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KRONECKER FACTORIAL DESIGNS FOR MULTIWAY ELIMINATION OF HETEROGENEITY

RAHUL MUKERJEE[1]* AND MAUSUMI SEN[2]

[1]*Department of Mathematics, Faculty of Science, Chiba University, Yayoi-cho 1-33, Chiba 260, Japan; Stat-Math Division, Indian Statistical Institute, 203 Barrackpore Trunk Road, Calcutta 700035, India*
[2]*University of Kalyani, Kalyani 741235, Nadia, West Bengal, India*

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Abstract. This paper considers the application of Kronecker product for the construction of factorial designs, with orthogonal factorial structure, in a set-up for multiway elimination of heterogeneity. A technique involving the use of projection operators has been employed to show how a control can be achieved over the interaction efficiencies. A modification of the ordinary Kronecker product yielding smaller designs has also been considered. The results appear to have a fairly wide coverage.

Key words and phrases: Efficiency, Kronecker product, orthogonal array, orthogonal factorial structure, projection.

1. Introduction

A factorial design is said to have the orthogonal factorial structure (OFS) if the adjusted treatment sum of squares admits an orthogonal splitting into components corresponding to different factorial effects. The construction problem for factorial experiments in a block design with OFS has received considerable attention in recent years and broadly two general procedures emerged, namely, (a) the use of generalized cyclic designs (see John (1973), Dean and John (1975) and John and Lewis (1983) for a comprehensive list of references) and (b) the use of Kronecker or Kronecker-type products of varietal designs (see Mukerjee (1981, 1984, 1986) and Gupta (1983, 1985)). As for designs eliminating heterogeneity in several directions, however, it appears that much work yet remains to be done. Recently, John and Lewis (1983) extended the procedure (a) to row-column designs. The present paper aims at extending the procedure (b) to designs for multiway elimination of heterogeneity and hence, in particular, to row-column designs. For some early work in this connexion, see Zelen and Federer (1964).

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*Now at Division of Statistics, Northern Illinois University, Dekalb, IL 60115-2888, U.S.A.

In the procedure (b), an $s_1 \times s_2 \times \cdots \times s_m$ factorial design is constructed by taking a Kronecker or Kronecker-type product of m varietal designs involving $s_1, s_2, \ldots, s_m$ treatments, respectively. Since these varietal designs are usually easily available, the method has much flexibility. Further, the method is useful from a practical viewpoint provided the resulting factorial design has OFS and allows an efficient estimation of the contrasts belonging to the factorial effects of interest. A review of the literature on the procedure (b) shows that Mukerjee (1981, 1984) and Gupta (1983) considered methods of construction for factorial block designs with OFS employing Kronecker-type products controlling the main-effect efficiencies (see also Lewis and Dean (1985) in this context), while Gupta (1985) and Mukerjee (1986) explored the possibilities of controlling the interaction efficiencies as well. In the present paper, it is intended to extend all these results to a set-up for multiway elimination of heterogeneity. The principal new feature is that while the earlier results are based entirely on explicit evaluation of eigenvalues, in the set-up considered in this paper, such an explicit evaluation is difficult and, therefore, a more subtle approach involving projection operators has been used to simplify the derivation considerably. Also, compared to Gupta (1985), a broader definition of efficiency has been adopted and the results are all exact.

2. The method of Kronecker product

Throughout this paper, whether the design considered is varietal or factorial, the fixed effects model with independent, homoscedastic errors is assumed. For $1 \leq j \leq m$, let D_j be a varietal design for t-way heterogeneity elimination involving s_j treatments, n_j observations and having a design matrix

$$V_j = [Z_{j0}, Z_{j1}, \ldots, Z_{jt}] ,$$

where Z_{j0} is $n_j \times s_j$ and Z_{ja} is of order $n_j \times u_{ja}$ $(1 \leq a \leq t)$, u_{ja} being the number of classes according to the a-th way of heterogeneity elimination. For $0 \leq a \leq t$, in each row of Z_{ja} exactly one element equals unity and the rest equal zero. Hence,

$$Z_{j0} \mathbf{1}_{s_j} = Z_{j1} \mathbf{1}_{u_{j1}} = \cdots = Z_{jt} \mathbf{1}_{u_{jt}} = \mathbf{1}_{n_j} , \tag{2.1}$$

where $\mathbf{1}_n$ is an $n \times 1$ vector with all elements unity. The s_j columns of Z_{j0} correspond to the effects of the s_j treatments involved in D_j while for $1 \leq a \leq t$, the u_{ja} columns of Z_{ja} correspond to the effects of the u_{ja} classes according to the a-th way of heterogeneity elimination. Let D_j be equireplicate with common replication number r_j. Then, (2.2) $$n_j = r_j s_j\ , \quad Z'_{j0}\mathbf{1}_{n_j} = r_j\mathbf{1}_{s_j}\ , \quad Z'_{j0}Z_{j0} = r_j I_{s_j}\ ,$$

where I_s is the $s\times s$ identity matrix. The reduced normal equations for the treatment effects in D_j have the coefficient matrix

(2.3) $$C_j = Z'_{j0}(\mathrm{pr}^{\perp}(Z_j))Z_{j0}\ ,$$

where

(2.4) $$Z_j = [Z_{j1},\ldots, Z_{jt}]\ ,$$

and for any matrix L, $\mathrm{pr}(L)=L(L'L)^{-}L'$, $\mathrm{pr}^{\perp}(L)=I-\mathrm{pr}(L)$, and $(L'L)^{-}$ is any generalized inverse of $L'L$.

