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Krull-Schmidt reduction for principal bundles

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Abstract. We prove a principal bundle analog of a theorem on vector bundles that says that a vector bundle on a projective variety is isomorphic to a unique direct sum of indecomposable vector bundles (unique up to a permutation of the direct summands).

1. Introduction

An algebraic vector bundle V is called indecomposable if it is not isomorphic to a direct sum of vector bundles of positive rank. A well known theorem of Atiyah states that any vector bundle V over an irreducible projective variety can be expressed as a direct sum of indecomposable vector bundles, and furthermore, in any two such expressions of V, the same indecomposable components appear the same number of times [At1].

Let G be a linear algebraic group over an algebraically closed field k of characteristic zero. In Section 2 we define a notion of L-indecomposability (here L stands for *Levi*) of a principal G-bundle over an irreducible projective variety over k. For $G = \mathrm{GL}(n,k)$, the notion of L-indecomposability of a G-bundle coincides with the notion of indecomposability of the associated rank n vector bundle.

Let G be a reductive group. By a Levi subgroup of G we will mean a reductive subgroup H of some parabolic subgroup P of G such that H projects isomorphically onto P quotiented by its unipotent radical $R_u(P)$.

We prove that any principal G-bundle E admits a reduction of structure group $E_H \subset E$ to a Levi subgroup H of G with the property that E_H is an L-indecomposable H-bundle (see Theorem 3.2). Furthermore, if E_{H_1} is another such reduction, then H is isomorphic to H_1 by an inner automorphism of G and the principal H-bundle E_H is isomorphic to E_{H_1} (with respect to such an isomorphism of H with H_1) (see Theorem 3.4). In fact, there is an automorphism F of the G-bundle E and an element $g \in G$ such that $F(E_H) = E_{H_1}g$ (see Proposition 3.3). For $G = \mathrm{GL}(n,k)$, this immediately implies the above theorem of [At1] for the vector bundle associated by the standard representation.

If $k = \mathbb{C}$ and the base is a compact connected complex manifold, then the above results remain valid (Remark 3.5).

The method of proof in [At1] is special to vector bundles. The method used here gives a new proof even in the case of vector bundles.

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2. The L-indecomposable principal bundles

Let k be an algebraically closed field of characteristic zero. Unless mentioned otherwise all the objects that we consider are over k.

Let G be a connected reductive linear algebraic group over k. A closed algebraic subgroup $P \subset G$ will be called a *parabolic subgroup* if G/P is complete. So G itself is considered as a parabolic subgroup of G.

For a parabolic subgroup P, its unipotent radical will be denoted by $R_u(P)$. So, $P/R_u(P)$ is reductive. There is a reductive subgroup $L(P)$ of P that projects isomorphically to $P/R_u(P)$. In fact, if we fix a maximal torus T of G contained in P, then $L(P)$ can be taken to be the maximal reductive subgroup of P invariant under the adjoint action of T on P. Any two reductive algebraic subgroups of P that project isomorphically to $P/R_u(P)$ are conjugate in P. If $P = G$, then $L(P) = G$.

By a *Levi subgroup* of G we will mean a reductive algebraic subgroup of some parabolic subgroup P of G that projects isomorphically to $P/R_u(P)$.

Let M be an irreducible projective variety over k. Let H be a linear algebraic group over k. A *principal H-bundle* over M is an algebraic variety E equipped with a surjective affine morphism $p : E \to M$ satisfying the following conditions:

(1) the morphism p is flat;

(2) the variety E is equipped with a right action $\psi : E \times H \to E$ of H such that $p \circ \psi = p \circ \pi_1$, where π_1 denotes the natural projection of $E \times H$ to E;

(3) the map

$$(\pi_1, \psi) : E \times H \to E \times_M E$$

to the fiber product is an isomorphism.

Given an H-bundle E and an algebraic subgroup $H_0 \subset H$, a *reduction of structure group* of E to H_0 is a section of the projection $E/H_0 \to M$.

Given such a section

$$\sigma : M \to E/H_0,$$

consider the inverse image $q^{-1}\big(\sigma(M)\big) \subset E$, where $q : E \to E/H_0$ is the quotient map. It is easy to see that $q^{-1}(\sigma(M))$ is preserved by the action of H_0 on E, and furthermore, $q^{-1}(\sigma(M))$ is a principal H_0-bundle over M.

