclude with a brief and sketchy discussion of two further systems.

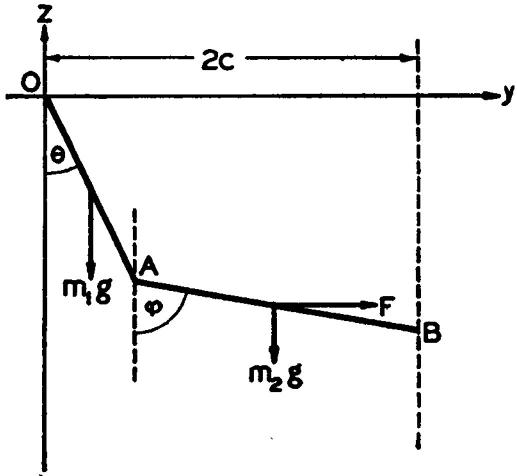

FIG. 3. System II.

Illustrative example II. Let us consider the system illustrated in Fig. 3. It is the same as system I above except that the end of the lower rod is constrained to move without friction on a fixed vertical line distant $2c$ from 0.

System I has already been investigated. It is now a known system. Instead, therefore, of taking the primitive of system II to be the same as the primitive of system I, we may take it to be system I itself.

The new constraint reduces the number of degrees of freedom to one and may be represented mathematically by the condition

$$a_1 \sin \theta + a_2 \sin \varphi = c. \tag{25}$$

The subspace now can be written parametrically as

$$\theta = \bar{\theta}, \qquad \varphi = \sin^{-1}\left\{\frac{c - a_1 \sin \bar{\theta}}{a_2}\right\}. \tag{26}$$

The transformation matrix C is given by

$$C = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}, \qquad \lambda = \frac{\partial \varphi}{\partial \bar{\theta}} = \frac{-a_1 \cos \bar{\theta}}{\sqrt{a_2^2 - (c - a_1 \sin \bar{\theta})^2}}. \tag{27}$$

By implicit differentiation of (25) we may also obtain the useful relation

$$a_1 \cos \theta + a_2 \lambda \cos \varphi = 0. \tag{28}$$

The new metrical tensor $\bar{m}_{\sigma\tau}$ (which here has only one component) may be obtained by the usual transformation formula, or the new expression for T may be obtained directly from (24) by substituting for φ and $\dot{\varphi}$ in terms of $\bar{\theta}$ and $\dot{\bar{\theta}}$ by means of (26), the latter method being the simpler in this particular problem. The generalized force is

$$\bar{f}_\sigma = [1 \ \ \lambda]\begin{bmatrix} -(m_1 + 2m_2)ga_1 \sin \bar{\theta} + 2aF \cos \bar{\theta} \\ a_2F \cos \varphi - a_2m_2g \sin \varphi \end{bmatrix}$$

$$= -(m_1 + 2m_2)ga_1 \sin \bar{\theta} + 2a_1F \cos \bar{\theta} + \lambda a_2F \cos \varphi - \lambda a_2m_2g \sin \varphi$$

$$= -(m_1 + 2m_2)ga_1 \sin \bar{\theta} + a_1F \cos \bar{\theta} + \frac{a_1m_2g(c - a_1 \sin \bar{\theta}) \cos \bar{\theta}}{\sqrt{a_2^2 - (c - a_1 \sin \bar{\theta})^2}}, \tag{29}$$

the terms in F partially cancelling in view of (28). The equation of motion may now be written down in the usual manner. Solving it is another matter!