The Kronecker product of $D_1,\ldots, D_m$ is a design D (for t-way heterogeneity elimination) involving $\prod_{j=1}^{m} s_j$ ($=v$, say) treatments, $\prod_{j=1}^{m} n_j$ observations and having a design matrix

(2.5) $$V = \left[\bigotimes_{j=1}^{m} Z_{j0}, \bigotimes_{j=1}^{m} Z_{j1},\ldots, \bigotimes_{j=1}^{m} Z_{jt}\right],$$

where $\otimes$ stands for Kronecker product, the columns of $\bigotimes_{j=1}^{m} Z_{j0}$ correspond to the effects of the Πs_j treatments and for $1\le a\le t$, the columns of $\bigotimes_{j=1}^{m} Z_{ja}$ correspond to the classes according to the a-th way of heterogeneity elimination. Physically, this means that if, for $1\le j\le m$, the treatment i_j occurs in the $(l_{j1},\ldots, l_{jt})$-th "cell" of D_j, then the treatment $(i_1,\ldots, i_m)$ occurs in the $((l_{11},\ldots, l_{m1}), (l_{12},\ldots, l_{m2}),\ldots, (l_{1t},\ldots, l_{mt}))$-th "cell" of D. The Πs_j treatments in D may be interpreted as factorial level combinations and, in this sense, D represents an $s_1\times s_2\times\cdots\times s_m$ factorial design for t-way elimination of heterogeneity.

Example 2.1. Let $m=2$, $t=2$, $s_1=3$, $s_2=4$, and D_1, D_2 be row-column designs such that

D_1:

0	1	2
1	2	0

,

D_2:

0	3	1	-
2	1	-	0
3	-	2	1
-	2	0	3

.

Then their Kronecker product D is a 3×4 factorial design laid out in 8 rows and 12 columns as shown below.

D:

00	03	01	–	10	13	11	–	20	23	21	–
02	01	–	00	12	11	–	10	22	21	–	20
03	–	02	01	13	–	12	11	23	–	22	21
–	02	00	03	–	12	10	13	–	22	20	23
10	13	11	–	20	23	21	–	00	03	01	–
12	11	–	10	22	21	–	20	02	01	–	00
13	–	12	11	23	–	22	21	03	–	02	01
–	12	10	13	–	22	20	23	–	02	00	03

It may be noted that the rows and/or columns may be incomplete. Moreover, as in this example, some cells may be left empty.

Analogously to (2.3), the $v \times v$ coefficient matrix of the reduced normal equations for treatment effects in D is given by

$$C = \left(\bigotimes_{j=1}^{m} Z_{j0} \right)' (\mathrm{pr}^{\perp}(Z)) \left(\bigotimes_{j=1}^{m} Z_{j0} \right), \tag{2.6}$$

where

$$Z = \left[\bigotimes_{j=1}^{m} Z_{j1}, \ldots, \bigotimes_{j=1}^{m} Z_{jt} \right]. \tag{2.7}$$

In order to show that D, as a factorial design, has OFS, the following concepts and lemmas will be helpful. The proof of the first lemma is available in Mukerjee (1980) and hence omitted here.

Let Ω denote the set of non-null m-component (0, 1)-vectors. For any $x=(x_1, \ldots, x_m) \in \Omega$, define

$$G^x = \bigotimes_{j=1}^{m} G_j^{x_j}, \tag{2.8}$$

where

$$G_j^{x_j} = \begin{cases} I_{s_j} & \text{if} \quad x_j = 1, \\ \mathbf{1}_{s_j}\mathbf{1}'_{s_j} & \text{if} \quad x_j = 0. \end{cases} \tag{2.9}$$

LEMMA 2.1 (Mukerjee (1980)). *The design D has OFS if and only if for every $x \in \Omega$, G^x commutes with C (i.e., CG^x is symmetric).*

LEMMA 2.2. *Let $A_{10}, A_{11}, \ldots, A_{1t}$ be matrices with the same number of rows and $A_{20}, A_{21}, \ldots, A_{2t}$ be matrices with the same number of rows. Let $A=[A_{11} \otimes A_{21}, \ldots, A_{1t} \otimes A_{2t}]$, $A_1=[A_{11}, \ldots, A_{1t}]$. Assume that*

$$\mu(A_{20}) \subset \bigcap_{a=1}^{t} \mu(A_{2a}),$$

where for any matrix L, $\mu(L)$ denotes the column space of L. Then

$$\mathrm{pr}(A)(A_{10}\otimes A_{20}) = \{(\mathrm{pr}(A_1))A_{10}\}\otimes A_{20} .$$

PROOF. Clearly, there exist matrices B_a $(1\leq a\leq t)$ and a matrix $\Delta_1=[\Delta'_{11},\ldots,\Delta'_{1t}]'$, where the number of rows of Δ_{1a} equals the number of columns of A_{1a} such that

$$A_{20} = A_{2a}B_a \quad (1 \leq a \leq t); \quad A'_1A_1\Delta_1 = A'_1A_{10} , \tag{2.10}$$

so that

$$\mathrm{pr}(A_1)A_{10} = A_1\Delta_1 = \sum_{a=1}^{t} A_{1a}\Delta_{1a} . \tag{2.11}$$

Defining $\Delta=[\Delta'_{11}\otimes B'_1,\ldots,\Delta'_{1t}\otimes B'_t]'$, after a little algebra using (2.10), it follows that $A'A\Delta=A'(A_{10}\otimes A_{20})$, and hence, by (2.10) and (2.11),

$$\begin{aligned}\mathrm{pr}(A)(A_{10}\otimes A_{20}) &= A\Delta = \sum_{a=1}^{t}(A_{1a}\otimes A_{2a})(\Delta_{1a}\otimes B_a)\\ &= \sum_{a=1}^{t}(A_{1a}\Delta_{1a})\otimes(A_{2a}B_a) = \{\mathrm{pr}(A_1)A_{10}\}\otimes A_{20} ,\end{aligned}$$

completing the proof.