For a G-bundle E over M, where G is a connected linear algebraic group over k, consider the quotient

$$\mathrm{Ad}(E) := \frac{E \times G}{G}$$

for the diagonal action of G, with G acting on itself (from the right) by inner conjugation. The conjugation action (from the right) of any $g_0 \in G$ on G is defined by $g \mapsto g_0^{-1} g g_0$.

Note that $\mathrm{Ad}(E)$ is a group scheme over M with fibers isomorphic to G. More precisely, the fibers of $\mathrm{Ad}(E)$ are identified with G up to an inner conjugation. Let $\mathrm{Aut}(E)$ denote the space of all global sections of the adjoint bundle $\mathrm{Ad}(E)$. Therefore, $\mathrm{Aut}(E)$ gets the structure of a group. Since M is a complete variety, $\mathrm{Aut}(E)$ is an algebraic group over k. In fact, $\mathrm{Aut}(E)$ is the group of all G bundle automorphisms of E over M [Gr], p. 82.

Any $\tau \in \mathrm{Aut}(E)$ gives an automorphism of E that commutes with the action of G. Indeed, for any point $x \in M$, if $(z, g) \in p^{-1}(x) \times G \subset E \times G$ represents $\tau(x)$, then the automorphism of E defined by τ sends z to zg.

Let

$$\mathrm{Aut}^0(E) \subset \mathrm{Aut}(E) \tag{2.1}$$

denote the connected component of $\mathrm{Aut}(E)$ containing the identity element. Let

$$T(E) \subset \mathrm{Aut}^0(E) \tag{2.2}$$

be a maximal torus of $\mathrm{Aut}^0(E)$. Any two maximal tori of $\mathrm{Aut}^0(E)$ are conjugate in $\mathrm{Aut}^0(E)$ [Bo], p. 148, Corollary 11.3(1). In particular, they are isomorphic.

Let $Z(G)$ denote the center of G. The connected component of $Z(G)$ containing the identity element will be denoted by $Z_0(G)$.

Definition 2.1. A G-bundle E over M is called L-*indecomposable* if a maximal torus $T(E)$ of $\mathrm{Aut}^0(E)$ is isomorphic to $Z_0(G)$, the connected component of the center of G. If a G-bundle E over M is not L-indecomposable then we say that E is L-*decomposable*.

Remark 2.2. Note that the center $Z(G)$ is contained in $\mathrm{Aut}(E)$. Indeed, as action of any $g \in Z(G)$ on E commutes with the action of G, it defines an automorphism of E. Therefore, $Z_0(G)$ is contained in $\mathrm{Aut}^0(E)$. It is easy to see that $Z_0(G)$ is contained in the center of $\mathrm{Aut}^0(E)$, and hence it is contained in any maximal torus of $\mathrm{Aut}^0(E)$. Therefore, if $Z_0(G)$ is isomorphic to a maximal torus $T(E)$ of $\mathrm{Aut}^0(E)$, then it is a maximal torus of $\mathrm{Aut}^0(E)$ by the inclusion map.

Remark 2.3. If G is a torus then the above definition implies that any G-bundle is L-*indecomposable*.

Proposition 2.4. *A principal G-bundle E over M is* L*-indecomposable if and only if the quotient* $\mathrm{Aut}^0(E)/Z_0(G)$ *is a unipotent group.*

Let G be a connected reductive linear algebraic group. A principal G-bundle E over M is L*-indecomposable if and only if E does not admit a reduction of structure group to a proper Levi subgroup of G.*

Proof. If a maximal torus of $\mathrm{Aut}^0(E)$ is isomorphic to $Z_0(G)$, as $Z_0(G)$ is in the center of $\mathrm{Aut}^0(E)$, the maximal torus of the quotient $\mathrm{Aut}^0(E)/Z_0(G)$ is the trivial group. In other words, $\mathrm{Aut}^0(E)/Z_0(G)$ is a unipotent group. In the converse direction, since $Z_0(G)$ is contained in any maximal torus of $\mathrm{Aut}^0(E)$, if $\mathrm{Aut}^0(E)/Z_0(G)$ is unipotent, its maximal torus being trivial, $Z_0(G)$ coincides with a maximal torus of $\mathrm{Aut}^0(E)$. This proves the first part of the proposition.