Illustrative example III. The preceding example made use of system I in the role of a known system, and essentially dealt with the imposition of a constraint on that system. One may, however, join several known systems together by the Kron method, and this is, in fact, the procedure of principal importance in practical problems. To illustrate the idea, let us outline the method of attack on the system shown in Fig. 4,

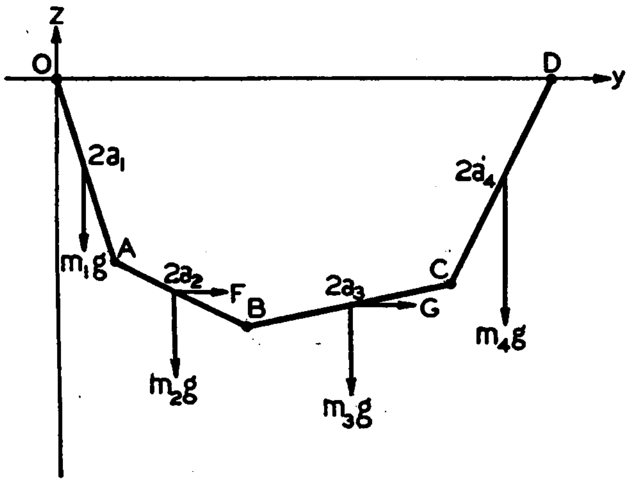

FIG. 4. System III.

the points O, D being fixed, and motion being confined to a vertical plane. This system may be regarded as system I interconnected with another system of the same type. Thus the primitive may be taken to consist of two systems of the type I, as shown in Fig. 5. For each system of type I the metrical tensor and force vector are known, being of the form (22) and (23). For brevity, we denote them in shape only by the following symbols:

$$\left[\begin{array}{c|c} (1) & \\ \hline & \end{array}\right], \quad \left[\begin{array}{c} (1') \\ \hline \\ \end{array}\right]; \qquad \left[\begin{array}{c|c} (2) & \\ \hline & \end{array}\right], \quad \left[\begin{array}{c} (2') \\ \hline \\ \end{array}\right].$$

Then for the whole primitive system the corresponding quantities are

$$\left[\begin{array}{cc|cc} (1) & & 0 & 0 \\ & & 0 & 0 \\ \hline 0 & 0 & (2) & \\ 0 & 0 & & \end{array}\right], \qquad \left[\begin{array}{c} (1') \\ \\ \hline (2') \\ \\ \end{array}\right],$$

which, of course, may be written down at once. The configuration space of the present primitive system is the direct product of the configuration spaces of the two systems of type I.

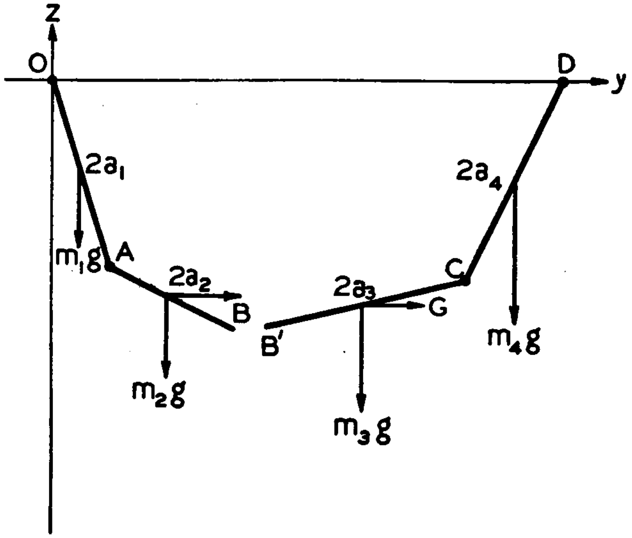

FIG. 5. Primitive system of system III.

The interconnection of the two systems at B introduces a single constraint, and by the method of subspaces the equations of motion of system III may be obtained in a routine manner.

The problems discussed above involved only very simple interconnections. When the interconnections are numerous and complicated, and the elements interconnected are themselves known complex dynamical, electrodynamical, or hydrodynamical systems, Kron's method of subspaces assumes the highest practical importance. In concluding the author wishes to thank Mr. Kron for many stimulating discussions of his work extending over several years.

Added April 12, 1944. While the general mathematical theory underlying Kron's method of subspaces as applied to dynamical systems is implied in the "Formal Proof" of the present paper, it is not there given in explicit detail. Professor Synge of the Ohio State University has suggested that a more explicit proof be included which goes directly to the mathematical basis of the method, and has kindly communicated the following outline of a method of proof from a different point of view which will be of interest to mathematicians wishing to see clearly what is fundamentally involved mathematically.

