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KRULL DIMENSION IN POWER SERIES RINGS

BY

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ABSTRACT. Let R denote a commutative ring with identity. If there exists a chain $P_0 \subset P_1 \subset \cdots \subset P_n$ of $n+1$ prime ideals of R, where $P_n \neq R$, but no such chain of $n+2$ prime ideals, then we say that R has dimension n. The power series ring $R[[X]]$ may have infinite dimension even though R has finite dimension.

1. **Introduction.** We shall write $\dim R = n$ to denote that R has dimension n. Seidenberg, in [6] and [7], has investigated the theory of dimension in rings of polynomials. In particular, he has shown in [6] that if $\dim R = n$, then $n + 1 \leq \dim R[X] \leq 2n + 1$, where X is an indeterminate over R. One might now ask whether it is also true that $n + 1 \leq \dim R[[X]] \leq 2n + 1$. It is easy to show that $n + 1 \leq \dim R[[X]]$ when $\dim R = n$. In [3] Fields has considered the theory of dimension in power series rings over valuation rings. Using results obtained by Fields, Arnold and Brewer have noted in [1] that $\dim V[[X]] \geq 4$ for any rank one nondiscrete valuation ring V. Thus, if $\dim R = n$, then $2n + 1$ is not, in general, an upper bound for $\dim R[[X]]$. In this paper we show that we may have $\dim R[[X]] = \infty$ even though R has finite dimension. Our main result is Theorem 1, which gives sufficient conditions on a ring R in order that $\dim R[[X]] = \infty$. In fact, the conditions given insure the existence of an infinite ascending chain of prime ideals in $R[[X]]$.

Throughout this paper, R denotes a commutative ring with identity, ω is the set of natural numbers, and ω_0 is the set of nonnegative integers. If $f(X) = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$, then we denote by A_f the ideal of R generated by the coefficients of $f(X)$. For an ideal A of R, we let $A[[X]] = \{f(x) = \sum_{i=0}^{\infty} a_i X^i \mid a_i \in A$ for each $i \in \omega_0\}$ and we define $AR[[X]]$ to be the ideal of $R[[X]]$ which is generated by A. Thus, $AR[[X]] = \{f(X) \mid A_f \subseteq B$ for some finitely generated ideal B of R, with $B \subseteq A\}$. We shall say that the ideal A is an ideal of *strong finite type* (or an SFT-ideal) provided there is a finitely generated ideal $B \subseteq A$ and $k \in \omega$ such that $a^k \in B$ for each $a \in A$. If each ideal of R is an SFT-ideal, then we say that R *satisfies the* SFT*-property.* Throughout, our notation and terminology are essentially that of [4].

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2. **Main Theorem.** Let R be a ring which does not satisfy the SFT-property. If M is an ideal of R which is not an SFT-ideal, then we may choose a sequence $\{a_i\}_{i=0}^{\infty}$ of elements of M so that $a_{k+1}^{k+1} \notin (a_0, \cdots, a_k)$ for each $k \in \omega_0$. Set $A_k = (a_0, \cdots, a_k)$ and let $A = \bigcup_{k=0}^{\infty} A_k$. For each $m \in \omega$, we now choose a sequence $\{a_{m,i}\}_{i=0}^{\infty}$ of elements of A as follows. For $m = 1$, we take $a_{1,i} = a_i$ for each $i \in \omega_0$. Having defined the sequence $\{a_{m,i}\}_{i=0}^{\infty}$ for $1 \leq m < n$, we define the sequence $\{a_{n,i}\}_{i=0}^{\infty}$ by taking $a_{n,i} = a_{n-1,i^2+1}$ for each $i \in \omega_0$. For each $n \in \omega$ we set $f_{(n)} = \sum_{i=0}^{\infty} a_{n,i} X^i$.