To prove the second part of the proposition assume that E admits a reduction of structure group to a proper Levi subgroup $H \subsetneq G$. Fix such a reduction $E_H \subset E$. Note that $\mathrm{Aut}^0(E_H)$ is a subgroup of $\mathrm{Aut}^0(E)$. Indeed, since E_H is a reduction of structure group of E, we have $E \cong E_H \times^H G$. Therefore, an automorphism of E_H extends to an automorphism of E by using the identity automorphism of G.

If H projects isomorphically onto a proper Levi factor of a parabolic subgroup of G, then $Z_0(H)$, namely the connected component of the center of H containing the identity element, is strictly larger than $Z_0(G)$. Since $Z_0(H) \subset \mathrm{Aut}^0(E_H) \subset \mathrm{Aut}^0(E)$, this implies that E is L-decomposable.

In the converse direction, assume that E is L-decomposable. This implies that G is nonabelian.

Take any automorphism $\psi \in T(E)$, where $T(E)$ as in (2.2) is a maximal torus of $\mathrm{Aut}^0(E)$. For any point $x \in M$, the evaluation map

$$e_x : \mathrm{Aut}^0(E) \to \mathrm{Ad}(E)_x \cong G$$

is a morphism of algebraic groups. Consequently, $e_x(\psi)$ is a semisimple element of G (see [St], p. 32, Proposition 3).

Now, the space of all conjugacy classes of semisimple elements in G is parametrized by the quotient $T/W(T)$, where $T \subset G$ is a maximal torus and $W(T)$ is the corresponding Weyl group. Note that $T/W(T)$ is an affine variety. Since the variety M is complete and irreducible, the conjugacy class of $e_x(\psi)$ is independent of x. Note that the conjugacy classes of $\mathrm{Ad}(E)_x$ are naturally in a bijective correspondence with the conjugacy classes of G.

Consider the collection of all elements g in a *given* maximal torus of G with the property that the centralizer of g is *not* a proper Levi subgroup of G. This collection, which we will denote by $\mathscr{B}$, is closed under multiplication by $Z(G)$. It is known that the quotient $\mathscr{B}/Z(G)$ is a finite set (see [DM], p. 113).

Observe that the image $e_x\big(T(E)\big)$ contains the center $Z(G)$ *properly* provided we have

$\dim T(E) > \dim Z(G)$. This is because if the image $e_x(T(E)) = Z(G)$ (using $\mathrm{Ad}(E)_x \cong G$), then from the fact that the conjugacy class of the image is independent of the point x it follows that we have $T(E) = Z(G)$ as a subgroup of $\mathrm{Aut}^0(E)$.

In view of the above observations and the given condition $\dim T(E) > \dim Z(G)$ (as E is L-decomposable), we can choose $\psi \in T(E)$ in such a way that the centralizer of $e_x(\psi)$ is a *proper Levi subgroup* of $\mathrm{Ad}(E)_x \cong G$. (This fact can also be deduced from the proof of [St], p. 98, Proposition 4.) So we will assume that the centralizer of $e_x(\psi)$ is a proper Levi subgroup of G. From the earlier remark on the independence of the conjugacy class of $e_x(\psi)$ on the choice of x it follows that this condition does not depend on the choice of x.

Fix an element $g_0 \in G$ in the conjugacy class in G defined by $e_x(\psi)$. Let $H_{g_0} \subset G$ be the centralizer of g_0 in G, which is a proper Levi subgroup.

Let

$$\psi : E \to E \tag{2.3}$$

be the automorphism in $T(E)$ chosen above. So for any point $z \in E$, we have $\psi(z) = zg(z)$, where ψ as in (2.3) and

$$z \mapsto g(z) \in G$$

is a morphism of varieties from E to G.

Let

$$\mathcal{S} \subset E \tag{2.4}$$

be the subvariety defined by all $z \in E$ with $g(z) = g_0$, where g_0 is the fixed element.

It is easy to see that for any point $x \in M$, the intersection $\mathcal{S} \cap E_x$ is nonempty and it is closed under the action of the centralizer H_{g_0} (here E_x denotes the fiber of E over x). Indeed, this is an immediate consequence of the identity

$$g(z\gamma) = \gamma^{-1} g(z)\gamma,$$

where $z \in E$ and $\gamma \in G$. Note that for each $z \in E$, the element $g(z)$ is in the conjugacy class (in G) defined by $e_x(\psi)$ (which is independent of x).