THEOREM 2.1. *The design D has OFS.*

PROOF. Take any $x=(x_1,\ldots, x_m) \in \Omega$. Without loss of generality (by a renaming of factors, if necessary) it may be assumed that $x_j=1$ $(1\leq j\leq f)$; $=0$ $(f+1\leq j\leq m)$. Then by (2.7) and (2.8),

$$\begin{aligned}&G^x = G^{(1)}\otimes G^{(2)}, \quad \bigotimes_{j=1}^{m} Z_{ja} = Z_a^{(1)}\otimes Z_a^{(2)} \quad (0 \leq a \leq t) ,\\ &Z = [Z_1^{(1)}\otimes Z_1^{(2)},\ldots, Z_t^{(1)}\otimes Z_t^{(2)}] ,\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}&G^{(1)} = \bigotimes_{j=1}^{f} I_{s_j}, \quad G^{(2)} = \bigotimes_{j=f+1}^{m} \mathbf{1}_{s_j}\mathbf{1}'_{s_j} ,\\ &Z_a^{(1)} = \bigotimes_{j=1}^{f} Z_{ja}, \quad Z_a^{(2)} = \bigotimes_{j=f+1}^{m} Z_{ja} \quad (0 \leq a \leq t) .\end{aligned} \tag{2.13}$$

By (2.12) and (2.13),

$$\left(\bigotimes_{j=1}^{m} Z_{j0}\right)G^x = \{Z_0^{(1)}G^{(1)}\}\otimes\{Z_0^{(2)}G^{(2)}\} = Z_0^{(1)}\otimes\{Z_0^{(2)}G^{(2)}\} . \tag{2.14}$$

Also, by (2.1) and (2.13), for $1 \leq a \leq t$,

$$Z_a^{(2)}\left\{\bigotimes_{j=f+1}^{m} \mathbf{1}_{u_{ja}}\mathbf{1}'_{s_j}\right\} = \bigotimes_{j=f+1}^{m} \{\mathbf{1}_{n_j}\mathbf{1}'_{s_j}\} = \bigotimes_{j=f+1}^{m} \{Z_{j0}\mathbf{1}_{s_j}\mathbf{1}'_{s_j}\} = Z_0^{(2)}G^{(2)}\,, \tag{2.15}$$

so that $\mu(Z_0^{(2)}G^{(2)}) \subset \bigcap_{a=1}^{t} \mu(Z_a^{(2)})$. Therefore, by (2.12), (2.14), (2.15) and Lemma 2.2,

$$\begin{aligned} \mathrm{pr}(Z)\left(\bigotimes_{j=1}^{m} Z_{j0}\right) G^x &= \{\mathrm{pr}(Z^{(1)})\,Z_0^{(1)}\}\otimes\{Z_0^{(2)}G^{(2)}\} \\ &= \{\mathrm{pr}(Z^{(1)})\,Z_0^{(1)}\}\otimes\left\{\bigotimes_{j=f+1}^{m} \mathbf{1}_{n_j}\mathbf{1}'_{s_j}\right\}, \end{aligned} \tag{2.16}$$

where

$$Z^{(1)} = [Z_1^{(1)}, \ldots, Z_t^{(1)}]\,. \tag{2.17}$$

By (2.2), (2.6), (2.12), (2.13), (2.16) and the standard rules for operations with Kronecker products, it follows that

$$\begin{aligned} CG^x &= \left(\bigotimes_{j=1}^{m} Z_{j0}\right)'\{I - \mathrm{pr}(Z)\}\left(\bigotimes_{j=1}^{m} Z_{j0}\right) G^x \\ &= \left(\bigotimes_{j=1}^{m} r_j I_{s_j}\right) G^x - \left(\bigotimes_{j=1}^{m} Z_{j0}\right)'\mathrm{pr}(Z)\left(\bigotimes_{j=1}^{m} Z_{j0}\right) G^x \\ &= \left(\prod_{j=1}^{m} r_j\right) G^x - \{Z_0^{(1)\prime}\mathrm{pr}(Z^{(1)})Z_0^{(1)}\}\otimes\left\{\bigotimes_{j=f+1}^{m} r_j\mathbf{1}_{s_j}\mathbf{1}'_{s_j}\right\}, \end{aligned} \tag{2.18}$$

which is evidently symmetric. Therefore, the result follows from Lemma 2.1.

3. The results on efficiency

We adopt a general definition of efficiency as indicated below. For every p $(0 \leq p \leq \infty)$ and every positive integer q, let $h_p^{(q)}$ be an extended real-valued function defined over the class $\Gamma^{(q)}$ of $q \times q$ non-negative definite (n.n.d.) matrices such that for any $B \in \Gamma^{(q)}$ with eigenvalues $\lambda_i(B)$ $(1 \leq i \leq q)$,

$$h_p^{(q)}(B) = \begin{cases} \left\{\prod_{i=1}^{q} \lambda_i(B)\right\}^{1/q} & \text{when} \quad p = 0\,, \\ \left\{q^{-1}\sum_{i=1}^{q} (\lambda_i(B))^{-p}\right\}^{-1/p} & \text{when} \quad 0 < p < \infty\,, \\ \min_{1 \leq i \leq q} \lambda_i(B) & \text{when} \quad p = \infty\,, \end{cases}$$

provided the $\lambda_i(B)$'s are all positive. If $\lambda_i(B)$'s are not all positive, then $h_p^{(q)}(B)=0$ $(0\leq p\leq\infty)$. For $1\leq j\leq m$, let P_j be an $(s_j-1)\times s_j$ matrix such that $[s_j^{-1/2}\mathbf{1}_{s_j}, P_j']'$ is an orthogonal matrix. Then (cf. Kiefer (1975)) the Φ_p-efficiency of the varietal design D_j is given by, say,

$$H_p^j = r_j^{-1}\, h_p^{(s_j-1)}\,(P_jC_jP_j') \quad (0\leq p\leq\infty)\,, \tag{3.1}$$

C_j being as in (2.3). Clearly, if $p=0$, 1, ∞, then Φ_p-efficiency reduces to the standard D-, A-, E-efficiencies, respectively.