Definition 1. Suppose that $g(X) \in R[[X]]$, $g(X) = \sum b_i X^i$, and let n, m, μ, r be integers such that $m \geq n \geq 1$, and $r \geq 0$. We shall say that the tuple (g, m, μ, r) has property (n) if for $i \geq r$ there exists an integer t_i such that the following hold, where we assume that $a_{m,i} = a_{n,k_i} = a_{1,s_i}$.

(i) $b_{t_i} = a_{m,i}^{\mu} + \alpha$ for some $\alpha \in A_{s_i - 1}$.

(ii) $t_i \leq \mu k_i$.

(iii) $b_j \in A_{s_i - 1}$ for $0 \leq j < t_i$.

For $n \in \omega$, we set $S_n = \{g(X) \in R[[X]] \mid (g, m, \mu, r)$ has property (n) for some m, $\mu \in \omega$ and $r \in \omega_0\}$. S_n is nonempty since $(f_{(n)}, n, 1, 0)$ satisfies property (n).

Lemma 1. *If n, $n_1 \in \omega$ are such that $n \geq n_1$, then $S_n \subseteq S_{n_1}$.*

Proof. Suppose that $g(X) \in S_n$ and that (g, m, μ, r) has property (n). We wish to see that (g, m, μ, r) also has property (n_1). But properties (i) and (iii) of Definition 1 already hold since they are independent of the choice of n. To see that (ii) holds, suppose that $i \geq r$ and that $a_{m,i} = a_{n,k_i} = a_{n_1,v_i}$. Then $k_i \leq v_i$, and hence $t_i \leq \mu k_i \leq \mu v_i$. It follows that $g(X) \in S_{n_1}$.

Lemma 2. *For each $n \in \omega$, S_n is a multiplicatively closed subset of $R[[X]]$.*

Proof. Let $g(X) \in R[[X]]$, $g(X) = \sum_{i=0}^{\infty} b_i X^i$. We first show that if (g, m, μ, r) has property (n) and if $m_1 \geq m$, then (g, m_1, μ, r) also has property (n). Thus, suppose that $i \geq r$ and that $a_{m_1,i} = a_{m,j_i} = a_{n,k_i} = a_{1,s_i}$. Since $j_i \geq i \geq r$, there exists an integer t_{j_i} such that:

(i) $b_{t_{j_i}} = a_{m,j_i}^{\mu} + \alpha$ for some $\alpha \in A_{s_i - 1}$.

(ii) $t_{j_i} \leq \mu k_i$.

(iii) $b_\lambda \in A_{s_i - 1}$ for $0 \leq \lambda < t_{j_i}$.

Taking $r_i = t_{j_i}$ and using the fact that $a_{m_1,i} = a_{m,j_i}$, we see that r_i satisfies properties (i), (ii) and (iii) of Definition 1 so (g, m_1, μ, r) has property (n).

Now let $g(X)$, $h(X) \in S_n$, where $g(X) = \sum_{i=0}^{\infty} b_i X^i$ and $h(X) = \sum_{i=0}^{\infty} c_i X^i$, and suppose that (g, m_1, μ_1, r_1) and (h, m_2, μ_2, r_2) satisfy property (n). By the preceding remarks, we may assume that $m_1 = m_2$ and, clearly, we may assume that $r_1 = r_2$. Set $m = m_1 = m_2$ and $r = r_1 = r_2$. We wish to show that $(gh, m, \mu_1 + \mu_2, r)$

has property (n). Suppose that $i \geq r$ and that $a_{m,i} = a_{n,k_i} = a_{1,s_i}$. By assumption there exist integers t_i and r_i such that $b_{t_i} = a_{m,i}^{\mu_1} + \alpha$ and $c_{r_i} = a_{m,i}^{\mu_2} + \beta$ for some $\alpha, \beta \in A_{s_i - 1}$. Moreover, b_λ, $c_\delta \in A_{s_i-1}$ for $0 \leq \lambda < t_i$ and $0 \leq \delta < r_i$. If $g(X)h(X) = \sum_{j=0}^{\infty} \xi_j X^j$, then

$$\xi_{t_i + r_i} = b_{t_i} c_{r_i} + \sum_{\lambda+\delta = t_i + r_i;\, \lambda \neq t_i;\, \delta \neq r_i} b_\lambda c_\delta .$$