From this identity it also follows that H_{g_0} acts transitively on the fibers of the projection of $\mathcal{S}$ to M. In other words, the subvariety $\mathcal{S}$ defines a reduction of structure group of the principal G-bundle E to the proper Levi subgroup H_{g_0}. This completes the proof of the proposition. $\square$

Remark 2.5. The above proposition justifies the "L" in L-indecomposability. An L-indecomposable G-bundle may admit a reduction of structure group to a proper reductive subgroup H of G with a maximal torus of H being a proper subgroup of a maximal torus of G. This happens for example when G is a torus and the G-bundle is trivial.

In the next section we will construct a reduction of the structure group of an L-decomposable G-bundle to a proper Levi subgroup H such that the principal H-bundle on M so obtained is L-indecomposable.

3. The Remak reduction

In this section G is a connected reductive linear algebraic group and E is a principal G-bundle over M.

As in (2.2), let $T(E)$ be a maximal torus in $\mathrm{Aut}^0(E)$. The group $T(E)$ gives a subgroup of G up to conjugation. To explain this, note that fixing a point $z \in E_x$, the group $\mathrm{Ad}(E)_x$ gets identified with G. Using this identification, the evaluation map $e_x : \mathrm{Aut}^0(E) \to \mathrm{Ad}(E)_x$ (that sends a section of $\mathrm{Ad}(E)$ to its evaluation at x) sends the subgroup $T(E) \subset \mathrm{Aut}^0(E)$ *isomorphically* to a subgroup of G. That the evaluation on $T(E)$ is an isomorphism follows from the fact that the conjugacy class of $e_x(s)$, $s \in T(E)$, is independent of x—see proof of Proposition 2.4. Indeed, if the evaluation of $\psi \in T(E)$ at some point x is the identity map of the fiber E_x, then $\psi(y)$ is the identity map of E_y for all y.

Let $T_E \subset G$ be a subgroup in the conjugacy class defined by $T(E)$. So T_E is contained in a maximal torus of G. Let H denote the centralizer of T_E in G. Note that H is a Levi subgroup of G. Clearly, H coincides with G if and only if T_E coincides with $Z_0(G)$.

Definition 3.1. With above notation the Levi subgroup H (which is determined up to inner conjugation) will be called the *Remak subgroup* for E.

It follows immediately from the above definition that the Remak subgroup for E coincides with G if and only if the principal G-bundle E is L-indecomposable.

Theorem 3.2. *The principal G-bundle E admits a reduction of structure group to its Remak subgroup H. More precisely, once a maximal torus $T(E)$ in* $\mathrm{Aut}^0(E)$ *is fixed, E admits a natural reduction to H. Furthermore, a principal H-bundle constructed this way from E is* L-*indecomposable.*

Proof. Choose and fix a point $z_0 \in E_x$. Using z_0, an isomorphism of $\mathrm{Ad}(E)_x$ with G is obtained. Now, take T_E to be the image of $T(E)$ in G by this isomorphism composed with the evaluation at x. Let

$$f : T(E) \to T_E \tag{3.1}$$

be the resulting isomorphism (which depends on the initial choice of z_0).

Now, as in the proof of Proposition 2.4, set

$$\mathscr{S} \subset E$$

to be the subset defined by the property that for any $z \in \mathscr{S}$, the equality

$$\sigma(z) = zf(\sigma)$$

is valid for every $\sigma \in T(E)$. Here f is the map defined in (3.1); the map $\sigma : E \to E$ is the automorphism defined by $\sigma \in T(E)$.

It is straightforward to check that $\mathscr{S}$ is closed under the action of the centralizer of T_E in G. In fact, $\mathscr{S}$ defines a reduction of structure group of E to H.

Let E_H denote the principal H-bundle defined by the subvariety $\mathscr{S}$. Replace the fixed point z_0 by $z_0 g$, where $g \in G$, but keep the maximal torus $T(E)$ fixed. If

$$\beta : \mathrm{Ad}(E)_x \to G$$

was the previous isomorphism for z_0, then the new isomorphism for $z_0 g$ coincides with $\mathrm{Ad}(g^{-1}) \circ \beta$, where $\mathrm{Ad}(g^{-1})$ is the automorphism of G defined by

$$\mathrm{Ad}(g^{-1})(h) = g^{-1} h g.$$

Therefore, T_E is replaced by

$$T_E' := \mathrm{Ad}(g^{-1})(T_E)$$

and H is replaced by $H' := \mathrm{Ad}(g^{-1})(H)$. This implies that $\mathscr{S}$ is replaced by $\mathscr{S}' := \mathscr{S} g$.

