KRULL DIMENSION AND NOETHERIANNESS

BY

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This paper essentially deals with the following question: Over what rings are all modules with Krull dimension (as defined in [1]; Gordon-Robson A.M.S. memoirs) noetherian. The problem originates with [2] Theorem 4.2 where it is shown that V-rings of Krull-dimension 1 are, among many other things, noetherian. In attempting to generalize this to higher dimensions it transpires that all modules with Krull dimension over a V-ring are noetherian and this leads to the question of other rings having this property. We tackle this mainly in Section 2 devoting Section 1 to proving the following somewhat independent result about noetherian V-rings: a ring R is a noetherian V-ring if and only if every R-module M has a minimal generating set and given a submodule N of M every minimal generating set of N can be extended to a minimal generating set of M. In Section 2 we prove (Theorem 2.8) that over a ring R, every module with Krull-dimension is noetherian if and only if every non-noetherian module has a proper non-noetherian submodule. Constructing an analogue of $Z_p\infty$ we show that there are non-noetherian modules with Krull-dimension over a polynomial ring and also study the case of group rings.

It has been pointed out to me that M. Teply has proved that V-rings with Krull dimension are noetherian using Proposition 1.5 and the resulting isomorphism constructed in the proof of our Theorem 1.6, though the work is unpublished. I would also like to acknowledge my indebtedness to Dr. K. Varadarajan for his valuable advice and the many improvements he suggested.

All the rings in this paper possess a unit and all modules are left unital. All properties will be assumed to be left properties e.g., "ideal" will mean "left ideal". The symbol $\langle\ \rangle$ will denote "module generated by by" i.e., $\langle C\rangle$ will mean the module generated by C, $\sim$ will denote set theoretic complement and $\{b\}$ the singleton set consisting of b.

1. Irredundant and redundant subsets of a module

DEFINITION 1.1. Let M be a module, B a subset of M. We say B is irredundant iff $A \subseteq B$, $\langle A\rangle = \langle B\rangle \Rightarrow A = B$. If B is not irredundant we call it redundant.

Remarks 1.2. (i) If $B \subseteq M$ is irredundant and $A \subseteq B$ then A is irredundant.

(ii) If $\{B_\alpha\}_{\alpha\in J}$ is a family of irredundant subsets of M totally ordered by inclusion then $\bigcup_\alpha B_\alpha$ is irredundant.

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(iii) B is redundant iff for some subset $A \subseteq B$, $\langle A \rangle = \langle A \sim \{a\} \rangle$ for some $a \in A$.

We need the following lemma which is straightforward to verify.

LEMMA 1.3. *Let B be an irredundant subset of M, $\{M_b\}_{b \in B}$ a collection of maximal submodules of M satisfying $b \notin M_b$ and $\langle B \sim \{b\} \rangle \subset M_b$. Let $I = \langle B \rangle$, $N = \bigcap_{b \in B} M_b$ and*

$$j: \frac{M}{N} \to \prod_{b \in B} \frac{M}{M_b}$$

the natural imbedding (the bth coordinate of $j(x + N)$ is $x + M_b$). Then j maps $(I + N)/N$ isomorphically onto $\oplus_{b \in B} M/M_b$.

DEFINITION 1.4. A ring is called a V-ring iff every simple R-module is injective.

The following is well known [2; Theorem 2.1].

PROPOSITION 1.5. *Let R be a ring M any R-module. Let $I \subset M$ be any submodule and $a \in M \sim I$. Then R is a V-ring iff there exists a maximal submodule $N \subset M$ with $a \notin N$ and $I \subset N$.*

THEOREM 1.6. *The following are equivalent for a ring R.*

(i) *R is a noetherian V-ring.*

(ii) *Given any irredundant generating set of a submodule N of any module M, it can be extended to an irredundant generating set for M. In particular for every R-module M there exists an irredundant generating set.*

Proof (i) $\Rightarrow$ (ii). Let C be an irredundant, generating set for N. Let

$$E = \{B \mid C \subseteq B \subseteq M \text{ with } B \text{ irredundant}\}.$$

E is non-empty, and when partially ordered by inclusion, by 1.2 (ii) and Zorn's lemma, has a maximal element say B. Suppose $\langle B \rangle \neq M$. From 1.5 and the irredundancy of B, there exist maximal submodules $\{M_b\}_{b \in B}$ of M with $b \notin M_b$ and $\langle B \sim \{b\} \rangle \subseteq M_b$. Let I, N, J be as in 1.3. The proof now breaks into two cases.