But if $\lambda \neq t_i$ and $\delta \neq r_i$, then either $\lambda < t_i$ or $\delta < r_i$. Consequently, either $b_\lambda \in A_{s_i-1}$ or $c_\delta \in A_{s_i-1}$. Since $b_{t_i} c_{r_i} = a_{m,i}^{\mu_1+\mu_2} + \alpha a_{m,i}^{\mu_2} + \beta a_{m,i}^{\mu_1} + \alpha\beta$, it follows that $\xi_{t_i+r_i} = a_{m,i}^{\mu_1+\mu_2} + \gamma$ for some $\gamma \in A_{s_i-1}$. By assumption, we have $t_i \leq \mu_1 k_i$ and $r_i \leq \mu_2 k_i$. Therefore, $t_i + r_i \leq (\mu_1 + \mu_2)k_i$. Finally, if $0 \leq \lambda < t_i + r_i$, then $\xi_\lambda = \sum_{j=0}^{\lambda} b_j c_{\lambda - j} \in A_{s_i-1}$ since either $j < t_i$ or $\lambda - j < r_i$.

Lemma 3. *Let $n, \nu \in \omega$ be such that $n > \nu$. If $g(X) \in S_n$, then $g(X) + h(X) f_{(\nu)}(X) \in S_n$ for arbitrary $h(X) \in R[[X]]$.*

Proof. Suppose that $g(X) = \sum_{i=0}^{\infty} b_i X^i$ and that (g, m, μ, r) has property (n). Let $\eta = \min\{i \in \omega_0 \mid a_{m,i} = a_{n,k_i} \Rightarrow k_i \geq \mu\}$ and set $r_1 = \max\{r, \eta\}$. If $q(X) = g(X) + h(X) f_{(\nu)}(X) = \sum_{i=0}^{\infty} \xi_i X^i$, then we wish to show that (q, m, μ, r_1) satisfies property (n). Thus, suppose that $i \geq r_1$ and that $a_{m,i} = a_{n,k_i} = a_{\nu,\lambda_i} = a_{1,s_i}$. By assumption, there exists an integer t_i such that $b_{t_i} = a_{m,i}^{\mu} + \alpha$ for some $\alpha \in A_{s_i-1}$ and such that $t_i \leq \mu k_i \leq k_i^2$. Since $\lambda_i \geq k_i^2 + 1$, it follows that $a_{\nu,j} \in A_{s_i-1}$ for $0 \leq j \leq t_i$. Consequently, if $h(X) = \sum_{j=0}^{\infty} c_j X^j$ and $h(X) f_{(\nu)}(X) = \sum_{i=0}^{\infty} \gamma_j X^j$, then $\gamma_{t_i} = \sum_{j=0}^{t_i} a_{\nu,j} c_{t_i - j} \in A_{s_i-1}$. Therefore, $\xi_{t_i} = b_{t_i} + \gamma_{t_i} = a_{m,i}^{\mu} + \alpha + \gamma_{t_i}$ and (i) of Definition 1 is satisfied. We already have that $t_i \leq \mu k_i$, so (ii) is also satisfied. To see that (iii) holds, suppose that $0 \leq \delta < t_i$. By assumption, we have that $b_\delta \in A_{s_i-1}$. Also, $\gamma_\delta = \sum_{j=0}^{\delta} a_{\nu,j} c_{t_i-j} \in A_{s_i-1}$, since $j \leq \delta < t_i \leq k_i^2$ implies that $a_{\nu,j} \in A_{s_i-1}$. Consequently, $\xi_\delta = b_\delta + \gamma_\delta \in A_{s_i-1}$ and our proof is complete.