If $F_Q \subset F$ is a reduction of structure group of a G-bundle F to a subgroup $Q \subset G$, then for any $g \in G$, the translation $F_Q g$ (of F_Q by g) is a reduction of the structure group of F to the subgroup $g^{-1} Q g$ of G. These two reductions are identified and we will not distinguish between them. Therefore, the reduction of structure group E_H of E to H defined by z_0 coincides with the reduction of structure group of E to H' defined by $z_0 g$.

To prove that the above H-bundle E_H is L-indecomposable, first recall that $\mathrm{Aut}^0(E_H)$ is a subgroup of $\mathrm{Aut}^0(E)$. If E_H is L-decomposable, then a maximal torus of $\mathrm{Aut}^0(E_H)$ is larger (in dimension) than the center of H. In that case, a maximal torus of $\mathrm{Aut}^0(E)$ will be larger (in dimension) than the center of H, as $\mathrm{Aut}^0(E_H)$ is a subgroup of $\mathrm{Aut}^0(E)$. On the contrary, the maximal torus $T(E) \cong T_E$ is contained in the center of H. This is a contradiction; hence the H-bundle E_H must be L-indecomposable. This completes the proof of the theorem. □

About the dependence of the reduction in Theorem 3.2 on the choice of the maximal torus $T(E)$, we recall that any other maximal torus of $\mathrm{Aut}^0(E)$ is a conjugate of $T(E)$. For any

$$\gamma \in \mathrm{Aut}^0(E),$$

consider the maximal torus

$$T'(E) := \gamma T(E) \gamma^{-1}$$

of $\mathrm{Aut}^0(E)$. For convenience replace the fixed point $z_0 \in E_x$ in the proof of Theorem 3.2 by the new fixed point $z_0' := \gamma(z_0)$.

For this new choice of maximal torus $\gamma T(E)\gamma^{-1}$ and z_0', the torus $T_E \subset G$ remains unchanged (same as for $T(E)$ and z_0), and hence the centralizer H is also unchanged. It is straightforward to check that $\mathscr{S}$ is replaced by $\mathscr{S}' := \gamma(\mathscr{S})$. In other words,

$$E_H' := \gamma(E_H) \subset E$$

is the new reduction.

Therefore, if we replace the maximal torus $T(E)$ by $T'(E)$, then the new reduction differs from the earlier one by the automorphism γ of E.

For the dependence of the reduction in Theorem 3.2 on the choice of the torus T_E of G, replace T_E by $T_E' := g^{-1}T_E g$, with $g \in G$. Replace the fixed point z_0 by $z_0 g$ but keep $T(E)$ fixed. Then, as we saw in the proof of Theorem 3.2, $\mathscr{S}$ gets replaced by $\mathscr{S}g$. Consequently, the new reduction is identified with the initial one (the identification is in the sense mentioned in the proof of Theorem 3.2).

Therefore, we have the following proposition:

Proposition 3.3. *Up to an automorphism of E, Theorem* 3.2 *gives a unique reduction of structure group to a Levi subgroup.*

Let G_1 (respectively, G_2) be a connected reductive linear algebraic group and E_{G_1} (respectively, E_{G_2}) a G_1-bundle (respectively, G_2-bundle) over M. Then the fiber product $E_{G_1} \times_M E_{G_2}$ is a principal $G_1 \times G_2$-bundle over M. Let

$$E_{H_1} \subset E_{G_1}$$

(respectively, $E_{H_2} \subset E_{G_2}$) be the reduction of structure group obtained in Theorem 3.2. It is easy to see that the reduction of structure group

$$E_{H_1} \times_M E_{H_2} \subset E_{G_1} \times_M E_{G_2}$$

to $H_1 \times H_2$ coincides with the one given by Theorem 3.2 for $E_{G_1} \times_M E_{G_2}$.

Now we will prove a converse to Theorem 3.2. The converse says that if $E_L \subset E$ is a reduction of structure group of the G-bundle to a Levi subgroup L of G and the principal L-bundle E_L is L-indecomposable, then the reduction E_L coincides with the reduction obtained in Theorem 3.2 for some choice of the maximal torus $T(E)$. We noted in Proposition 3.3 that choosing a different maximal torus $T(E)$ of $\mathrm{Aut}^0(E)$ corresponds to changing the reduction of structure group by applying an automorphism of the E.