Turning to the factorial set-up, for any $x=(x_1,\ldots, x_m)\in\Omega$, define

$$P^x = \bigotimes_{j=1}^{m} P_j^{x_j}\,, \tag{3.2}$$

where for $1\leq j\leq m$,

$$P_j^{x_j} = \begin{cases} P_j & \text{if} \quad x_j = 1\,, \\ s_j^{-1/2}\,\mathbf{1}_{s_j}' & \text{if} \quad x_j = 0\,. \end{cases} \tag{3.3}$$

Let τ be a $v\times 1$ vector of (factorial) treatment effects in D. Then (cf. Kurkjian and Zelen (1963) and Mukerjee (1981)) it may be seen that $P^x\tau$ represents a full set of orthonormal contrasts belonging to the factorial effect $F_1^{x_1}\cdots F_m^{x_m}$ $(=\zeta(x)$, say), where the m factors are denoted by $F_1,\ldots, F_m$. Let $\alpha(x)=\prod_{j=1}^{m}(s_j-1)^{x_j}$ be the number of rows of P^x and A_x denote the $\alpha(x)\times\alpha(x)$ coefficient matrix of the reduced normal equations for estimating $P^x\tau$ in D (cf. Kiefer (1975)). Then the Φ_p-efficiency of D with respect to the factorial effect $\zeta(x)$ is given by, say,

$$E_p^x = r^{-1}\, h_p^{(\alpha(x))}(A_x) \quad (0\leq p\leq\infty)\,, \tag{3.4}$$

where $r=\prod_{j=1}^{m} r_j$ is the number of replications in D.

The following lemmas will be helpful. Lemmas 3.1 and 3.2 are well-known while Lemma 3.3 follows from Poincare's separation theorem (see e.g., Rao (1973*a*, Chapter 1)).

LEMMA 3.1. *Let A_{1j}, A_{2j} be matrices such that $\mu(A_{1j})\subset\mu(A_{2j})$ $(1\leq j\leq\omega)$. Then*

$$\mu\Big(\bigotimes_{j=1}^{\omega} A_{1j}\Big)\subset\mu\Big(\bigotimes_{j=1}^{\omega} A_{2j}\Big)\,.$$

LEMMA 3.2. *If $\mu(A)\subset\mu(B)$, then* $\mathrm{pr}(B)-\mathrm{pr}(A)$ *is n.n.d.*

LEMMA 3.3. *For $q\times q$ n.n.d. matrices A, B, if $A-B$ is n.n.d., then* $h_p^{(q)}(A)\geq h_p^{(q)}(B)$ $(0\leq p\leq\infty)$.

The next theorem provides lower bounds for the efficiencies with respect to different factorial effects in D in terms of the coefficiencies of the varietal designs $D_1,\ldots, D_m$.

THEOREM 3.1. *For every $x=(x_1,\ldots, x_m)\in\Omega$, and every p $(0\leq p\leq\infty)$,* $E_p^x\geq\max_{1\leq j\leq m}\{x_jH_p^j\}$.

PROOF. By Theorem 2.1, the factorial design D has OFS and hence (cf. Mukerjee (1986)) for every $x\in\Omega$, $A_x=P^xCP^{x\prime}$, C being as in (2.6). As before, let without loss of generality $x=(x_1,\ldots, x_m)$ where $x_j=1$ $(1\leq j\leq f)$; $=0$ $(f+1\leq j\leq m)$. Then by (2.8), (2.9), (3.2) and (3.3),

$$P^x=\left(\prod_{j=f+1}^{m}s_j\right)^{-3/2}\left\{P^{(1)}\otimes\left(\bigotimes_{j=f+1}^{m}\mathbf{1}'_{s_j}\right)\right\}G^x\ , \tag{3.5}$$

where

$$P^{(1)}=\bigotimes_{j=1}^{f}P_j\ . \tag{3.6}$$

By (2.8), (2.9), (2.18) and (3.5), it follows after some simplification that

$$A_x=P^xCP^{x\prime}=\left(\prod_{j=f+1}^{m}r_j\right)\left\{\left(\prod_{j=1}^{f}r_j\right)I_{\alpha(x)}-P^{(1)}Z_0^{(1)\prime}\operatorname{pr}(Z^{(1)})\,Z_0^{(1)}P^{(1)\prime}\right\}, \tag{3.7}$$

where $Z^{(1)}$ is as in (2.17).