We now state our main result.

Theorem 1. *Let R be a commutative ring with identity. The following conditions are equivalent and imply that $R[[X]]$ has infinite dimension.*

(1) *R does not satisfy the SFT-property.*

(2) *There exists an ideal A of R such that $A[[X]] \not\subseteq \sqrt{AR[[X]]}$.*

(3) *There exists a prime ideal P of R such that $P[[X]] \neq \sqrt{PR[[X]]}$.*

Proof. Assume that (1) holds. We shall first prove that $\dim R[[X]] = \infty$. Let the ideal A be as previously defined. We wish to see that $AR[[X]] \cap S_1 = \emptyset$. Thus, let $g(X) \in AR[[X]] \cap S_1$, $g(X) = \sum_{i=0}^{\infty} b_i X^i$. Then $A_g \subseteq C$ for some finitely generated ideal C of R, where $C \subseteq A$. Consequently, there exists $k \in \omega_0$ such that $A_g \subseteq A_k$. Suppose that (g, m, μ, r) has property (1) and that r has been chosen so that if $i \geq r$ and $a_{m,i} = a_{1,s_i}$, then $s_i > \max\{\mu, k\}$. If t_i is such that $b_{t_i} = a_{m,i}^{\mu} + \alpha$ for some $\alpha \in A_{s_i-1}$, then we have $a_{m,i}^{\mu} + \alpha \in A_k \subseteq A_{s_i-1}$.

Therefore, $a_{m,i}^{\mu} \in A_{s_i-1}$, a contradiction since $a_{m,i}^{s_i} = a_{i,s_i}^{s_i} \notin A_{s_i-1}$ and $s_i > \mu$. (Since $f_{(1)} \in S_1$, it follows that $f_{(1)} \in A[[X]] - \sqrt{AR[[X]]}$. Thus we see that (1) implies (2).) But $S_1 \cap AR[[X]] = \emptyset$ implies the existence of a prime ideal P_1 of $R[[X]]$ such that $AR[[X]] \subseteq P_1$ and $P_1 \cap S_1 = \emptyset$. Suppose there exists a chain $P_1 \subset \cdots \subset P_n$ of prime ideals of $R[[X]]$ such that $P_n \cap S_n = \emptyset$, and let $C_n = P_n + (f_{(n)}(X))$. If $g(X) \in S_{n+1}$, then by Lemma 3, $g(X) + h(X)f_{(n)}(X) \in S_{n+1} \subseteq S_n$ for arbitrary $h(X) \in R[[X]]$. It follows that $g(X) + h(X)f_{(n)}(X) \notin P_n$ and hence that $g(X) \notin C_n$. Thus, $C_n \cap S_{n+1} = \emptyset$ and there exists a prime ideal P_{n+1} such that $P_n \subset C_n \subseteq P_{n+1}$ and $P_{n+1} \cap S_{n+1} = \emptyset$. We see by induction that $\dim R[[X]] = \infty$.

To see that (2) implies (3), we note that if $A[[X]] \not\subseteq \sqrt{AR[[X]]}$, then there exists a prime ideal Q of $R[[X]]$ such that $AR[[X]] \subseteq Q$ but $A[[X]] \not\subseteq Q$. If $P = Q \cap R$, then $P \supseteq A$ and hence $P[[X]] \supseteq A[[X]]$. Therefore, $Q \supseteq PR[[X]]$ but $Q \not\supseteq P[[X]]$. It follows that $P[[X]] \neq \sqrt{PR[[X]]}$. In order to show that (3) implies (1), we require the following lemma.