Case (1). $I \not\supset N$. Pick $u \in N$, $u \notin I$. Then $B' = B \cup \{u\}$ is irredundant by 1.2 (iii) since $u \notin \langle B \rangle = I$ and $b \notin \langle B' \sim \{b\} \rangle \subseteq \langle B \sim \{b\} \rangle + N \subseteq M_b$, contradicting maximality of B.

Case (2). $I \supset N$. By 1.3, $(I + N)/N$ is isomorphic to $\oplus_{b \in B} M/M_b$ which is injective since R is a noetherian V-ring and each M/M_b is simple. Therefore $(I + N)/N = I/N$ $(I \supset N)$ is a direct summand of M/N. Let $M/N = (I/N) \oplus (I'/N)$ where I' is a submodule of M. Then $I \cap I' = N$ and $I' \neq 0$ since

$I \neq M$ by hypothesis. Let $0 \neq u \in I'$. Again $B' = B \cup \{u\}$ is irredundant by 1.2 (iii) since $u \notin \langle B \rangle = I$ and

$$\langle B' \sim \{b\} \rangle \cap I \subseteq (\langle B \sim \{b\} \rangle + I') \cap I \subseteq \langle B \sim \{b\} \rangle + N \subseteq M_b,$$

and hence $b \notin \langle B' \sim \{b\} \rangle$ for all $b \in B$. This contradicts maximality of B, and hence we conclude that $\langle B \rangle = M$ proving (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i). Let $\{S_k\}_{k \geq 1}$ be a countable family of simple R-modules, $S = \bigoplus_{k \geq 1} S_k$ and E the injective hull of S. We show $E = S$ and hence conclude by [3; Proposition 1] that R is a noetherian V-ring. Suppose $E \neq S$. Let $0 = x_k \in S_k$ for $k \geq 1$. Then $C = \{x_k\}_{k \geq 1}$ is an irredundant generating for S and let $D \supsetneqq C$ be an irredundant generating set for E. Let $x \in D \sim C$. Since $x \neq 0$ and S is essential in E, there exist λ, $\lambda_i \in R$, $1 \leq i \leq n$ with $\lambda x = \sum_{i=1}^{n} \lambda_i x_i$ where $\lambda_n x_n \neq 0$. Hence there exists $\mu \in R$ with $\mu \lambda_n x_n = x_n$, so $\mu(\lambda x - \sum_{i=1}^{n-1} \lambda_i x_i) = x_n$ contradicting irredundancy of D. Hence $E = S$.

2. Tall rings

DEFINITION 2.1. A module M is said to be tall iff there exists a submodule $N \subset M$ with both N and M/N non-noetherian.

Clearly, if either a submodule or quotient module of M is tall then M itself is tall.

LEMMA 2.2. *Let M have an infinite irredundant set. Then M is tall.*

Proof. Let B be the infinite irredundant set and let $B = C \cup D$ where C and D are infinite and $C \cap D = \emptyset$. Let $N = \langle C \rangle$. It is then clear that N and M/N are non-noetherian.

DEFINITION 2.3. A ring R will be called a tall ring iff every non-neotherian R-module is tall.

Given any module M over a ring R we define two submodules $G(M)$ and $H(M)$ of M as follows:

DEFINITION 2.4. If M is noetherian we set $G(M) = H(M) = 0$. In case M is not noetherian

$$G(M) = \bigcap \{N \mid N \text{ a submodule of } M \text{ with } M/N \text{ noetherian}\}$$
$$H(M) = \bigcap \{N \mid N \text{ a non-noetherian submodule of } M\}.$$

When M is not noetherian, and N a submodule of M such that M/N is noetherian, then clearly N is not noetherian. It follows that $H(M) \subset G(M)$. Also if I is a maximal submodule of M then M/I is simple and hence noetherian. Hence $H(M) \subset G(M) \subset J(M)$ where $J(M)$ is the Jacobson radical of M.