By Lemma 3.1, $\mu\left(\bigotimes_{j=1}^{f}Z_{ja}\right)\subset\mu\left(Z_{1a}\otimes\left(\bigotimes_{j=2}^{f}I_{n_j}\right)\right)$, $1\leq a\leq t$, so that by (2.4), (2.13) and (2.17), $\mu(Z^{(1)})\subset\mu\left(Z_1\otimes\left(\bigotimes_{j=2}^{f}I_{n_j}\right)\right)$. Consequently, by Lemma 3.2,

$$\operatorname{pr}\left\{Z_1\otimes\left(\bigotimes_{j=2}^{f}I_{n_j}\right)\right\}-\operatorname{pr}(Z^{(1)})=\{\operatorname{pr}(Z_1)\}\otimes\left(\bigotimes_{j=2}^{f}I_{n_j}\right)-\operatorname{pr}(Z^{(1)}) \tag{3.8}$$

is n.n.d. Now by (2.2), (2.3), (2.13) and (3.6), and the definition of the matrices P_j,

$$\begin{aligned}(P_1C_1P_1')\otimes\left(\bigotimes_{j=2}^{f}r_jI_{s_j-1}\right)&=\left(\prod_{j=1}^{f}r_j\right)I_{\alpha(x)}\\&\quad-(P_1Z_{10}'\operatorname{pr}(Z_1)Z_{10}P_1')\otimes\left(\bigotimes_{j=2}^{f}P_jZ_{j0}'\,I_{n_j}Z_{j0}\,P_j'\right)\end{aligned}$$

$$= \left(\prod_{j=1}^{f} r_j\right) I_{\alpha(x)}$$
$$- P^{(1)} Z_0^{(1)\prime} \left\{(\mathrm{pr}(Z_1)) \otimes \left(\bigotimes_{j=2}^{f} I_{n_j}\right)\right\} Z_0^{(1)} P^{(1)\prime} ,$$

so that by (3.7),

$$A_x - \left(\prod_{j=f+1}^{m} r_j\right)\left\{(P_1 C_1 P_1') \otimes \left(\bigotimes_{j=2}^{f} r_j I_{s_j-1}\right)\right\}$$
$$= \left(\prod_{j=f+1}^{m} r_j\right) P^{(1)} Z_0^{(1)\prime} \left[\{\mathrm{pr}(Z_1)\} \otimes \left(\bigotimes_{j=2}^{f} I_{n_j}\right) - \mathrm{pr}(Z^{(1)})\right] Z_0^{(1)} P^{(1)\prime} ,$$

which is n.n.d. in view of the n.n.d.-ness of the right-hand member of (3.8).

Consequently, by Lemma 3.3,

$$h_p^{(\alpha(x))}(A_x) \geq h_p^{(\alpha(x))} \left[\left(\prod_{j=f+1}^{m} r_j\right)\left\{(P_1 C_1 P_1') \otimes \left(\bigotimes_{j=2}^{f} r_j I_{s_j-1}\right)\right\}\right]$$
$$= \left(\prod_{j=2}^{m} r_j\right) h_p^{(s_1-1)} (P_1 C_1 P_1') \quad (0 \leq p \leq \infty).$$

Dividing the above by $\prod_{j=1}^{m} r_j$, it is immediate from (3.1) and (3.4) that $E_p^x \geq H_p^1$ ($0 \leq p \leq \infty$). Similarly, $E_p^x \geq H_p^j$ ($1 \leq j \leq f$; $0 \leq p \leq \infty$), and hence

$$E_p^x \geq \max_{1 \leq j \leq m} \{x_j H_p^j\} \quad (0 \leq p \leq \infty) ,$$

since $x_j = 0$ ($f+1 \leq j \leq m$). This completes the proof.

Remark. In view of Theorem 3.1, by choosing $D_1, \ldots, D_m$ suitably and then applying the method of Kronecker product, one can control and hence remain assured of the factorial effect efficiencies in D, in terms of the efficiencies of the varietal designs $D_1, \ldots, D_m$. This is important, since in practice it is often much easier to construct varietal designs rather than factorial designs. Theorems 2.1 and 3.1 make the task of construction of factorial designs, for multiway elimination of heterogeneity, rather simple. It is just enough to start from varietal designs $D_1, \ldots, D_m$ and to take their Kronecker product. Then by Theorem 2.1, the resulting factorial design D has OFS, whereas Theorem 3.1 guarantees that the factorial effect efficiencies in D will be high, provided $D_1, \ldots, D_m$ are efficient varietal designs.

In particular, if $\zeta(x)$ represents a main effect (i.e., $f=1$), then it is easy to see, from the proof of Theorem 3.1, that equality holds in the lower bound given by Theorem 3.1. On the other hand, if $\zeta(x)$ represents an interaction involving two or more factors, then very often, one gets the satisfying observation that the actual value of E_p^x is much greater than the corresponding lower bound. For example, for the designs in Example 2.1, it may be seen that D_1, D_2 are balanced with $H_p^1=0.75$, $H_p^2=0.6667$ $(0\leq p\leq\infty)$. By Theorem 3.1, therefore, for the resulting factorial design D, one obtains $E_p^{10}\geq 0.75$, $E_p^{01}\geq 0.6667$, $E_p^{11}\geq 0.75$. Actual computation shows that for E_p^{10}, E_p^{01}, these lower bounds are attained while the true value of E_p^{11} is as high as 0.975. Hence, the method is expected to be particularly useful when emphasis lies on the efficient estimation of the interaction contrasts.

4. The restricted Kronecker product

Although Theorems 2.1 and 3.1 make the method of Kronecker product attractive from theoretical considerations, one practical difficulty may arise with this method, when the number of factors, m, is large in the sense that the number of observations in D, namely $\prod_{j=1}^{m} n_j$, may then become prohibitively large. To overcome this difficulty, one may consider a method of construction which guarantees OFS but exercises a control only over the lower order interaction efficiencies. Such an approach appears to be reasonable since, especially when the number of factors is large, not much interest usually lies in the higher order interactions. To that effect, we consider below a modified version of the method of Kronecker product.