Lemma 4. *Let A be an ideal of R and suppose that there exists $k \in \omega$ such that $a^k = 0$ for each $a \in A$. If $f(X) \in A[[X]]$, then $f(X)$ is nilpotent.*

Proof. We first prove the existence of an integer m such that $m\xi = 0$ for all $\xi \in A^m$. Suppose we have integers $\mu, \nu_1, \cdots, \nu_t$ such that $\mu a_1^{\nu_1} \cdots a_t^{\nu_t} = 0$ for all $a_1, \cdots, a_t \in A$ (certainly this condition is satisfied if $\mu = t = 1$ and $\nu_1 = k$) and suppose that $\nu_i \geq 2$ for some i, $1 \leq i \leq t$. For convenience, we suppose that $\nu_1 \geq 2$. Now let $b_0, b_1, \cdots, b_t \in A$. By assumption, we have that

$$0 = \mu(b_0 + b_1)^{\nu_1} b_2^{\nu_2} \cdots b_t^{\nu_t} = \mu b_0^{\nu_1 - 2}(b_0 + b_1)^{\nu_1} b_2^{\nu_2} \cdots b_t^{\nu_t} = \sum_{j=0}^{\nu_1} \xi_j,$$

where $\xi_j = \mu\binom{\nu_1}{j} b_0^{2\nu_1 - j - 2} b_1^j b_2^{\nu_2} \cdots b_t^{\nu_t}$. If $0 \leq j \leq \nu_1 - 2$, then $2\nu_1 - j - 2 \geq \nu_1$ so that $\xi_j = 0$. Also, $\xi_{\nu_1} = b_0^{\nu_1 - 2}(\mu b_1^{\nu_1} \cdots b_t^{\nu_t}) = 0$. It follows that $0 = \xi_{\nu_1 - 1} = \mu\nu_1 b_0^{\nu_1 - 1} b_1^{\nu_1 - 1} b_2^{\nu_2} \cdots b_t^{\nu_t}$. By a finite number of repetitions of this procedure, we may find integers μ and t such that $\mu a_1 \cdots a_t = 0$ for all $a_1, \cdots, a_t \in A$. If we set $m = \mu t$, then $mA^m = (0)$. Now let $f(X) \in A[[X]]$, $f(X) = \sum_{i=0}^{\infty} a_i X^i$. Following a proof given by Fields [2, Theorem 1] we suppose that $m = p$ is a prime integer. Then $(f(X))^{p^k} = \sum_{i=0}^{\infty} a_i^{p^k} X^{ip^k} = 0$. If m is not prime and $m = p_1^{e_1} \cdots p_t^{e_t}$ is a prime factorization for m, then let $\phi_j : R[[X]] \to (R/p_j A^{p_j})[[X]]$ be the canonical homomorphism for $1 \leq j \leq t$. By the previous case for m a prime, we have $0 = [\phi_j(f(X))]^{p_j^k}$, that is $(f(X))^{p_j^k} \in p_j A^{p_j}[[X]]$. If $n = (p_1^{e_1 k} + \cdots + p_t^{e_t k})m$, then

$$(f(X))^n - [((f(X))^{p_1^k})^{e_1} \cdots ((f(X))^{p_t^k})^{e_t}]^m \in [(p_1 A^{p_1})^{e_1}[[X]] \cdots (p_t A^{p_t})^{e_t}[[X]]]^m \subseteq mA^m[[X]] = (0).$$

To complete the proof of Theorem 1, suppose that B is an ideal of R which is an SFT-ideal. By definition, there exists $k \in \omega$ and a finitely generated ideal $C \subseteq B$ such that $b^k \in C$ for all $b \in B$. Setting $\bar{R} = R/C$ and $\bar{B} = B/C$, it follows from Lemma 4 that $f(X)$ is nilpotent for each $f(X) \in \bar{B}[[X]]$. Therefore, if $g(X) \in B[[X]]$, then $g(X) \in \sqrt{C[[X]]} = \sqrt{CR[[X]]} \subseteq \sqrt{BR[[X]]}$. Consequently, if P is a prime ideal of R such that $P[[X]] \not\subseteq \sqrt{PR[[X]]}$, then P is not an SFT-ideal. This proves that (3) implies (1) and the theorem follows.