LEMMA 2.5. *If $M/G(M)$ is noetherian and $H(G(M)) \neq G(M)$ then M is tall.*

Proof. From $H(G(M)) \neq G(M)$ we see immediately that $G(M) \neq 0$ and hence that M is not noetherian. Since $H(G(M)) \neq G(M)$ let $P \subset_{\neq} G(M)$ be such that P is not noetherian. $M/G(M)$ is noetherian by assumption so $G(M)/P$ noetherian $\Rightarrow M/P$ noetherian $\Rightarrow G(M) \subset P$ a contradiction. Since $G(M)/P \subset M/P$, M is tall.

PROPOSITION 2.6. *Let M be a module that is not finitely generated. Suppose that $H(G(M/I)) = G(M/I)$ for any finitely generated submodule I of M, with the property that $G(M/I)$ is non-noetherian. Suppose further that $G(M) = 0$. Then M is tall.*

Proof. The proof splits into two cases.

Case (1). Suppose for some finitely generated submodule I, $G(M/I)$ is non-noetherian. Let $P = (M/I)/(G(M/I))$. If P is non-noetherian clearly M/I is tall. If P is noetherian, by 2.5, since $H(G(M/I)) \neq G(M/I)$, M/I is tall. Hence M itself is tall.

Case (2). Suppose $G(M/I)$ is noetherian for every finitely generated submodule I of M. In this case, we show that there exists an infinite irredundant subset of M. Let $0 \neq a_1 \in M$. Then clearly, the singleton $\{a_1\}$ is an irredundant subset of M. Since $G(M) = 0$, there exists a submodule N_1 of M with M/N_1 noetherian and $a_1 \notin N_1$. Taking this as a first step, assume inductively that we have determined distinct elements $a_1, \ldots, a_r$ of M, submodules $N_1, \ldots, N_r$ of M satisfying the following conditions:

(i) $\{a_1, \ldots, a_r\}$ is an irredundant set of M;
(ii) M/N_i is neotherian;
(iii) $a_i \notin N_i$;
(iv) $a_j \in N_i$ whenever $1 \leq i \neq j \leq r$.

Let $N = \bigcap_{i=1}^r N_i$, $M/I = \overline{M}$ and $\eta\colon M \to \overline{M}$ the canonical quotient map. M/N is noetherian since it imbeds into $\prod_{i=1}^r M/N_i$. $\overline{M}$ is not noetherian since I is finitely generated and M is not, and hence $\eta(N)$ is not noetherian. So there exists $\bar{a} \in \eta(N)$, $\bar{a} \notin G(\overline{M})(G(\overline{M})$ noetherian by hypothesis). Let $\bar{a} \notin \overline{P}$ where $\overline{P} \subset \overline{M}$ is such that $\overline{M}/\overline{P}$ is noetherian. Pick $a_{r+1} \in \eta^{-1}(\bar{a})$ and set $N_{r+1} = \eta^{-1}(\overline{P})$. It is easy to verify that the induction postulates (i) to (iv) are satisfied for $i = r + 1$. Let $B_r = \langle a_1, \ldots, a_r \rangle$ for $r \geq 1$ and set $B = \bigcup_{r \geq 1} B_r$. Then B is an infinite irredundant set and M is tall by 2.2.

We recall the definition of Krull-dimension as outlined by Gordon and Robson [1]. The Krull dimension (written K-dim) of a module is defined by transfinite recursion as follows: K-dim $M = -1$ when $M = 0$. Given an ordinal α, and assuming that the concept K-dim $N < \alpha$ is already defined, then K-dim M is defined to be α if K-dim $M \nless \alpha$ and there exists no descending sequence $M = I_0 \supset I_1 \supset \cdots$ of submodules of M with K-dim $(I_{i-1}/I_i) \nless \alpha$ for $i \geq 1$.

THEOREM 2.7. *The following conditions are equivalent for a ring R.*

(i) *Every R-module with Krull dimension is noetherian.*

(ii) *R is a tall ring.*

(iii) *Every non-noetherian R-module has a proper non-noetherian submodule.*

Proof. (i) ⇒ (ii). To show that R is a tall ring, it suffices to show that if M is an R-module with the property that for any submodule N of M, one of N or M/N is noetherian, then M itself is noetherian. If we assume (1) we have only to show that such an M has Krull dimension. Let

$$\begin{aligned} \alpha &= \sup \{K\text{-dim } N \mid N \subset M,\ N \text{ noetherian}\} \\ \beta &= \sup \{K\text{-dim } M/N \mid N \subset M,\ M/N \text{ noetherian}\} \\ \gamma &= \sup (\alpha, \beta). \end{aligned}$$

Given any descending sequence $M = M_0 \supset M_1 \supset \cdots$ it is clear that K-dim $(M_{i-1}/M_i) \leq \gamma$ for $i \geq 1$ and hence that M has Krull dimension.