With notations as in Section 2, suppose for $1\leq j\leq m$ and $0\leq a\leq t$, it is possible to partition Z_{ja} as

$$Z_{ja} = [Z'_{ja1}, Z'_{ja2}, \ldots, Z'_{jaw_j}]' , \tag{4.1}$$

where for $1\leq l\leq w_j$, Z_{jal} has $n_j w_j^{-1}$ $(=\beta_j$, say) rows, such that

$$\mathbf{1}'_{\beta_j} Z_{ja1} = \mathbf{1}'_{\beta_j} Z_{ja2} = \cdots = \mathbf{1}'_{\beta_j} Z_{jaw_j} \quad (= \psi'_{ja}, \text{ say}) \qquad (1 \leq a \leq t;\ 1 \leq j \leq m) , \tag{4.2}$$

$$\mathbf{1}'_{\beta_j} Z_{j0l} = (r_j w_j^{-1}) \mathbf{1}'_{s_j} \quad (1 \leq l \leq w_j;\ 1 \leq j \leq m). \tag{4.3}$$

By (2.1) and (4.1), for $1\leq l\leq w_j$ and $1\leq j\leq m$,

$$Z_{j0l}\mathbf{1}_{s_j} = Z_{j1l}\mathbf{1}_{u_{j1}} = \cdots = Z_{jtl}\mathbf{1}_{u_{jt}} = \mathbf{1}_{\beta_j} . \tag{4.4}$$

Also, recalling that for $1\leq j\leq m$, in each row of Z_{j0} exactly one element equals unity and the rest equal zero, it follows from (4.1) and (4.3) that

$$Z'_{j0l} Z_{j0l} = (r_j w_j^{-1}) I_{s_j} \quad (1 \leq l \leq w_j;\ 1 \leq j \leq m) . \tag{4.5}$$

Physically, the partitioning (4.1) means that for $1\leq j\leq m$, the varietal design D_j

is partitioned into w_j subdesigns such that each subdesign involves β_j observations, in each subdesign, each of the s_j treatments is replicated $r_j w_j^{-1}$ times and the condition (4.2) holds. In many practical situations, such a partitioning can be attained in a natural way. An illustrative example in this connexion will be presented at the end of this section.

In the following, for matrices $L_1, \ldots, L_\omega$ having the same number of columns, we define $\bigcup_{i=1}^{\omega} L_i = [L_1', \ldots, L_\omega']'$. Then the restricted Kronecker product of order $g(\leq m)$ of $D_1, \ldots, D_m$ is a design $D^{(g)}$ involving $\prod_{j=1}^{m} s_j$ treatments and having a design matrix

$$V^{(g)} = \bigcup_{(\gamma_1, \ldots, \gamma_m)' \in T} \left[\bigotimes_{j=1}^{m} Z_{j0\gamma_j}, \bigotimes_{j=1}^{m} Z_{j1\gamma_j}, \ldots, \bigotimes_{j=1}^{m} Z_{jt\gamma_j} \right],$$

the union being taken over only a subset T of the $\prod_{j=1}^{m} w_j$ possible combinations $(\gamma_1, \ldots, \gamma_m)'$ such that the combinations included in T, written as columns, form an orthogonal array (possibly with variable symbols) with m rows, strength g and $w_1, \ldots, w_m$ symbols (cf. Rao (1973*b*)). As before, $D^{(g)}$ may be interpreted as an $s_1 \times \cdots \times s_m$ factorial design for t-way elimination of heterogeneity and if N be the cardinality of T, then the number of observations required in $D^{(g)}$ is easily seen to be $N\left(\prod_{j=1}^{m} n_j w_j^{-1}\right)$ which is less than the number of observations, $\prod_{j=1}^{m} n_j$, in the ordinary Kronecker product design D, whenever the orthogonal array T is non-trivial, i.e., whenever $N < \prod_{j=1}^{m} w_j$. Note that in $D^{(g)}$ each of the v (factorial) treatments is replicated $N\left(\prod_{j=1}^{m} r_j w_j^{-1}\right)$ $(=r^{(g)}$, say) times. In particular, if $g=m$, then the restricted Kroncker product reduces to ordinary Kronecker product. Theorems 4.1 and 4.2 below extend Theorems 2.1 and 3.1 to the present set-up.

THEOREM 4.1. *The design $D^{(g)}$ has OFS.*

PROOF. Defining

$$Q_a = \bigcup_{(\gamma_1, \ldots, \gamma_m)' \in T} \left\{ \bigotimes_{j=1}^{m} Z_{ja\gamma_j} \right\} \quad (0 \leq a \leq t); \quad Q = [Q_1, \ldots, Q_t],$$

the $v \times v$ coefficient matrix of the reduced normal equations for the (factorial) treatment effects in $D^{(g)}$ is given by, say,

$$C^{(g)} = Q_0' \operatorname{pr}^{\perp}(Q)\, Q_0, \tag{4.6}$$

which is analogous to (2.6). In order to apply Lemma 2.1, one must show that $C^{(g)}G^x$ is symmetric for every $x \in \Omega$. Without loss of generality, let $x=(x_1,\ldots, x_m)$, where $x_j=1$ $(1\leq j\leq f)$; $=0$ $(f+1\leq j\leq m)$. Then as in (2.12), $G^x=G^{(1)}\otimes G^{(2)}$, where $G^{(1)}$, $G^{(2)}$ are defined by (2.13). Let

$$(4.7) \qquad Q_a^{(1)} = \bigcup_{(\gamma_1,\ldots,\gamma_m)'\in T} \left\{ \bigotimes_{j=1}^{f} Z_{ja\gamma_j} \right\} \quad (0 \leq a \leq t); \qquad Q^{(1)} = [Q_1^{(1)},\ldots, Q_t^{(1)}] .$$