Let (a, b) (e, f) (i, j) take three different ranges of values, with the usual summation convention for each. Let there be two independent holonomic dynamical systems:

I. Coordinates: x^a,
K. Energy: $T_{(1)} = \frac{1}{2} m_{ab} \dot{x}^a \dot{x}^b$,
Generalized forces: X_a.

II. Coordinates: x^e,
K. Energy: $T_{(2)} = \frac{1}{2} m_{ef} \dot{x}^e \dot{x}^f$,
Generalized forces: X_e.

Let us define, with $D = d/dt$,

$$\begin{aligned} S_a &= D(\partial T_{(1)}/\partial \dot{x}^a) - \partial T_{(1)}/\partial x^a, \\ S_e &= D(\partial T_{(2)}/\partial \dot{x}^e) - \partial T_{(2)}/\partial x^e. \end{aligned} \tag{A}$$

Then the equations of motion of I and II are $S_a = X_a$, $S_e = X_e$.

Now establish constraints between I and II, the reactions of constraint being workless. These constraints may be written

$$x^a = x^a(x^i), \qquad x^e = x^e(x^i),$$

where x^i are the generalized coordinates of the system III resulting from the combination. Write

$$C_i^a = \partial x^a/\partial x^i, \qquad C_i^e = \partial x^e/\partial x^i.$$

We have then

$$\dot{x}^a = C_i^a \dot{x}^i, \qquad \dot{x}^e = C_i^e \dot{x}^i.$$

It is easy to prove that

$$\begin{aligned} DC_i^a &= \partial \dot{x}^a/\partial x^i, \qquad DC_i^e = \partial \dot{x}^e/\partial x^i, \\ C_i^a &= \partial \dot{x}^a/\partial \dot{x}^i, \qquad C_i^e = \partial \dot{x}^e/\partial \dot{x}^i. \end{aligned} \tag{B}$$

Let X'_a, X'_e be the reactions due to the constraint. We have

$$X'_a \delta x^a + X'_e \delta x^e = 0$$

for any displacement satisfying the constraints, i.e., for

$$\delta x^a = C_i^a \delta x^i, \qquad \delta x^e = C_i^e \delta x^i.$$

Hence

$$X'_a C_i^a + X'_e C_i^e = 0.$$

Now the equations of motion of I and II under the constraint are

$$S_a = X_a + X'_a, \qquad S_e = X_e + X'_e,$$

and hence

$$S_a C_i^a + S_e C_i^e = X_a C_i^a + X_e C_i^e. \tag{C}$$

We have for system III:
Coordinates: x^i,
K. Energy: $T_{(3)} = \frac{1}{2} m_{ij} \dot{x}^i \dot{x}^j$,
Generalized forces: X_i.

Let us define

$$S_i = D(\partial T_{(3)}/\partial \dot{x}^i) - \partial T_{(3)}/\partial x^i.$$

Our problem is to show how S_i, X_i are to be computed in terms of the elements of I and II. (We know from dynamical theory that $S_i = X_i$, but we can forget this knowledge, as we prove it incidentally below.)

We have

$$T_{(3)} = T_{(1)} + T_{(2)},$$

and hence

$$m_{ij} = m_{ab} C_i^a C_j^b + m_{ef} C_i^e C_j^f. \tag{D}$$

It is easily seen by direct transformation that

$$S_i = S_a C_i^a + S_e C_i^e. \tag{E}$$

By considerations of work we have

$$X_i \delta x^i = X_a \delta x^a + X_e \delta x^e,$$

and so

$$X_i = X_a C_i^a + X_e C_i^e. \tag{F}$$

Hence by equation (C), we have, as equations of motion of III, $S_i = X_i$, where S_i and X_i are given by (E) and (F). The transformation of the metric is given by (D).

We may sum up essentially by saying: *Metric and force transform by* (D) *and* (F) *when two systems are linked by workless constraints*. The extension to the linkage of any number of systems is immediate.