(ii) ⇒ (i). The proof is by transfinite induction. The only module of Krull-dimension -1 is 0 and clearly 0 is noetherian. Let α be any ordinal and assume that all modules with Krull dimension $<\alpha$ have been shown to be noetherian. Now suppose M is a module of Krull-dimension α. If M is not noetherian using the fact that R is a tall ring, we can construct a descending sequence $M = M_0 \supset M_1 \supset M_2 \supset \cdots$ of submodules of M with M_i and M_{i-1}/M_i both non-noetherian, for $i \geq 1$. Then K-dim $(M_{i-1}/M_i) \not\geq \alpha$, since M_{i-1}/M_i is not noetherian and this contradicts the fact that K-dim $M = \alpha$.

(ii) ⇒ (iii). Immediate by definition.

(iii) ⇒ (ii). Let M be a non-noetherian R-module. We have to show that M is tall. We consider two cases.

Case (1). $M/G(M)$ noetherian. Then $G(M) \neq 0$. Hence $H(G(M)) \neq G(M)$ and by Lemma 2.5 M is tall.

Case (2). $M/G(M)$ non-noetherian. Let us write A for $M/G(M)$. Then $G(A) = 0$ and if $I \subseteq A$ is a submodule with $G(A/I) \neq 0$ then $H(G(A/I)) \neq G(A/I)$ (by hypothesis). By 2.6, A is tall and hence M is tall. This proves (iii) ⇒ (ii).

The abelian group $Z_p\infty$ is not noetherian, but has no proper noetherian submodules. Thus Z is not a tall ring. We will show that the construction of $Z_p\infty$ can be imitated over a polynomial ring and hence show that no polynomial ring is a tall ring.

PROPOSITION 2.8. *For any ring R the polynomial ring $R[(X_\alpha)_{\alpha \in J}]$ is any set of indeterminates $(X_\alpha)_{\alpha \in J}$ ($J \neq \emptyset$) is not a tall ring.*

Proof. It suffices to show that the polynomial ring $R[X]$ in one indeterminate X is not a tall ring. Let S be a simple R-module. Let $u = \eta(1)$ where $\eta\colon R \to S$ is the canonical quotient map. Let $S_i = S$ for all integers $i \geq 1$ and $T = \bigoplus_{i \geq 1} S_i$. Let u_i be the element of T whose ith coordinate is u and all other coordinates zero. Define $Xu_1 = 0$, $Xu_i = u_{i-1}$ for $i \geq 2$ and $X\lambda u_i = \lambda X u_i$ for any $\lambda \in R$. Since the u_i generate T, we get a well defined action of $R[X]$ on T. It is straight forward to see that T is not noetherian as an $R[X]$ module and that every proper submodule of T is noetherian as an $R[X]$ module. Hence $R[X]$ is not a tall ring.

Let R be a subring of S. We determine some sufficient conditions for the implication R is a tall ring $\Rightarrow S$ is a tall ring to hold. In the course of the proof we need the following lemma which is easy to verify.

LEMMA 2.9. *Let M be an R-module A, B, C, D submodules of M with $A \supset B$, $C \supset D$. Then there exists an epimorphism*

$$\frac{A + C}{B + D} \to \frac{A}{B + AC} \oplus \frac{C}{D + AC}.$$

We now fix the following notation: R is a subring of S containing the identity of S. $G = \{1 = g_0, g_1, \ldots, g_n\}$ is a finite subset of the centralizer of R in S, and as an R module, $S = \langle G \rangle$. If M is any S-module and A an R-submodule of M, then we define R-submodules A_i $(0 \leq i \leq n)$ of M by $A_i = \sum_{0 \leq j \leq i} g_i A$. Clearly $A = A_0 \subset A_1 \subset \cdots \subset A_n$, and A_n is an S-submodule of M. Setting $g_i^{-1}(A) = \{m \in M \mid g_i m \in A\}$, it is clear that $g_i^{-1}A$ is an R-submodule of M.