Clearly, there exists a matrix $\Delta_1=[\Delta_{11}',\ldots, \Delta_{1t}']'$, where the number of rows of Δ_{1a} equals the number of columns of $Q_a^{(1)}$ $(1\leq a\leq t)$ such that

$$(4.8) \qquad Q^{(1)\prime}Q^{(1)}\Delta_1 = Q^{(1)\prime}Q_0^{(1)}G^{(1)}.$$

Now if one defines

$$(4.9) \qquad \Delta = \left[\Delta_{11}'\otimes\left(\bigotimes_{j=f+1}^{m} \mathbf{1}_{u_{j1}}\mathbf{1}_{s_j}'\right)',\ldots, \Delta_{1t}'\otimes\left(\bigotimes_{j=f+1}^{m} \mathbf{1}_{u_{jt}}\mathbf{1}_{s_j}'\right)'\right]',$$

(recall that u_{ja} is the number of columns in Z_{ja}) and applies (2.13), (4.2), (4.4) and (4.8), it follows after considerable algebra that

$$(4.10) \qquad Q'Q\Delta = Q'Q_0G^x,$$

since both sides of (4.10) equal to

$$\bigcup_{a=1}^{t}\left[\left\{\sum_{(\gamma_1,\ldots,\gamma_m)'\in T} \bigotimes_{j=1}^{f}(Z_{ja\gamma_j}'Z_{j0\gamma_j})\right\}\otimes\left\{ \bigotimes_{j=f+1}^{m} (\psi_{ja}\mathbf{1}_{s_j}')\right\}\right] .$$

The details of this derivation follow essentially along the line of proof of Lemma 2.2 but are omitted here to save space. From (4.9) and (4.10),

$$(4.11) \quad Q_0'\mathrm{pr}(Q)Q_0G^x = Q_0'Q\Delta = \{Q_0^{(1)\prime}\mathrm{pr}(Q^{(1)})\, Q_0^{(1)}\}\otimes\left\{ \bigotimes_{j=f+1}^{m} (r_jw_j^{-1}\mathbf{1}_{s_j}\mathbf{1}_{s_j}')\right\} ,$$

again after some algebra based on applications of (4.3) and (4.4). Evidently, $Q_0'\mathrm{pr}(Q)Q_0G^x$ is symmetric. Also by (4.5) and the definition of Q_0,

$$(4.12) \qquad Q_0'Q_0 = N\left(\prod_{j=1}^{m} r_jw_j^{-1}\right) \bigotimes_{j=1}^{m} I_{s_j} ,$$

as defined earlier N being the cardinality of T. From (4.12), $Q_0'Q_0G^x$ is symmetric. Hence, by (4.6), $C^{(g)}G^x$ is symmetric, completing the proof.

For our next result, the notations are as in Section 3, the only change being that for any $x \in \Omega$, the coefficient matrix of the reduced normal equations for estimating $P^x\tau$ in $D^{(g)}$ is denoted by $A_x^{(g)}$ and accordingly,

$$E_p^x(g) = r^{(g)-1} h_p^{(\alpha(x))}(A_x^{(g)})$$

represents the Φ_p-efficiency of $D^{(g)}$ with respect to the factorial effect $\zeta(x)$.

THEOREM 4.2. *For $x=(x_1,\dots, x_m) \in \Omega$ and every p $(0 \le p \le \infty)$*

$$E_p^x(g) \ge \max_{1\le j\le m} \{x_j H_p^j\} ,$$

provided among $x_1,\dots, x_m$ at most g are unity.

PROOF. Without loss of generality take $x=(x_1,\dots, x_m)$ where $x_j=1$ $(1\le j\le f)$; $=0$ $(f+1\le j\le m)$ and $f\le g$. Since the combinations $(\gamma_1,\dots, \gamma_m)'$ included in T form an orthogonal array with N assemblies and strength g (and hence with strength f, for $f\le g$), it follows that for every $(\gamma_1,\dots, \gamma_f)'$ $(1\le \gamma_j \le w_j;$ $1\le j\le f)$ there are exactly $N\left(\prod_{j=1}^{f} w_j\right)^{-1}$ combinations in T with the first f entries equal to $\gamma_1,\dots, \gamma_f$, provided $f\le g$. Hence by (4.1) and (4.7), for any a, k $(0\le a,$ $k\le t)$,

$$\begin{aligned}
(4.13)\qquad Q_a^{(1)\prime} Q_k^{(1)} &= \sum_{(\gamma_1,\dots,\gamma_m)'\in T} \left\{ \bigotimes_{j=1}^{f} (Z'_{ja\gamma_j} Z_{jk\gamma_j}) \right\} \\
&= \left(N\Big/\prod_{j=1}^{f} w_j\right) \sum_{\gamma_1=1}^{w_1} \cdots \sum_{\gamma_f=1}^{w_f} \left\{ \bigotimes_{j=1}^{f} (Z'_{ja\gamma_j} Z_{jk\gamma_j}) \right\} \\
&= \left(N\Big/\prod_{j=1}^{f} w_j\right) \bigotimes_{j=1}^{f} \left\{ \sum_{\gamma_j=1}^{w_j} Z'_{ja\gamma_j} Z_{jk\gamma_j} \right\} \\
&= \left(N\Big/\prod_{j=1}^{f} w_j\right) \bigotimes_{j=1}^{f} \{Z'_{ja} Z_{jk}\} = \left(N\Big/\prod_{j=1}^{f} w_j\right) Z_a^{(1)\prime} Z_k^{(1)} ,
\end{aligned}$$