If $\dim R = n$, then it is natural to ask whether the conditions given in Theorem 1 are necessary in order that $\dim R[[X]] = \infty$. Another interesting question which arises is whether the following conditions are equivalent:

(1) $\dim R[[X]] \neq n + 1$.

(2) $\dim R[[X]] = \infty$.

We show that both these questions can be answered affirmatively if $\dim R = 0$.

Theorem 2. *Let R be a commutative ring with identity and suppose that $\dim R = 0$. Then the following statements are equivalent:*

(1) $\dim R[[X]] \neq 1$.

(2) $\dim R[[X]] = \infty$.

(3) *R contains a maximal ideal M such that* $M[[X]] \not\subseteq \sqrt{MR[[X]]}$.

Proof. We have already seen that (3) implies (2) and clearly, (2) implies (1). Suppose that (1) holds and let $Q_0 \subset Q_1 \subset Q_2 \subset R[[X]]$ be a chain of prime ideals of $R[[X]]$. If $M = Q_0 \cap R$, then M is a maximal ideal of R so we have $M = Q_0 \cap R = Q_1 \cap R = Q_2 \cap R$. Now $Q_0 \not\supseteq M[[X]]$ since $R[[X]]/M[[X]] \cong (R/M)[[X]]$ is a rank one discrete valuation ring. But by [1, Proposition 1], either $Q_0 \subset M[[X]]$ or $Q_0 \supseteq M[[X]]$. Therefore, $MR[[X]] \subseteq Q_0 \subset M[[X]]$ and $M[[X]] \not\subseteq \sqrt{MR[[X]]}$.

3. **Examples.** We conclude by providing three examples of finite dimensional rings R such that $\dim R[[X]] = \infty$.

Example 1. If V is a rank one nondiscrete valuation ring, then $\dim V[[X]] = \infty$. More generally, if V is a valuation ring which contains an idempotent prime ideal P, then P is not an SFT-ideal so $\dim V[[X]] = \infty$.

Example 2. An integral domain D is said to be *almost Dedekind* provided D_M is a Noetherian valuation ring for each maximal ideal M of D. Let D be any almost Dedekind domain which is not Dedekind [4, p. 586], and let M be a maximal ideal of D which is not finitely generated. It follows from Theorem 29.4 of [4, p. 411] that M is not the radical of a finitely generated ideal. Thus, M is not an SFT-ideal and $\dim D[[X]] = \infty$. More generally, *if R is a commutative ring with identity which does not have Noetherian prime spectrum, then* $\dim R[[X]] = \infty$. This is an immediate consequence of Corollary 2.4 of [5] which states that a ring R has Noetherian prime spectrum if and only if each prime ideal of R is the radical of a finitely generated ideal. Example 1 and the following example illustrate that we may have $\dim R[[X]] = \infty$ even though R has Noetherian prime spectrum.

Example 3. Let $\{Y_i\}_{i=0}^{\infty}$ be a collection of indeterminates over Q, the field of rationals, and set $R = Q[Y_0, Y_1, \cdots]/(Y_0^n, Y_1^n, \cdots)$, where n is a positive integer and $n \geq 2$. We note that $\dim R = 0$ and that $M = (\bar{Y}_0, \bar{Y}_1, \cdots)$ is the unique proper prime ideal of R. If $f(X) = \sum_{i=0}^{\infty} \bar{Y}_i X^i$, then Fields proves in [2] that $f(X)$ is not nilpotent. If $g(X) \in MR[[X]]$, then $g(X) = \sum_{i=0}^{t} \bar{Y}_i h_i(X)$ for some $t \in \omega$ and $h_i(X) \in R[[X]]$. Since $\bar{Y}_i^n = 0$ for $0 \leq i \leq t$, it follows that $g(X)$ is nilpotent. Consequently, $f(X) \notin \sqrt{MR[[X]]}$ so, by Theorem 1, $\dim R[[X]] = \infty$.

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