PROPOSITION 2.10. *Let R, S, G be as above, with R a tall ring and M an S-module. If M is non-noetherian when considered as an R-module in the natural way, then there exists an S-submodule N of M with both N and M/N non-noetherian as R-modules.*

Proof. We will construct R-submodules $A^{(0)}, A^{(1)}, \ldots, A^{(n)}$ of M such that $A_i^{(i)}$ and $M/A_i^{(i)}$ are both non-noetherian as R-modules. Since M is non-noetherian and R is a tall ring, there exists an R-submodules $A^{(0)} = A_0^{(0)}$ of M with $A^{(0)}$ and $M/A^{(0)}$ non-noetherian. Suppose $A^{(0)}, \ldots, A^{(r)}$ have been constructed with the required property. If $(A_r^{(r)} + g_{r+1}A_r)/A_r^{(r)}$ is noetherian, set $A^{(r+1)} = A^{(r)}$. It is easy to see that $A_{r+1}^{(r+1)}$ and $M/A_{r+1}^{(r+1)}$ are non-noetherian. Now suppose

$$\frac{A_r^{(r)} + g_{r+1}A^{(r)}}{A_r^{(r)}} \simeq \frac{g_{r+1}A^{(r)}}{A_r^{(r)} \cap g_{r+1}A^{(r)}}$$

is non-noetherian. Then the proof splits into two cases.

Case (1). $B = A_r^{(r)} \cap g_{r+1}A^r$ is not noetherian. Then there exists a non-noetherian submodule C of B with B/C also non-noetherian. We set $A^{(r+1)} = g_{r+1}^{-1}(C)$. Then $A^{(r+1)} \subset A^{(r)}$, and $A^{(r+1)}$ is non-noetherian as an R-module.

Case (2). $B = A_r^{(r)} \cap g_{r+1}A^r$ is noetherian. In this case, since $g_{r+1}A_r$ is not noetherian and R is a tall ring we can get a non-noetherian submodule C of $g_{r+1}A^{(r)}$ with $C \supset B$ and $g_{r+1}A^{(r)}/C$ non-noetherian. Again we set $A^{(r+1)} =$

$g_{r+1}^{-1}(C)$, and $A^{(r+1)}$ is non-noetherian. By lemma 2.9 there exists an epimorphism

$$\frac{A_r^{(r)} + g_{r+1}A^{(r)}}{A_r^{(r+1)} + g_{r+1}A^{(r+1)}} \to \frac{A_r^{(r)}}{A_r^{(r+1)} + B} \oplus \frac{g_{r+1}A^{(r)}}{g_{r+1}A^{(r+1)} + B}$$

and in both cases the second component of the direct sum is non-noetherian.

THEOREM 2.11. *Let R be a subring of S and let S be generated as an R-module by a finite subset of the centralizer of R in S. Then if R is a tall ring so too is S.*

Proof. Let M be a non-noetherian S-module. M is a non-noetherian R-module and by Proposition 2.10 has an S-submodule N such that N and M/N are non-noetherian as R-modules. In particular $M \neq N$. Again by applying Proposition 2.10 to N, we get an S-submodule N_1 such that N_1 and N/N_1 are non-noetherian and hence $N_1 \subset_{\neq} N$. A simple induction allows us to construct a chain $N_1 \subset_{\neq} N_2 \subset_{\neq} \cdots \subseteq N$ which allows us to conclude that N is not noetherian. Hence by 2.7 S is a tall ring.

We conclude with some examples of tall rings.

(1) Every V-ring and every perfect ring R is a tall ring. In both these cases, given a module $M \neq 0$, the Jacobson radical $J(M) \neq M$. Since $H(M) \subset G(M) \subset J(M)$ it follows from 2.7 that R is a tall ring.

(2) Let A be an infinite direct product of copies of Z_2. Then A is a V-ring, but A is not perfect ($J(A) = 0$). Let G be a finite group of order 2. Then $A(G)$ is a tall ring from example (1) and Theorem 2.7, but $A[G]$ is neither perfect (epimorphic images of perfect rings are perfect) nor a V-ring from [2; Corollary 6.7].

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