Theorem 3.4. *Let L be a Levi subgroup of G and $E_L \subset E$ a reduction of structure group of a principal G-bundle to the subgroup L. If E_L is* L*-indecomposable, then there is a maximal torus $T(E)$ of* $\mathrm{Aut}^0(E)$ *such that the reduction of structure group of E corresponding to $T(E)$, obtained in Theorem* 3.2, *coincides with E_L.*

Proof. Let $Z_0(L)$ denote the connected component of the center of the Levi subgroup L containing the identity element. So $Z_0(L)$ acts as automorphisms of the principal

L-bundle E_L. Since the L-bundle E_L is L-indecomposable, the group $Z_0(L)$ coincides with a maximal torus of $\mathrm{Aut}^0(E_L)$.

Now recall that $\mathrm{Aut}^0(E_L)$ is a subgroup of $\mathrm{Aut}^0(E)$. We will show that for any automorphism

$$\sigma \in \mathrm{Aut}^0(E) \backslash \mathrm{Aut}^0(E_L) \tag{3.2}$$

in the complement, there is an element in $h \in Z_0(L) \subset \mathrm{Aut}^0(E)$ such that σ does not commute with h.

To prove this, take any point $y \in M$. If we fix a point in the fiber $(E_L)_y$ of E_L over y, then the fiber $\mathrm{Ad}(E_L)_y$ is identified with L as well as $\mathrm{Ad}(E)_y$ is identified with G (the trivialization of E_y is done using the image of y by the inclusion map of E_L in E). With these identifications, the above inclusion of $\mathrm{Ad}(E_L)_y$ in $\mathrm{Ad}(E)_y$ is simply the inclusion of L in G. The homomorphism of $Z_0(L)$ to $\mathrm{Ad}(E_L)$ coincides with the inclusion map of $Z_0(L)$ in L.

Note that for any automorphism σ as in (3.2), there exists a point $y \in M$ such that σ does not preserve the subvariety $(E_L)_y$ of E_y. On the other hand, the centralizer of $Z_0(L)$ in G coincides with L. Therefore, it follows immediately that there is an element $h \in Z_0(L)$ with the property that

$$(h\sigma)|_{E_y} \neq (\sigma h)|_{E_y}.$$

Consequently, $h\sigma \neq \sigma h$.

The above assertion that for any given element in $\mathrm{Aut}^0(E) \backslash \mathrm{Aut}^0(E_L)$ there is an element in $Z_0(L)$ not commuting with it immediately implies that $Z_0(L)$ is a maximal torus of $\mathrm{Aut}^0(E)$.

If we set $T(E)$ in Theorem 3.2 to be this maximal torus $Z_0(L)$ and take the base point z_0 in the proof of Theorem 3.2 to be in E_L, then the reduction of structure group of E to the centralizer of $Z_0(L)$ in G (which is L) constructed in Theorem 3.2 coincides with the reduction E_L. Indeed, this is an immediate consequence of the construction in Theorem 3.2. This completes the proof of the theorem. □

Note that since a Levi subgroup of L, where L is a Levi subgroup of G, is also a Levi subgroup of G, an L-bundle E_L obtained by a reduction of structure group of a principal G-bundle E is L-indecomposable if E_L does not admit a further reduction to a Levi subgroup of G contained in L.

A reductive subgroup of G is called *irreducible* if it is not contained in a proper parabolic subgroup of G. A reducible subgroup is contained in the Levi subgroup of a proper parabolic subgroup of G. Therefore, for an L-indecomposable bundle E admitting a reduction of structure group to a reductive subgroup H, the inclusion of H in G is irreducible.

Remark 3.5. If $k = \mathbb{C}$ and M is a compact connected complex manifold, then all the results obtained in Sections 2 and 3 remain valid for holomorphic principal bundles (in place of algebraic ones). Indeed, the only property of M that we have used is that the space of global functions on it are the constant ones.

Remark 3.6. Let M be a connected smooth projective curve over an algebraically closed field k of characteristic zero. An indecomposable vector bundle V over M admits a holomorphic connection if and only if $\text{degree}(V) = 0$ [At2], p. 203, Proposition 19. Now, let E be an L-indecomposable G-bundle over M, where G is a connected reductive linear algebraic group. The G-bundle E admits a holomorphic connection if and only if for any character χ of G the associated line bundle $E_\chi := (E \times \mathbb{C})/G$ (the action of G on the right of $\mathbb{C}$ is defined using χ) over M is of degree zero (see last paragraph of Section 1 (p. 335) of [AB]).

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