whenever $f\le g$. In the above, $Z_a^{(1)}$, $Z_k^{(1)}$ are as defined in (2.13). By (2.13), (2.17), (4.7) and (4.13), it now follows that

$$Q^{(1)\prime} Q^{(1)} = \left(N\Big/\prod_{j=1}^{f} w_j\right) Z^{(1)\prime} Z^{(1)}, \qquad Q^{(1)\prime} Q_0^{(1)} = \left(N\Big/\prod_{j=1}^{f} w_j\right) Z^{(1)\prime} Z_0^{(1)} ,$$

whenever $f\le g$. Therefore, for $f\le g$,

$$Q_0^{(1)\prime} \mathrm{pr}(Q^{(1)})\, Q_0^{(1)} = \left(N\Big/\prod_{j=1}^{f} w_j\right) \{Z_0^{(1)\prime} \mathrm{pr}(Z^{(1)})\, Z_0^{(1)}\} .$$

Hence by (4.6), (4.11) and (4.12), for $f\le g$,

$$C^{(g)}G^x = \left\{N \prod_{j=1}^{m} (r_j w_j^{-1})\right\} G^x - \left(N \Big/ \prod_{j=1}^{f} w_j\right)\{Z_0^{(1)\prime}\mathrm{pr}(Z^{(1)})\, Z_0^{(1)}\} \otimes \left\{ \bigotimes_{j=f+1}^{m} (r_j w_j^{-1}\, \mathbf{1}_{s_j} \mathbf{1}_{s_j}') \right\},$$

which is analogous to (2.18). The rest of the proof may now be completed proceeding along the line of Proof of Theorem 3.1.

In view of Theorem 4.2, applying the method of restricted Kronecker product of order g, one can control and hence remain assured of the factorial effect efficiencies in $D^{(g)}$, for effects involving up to g factors, in terms of the efficiencies of $D_1, \ldots, D_m$. In particular, if it is desired only to control the main effect efficiencies, then $g=1$ and T should represent an orthogonal array of strength 1, which can be obtained very easily. If, in addition, it is desired to control the two-factor interaction efficiencies, then $g=2$ and T should be an orthogonal array of strength 2. This also poses no major combinatorial problem since orthogonal arrays of strength 2 are available in plenty (see e.g., Raghavarao (1971)).

As indicated earlier, in many situations there exists a natural way of attaining the partitioning (4.1) such that (4.2) and (4.3) are satisfied. For example, considering a set-up of row-column designs (i.e., $t=2$), suppose D_j is a complete or an incomplete latin square which can be partitioned into disjoint transversals such that each transversal contains each of the s_j treatments in D_j exactly once. Then these transversals provide a natural way of attaining a partitioning (4.1) such that (4.2) and (4.3) hold. These considerations indicate that the method of restricted Kronecker product has a wide applicability. The following example serves as an illustration.

Example 4.1. To construct a $4 \times 5 \times 7$ factorial row-column design, take D_1, D_2 and D_3 as incomplete latin squares given by

D_1:

0	2	3	–
3	1	–	2
1	–	2	0
–	0	1	3

, D_2:

–	1	–	3	4
1	–	3	–	0
2	3	–	0	–
–	4	0	–	2
4	–	1	2	–

, D_3:

0	–	2	–	–	5	–
–	2	–	4	–	–	0
2	–	4	–	6	–	–
–	4	–	6	–	1	–
–	–	6	–	1	–	3
5	–	–	1	–	3	–
–	0	–	–	3	–	5

.

Here $s_1=4$, $s_2=5$ and $s_3=7$. In each of these squares the cells will be denoted by ordered pairs (y_1, y_2) $(y_1, y_2=1, 2, \ldots)$. Then a partitioning of D_1 as in (4.1) which satisfies (4.2) and (4.3) (with $w_1=3$) is given by the three sets of cells: $\{(1,1),(2,2),(3,3),(4,4)\}$, $\{(1,2),(2,1),(3,4),(4,3)\}$, $\{(1,3),(2,4),(3,1),(4,2)\}$.
A similar partitioning of D_2 with $w_2=3$ is given by the three sets of cells:

$\{(1,2),(2,3),(3,4),(4,5),(5,1)\}$, $\{(1,4),(2,5),(3,1),(4,2),(5,3)\}$, $\{(1,5),(2,1),(3,2),(4,3),(5,4)\}$,
and a partitioning of D_3 with $w_3=3$ is given by the three sets of cells: $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7)\}$, $\{(1,3),(2,4),(3,5),(4,6),(5,7),(6,1),(7,2)\}$, $\{(1,6),(2,7),(3,1),(4,2),(5,3),(6,4),(7,5)\}$.
Note that $w_1=w_2=w_3=3$. Hence taking $T=\{(1,1,1),(1,2,2),(1,3,3),(2,1,2),(2,2,3),(2,3,1),(3,1,3),(3,2,1),(3,3,2)\}$, which is an orthogonal array of strength 2, and applying the method of restricted Kronecker product (with $g=2$), one can get a $4\times5\times7$ factorial row-column design, say $D^{(2)}$, which has OFS and in which the main effect and two-factor interaction efficiencies are controlled in the sense of Theorem 4.2. Note that the cardinality of T is 9 while $\prod_{j=1}^{3} w_j=27$, so that the number of observations required in $D^{(2)}$ is only one-thirds the number of observations required in the ordinary Kronecker product of D_1, D_2 and D_3.

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